DE GRUYTER Adv. Nonlinear Stud. 2016; 16 (3):491–498

Research Article

José Carmona and Pedro J. Martínez-Aparicio*

A Singular Semilinear Elliptic Equation with a Variable Exponent

DOI: 10.1515/ans-2015-5039
Received July 28, 2015; revised October 26, 2015; accepted November 8, 2015

Abstract: In this paper we consider singular semilinear elliptic equations with a variable exponent whose model problem is

$$\begin{cases} -\Delta u = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is an open bounded set of $\mathbb{R}^N$, $\gamma(x)$ is a positive continuous function and $f(x)$ is a positive function that belongs to a certain Lebesgue space. We prove that there exists a solution to this problem in the natural energy space $H_0^1(\Omega)$ when $\gamma(x) \leq 1$ in a strip around the boundary. For another case, we prove that the solution belongs to $H^1_{\mathrm{loc}}(\Omega)$ and that it is zero on the boundary in a suitable sense.

Keywords: Semilinear Equations, Singularity, Existence, Positive Solutions, Variable Exponent

MSC 2010: 35A01, 35B09, 35B45, 35D30, 35J25, 35J61, 35J75

Communicated by: Jean Mawhin

1 Introduction

We are concerned with the existence of solutions for the following boundary value problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is an open and bounded subset of $\mathbb{R}^N$ ($N \geq 3$), M is a bounded elliptic matrix, i.e., there exist $0 < \alpha \leq \beta$ such that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta \tag{1.2}$$

for every $\xi \in \mathbb{R}^N$ and for almost every x in Ω, $\gamma(x) \in C^1(\overline{\Omega})$ is a positive function and $f(x)$ is a positive function that belongs to a certain Lebesgue space.

Problem (1.1) arises in certain problems in fluid mechanics and pseudoplastic flow in dimension $N = 1$ (see [13]). Regarding the literature, the problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been extensively studied in the past. In [5] Fulks and Maybee considered some singular problems including the case $g(x, s) = f(x)e^{1/s}$ or $g(x, s) = \frac{f(x)}{s^\gamma}$ for a regular function $f(x)$ and some positive constant γ, and

José Carmona: Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120 Almería, Spain, e-mail: jcarmona@ual.es
***Corresponding author: Pedro J. Martínez-Aparicio:** Departamento de Matemática Aplicada y Estadística, Campus Alfonso XIII, Universidad Politécnica de Cartagena, 30203 Murcia, Spain, e-mail: pedroj.martinez@upct.es

they proved the existence of classical solutions for $M(x)$ being the identity matrix. Similar results, with different proofs, were obtained in [4, 16] for a general regular matrix $M(x)$ and a regular function $g(x, s)$, which is uniformly bounded for $s > 1$ with $\lim_{s\to 0} g(x, s) = +\infty$ uniformly for $x \in \overline{\Omega}$. Here classical solution means a $C^2(\Omega) \cap C_0(\overline{\Omega})$ solution. Moreover, with less regularity, M uniformly elliptic and $g \in C(\overline{\Omega} \times (0, +\infty))$, the existence of $W^{2,q}_{\text{loc}}(\Omega) \cap C_0(\overline{\Omega})$ solution ($q > N$) was proved in [4]. Furthermore, in the case where $g(x, s)$ does not depend on x, some estimates for the solutions near the boundary were obtained in [4]. For example, in the particular case $g(s) = \frac{1}{s^\gamma}$ with $\gamma > 1$ it was proved that the behavior of the solution near the boundary is like $d(x)^{2/(1+\gamma)}$, where $d(x)$ denotes the distance to the boundary. Thus, in this case one can not expect $C^1(\overline{\Omega})$ solutions, but it was proved that the power $\frac{1+\gamma}{2}$ of the solution is Lipschitz continuous in $\overline{\Omega}$.

In [11] this behavior near the boundary of the solution was extended to the case $g(x, s) = \frac{f(x)}{u^\gamma}$ for a regular strictly positive function f (and improved for f that behaves as $d(x)^\delta$ near the boundary for some $\delta > -2$, in particular, f may not even be in $L^1(\Omega)$), showing that the problem can have a classical solution but not a weak solution. In fact, they proved that the solution is in $W^{1,2}(\Omega)$ if and only if $\gamma < 3$. Later on, in [9, 10] these results were generalized for $\Omega = \mathbb{R}^N$. In [17] Zhang and Cheng studied the case where $g(x, s) = f(x)g_1(s)$ with f Hölder continuous and $f(x) \sim d(x)^\delta$ near the boundary for some $\delta \in \mathbb{R}$. For $g_1(s) = \frac{1}{s^\gamma}$, they proved that there is no classical solution for $\delta \le -2$ while for $\delta > -2$ there is (which belongs to $H^1_0(\Omega)$ if and only if $\gamma - 2\delta < 3$).

