Research Article

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Fractional Schrödinger–Poisson Systems with a General Subcritical or Critical Nonlinearity

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Abstract: We consider a fractional Schrödinger–Poisson system with a general nonlinearity in the subcritical and critical case. The Ambrosetti–Rabinowitz condition is not required. By using a perturbation approach, we prove the existence of positive solutions. Moreover, we study the asymptotics of solutions for a vanishing parameter.

Keywords: Schrödinger–Poisson Systems, Variational Methods, Critical Growth

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1 Introduction and Main Results

We are concerned with the fractional nonlinear Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda\phi u = g(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $(-\Delta)^\alpha$ is the fractional Laplacian operator for $\alpha = s, t \in [0, 1]$. The fractional Schrödinger equation was introduced by Laskin [28] in the context of fractional quantum mechanics for the study of particles on stochastic fields modeled by Lévy processes. The operator $(-\Delta)^\alpha$ can be seen as the infinitesimal generator of Lévy stable diffusion processes (see Applebaum [3]). If $\lambda = 0$, then (1.1) reduces to the nonlinear fractional scalar field equation

$$(-\Delta)^s u = g(u) \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

This equation is related to the standing waves of the time-dependent fractional scalar field equation

$$i\phi_t - (-\Delta)^s\phi + g(\phi) = 0 \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

which is a physically relevant generalization of the classical nonlinear Schrödinger equation. In fact, up to replacing $(-\Delta)^\alpha$ with $(1-\alpha)(-\Delta)^\alpha$, the operators in the above equations converge to $-\Delta$, in a suitable sense, due to the results in Bourgain, Brezis and Mironescu [9]. Here, i is the imaginary unit and t denotes the time variable. For power-type nonlinearities, the fractional Schrödinger equation (1.3) was derived in [28] by replacing the Brownian motion in the path integral approach with the so-called Lévy flights (see, e.g., Metzler and Klafter [30]). Hence, the equation we want to study appears as a perturbation of a physically meaningful equation. Also, Frank and Lenzmann [21, 22] obtained deep results on the uniqueness and the non-degeneracy of ground states for (1.2) in the case when $g(u) = |u|^{p-2}u - u$ for subcritical p; see also

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Secchi and Squassina [34], where the soliton dynamics for (1.3) with an external potential was investigated. In [24], Giammetta studied the evolution equation associated with the one-dimensional system

$$\begin{cases} -\Delta u + \lambda\phi u = g(u) & \text{in } \mathbb{R}, \\ (-\Delta)^t \phi = \lambda u^2 & \text{in } \mathbb{R}. \end{cases} \tag{1.4}$$

In this case, the diffusion is fractional only in the Poisson equation. Our system is more general and contains this as a particular case. If $\mathcal{K}_\alpha(x) = |x|^{\alpha-N}$, the equation

$$\sqrt{-\Delta}u + u = (\mathcal{K}_2 * |u|^2)u, \quad u \in H^{1/2}(\mathbb{R}^3),\ u > 0,$$

is studied in Frank and Lenzmann [20] and in Elgart and Schlein [19] it is shown that the dynamical evolution of boson stars is described by the nonlinear evolution equation

$$i\partial_t\psi = \sqrt{-\Delta + m^2}\psi - (\mathcal{K}_2 * |\psi|^2)\psi, \quad m \geq 0,$$

for a field $\psi : [0, T) \times \mathbb{R}^3 \to \mathbb{C}$ (see also Fröhlich, Jonsson and Lenzmann [23]). The square root of the Laplacian also appears in the semi-relativistic Schrödinger–Poisson–Slater system (see Bellazzini, Ozawa and Visciglia [6] and also the model studied in D'Avenia, Siciliano and Squassina [16]).

Observe that if we formally take $s = t = 1$, then (1.1) reduces to the classical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \lambda\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \lambda u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

which describes systems of identically charged particles interacting with each other in the case when magnetic effects can be neglected (see Benci and Fortunato [7]). In recent years, the Schrödinger–Poisson system (1.5) has been widely studied by many researchers. Here, we would like to cite some related results, for example, Cerami and Vaira [11] for positive solutions, Azzollini and Pomponio [5] for ground state solutions, D'Aprile and Wei [15] for semi-classical states, and Ianni [25] for sign-changing solutions. See also Ambrosetti [2] and the references therein. In [4], Azzollini, d'Avenia and Pomponio were concerned with (1.5) under the Berestycki–Lions conditions (H2)–(H4) with $s = 1$ (see below). They proved that (1.5) admits a positive radial solution if $\lambda > 0$ small enough. For the critical case, we refer to [38] and to the recent work [39] by the authors of the present work.

1.1 Main Results

In this paper, we are mainly concerned with positive solutions of (1.1). First, we consider the subcritical case with the Berestycki–Lions conditions. More precisely, we assume the following hypotheses on g.

(H1) $g \in C^1(\mathbb{R}, \mathbb{R})$.

(H2) $-\infty < \liminf_{\tau\to 0} \frac{g(\tau)}{\tau} \leq \limsup_{\tau\to 0} \frac{g(\tau)}{\tau} = -m < 0$.

(H3) $\limsup_{\tau\to\infty} \frac{g(\tau)}{\tau^{2_s^*-1}} \leq 0$, where $2_s^* = \frac{6}{3-2s}$.

(H4) There exists $\xi > 0$ such that $G(\xi) := \int_0^\xi g(\tau)\,d\tau > 0$.

Our first result is the following theorem.

Theorem 1.1. *Suppose that g satisfies* (H1)–(H4) *and* $2t + 4s \geq 3$. *Then, the following hold.*

(i) *There exists* $\lambda_0 > 0$ *such that, for every* $\lambda \in (0, \lambda_0)$, (1.1) *admits a nontrivial positive radial solution* $(u_\lambda, \phi_\lambda)$.

(ii) *Along a subsequence,* $(u_\lambda, \phi_\lambda)$ *converges to* $(u, 0)$ *in* $H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ *as* $\lambda \to 0$, *where u is a radial ground state solution of* (1.2).