Regarding existence and regularity results for the case where $g(x, s) = \frac{f(x)}{u^\gamma}$, for $f \in L^m(\Omega)$, we refer to the papers [1–3, 14]. For existence and homogenization results for this kind of problems, we refer to the papers [6–8].

In [2] Boccardo and Orsina studied problem (1.1) with $\gamma(x) = \gamma$ a positive constant and f in a certain Lebesgue space. They proved some existence and regularity results depending on γ and on the summability of f. Specifically, they took an increasing sequence u_n of solutions to nonsingular problems of the form

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n(x)}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n(x) = \min\{f(x), n\}$. For any $\omega \subset\subset \Omega$, this sequence satisfies the following property:

$$u_n(x) \ge u_{n-1}(x) \ge \cdots \ge u_1(x) \ge c_\omega > 0 \quad \text{for all } x \in \omega. \tag{1.3}$$

In order to prove it, they use, strongly, that f/s^γ is non increasing for $s > 0$ and, as a main tool, the strong maximum principle. Note that (1.3) provides the existence of the limit $u = \sup_n u_n$ (eventually it may take infinite values), which is strictly away from zero on any compact set ω of Ω. In addition, (1.3) implies that, on every such set ω, the sequence $f_n(x)/(u_n + \frac{1}{n})^\gamma$ is dominated by a function that belongs to $L^1(\omega)$. Thus, in order to prove that u is a solution in the sense of distributions, Boccardo and Orsina proved some a priori estimates for u_n. More precisely, for $f \in L^1(\Omega)$ and $\gamma = 1$, they proved an a priori estimate in $H^1_0(\Omega)$. The same was proved in the case $\gamma < 1$ but it needed more summability on f, namely $f \in L^m(\Omega)$ with $m = \frac{2N}{N+2+\gamma(N-2)}$. Finally, for $\gamma > 1$ it is not possible to obtain a priori estimates in $H^1(\Omega)$ and it was proved that for $f \in L^1(\Omega)$, a convenient power of u_n is bounded in $H^1_0(\Omega)$. In [1], under more restrictive hypothesis on f, Arcoya and Moreno-Mérida improved the meaning of the boundary condition and obtained energy solutions if $f \in L^m(\Omega)$ with $m > 1$ and $1 < \gamma < \frac{3m-1}{m+1}$. Recently, Oliva and Petitta [14] considered the same problem adding a nonnegative bounded Radon measure on the right-hand side, and they established the existence and uniqueness of the solution in a weak sense and under minimal assumptions on the data.

Giachetti, Martínez-Aparicio and Murat [6] have studied the model problem (1.1) with $\gamma(x) = \gamma \le 1$. They proved some existence, stability and homogenization results without assuming, for a more general nonlinearity $F(x, s)$, that it is nonincreasing in the s variable and without using the strong maximum principle in the proofs of their results. They studied the case $\gamma > 1$ in the papers [7] and [8], where the singularity has a stronger behavior and no global energy estimates are available for the solutions. This makes the problem harder, in particular from the point of view of homogenization. For this reason, they introduced a convenient framework where they proved existence, stability, uniqueness and homogenization results.

In the present paper we deal with a variable exponent, and we may have a region inside Ω where $\gamma(x) \le 1$ and another region where $\gamma(x) > 1$. Some existence and regularity results have been obtained in [3, 6–8] but, to our knowledge, the role played by the behavior of $\gamma(x)$ near the boundary has not been studied. We are inspired by [2] and we work with approximations, by considering an increasing sequence u_n of solutions to nonsingular problems. Our main results here imply that what matters for the a priori estimate of u_n is the behavior of $\gamma(x)$ near the boundary. It will be possible to prove a priori estimates in $H_0^1(\Omega)$ if $\gamma(x) \leq 1$ for every x in a strip around $\partial\Omega$ and inside Ω. We can see easily in the proof of Proposition 3.1 below how we use the condition on $\gamma(x)$ near the boundary to prove the existence result in this case.

In another case, we prove that $u_n^{(\gamma^*+1)/2}$ is bounded in $H_0^1(\Omega)$ for some $\gamma^* > 1$ (see Proposition 3.3), and if $\gamma(x) \leq 1$ for every x in a strip around $\Gamma \subset \partial\Omega$ and inside Ω, then the solutions belongs to

$$H_\Gamma^1(\omega) = \{u \in H^1(\omega) : u|_{\partial\omega\cap\Gamma} = 0\}$$

for every open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega \cup \Gamma$.

Our main results are the following two theorems.

Theorem 1.1. *Let $f \in L^{2N/(N+2)}(\Omega)$, $\gamma(x) < 1$ on $\partial\Omega$ or $\gamma(x) \leq 1$ on $\partial\Omega$ with $\frac{\partial\gamma(x)}{\partial n_e} \geq 0$, and assume that* (1.2) *holds. Then, there exists a solution $u \in H_0^1(\Omega)$ to problem* (1.1).

In order to show the existence of a solution, we will use the fact that $\gamma(x) \leq 1$ for every x in a strip Ω_δ around $\partial\Omega$ and inside Ω, i.e.,

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\} \quad \text{for } \delta > 0 \text{ fixed.}$$

Our hypothesis on $\gamma(x)$ in Theorem 1.1 guarantees this fact. Note that we can extend the result to functions $\gamma(x)$ such that $\gamma(x) < 1$ on $A \subset \partial\Omega$ and $\gamma(x) = 1$ on $\partial\Omega \setminus A$ with $\frac{\partial\gamma(x)}{\partial n_e} \geq 0$ there.

Theorem 1.2. *Assume that for some $\gamma^* > 1$ and some $\delta > 0$ we have that $\|\gamma\|_{L^\infty(\Omega_\delta)} \leq \gamma^*$, and that* (1.2) *holds. Assume also that $f \in L^m(\Omega)$ with $m = \frac{N(\gamma^*+1)}{N+2\gamma^*}$. Then, there exists a solution $u \in H_{\text{loc}}^1(\Omega)$ to problem* (1.1) *such that $u^{(\gamma^*+1)/2} \in H_0^1(\Omega)$. Furthermore, if there exists $\Gamma \subset \partial\Omega$ such that $\gamma(x) \leq 1$ in the set $\{x \in \Omega : \operatorname{dist}(x, \Gamma) < \nu\}$ for some $\nu > 0$, then $u \in H_\Gamma^1(\omega)$ for every open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega \cup \Gamma$.*

We observe that if $p = \|\gamma\|_{L^\infty(\overline{\Omega})}$, then

$$\frac{f(x)}{u^{\gamma(x)}} \leq f(x)\Big(\frac{1}{u^p} + 1\Big).$$

In the case where $M(x)$ is the identity matrix, [3, Theorem 2.5] dictates that if $p = 1$ and $f \in L^m(\Omega)$ with $1 \leq m$, then (1.1) admits a solution $u \in H_0^1(\Omega)$. On the other hand, if $p > 1$, then (1.1) admits a solution $u \in H_{\text{loc}}^1(\Omega)$ with $u^{(p+1)/2} \in H_0^1(\Omega)$. In both cases it is proved that $c \cdot \operatorname{dist}(x, \partial\Omega) \leq u(x)$ for almost every $x \in \Omega$ and for some positive constant c.

The plan of the paper is the following. We dedicate Section 2 to several aspects, we consider the approximated problems, we prove the existence of the approximated solutions in $H_0^1(\Omega)$ and we show (1.3) as in [2]. We prove the keystones of the existence results in Section 3, namely Proposition 3.1 and Proposition 3.3. Section 4 is devoted to passing to the limit in the approximated problem using all the properties that we have proved in the previous sections.

Notations.
- For every $s \in \mathbb{R}$, we consider the positive and negative parts given by $s^+ = \max\{s, 0\}$ and $s^- = \min\{s, 0\}$, respectively.
- For any $k > 0$, we set $T_k(s) = \min(k, \max(s, -k))$ and $G_k(s) = s - T_k(s)$.
- We define the set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for $\delta > 0$ fixed.
- We denote by $|E|$ the Lebesgue measure of a measurable set E in $\mathbb{R}^{\mathbb{N}}$.
- For $1 \leq p \leq +\infty$, $\|u\|_p$ denotes the usual norm of a function $u \in L^p(E)$.
- We equip the standard Sobolev space $H_0^1(E)$ with the usual norm $\|u\| = (\int_E |\nabla u|^2)^{1/2}$.
- For any $1 < p < N$, $p^* = \frac{Np}{N-p}$ denotes the Sobolev conjugate exponent of p.
- $\mathcal{S}$ denotes the best Sobolev constant, i.e.,

$$\mathcal{S} = \sup_{\|u\|_{H_0^1(\Omega)}=1} \|u\|_{L^{2^*}(\Omega)}.$$

- We recall that, for $1 < p < \infty$, the dual space of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugated exponent of p.