Remark 1.2. The hypotheses (H2)–(H4) are the so-called Berestycki–Lions conditions, which were introduced in Berestycki and Lions [8] for the derivation of the ground state of (1.2) with $s = 1$. Under (H1)–(H4), Chang and Wang [12] proved the existence of ground state solutions to (1.2) for $s \in (0, 1)$. The hypothesis (H1) is only used to get the better regularity of solutions to (1.2), which guarantees the Pohožaev identity. By the Pohožaev identity, (H4) is necessary.

Remark 1.3. The hypothesis $2t + 4s \geq 3$ is just used to guarantee that the Poisson equation $(-\Delta)^t \phi = \lambda u^2$ makes sense, due to the fact that $\mathcal{D}^{t,2}(\mathbb{R}^3) \hookrightarrow L^{2_t^*}(\mathbb{R}^3)$. For details, see Section 2 below.

In the variational approach to the study of elliptic problems, the Palais–Smale condition ((PS) condition for short) plays a crucial role. To verify the (PS) condition, the so-called Ambrosetti–Rabinowitz condition

$$\mu \int_0^\tau f(\xi)\,\mathrm{d}\xi \leq \tau f(\tau), \quad \tau \in \mathbb{R} \setminus \{0\},\ \mu > 2, \tag{AR}$$

has been frequently used in the literature. The main role of (AR) is to guarantee the boundedness of the (PS) sequence in some suitable Sobolev space. More recently, Pucci, Xiang and Zhang [32] considered fractional p-Laplacian equations of Schrödinger–Kirchhoff type

$$M\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + g(x). \tag{1.6}$$

With the use of (AR), they established the existence of multiple solutions to (1.6) via the Ekeland variational principle and the mountain pass theorem. In fact, (AR) is a technical assumption. Many mathematicians have tried to remove or weaken it. In [8], Berestycki and Lions considered the autonomous scalar field equation. Without using (AR), they proved the existence of ground state solutions by the constraint variational method. However, it is not easy to use the idea in [8] in order to deal directly with non-autonomous problems. In [26], Jeanjean introduced a monotonicity trick to overcome the difficulty due to the lack of (AR) in the non-autonomous case. In [39], without (AR), the authors of the present work considered the existence and the concentration of positive solutions to (1.1) in the critical case for $s = t = 1$. It is natural to wonder if similar results can hold for the critical fractional case. This is just our second goal of the present paper. In the critical case, we assume the following hypotheses on g.

(H2)′ $\lim_{\tau\to 0} \frac{g(\tau)}{\tau} = -a < 0$.

(H3)′ $\lim_{\tau\to\infty} \frac{g(\tau)}{\tau^{2_s^*-1}} = b > 0$.

(H4)′ There exists $\mu > 0$ and $q < 2_s^*$ such that $g(\tau) - b\tau^{2_s^*-1} + a\tau \geq \mu\tau^{q-1}$ for all $\tau > 0$.

Our second result is the following theorem.

Theorem 1.4. *Suppose that g satisfies* (H1) *and* (H2)′–(H4)′. *Then, the following hold.*

(i) *The limit problem* (1.2) *admits a ground state solution if* $\max\{2_s^* - 2, 2\} < q < 2_s^*$.

(ii) *If* $2t + 4s \geq 3$, *then there exists* $\lambda_0 > 0$ *such that, for every* $\lambda \in (0, \lambda_0)$, (1.1) *admits a nontrivial positive radial solution* $(u_\lambda, \phi_\lambda)$ *if* $\max\{2_s^* - 2, 2\} < q < 2_s^*$.

(iii) *Along a subsequence,* $(u_\lambda, \phi_\lambda)$ *converges to* $(u, 0)$ *in* $H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ *as* $\lambda \to 0$, *where u is a radial ground state solution of* (1.2).

Remark 1.5. In the case $s = 1$, the hypotheses (H2)′–(H4)′ were introduced in Zhang and Zou [40] (see also Alves, Souto and Montenegro [1]) to obtain the ground state of the scalar field equation $-\Delta u = g(u)$ in $\mathbb{R}^N$. In [36], Shang and Zhang considered the fractional problem (1.2) in the critical case (see also Shang, Zhang and Yang [37]). With the help of the monotonicity of $\tau \mapsto g(\tau)/\tau$, the ground state solutions were obtained by using the Nehari approach. To the best of our knowledge, there are few results in the literature about the ground states of the critical fractional problem (1.2) with a general nonlinearity, particularly without the Ambrosetti–Rabinowitz condition and the monotonicity of $g(\tau)/\tau$. Theorem 1.4 seems to be the first result in this direction.

Remark 1.6. Without loss generality, from now on, we assume that $a = b = \mu = 1$.

We conclude by fixing some notation that we will use throughout the paper. We define the norm

$$\|u\|_p := \Big(\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x\Big)^{1/p}, \quad p \in [1, \infty),$$

the value

$$2_\alpha^* := \frac{6}{3-2\alpha}, \quad \alpha \in (0,1),$$

and we let $\hat{u} = \mathcal{F}(u)$ denote the Fourier transform of u.

In the rest of the paper, we use the perturbation approach to prove Theorem 1.1 and Theorem 1.4. Similar arguments can also be found in [39]. The paper is organized as follows. In Section 2, we introduce the functional framework and some preliminary results. In Section 3, we construct the min-max level. In Section 4, we use a perturbation argument to complete the proof of Theorem 1.1 and we give the proof of Theorem 1.4.

2 Preliminaries and Functional Setting

2.1 Fractional-Order Sobolev Spaces

The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ of a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^\alpha \phi)(\xi) = |\xi|^{2\alpha} \mathcal{F}(\phi)(\xi), \quad \xi \in \mathbb{R}^3,$$

where $\mathcal{F}$ is the Fourier transform, i.e.,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-2\pi i \xi \cdot x)\phi(x)\,\mathrm{d}x,$$

where i is the imaginary unit. If ϕ is smooth enough, it can be computed by the singular integral

$$(-\Delta)^\alpha \phi(x) = c_\alpha \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{\phi(x) - \phi(y)}{|x-y|^{3+2\alpha}}\,\mathrm{d}y, \quad x \in \mathbb{R}^3,$$

where c_α is a normalization constant and P.V. stands for the principal value.

For any $\alpha \in (0,1)$, we consider the fractional-order Sobolev space

$$H^\alpha(\mathbb{R}^3) = \Big\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\hat{u}|^2\,\mathrm{d}\xi < \infty\Big\}$$

endowed with the norm

$$\|u\|_\alpha = \Big(\int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})|\hat{u}|^2\,\mathrm{d}\xi\Big)^{1/2}, \quad u \in H^\alpha(\mathbb{R}^3),$$

and with the inner product

$$(u,v) = \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})\hat{u}\bar{\hat{v}}\,\mathrm{d}\xi, \quad u, v \in H^\alpha(\mathbb{R}^3).$$

It is easy to see that the inner products

$$u, v \mapsto \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})\hat{u}\bar{\hat{v}}\,\mathrm{d}\xi \quad \text{and} \quad u, v \mapsto \int_{\mathbb{R}^3} (uv + (-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v)\,\mathrm{d}x$$

on $H^s(\mathbb{R}^3)$ are equivalent (see [36]). The homogeneous Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$\mathcal{D}^{\alpha,2}(\mathbb{R}^3) = \{u \in L^{2_\alpha^*}(\mathbb{R}^3) : |\xi|^\alpha \hat{u} \in L^2(\mathbb{R}^3)\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|^2_{\mathcal{D}^{\alpha,2}} = \|(-\Delta)^{\alpha/2}u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\hat{u}|^2\,\mathrm{d}\xi, \quad u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3),$$

and the inner product

$$(u,v)_{\mathcal{D}^{\alpha,2}} = \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v\,\mathrm{d}x, \quad u,v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3).$$

For a further introduction on fractional-order Sobolev spaces, we refer the interested reader to Di Nezza, Palatucci and Valdinoci [17]. Let

$$H^s_r(\mathbb{R}^3) = \{u \in H^3(\mathbb{R}^3) : u(x) = u(|x|)\}.$$

Now, we introduce the following Sobolev embedding theorems.

Lemma 2.1 (Lions [29]). *For any $\alpha \in (0,1)$, $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\alpha]$ and compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for $q \in [1, 2^*_\alpha)$. Moreover, $H^\alpha_r(\mathbb{R}^3)$ is compactly embedded into $L^q(\mathbb{R}^3)$ for $q \in (2, 2^*_\alpha)$.*

Lemma 2.2 (Cotsiolis and Tavoularis [14], Di Nezza, Palatucci, and Valdinoci [17]). *For any $\alpha \in (0,1)$, $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^3)$, i.e., there exists $S_\alpha > 0$ such that*

$$\Big(\int_{\mathbb{R}^3} |u|^{2^*_\alpha}\,\mathrm{d}x\Big)^{2/2^*_\alpha} \le S_\alpha \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2\,\mathrm{d}x, \quad u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3).$$

2.2 The Variational Setting

Now, we study the variational setting of (1.1). By Lemma 2.1, for $2t + 4s \ge 3$, we have

$$H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3).$$

Then, for $u \in H^s(\mathbb{R}^3)$, by Lemma 2.2, the linear operator $P : \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$P(v) = \int_{\mathbb{R}^3} u^2 v \le \|u\|^2_{12/(3+2t)}\|v\|_{2^*_t} \le C\|u\|_s^2\|v\|_{\mathcal{D}^{t,2}}$$

is well defined on $\mathcal{D}^{t,2}(\mathbb{R}^3)$ and is continuous. Thus, it follows from the Lax–Milgram theorem that there exists a unique $\phi^t_u \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t\phi^t_u = \lambda u^2$. Moreover, for $x \in \mathbb{R}^3$, we have

$$\phi^t_u(x) := \lambda c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}\,\mathrm{d}y, \tag{2.1}$$

where we have set

$$c_t = \frac{\Gamma(\frac{3}{2} - 2t)}{\pi^{3/2}2^{2t}\Gamma(t)}.$$

Formula (2.1) is called the t-Riesz potential. Substituting (2.1) into (1.1), we can rewrite (1.1) in the equivalent form

$$(-\Delta)^s u + \lambda\phi^t_u u = g(u), \quad u \in H^s(\mathbb{R}^3). \tag{2.2}$$

We define the energy functional $\Gamma_\lambda : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Gamma_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi^t_u u^2\,\mathrm{d}x - \int_{\mathbb{R}^3} G(u)\,\mathrm{d}x$$

with

$$G(\tau) = \int_0^\tau g(\zeta)\,\mathrm{d}\zeta.$$

Obviously, the critical points of Γ_λ are the weak solutions of (2.2).

Definition 2.3. (i) We call $(u, \phi) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ a weak solution of (1.1) if u is a weak solution of (2.2).
(ii) We call $u \in H^s(\mathbb{R}^3)$ a weak solution of (2.2) if

$$\int_{\mathbb{R}^3} ((-\Delta)^{s/2}u(-\Delta)^{s/2}v + \lambda\phi_u^t uv)\,dx = \int_{\mathbb{R}^3} g(v)v\,dx \quad \text{for all } v \in H^s(\mathbb{R}^3).$$

Setting

$$T(u) := \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx,$$

we summarize some properties of ϕ_u^t and $T(u)$ which will be used later.