2 Preliminary Results

In order to deal with (1.1), as was pointed out in the introduction, we follow closely the approximate scheme of [2]. Thus, we consider the following approximating problems:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f_n(x) = T_n(f(x))$. We will give sufficient conditions to assure that, for every $n \in \mathbb{N}$, u_n is well defined, belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and the sequence u_n has a limit u which turns out to be a solution to problem (1.1) in the sense of the following definition.

Definition 2.1. We say that $u \in H^1_{\text{loc}}(\Omega)$ is a positive solution for (1.1) if $u > 0$ almost everywhere in Ω,

$$\frac{f(x)}{u^{\gamma(x)}} \in L^1_{\text{loc}}(\Omega)$$

and

$$\int_\Omega M(x)\nabla u \nabla \phi = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\phi \tag{2.2}$$

for every $\phi \in C_0^1(\Omega)$.

The next two lemmas, whose proofs follow as in [2], assure the existence of u_n and that u_n is uniformly bounded from below in compact sets of Ω. We include here the proofs for convenience of the reader.

Lemma 2.2. *Problem* (2.1) *has a nonnegative solution* $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. For every fixed $n \in \mathbb{N}$ we can deduce the existence of u_n by means of the Schauder's fixed point theorem applied to the operator $S\colon L^2(\Omega) \to L^2(\Omega)$ defined by $S(v) = w \in H_0^1(\Omega)$ for every $v \in L^2(\Omega)$, where w is the unique solution of (see [12])

$$\begin{cases} -\operatorname{div}(M(x)\nabla w) = \dfrac{f_n(x)}{(|v| + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\gamma(x) \in C^1(\overline{\Omega})$ we can define $\gamma^* = \|\gamma(x)\|_{L^\infty(\Omega)}$. Taking w as test function and using (1.2), Poincaré's and Hölder's inequalities, we have

$$\alpha\lambda_1 \int_\Omega w^2 \le \alpha \int_\Omega |\nabla w|^2 \le \int_\Omega M(x)\nabla w \nabla w = \int_\Omega \frac{f_n(x) w}{(|v| + \frac{1}{n})^{\gamma(x)}} \le n^{\gamma^*+1} \int_\Omega |w| \le n^{\gamma^*+1}|\Omega|^{1/2}\Big(\int_\Omega |w|^2\Big)^{1/2}.$$

In particular, a ball of large enough radius remains invariant for S. Moreover, from the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we deduce that S is continuous and compact on $L^2(\Omega)$. Thus, we can use Schauder's fixed point theorem to prove the existence of $u_n \in H_0^1(\Omega)$ solving the following problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = \dfrac{f_n(x)}{(|u_n| + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, taking u_n^- as test function, using (1.2) and taking into account that $\frac{f_n(x)}{(|u_n|+\frac{1}{n})^{\gamma(x)}} \ge 0$, we get

$$\alpha \int_\Omega |\nabla u_n^-|^2 \le \int_\Omega M(x)\nabla u_n \nabla u_n^- = \int_\Omega \frac{f_n(x)}{(|u_n| + \frac{1}{n})^{\gamma(x)}} u_n^- \le 0.$$

Therefore, $u_n^- \equiv 0$ and in particular, $u_n \ge 0$ and solves (2.1). As the right-hand side of (2.1) belongs to $L^\infty(\Omega)$, we can use [15, Theorem 4.2] to deduce that u_n belongs to $L^\infty(\Omega)$. □

Lemma 2.3. *The sequence u_n is increasing with respect to n, $u_n > 0$ in Ω and, for every $\omega \subset\subset \Omega$, there exists $c_\omega > 0$ (independent on n) such that*

$$u_n(x) \geq c_\omega > 0 \quad \text{for every } x \in \omega \text{ and every } n \in \mathbb{N}. \tag{2.3}$$

Proof. Observe that taking $(u_n - u_{n+1})^+$ as test function in the equations satisfied by u_n and u_{n+1} and subtracting, and then taking into account (1.2) and that $0 \leq f_n(x) \leq f_{n+1}(x)$, we get that

$$\begin{aligned}\alpha \int_\Omega |\nabla (u_n - u_{n+1})^+|^2 &\leq \int_\Omega M(x)\nabla(u_n - u_{n+1})\nabla(u_n - u_{n+1})^+ \\ &\leq \int_\Omega f_{n+1}(x)\Big(\frac{1}{(u_n + \frac{1}{n+1})^{\gamma(x)}} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}}\Big)(u_n - u_{n+1})^+ \\ &\leq 0.\end{aligned}$$