Lemma 2.4. *If $t, s \in (0,1)$ and $2t + 4s \geq 3$, then, for any $u \in H^s(\mathbb{R}^3)$, the following hold.*
(i) $u \mapsto \phi_u^t : H^s(\mathbb{R}^3) \mapsto \mathcal{D}^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets.*
(ii) $\phi_u^t(x) \geq 0$, $x \in \mathbb{R}^3$, *and* $T(u) \leq c\lambda\|u\|_s^4$ *for some* $c > 0$.
(iii) $T(u(\cdot/\tau)) = \tau^{3+2t}T(u)$ *for any* $\tau > 0$ *and* $u \in H^s(\mathbb{R}^3)$.
(iv) *If* $u_n \to u$ *weakly in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *weakly in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$.
(v) *If* $u_n \to u$ *weakly in* $H^s(\mathbb{R}^3)$, *then* $T(u_n) = T(u) + T(u_n - u) + o(1)$.
(vi) *If u is a radial function, so is* ϕ_u^t.

Proof. The proof is similar to that in [33], so we omit the details here. □

3 The Subcritical Case

3.1 The Modified Problem

It follows from Lemma 2.4 that Γ_λ is well defined on $H^s(\mathbb{R}^3)$ and is of class C^1. Since we are concerned with positive solutions of (2.2), similarly to [8] (see also [12]), we modify our problem first. Without loss of generality, we assume that

$$0 < \xi = \inf\{\tau \in (0,\infty) : G(\tau) > 0\},$$

where ξ is given in (H4). Let

$$\tau_0 = \inf\{\tau > \xi : g(\tau) = 0\} \in [\xi, \infty]$$

and define a function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau) & \text{for } \tau \in [0, \tau_0], \\ 0 & \text{for } \tau \geq \tau_0, \end{cases}$$

and $\tilde{g}(\tau) = 0$ for $\tau \leq 0$. If $u \in H^s(\mathbb{R}^3)$ is a solution of (2.2), where g is replaced by $\tilde{g}$, then, by the maximum principle (see Cabré and Sire [10]), we get that u is positive and $u(x) \leq \tau_0$ for any $x \in \mathbb{R}^3$, i.e., u is a solution of the original problem (2.2) with g. Thus, from now on, we can replace g by $\tilde{g}$, but still use the same notation g. In addition, for $\tau > 0$, let

$$g_1(\tau) = \max\{g(\tau) + m\tau, 0\} \quad \text{and} \quad g_2(\tau) = g_1(\tau) - g(\tau).$$

Then, we have $g_2(\tau) \geq m\tau$ for $\tau \geq 0$,

$$\lim_{\tau\to 0} \frac{g_1(\tau)}{\tau} = 0 \quad \text{and} \quad \lim_{\tau\to+\infty} \frac{g_1(\tau)}{\tau^{2_s^*-1}} = 0, \tag{3.1}$$

and, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$g_1(\tau) \leq \varepsilon g_2(\tau) + C_\varepsilon \tau^{2_s^*-1}, \quad \tau \geq 0. \tag{3.2}$$

Let

$$G_i(u) = \int_0^u g_i(\tau)\,\mathrm{d}\tau, \quad i = 1, 2.$$

Then, by (3.1) and (3.2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$G_1(\tau) \le \varepsilon G_2(\tau) + C_\varepsilon |\tau|^{2_s^*}, \quad \tau \in \mathbb{R}. \tag{3.3}$$

3.2 The Limit Problem

In the following, we will find solutions of (2.2) by seeking critical points of Γ_λ. If $\lambda = 0$, (2.2) becomes

$$(-\Delta)^s u = g(u), \quad u \in H^s(\mathbb{R}^3), \tag{3.4}$$

which is referred to as the limit problem of (2.2). We define an energy functional for the limit problem (3.4) by

$$L(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} G(u)\,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^3).$$

In [12], Chang and Wang proved that, with the same assumptions on g as in Theorem 1.1, there exists a positive ground state solution $U \in H_r^3(\mathbb{R}^3)$ of (3.4). Moreover, each such solution U of (3.4) satisfies the Pohožaev identity

$$\frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}U|^2\,\mathrm{d}x = 3\int_{\mathbb{R}^3} G(U)\,\mathrm{d}x. \tag{3.5}$$

Let S be the set of positive radial ground state solutions U of (3.4). Then, $S \neq \varnothing$ and we have the following compactness result which plays a crucial role in the proof of Theorem 1.1.

Proposition 3.1. *Under the assumptions in Theorem 1.1, S is compact in $H_r^s(\mathbb{R}^3)$.*

As shown in Cho and Ozawa [13], for general $s \in (0, 1)$, we do not have a similar radial lemma in $H_r^s(\mathbb{R}^3)$. So the Strauss compactness lemma (see [8]) is not applicable here. Before we prove Proposition 3.1, we begin with the following compactness lemma which is a special case of [12, Lemma 2.4.]

Lemma 3.2 (Chang and Wang [12]). *Assume that $Q \in C(\mathbb{R}, \mathbb{R})$ satisfies*

$$\lim_{\tau\to 0} \frac{Q(\tau)}{\tau^2} = \lim_{|\tau|\to\infty} \frac{Q(\tau)}{|\tau|^{2_s^*}} = 0$$

and there exists a bounded sequence $\{u_n\}_{n=1}^\infty \subset H_r^s(\mathbb{R}^3)$ for some $v \in L^1(\mathbb{R}^3)$ with

$$\lim_{n\to\infty} Q(u_n(x)) = v(x) \quad \text{a.e. } x \in \mathbb{R}^3.$$

Then, up to a subsequence, we have $Q(u_n) \to v$ strongly in $L^1(\mathbb{R}^3)$ as $n \to \infty$.

Proof of Proposition 3.1. Let $\{u_n\}_{n=1}^\infty \subset S$ and denote by E the least energy of (3.4). Then, for any n, u_n satisfies $L(u_n) = E$ and the Pohožaev identity (3.5), which implies that

$$E = \frac{s}{3}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^3} G(u_n)\,\mathrm{d}x = \frac{3-2s}{2s}E.$$

Obviously, $\{\|(-\Delta)^{s/2}u_n\|_2\}$ is bounded. It follows from Lemma 2.2 that $\{\|u_n\|_{2_s^*}\}$ is bounded. By (3.3), as we can see in [8], $\{\|u_n\|_2\}$ is bounded, which yields that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$. Without loss of generality, we can assume that there exists $u_0 \in H_r^s(\mathbb{R}^3)$ such that $u_n \to u_0$ weakly in $H_r^s(\mathbb{R}^3)$, strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$, and $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^3$.

In the following, we adopt some ideas from [8] to prove that $u_n \to u_0$ strongly in $H_r^s(\mathbb{R}^3)$. For $u \in H^s(\mathbb{R}^3)$, let

$$J(u) = \frac{s}{3} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \, dx \quad \text{and} \quad V(u) = \int_{\mathbb{R}^3} G(u) \, dx.$$

Then, we know that u_n is a minimizer of the constrained minimizing problem

$$\inf\Big\{J(u) : u \in H_r^s(\mathbb{R}^3),\, V(u) = \frac{3-2s}{2s} E\Big\}.$$

By (3.1) and Lemma 3.2 we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} G_1(u_n) = \int_{\mathbb{R}^3} G_1(u_0).$$

Then, by Fatou's Lemma,

$$V(u_0) \geq \frac{3-2s}{2s} E,$$

which implies that $u_0 \not\equiv 0$. Meanwhile, it is easy to see that $J(u_0) \leq E$. Similarly to [8], we know that u_0 satisfies

$$J(u_0) = E \quad \text{and} \quad V(u_0) = \frac{3-2s}{2s} E,$$

which yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_2(u_0).$$

By Fatou's Lemma, we know that $\|u_n\|_2 \to \|u_0\|_2$ as $n \to \infty$. Thus, $u_n \to u_0$ strongly in $H_r^s(\mathbb{R}^3)$. □

3.3 The Min-Max Level

Take $U \in S$ and let

$$U_\tau(x) = U\Big(\frac{x}{\tau}\Big), \qquad \tau > 0.$$

Then, by the definition of $\hat{U} = \mathcal{F}(U)$, we know that $\hat{U}(\,\cdot\,/\tau) = \tau^3 \hat{U}(t\,\cdot\,)$ and

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} U_\tau|^2 \, dx = \int_{\mathbb{R}^3} |\xi|^{2s} \Big|\hat{U}\Big(\frac{\xi}{\tau}\Big)\Big|^2 = \tau^{3-2s} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} U|^2 \, dx.$$

By the Pohožaev identity, we have

$$L(U_\tau) = \Big(\frac{\tau^{3-2s}}{2} - \frac{3-2s}{6}\tau^3\Big) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} U|^2.$$

Thus, there exists $\tau_0 > 1$ such that $L(U_\tau) < -2$ for $\tau \geq \tau_0$. Set

$$D_\lambda \equiv \max_{\tau \in [0,\tau_0]} \Gamma_\lambda(U_\tau).$$

By virtue of Lemma 2.4, we have $\Gamma_\lambda(U_\tau) = L(U_\tau) + O(\lambda)$. Note that since $\max_{\tau\in[0,\tau_0]} L(U_\tau) = E$, we get that $D_\lambda \to E$ as $\lambda \to 0^+$.

Moreover, similarly to [39], we can prove the following lemma, which is crucial in defining the uniformly bounded set of the mountain paths (see below).

Lemma 3.3. *There exist $\lambda_1 > 0$ and $\mathcal{C}_0 > 0$ such that, for any $0 < \lambda < \lambda_1$, we have*

$$\Gamma_\lambda(U_{\tau_0}) < -2, \quad \|U_\tau\|_s \leq \mathcal{C}_0 \quad \text{for all } \tau \in (0, \tau_0], \quad \|u\|_s \leq \mathcal{C}_0 \quad \text{for all } u \in S.$$

Now, for any $\lambda \in (0, \lambda_1)$, we define a min-max value C_λ as

$$C_\lambda = \inf_{\gamma \in \Upsilon_\lambda} \max_{\tau \in [0,\tau_0]} \Gamma_\lambda(\gamma(\tau)),$$

where

$$\Upsilon_\lambda = \{\gamma \in C([0,\tau_0], H_r^s(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(\tau_0) = U_{\tau_0},\ \|\gamma(\tau)\|_s \le \mathcal{C}_0 + 1,\ \tau \in [0,\tau_0]\}.$$

Obviously, for $\tau > 0$, we have

$$\|U_\tau\|_s^2 = \tau^{3-2s}\|(-\Delta)^{s/2}U\|_2^2 + \tau^3\|U\|_2^2.$$

Then, we can define $U_0 \equiv 0$ so $U_\tau \in \Upsilon_\lambda$. Moreover, we have

$$\limsup_{\lambda \to 0^+} C_\lambda \le \lim_{\lambda \to 0^+} D_\lambda = E.$$

Proposition 3.4. *We have* $\lim_{\lambda \to 0^+} C_\lambda = E$.

Proof. It suffices to prove that

$$\liminf_{\lambda \to 0^+} C_\lambda \ge E.$$

Now, we give the mountain pass value

$$b = \inf_{\gamma \in \Upsilon} \max_{\tau \in [0,1]} L(\gamma(\tau)),$$

where

$$\Upsilon = \{\gamma \in C([0,1], H_r^s(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(1) < 0\}.$$

It follows from [12, Lemma 3.2] that L satisfies the mountain pass geometry. As we can see in Jeanjean and Tanaka [27], b agrees with the least energy level of (3.4), i.e., $b = E$. Note that $\phi_u^t(x) \ge 0$ for $x \in \mathbb{R}^3$. Then, $\tilde{\gamma}(\cdot) = \gamma(\tau_0 \cdot) \in \Upsilon$ for any $\gamma \in \Upsilon_\lambda$ and it follows that $C_\lambda \ge b$, which concludes the proof. □

3.4 Proof of Theorem 1.1

Now, for $\alpha, d > 0$, define

$$\Gamma_\lambda^\alpha := \{u \in H_r^s(\mathbb{R}^3) : \Gamma_\lambda(u) \le \alpha\}$$

and

$$S^d = \Big\{u \in H_r^s(\mathbb{R}^3) : \inf_{v \in S} \|u - v\|_s \le d\Big\}.$$

In the following, we will find a solution $u \in S^d$ of (2.2) for sufficiently small $\lambda > 0$ and some $0 < d < 1$. The following proposition is crucial for obtaining a suitable (PS) sequence for Γ_λ and plays a key role in our proof.