The last inequality is due to the fact that $f_{n+1}(x) \geq 0$, $(u_n - u_{n+1})^+ \geq 0$ and

$$\Big(\frac{1}{(u_n + \frac{1}{n+1})^{\gamma(x)}} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^{\gamma(x)}}\Big) \leq 0 \quad \text{in } \{x \in \Omega : u_n(x) \geq u_{n+1}(x)\}.$$

Therefore, $(u_n - u_{n+1})^+ \equiv 0$, and thus

$$u_n \leq u_{n+1}. \tag{2.4}$$

On the other hand, we know that

$$\int_\Omega M(x)\nabla u_1 \nabla \phi = \int_\Omega \frac{f_1(x)}{(u_1 + 1)^{\gamma(x)}}\phi \geq \int_\Omega \frac{f_1(x)}{(\|u_1\|_{L^\infty(\Omega)} + 1)^{\gamma(x)}}\phi,$$

and since

$$\frac{f_1(x)}{(\|u_1\|_{L^\infty(\Omega)} + 1)^{\gamma(x)}} \not\equiv 0,$$

we deduce, using the strong maximum principle, that $u_1 > 0$ in Ω. Thus, $u_1(x) \geq c_\omega > 0$ for every $x \in \omega$ and every $n \in \mathbb{N}$. By (2.4) the proof is completed. □

Due to [15], we can prove easily that if $f \in L^m(\Omega)$ for some $m > \frac{N}{2}$, then the sequence of solutions of the approximated problem (2.1) is bounded in $L^\infty(\Omega)$, being an estimate independent on $\gamma(x)$.

Lemma 2.4. *Let $f \in L^m(\Omega)$ for some $m > \frac{N}{2}$. The sequence $\{u_n\}$ of solutions of problem (2.1) is bounded in $L^\infty(\Omega)$, i.e., there exists $C > 0$ independent of n and $\gamma(x)$ with*

$$\|u_n\|_\infty \leq C \quad \text{for all } n \in \mathbb{N}.$$

Proof. To prove an a priori estimate in $L^\infty(\Omega)$, let $k > 1$, we take $\phi = G_k(u_n)$ as test function in (2.1) and using (1.2) we obtain

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega M(x)\nabla G_k(u_n)\nabla G_k(u_n) = \int_\Omega \frac{f_n(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} G_k(u_n).$$

Using the fact that $u_n + \frac{1}{n} \geq k \geq 1$ on the set $\{u_n \geq k\}$, where $G_k(u_n) \neq 0$, we deduce that

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega f(x) G_k(u_n).$$

Now, by Stampacchia's method [15], from the last inequality follows the existence of $C > 0$ such that

$$\|u_n\|_\infty \leq C,$$

where the constant C does not depend on $\gamma(x)$. □

3 Estimates in the Sobolev Space

In this section we prove some properties that we will need in the proofs of the main results.

Observe that we only need $\gamma(x) \leq 1$ near the boundary in order to prove the following proposition.

Proposition 3.1. *Let $f \in L^{2N/(N+2)}(\Omega)$ and assume that there exists $\delta > 0$ with $\gamma(x) \leq 1$ in Ω_δ and that (1.2) holds. Then, the sequence $\{u_n\}$ of solutions of the problem (2.1) is bounded in $H_0^1(\Omega)$, i.e., there exists $C > 0$, independent of n, with*

$$\|u_n\|_{H_0^1(\Omega)} \leq C \quad \text{for all } n \in \mathbb{N}.$$

Remark 3.2. In order to show the existence of a solution, we will use the fact that $\gamma(x) \leq 1$ for every x in a strip around $\partial\Omega$ and inside Ω. Our hypothesis on $\gamma(x)$ in Theorem 1.1 guarantees this fact. Note that we can extend the result to functions $\gamma(x)$ such that $\gamma(x) < 1$ on $A \subset \partial\Omega$ and $\gamma(x) = 1$ on $\partial\Omega \setminus A$ with $\frac{\partial\gamma(x)}{\partial n_e} \geq 0$ there.