Proposition 3.5. *Let* $\{\lambda_i\}_{i=1}^\infty$ *be such that* $\lim_{i\to\infty} \lambda_i = 0$ *and* $\{u_{\lambda_i}\} \subset S^d$ *with*

$$\lim_{i\to\infty} \Gamma_{\lambda_i}(u_{\lambda_i}) \le E \quad \text{and} \quad \lim_{i\to\infty} \Gamma'_{\lambda_i}(u_{\lambda_i}) = 0.$$

Then, for d small enough, there is $u_0 \in S$*, up to a subsequence, such that* $u_{\lambda_i} \to u_0$ *in* $H_r^s(\mathbb{R}^3)$.

Proof. For convenience, we write λ for λ_i. Since $u_\lambda \in S^d$ and S is compact, we know that $\{u_\lambda\}$ is bounded in $H_r^s(\mathbb{R}^3)$. Then, by Lemma 2.4, we see that

$$\lim_{i\to\infty} L(u_\lambda) \le E \quad \text{and} \quad \lim_{i\to\infty} L'(u_\lambda) = 0.$$

It follows from [12, Lemma 3.3] that there is $u_0 \in H_r^s(\mathbb{R}^3)$, up to a subsequence, such that $u_\lambda \to u_0$ strongly in $H_r^s(\mathbb{R}^3)$. Obviously, $0 \notin S^d$ for d small. This implies that $u_0 \not\equiv 0$, $L(u_0) \le E$, and $L'(u_0) = 0$. Thus, $L(u_0) = E$, i.e., $u_0 \in S$, which completes the proof. □

By Proposition 3.5, for small $d \in (0, 1)$, there exist $\omega > 0$, $\lambda_0 > 0$ such that

$$\|\Gamma'_\lambda(u)\|_s \geq \omega, \quad u \in \Gamma_\lambda^{D_\lambda} \cap (S^d \setminus S^{d/2}),\ \lambda \in (0, \lambda_0). \tag{3.6}$$

Similarly to [39], we have the following proposition.

Proposition 3.6. *There exists $\alpha > 0$ such that, for small $\lambda > 0$,*

$$\Gamma_\lambda(\gamma(\tau)) \geq C_\lambda - \alpha \quad \textit{implies that} \quad \gamma(\tau) \in S^{d/2},$$

where $\gamma(\tau) = U(\cdot/\tau)$ for $\tau \in (0, \tau_0]$.

Proof. From Lemma 2.4 and the Pohožaev identity, we have

$$\Gamma_\lambda(\gamma(\tau)) = \Big(\frac{\tau^{3-2s}}{2} - \frac{3-2s}{6}\tau^3\Big) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} U|^2 + \lambda \tau^{3+2t} T(U).$$

Then,

$$\lim_{\lambda \to 0^+} \max_{\tau \in [0,\tau_0]} \Gamma_\lambda(\gamma(\tau)) = \max_{\tau \in [0,\tau_0]} \Big(\frac{\tau^{3-2s}}{2} - \frac{3-2s}{6}\tau^3\Big) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} U|^2 = E$$

and the conclusion follows. □

Similarly as in [39], thanks to (3.6) and Proposition 3.6, we can prove the following proposition, which assures the existence of a bounded (PS) sequence for Γ_λ.

Proposition 3.7. *For $\lambda > 0$ small enough, there exists $\{u_n\}_n \subset \Gamma_\lambda^{D_\lambda} \cap S^d$ such that $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$.*

Proof of Theorem 1.1. It follows from Proposition 3.7 that there exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, there exists $\{u_n\} \in \Gamma_\lambda^{D_\lambda} \cap S^d$ with $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$. Noting that S is compact in $H_r^s(\mathbb{R}^3)$, we get that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$. Assume that $u_n \to u_\lambda$ weakly in $H_r^s(\mathbb{R}^3)$. Then $\Gamma'_\lambda(u_\lambda) = 0$. It follows from the compactness of S that $u_\lambda \in S^d$ and $\|u_n - u_\lambda\|_s \leq 3d$ for n large. So, $u_\lambda \not\equiv 0$ for small $d > 0$. By Lemma 2.4, we have

$$\Gamma_\lambda(u_n) = \Gamma_\lambda(u_\lambda) + \Gamma_\lambda(u_n - u_\lambda) + o(1).$$

Noting that

$$G_2(\tau) \geq \frac{m}{2}\tau^2 \quad \text{for any } \tau \in \mathbb{R},$$

it follows from (3.3) that, for some $C > 0$,

$$\Gamma_\lambda(u_n - u_\lambda \geq \frac{1}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{s/2}(u_n - u_\lambda)|^2 + \frac{m}{4}|u_n - u_\lambda|^2)\, dx - C \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2_s^*}\, dx.$$

Then, by Lemma 2.2, for small $d > 0$, it is easy to verify that $\Gamma_\lambda(u_n - u_\lambda) \geq 0$ for large n. So $u_\lambda \in \Gamma_\lambda^{D_\lambda} \cap S^d$ with $\Gamma'_\lambda(u_\lambda) = 0$. Thus, u_λ is a nontrivial solution of (2.2). Finally, by Proposition 3.5, we can get the asymptotic behavior of u_λ as $\lambda \to 0^+$. □

4 The Critical Case

In this section, we consider the Schrödinger–Poisson system (1.1) in the critical case. First, we establish the existence of ground state solutions to the fractional scalar field equation (1.2) with a general critical nonlinear term. Then, by a perturbation argument, we seek solutions of (1.1) in some neighborhood of the ground states of (1.2).