Proof. Let us denote $\omega_\delta = \Omega \setminus \overline{\Omega}_\delta$ and recall that $u_n \geq c_{\omega_\delta}$ in ω_δ, where $0 < c_{\omega_\delta}$ is given by Lemma 2.3. Thus, taking u_n as test function in (2.1) and using (1.2), it follows that

$$\begin{aligned}
\alpha \int_\Omega |\nabla u_n|^2 &\leq \int_\Omega M(x)\nabla u_n \nabla u_n \\
&= \int_{\overline{\Omega}_\delta} \frac{f_n(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n + \int_{\omega_\delta} \frac{f_n(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n \\
&\leq \int_{\overline{\Omega}_\delta} f(x) u_n^{1-\gamma(x)} + \int_{\omega_\delta} \frac{f(x)}{c_{\omega_\delta}^{\gamma(x)}} u_n \\
&\leq \int_{\overline{\Omega}_\delta \cap \{u_n \leq 1\}} f(x) + \int_{\overline{\Omega}_\delta \cap \{u_n \geq 1\}} f(x) u_n + \int_{\omega_\delta} \frac{f(x)}{c_{\omega_\delta}^{\gamma(x)}} u_n \\
&\leq \|f\|_{L^1(\Omega)} + (1 + \|c_{\omega_\delta}^{-\gamma(x)}\|_{L^\infty(\Omega)}) \int_\Omega f(x) u_n.
\end{aligned}$$

Using Hölder's and Sobolev's inequalities, we deduce that

$$\alpha \|u_n\|_{H_0^1(\Omega)}^2 \leq \|f\|_{L^1(\Omega)} + \mathcal{S}(1 + \|c_{\omega_\delta}^{-\gamma(x)}\|_{L^\infty(\Omega)}) \|f\|_{L^{2N/(N+2)}(\Omega)} \|u_n\|_{H_0^1(\Omega)},$$

and this implies the existence of $C > 0$ such that

$$\|u_n\|_{H_0^1(\Omega)} \leq C \quad \text{for all } n \in \mathbb{N}.$$

Hence, we conclude that the sequence u_n is bounded in $H_0^1(\Omega)$. □

In the following result we only need to assume that $\gamma(x) > 1$ in $\Gamma \subset \partial\Omega$.

Proposition 3.3. *Assume that for some $\gamma^* > 1$ and $\delta > 0$ we have that $\|\gamma\|_{L^\infty(\Omega_\delta)} \leq \gamma^*$ and that (1.2) holds. Assume also that $f \in L^m(\Omega)$ with $m = \frac{N(\gamma^*+1)}{N+2\gamma^*}$. Then, $u_n^{(\gamma^*+1)/2}$ is bounded in $H_0^1(\Omega)$ and u_n is bounded in $H_{\mathrm{loc}}^1(\Omega)$. Moreover, if for some $\Gamma \subset \partial\Omega$ we have that $\gamma(x) \leq 1$ in the set $\{x \in \Omega : \operatorname{dist}(x, \Gamma) < \delta\}$, then u_n is bounded in $H_\Gamma^1(\omega)$ for every open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega \cup \Gamma$.*

Remark 3.4. We point out that $m = \frac{N(\gamma^*+1)}{N+2\gamma^*}$ tends to $\frac{2N}{N+2}$ if $\gamma^* \to 1$ and tends to $\frac{N}{2}$ if $\gamma^* \to \infty$.

Proof. We take $u_n^{\gamma^*}$ as test function in (2.1), and then from (1.2) and Lemma 2.3 we obtain

$$\begin{aligned}
\frac{4\gamma^* \alpha}{(\gamma^* + 1)^2} \int_\Omega |\nabla u_n^{(\gamma^*+1)/2}|^2 &= \gamma^* \alpha \int_\Omega |\nabla u_n|^2 u_n^{\gamma^*-1} \\
&\leq \int_{\overline{\Omega}_\delta} f(x) u_n^{\gamma^*-\gamma(x)} + \int_{\omega_\delta} \frac{f(x)}{c_{\omega_\delta}^{\gamma(x)}} u_n^{\gamma^*}
\end{aligned}$$

$$\leq \|f\|_{L^1(\Omega)} + (1 + \|c_{\omega_\delta}^{-\gamma(x)}\|_{L^\infty(\Omega)}) \int_\Omega f(x)(u_n^{(\gamma^*+1)/2})^{2\gamma^*/(\gamma^*+1)}$$
$$\leq \|f\|_{L^1(\Omega)} + \mathcal{S}(1 + \|c_{\omega_\delta}^{-\gamma(x)}\|_{L^\infty(\Omega)})\|f\|_{L^m(\Omega)}\|u_n^{(\gamma^*+1)/2}\|_{H_0^1(\Omega)}.$$

This completes the first part. Observe that the Sobolev embedding implies that u_n is also bounded in $L^{2^*(\gamma^*+1)/2}(\Omega)$.