4.1 The Limit Problem

In this subsection, we use the constraint variational approach to seek ground state solutions of (1.2). A similar argument also can be found in [8, 18, 40]. Let

$$T(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2\,dx \quad \text{and} \quad V(u) = \int_{\mathbb{R}^3} G(u)\,dx.$$

We recall that U is called a ground state solution of (1.2) if and only if $I(U) = m_0$, where

$$m_0 := \inf\{I(u) : u \in H^s(\mathbb{R}^3) \setminus \{0\} \text{ is a solution of (1.2)}\}$$

and

$$I(u) = T(u) - V(u).$$

The existence of ground states is reduced to looking at the constraint minimization problem

$$M := \inf\{T(u) : V(u) = 1, u \in H^s(\mathbb{R}^3)\} \tag{4.1}$$

and eventually removing the Lagrange multiplier by some appropriate scaling. Now, we state the main result in this subsection.

Theorem 4.1. *Let* $s \in (0,1)$ *and assume that* (H2)′–(H4)′ *hold along with*

(H0) $g \in C(\mathbb{R},\mathbb{R})$ *and* g *is odd, i.e.,* $g(-\tau) = -g(\tau)$ *for* $\tau \in \mathbb{R}$.

Then, (1.2) *admits a positive ground state solution.*

Remark 4.2. Since we are concerned with positive solutions of (1.2), (H0) can be replaced by

(H0)′ $g \in C(\mathbb{R}^+,\mathbb{R})$.

Moreover, similarly to Theorem 4.1, a similar result in $\mathbb{R}^N$ for $N > 2s$ can be also obtained.

Proof of Theorem 4.1. The proof follows the lines of that in [40]. For completeness, we give the details here.

Step 1. Let M be given by (4.1) and let S_s be the Sobolev best constant in Lemma 2.2 for $s \in (0,1)$. Then, we claim that

$$0 < M < \frac{1}{2}(2_s^*)^{(3-2s)/3}S_s.$$

First, we prove that $\{u \in H^s(\mathbb{R}^3) : V(u) = 1\} \neq \emptyset$. By [14, 35], S_s can be achieved by

$$U_\varepsilon(x) = \kappa\varepsilon^{-(3-2s)/2}\left(\mu^2 + \left|\frac{x}{\varepsilon S_s^{1/2s}}\right|^2\right)^{-(3-2s)/2}$$

for any $\varepsilon > 0$, where $\kappa \in \mathbb{R}$, $\mu > 0$ are fixed constants. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support B_2 such that $\varphi \equiv 1$ on B_1 and $0 \le \varphi \le 1$ on B_2, where $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$. Let $\psi_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$. From [35], it follows that

$$\int_{\mathbb{R}^3} |\psi_\varepsilon|^{2_s^*} = S_s^{3/2s} + O(\varepsilon^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |(-\Delta)^{s/2}\psi_\varepsilon|^2 = S_s^{3/2s} + O(\varepsilon^{3-2s}). \tag{4.2}$$

Letting

$$v_\varepsilon = \frac{\psi_\varepsilon}{\|\psi_\varepsilon\|_{2_s^*}},$$

we have

$$\|(-\Delta)^{s/2}v_\varepsilon\|_2^2 \le S_s + O(\varepsilon^{3-2s}).$$

Letting

$$\Gamma_\varepsilon := \frac{1}{q}\|v_\varepsilon\|_q^q - \frac{1}{2}\|v_\varepsilon\|_2^2,$$

by $(H4)'$ we have

$$V(v_\varepsilon) \geq \frac{1}{2_s^*} + \Gamma_\varepsilon.$$

In the following, we will show that

$$\lim_{\varepsilon \to 0} \frac{\Gamma_\varepsilon}{\varepsilon^{3-2s}} = +\infty. \tag{4.3}$$

By $\max\{2_s^* - 2, 2\} < q < 2_s^*$, we know that $(3-2s)q > 3$. Then, it is easy to see that there exist $C_1(s), C_2(s) > 0$ such that

$$\|v_\varepsilon\|_q^q \geq \frac{1}{\|\psi_\varepsilon\|_{2_s^*}^q} \int_{B_1} |U_\varepsilon|^q \geq C_1(s)\varepsilon^{3-(3-2s)q/2} \int_0^{1/(\varepsilon S_s^{1/(2s)})} \frac{r^2}{(\mu^2 + r^2)^{(3-2s)q/2}}\, dr = O(\varepsilon^{3-(3-2s)q/2})$$

and

$$\|v_\varepsilon\|_2^2 \leq \frac{1}{\|\psi_\varepsilon\|_{2_s^*}^2} \int_{B_2} |U_\varepsilon|^2 \leq C_2(s)\varepsilon^{2s} \int_0^{2/(\varepsilon S_s^{1/(2s)})} \frac{r^2}{(\mu^2 + r^2)^{3-2s}}\, dr = \begin{cases} O(\varepsilon^{2s}), & \text{for } s < \frac{3}{4}, \\ O\Big(\varepsilon^{2s} \ln \frac{1}{\varepsilon}\Big), & \text{for } s = \frac{3}{4}, \\ O(\varepsilon^{3-2s}), & \text{for } s > \frac{3}{4}. \end{cases}$$

Then, we obtain that

$$\Gamma_\varepsilon \geq O(\varepsilon^{3-(3-2s)q/2}) \quad \text{for } s \in (0,1).$$

Noting that $\max\{2_s^* - 2, 2\} < q < 2_s^*$, it is easy to verify that (4.3) is true. Thus, it follows that $V(v_\varepsilon) > 0$ for small $\varepsilon > 0$. By a scaling, we get that $\{u \in H^s(\mathbb{R}^3) : V(u) = 1\} \neq \varnothing$.