In order to prove that u_n is bounded in $H^1(\omega)$ for every $\omega \subset\subset \Omega$ we follow closely [2]. In fact, a careful analysis of the proof allow us to prove that if for some $\Gamma \subset \partial\Omega$ we have that $\gamma(x) \leq 1$ in the set $\{x \in \Omega : \mathrm{dist}(x, \Gamma) < \delta\}$, then u_n is bounded in $H_\Gamma^1(\omega)$ for every open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega \cup \Gamma$. Indeed, we take $\phi \in C^1(\overline{\Omega})$ with $\operatorname{supp}\phi \subset \Omega$ if $\gamma(x) > 1$ on $\partial\Omega$ and $\operatorname{supp}\phi \subset \Omega \cup \Gamma$ otherwise.

We use the notation $\Omega^* = \{\phi \neq 0\}$ and $\Omega^*_{\delta,\Gamma} = \{x \in \Omega^* : \mathrm{dist}(x, \partial\Omega^* \cap \Gamma) < \delta\}$. Recall that $\overline{\Omega}^* \subset \Omega \cup \Gamma$, and thus $\omega_{\delta,\Gamma} \equiv \Omega^* \setminus \overline{\Omega}^*_{\delta,\Gamma}$ is compactly embedded in Ω and $\gamma(x) \leq 1$ in $\overline{\Omega}^*_{\delta,\Gamma}$.

Taking $u_n\phi^2$ as test function, then by (1.2) and Lemma 2.3, we obtain

$$\begin{aligned}
\alpha \int_\Omega |\nabla u_n|^2\phi^2 + 2\int_\Omega M(x)u_n\phi\nabla u_n\nabla\phi &\leq \int_\Omega \frac{f_n(x)u_n\phi^2}{(u_n + \frac{1}{n})^{\gamma(x)}} \\
&\leq \int_{\overline{\Omega}^*_{\delta,\Gamma}} f(x)u_n^{1-\gamma(x)}\phi^2 + \int_{\omega_{\delta,\Gamma}} \frac{f(x)u_n\phi^2}{c_{\omega_{\delta,\Gamma}}^{\gamma(x)}} \\
&\leq \int_{\overline{\Omega}^*_{\delta,\Gamma}\cap\{u_n\leq 1\}} f(x)\phi^2 + \int_{\overline{\Omega}^*_{\delta,\Gamma}\cap\{u_n\geq 1\}} f(x)u_n\phi^2 + \int_{\omega_{\delta,\Gamma}} \frac{f(x)u_n\phi^2}{c_{\omega_{\delta,\Gamma}}^{\gamma(x)}} \\
&\leq \|f\phi^2\|_{L^1(\Omega)} + (1 + \|c_{\omega_{\delta,\Gamma}}^{-\gamma(x)}\|_{L^\infty(\Omega)}) \int_\Omega f(x)u_n\phi^2.
\end{aligned}$$

Using Young's inequality and (1.2), we also have that

$$2\int_\Omega M(x)u_n\phi\nabla u_n\nabla\phi \geq -2\beta\int_\Omega |u_n\phi||\nabla u_n||\nabla\phi| \geq -\frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 - \frac{2\beta^2}{\alpha}\int_\Omega |\nabla\phi|^2u_n^2.$$

Combining both inequalities and taking into account that $m \geq (\frac{2^*(\gamma^*+1)}{2})' \equiv s'$ yields

$$\frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\phi^2 \leq \|f\phi^2\|_{L^1(\Omega)} + (1 + \|c_{\omega_{\delta,\Gamma}}^{-\gamma(x)}\|_{L^\infty(\Omega)})\|\phi^2\|_{L^\infty(\Omega)}\|f\|_{L^{s'}(\Omega)}\|u_n\|_{L^s(\Omega)} + \frac{2\beta}{\alpha}\||\nabla\phi|^2\|_{L^\infty(\Omega)}\|u_n\|^2_{L^2(\Omega)},$$

which allow us to complete the proof using that $s > 2$ and the fact that $\|u_n\|_{L^2(\Omega)}$ and $\|u_n\|_{L^s(\Omega)}$ are bounded sequences. □

4 Proof of the Main Results

In this section we pass to the limit in the approximated problem (2.1).

Proof of Theorem 1.1. Since u_n is bounded in $H_0^1(\Omega)$, then by Proposition 3.1, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in H_0^1(\Omega)$. Thus, $u_n \to u$ strongly in $L^t(\Omega)$ with $t < 2^*$ and $u_n(x) \to u(x)$ almost everywhere in Ω.

Therefore, we have

$$\lim_{n\to+\infty} \int_\Omega M(x)\nabla u_n\nabla\phi = \int_\Omega M(x)\nabla u\nabla\phi \quad \text{for every } \phi \in C_0^1(\Omega).$$

Using that u_n satisfies (2.3), we can state, on the set where $\{u_n \geq c_\omega\}$ and $\phi \neq 0$, that

$$0 \leq \left|\frac{f_n(x)\phi}{(u_n + \frac{1}{n})^{\gamma(x)}}\right| \leq \|\phi c_\omega^{-\gamma(x)}\|_{L^\infty(\Omega)}f(x) \quad \text{for every } \phi \in C_0^1(\Omega).$$

Using now the dominated Lebesgue's theorem, we conclude that

$$\lim_{n\to+\infty}\int_\Omega \frac{f_n(x)}{(u_n+\frac{1}{n})^{\gamma(x)}} = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\phi.$$

Therefore, we have proved that $u_n \to u$ satisfies

$$\int_\Omega M(x)\nabla u \nabla\phi = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\phi \quad \text{for every } \phi \in C_0^1(\Omega). \qquad \square$$

Proof of Theorem 1.2. The first part of the proof follows exactly as the previous one using Proposition 3.3 instead of Proposition 3.1.

Observe that, for the second part of the theorem, since $u_n^{(\gamma^*+1)/2}$ is bounded in $H_0^1(\Omega)$, we have that $u^{(\gamma^*+1)/2} \in H_0^1(\Omega)$. Analogously, if there exists $\Gamma \subset \partial\Omega$ such that $\gamma(x) \le 1$ in the set $\{x \in \Omega : \text{dist}(x,\Gamma) < \nu\}$ for some $\nu > 0$, then u_n is bounded in $H_\Gamma^1(\omega)$, and thus $u \in H_\Gamma^1(\omega)$ for every open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega \cup \Gamma$. $\square$

Funding: The research was supported by MINECO grant MTM2015-68210-P, Junta de Andalucía FQM-194 (first author) and FQM-116 (second author), and Programa de Apoyo a la Investigación de la Fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia, reference 19461/PI/14 (second author).

References

[1] D. Arcoya and L. Moreno-Mérida, Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity, *Nonlinear Anal.* **95** (2014), 281–291.

[2] L. Boccardo and L. Orsina, Semilinear elliptic equations with singular nonlinearities, *Calc. Var. Partial Differential Equations* **37** (2010), 363–380.

[3] t G. M. Coclite and M. M. Coclite, On a Dirichlet problem in bounded domains with singular nonlinearity, *Discrete Contin. Dyn. Syst.* **33** (2013), 4923–4944.

[4] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Partial Differential Equations* **2** (1977), 193–222.

[5] W. Fulks and J. S. Maybee, A singular non-linear equation, *Osaka J. Math.* **12** (1960), 1–19.

[6] D. Giachetti, P. J. Martínez-Aparicio and F. Murat, A semilinear elliptic equation with a mild singularity at $u = 0$: Existence and homogenization, *J. Math. Pures Appl.*, to appear.

[7] D. Giachetti, P. J. Martínez-Aparicio and F. Murat, Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$, preprint (2014).

[8] D. Giachetti, P. J. Martínez-Aparicio and F. Murat, Homogenization of a Dirichlet semilinear elliptic problem with a strong singularity at $u = 0$ in a domain with many small holes, preprint (2014).

[9] A. V. Lair and A. W. Shaker, Entire solution of a singular semilinear elliptic problem, *J. Math. Anal. Appl.* **200** (1996), no. 2, 498–505.

[10] A. V. Lair and A. W. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, *J. Math. Anal. Appl.* **211** (1997), no. 2, 371–385.

[11] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.* **111** (1991), no. 3, 721–730.

[12] J. Leray and J. L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty–Browder, *Bull. Soc. Math. France* **93** (1965), 97–107.

[13] A. Nachman and A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), no. 2, 275–281.

[14] F. Oliva and F. Petitta, On singular elliptic equations with measure sources, *ESAIM Control Optim. Calc. Var.* **22** (2016), 289–308.

[15] G. Stampacchia, *Equations elliptiques du second ordre à coefficients discontinus*, Semin. Math. Super. 16, Les Presses de l'Université de Montréal, Montreal, 1966.

[16] C. A. Stuart, Existence and approximation of solutions of non-linear elliptic equations, *Math. Z.* **147** (1976), no. 1, 53–63.

[17] Z. Zhang and J. Cheng, Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems, *Nonlinear Anal.* **57** (2004), no. 3, 473–484.