Next, obviously, $M \in (0, +\infty)$. For small $\varepsilon > 0$, we have $V(v_\varepsilon) > 0$ and

$$M \leq \frac{T(v_\varepsilon)}{(V(v_\varepsilon))^{2/2_s^*}} \leq \frac{1}{2} \frac{\|(-\Delta)^{s/2} v_\varepsilon\|_2^2}{\big(\frac{1}{2_s^*} + \Gamma_\varepsilon\big)^{2/2_s^*}} \leq \frac{1}{2}(2_s^*)^{2/2_s^*} S_s \frac{1 + O(\varepsilon^{N-2s})}{(1 + 2_s^*\Gamma_\varepsilon)^{2/2_s^*}}.$$

If $p \geq 1$, then $(1+t)^p \leq 1 + p(1+t)^{1+p} t$ for all $t \geq -1$. From (4.3), it follows that

$$(1 + O(\varepsilon^{N-2s}))^{2_s^*/2} - 1 \leq \frac{2_s^*}{2}(1 + O(\varepsilon^{N-2s}))^{1+2_s^*/2} O(\varepsilon^{N-2s}) < 2_s^* \Gamma_\varepsilon$$

for small $\varepsilon > 0$, which yields $1 + O(\varepsilon^{N-2s}) < (1 + 2_s^*\Gamma_\varepsilon)^{2/2_s^*}$. Then,

$$M < \frac{1}{2}(2_s^*)^{(3-2s)/3} S_s.$$

Step 2. Here, we show that M can be achieved. Noting that g is odd and using the fractional Pólya–Szegő inequality (see Park [31]), without loss of generality, we can assume that there exists a positive minimizing sequence $\{u_n\} \subset H_r^s(\mathbb{R}^3)$ such that $V(u_n) = 1$ and $T(u_n) \to M$ as $n \to \infty$. By Lemma 2.2, it is easy to see that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$. By Lemma 2.1 we can assume that $u_n \to u_0$ weakly in $H^s(\mathbb{R}^3)$, strongly in $L^q(\mathbb{R}^3)$, and a.e. in $\mathbb{R}^3$. Setting $v_n = u_n - u_0$, we have $T(u_n) = T(v_n) + T(u_0) + o(1)$ and

$$\|u_n\|_{2_s^*}^{2_s^*} = \|v_n\|_{2_s^*}^{2_s^*} + \|u_0\|_{2_s^*}^{2_s^*} + o(1) \quad \text{and} \quad \|u_n\|_2^2 = \|v_n\|_2^2 + \|u_0\|_2^2 + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Letting $f(s) = g(s) - s^{2_s^*-1} + s$, it follows from Lemma 3.2 that

$$\int_{\mathbb{R}^3} F(u_n) = \int_{\mathbb{R}^3} F(u_0) + \int_{\mathbb{R}^3} F(v_n) + o(1).$$

So, $V(u_n) = V(v_n) + V(u_0) + o(1)$.

Next, we prove that u_0 is the minimizer for M. Setting $S_n = T(v_n)$, $S_0 = T(u_0)$, $V(v_n) = \lambda_n$, and $V(u_0) = \lambda_0$, we have $\lambda_n = 1 - \lambda_0 + o(1)$ and $S_n = M - S_0 + o(1)$. Under a scale change, we get that

$$T(u) \geq M(V(u))^{(3-2s)/3} \tag{4.4}$$

for all $u \in H^s(\mathbb{R}^3)$ and $V(u) \geq 0$. By (4.4) we have $\lambda_0 \in [0, 1]$. If $\lambda_0 \in (0, 1)$, then, again by (4.4), we have

$$M = \lim_{n\to\infty}(S_0 + S_n) \geq \lim_{n\to\infty} M((\lambda_0)^{(3-2s)/3} + (\lambda_n)^{(3-2s)/3}) = M((\lambda_0)^{(3-2s)/3} + (1-\lambda_0)^{(3-2s)/3}) > M(\lambda_0 + 1 - \lambda_0) = M,$$

which is a contradiction. On the other hand, if $\lambda_0 = 0$, then $S_0 = 0$, which implies that $u_0 = 0$. Then,

$$\limsup_{n\to\infty} \|v_n\|_{2_s^*}^2 \geq (2_s^*)^{(3-2s)/3}$$

and

$$M = \frac{1}{2}\lim_{n\to\infty} \|(-\Delta)^{s/2} v_n\|_2^2 \geq \frac{1}{2}(2_s^*)^{(3-2s)/3} \liminf_{n\to\infty} \frac{\|(-\Delta)^{s/2} v_n\|_2^2}{\|v_n\|_{2_s^*}^2} \geq \frac{1}{2}(2_s^*)^{(3-2s)/3} S_s,$$

which is again a contradiction. Then, we conclude that $\lambda_0 = 1$, i.e., M is achieved by u_0.

Finally, letting $U(\cdot) = u_0(\cdot/\sigma_0)$, where

$$\sigma_0 = \left(\frac{3-2s}{3} M\right)^{1/2},$$

we have that U is a ground state solution of (1.2). □

Remark 4.3. Furthermore, similarly to Chang and Wang [12], if we additionally assume that $g \in C^1(\mathbb{R}, \mathbb{R})$, then U satisfies the Pohožaev identity

$$\frac{3-2s}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} U|^2 \, dx = 3 \int_{\mathbb{R}^3} G(U) \, dx.$$

Similarly to [27, 40], U is also a mountain pass solution.

Let S_1 be the set of positive radial ground state solutions U of (1.2). Then, as in Step 2 in the proof of Theorem 4.1, we have the following compactness result.

Proposition 4.4. *Under the assumptions of Theorem 4.1, S_1 is compact in $H_r^s(\mathbb{R}^3)$.*

4.2 Proof of Theorem 1.4

In the following, we are ready to prove Theorem 1.4. Similarly to Section 3, take $U \in S_1$ and let

$$U_\tau(x) = U\left(\frac{x}{\tau}\right), \quad \tau > 0.$$

Then, there exists $\tau_1 > 1$ such that $I(U_\tau) < -2$ for $\tau \geq \tau_1$. Setting

$$D_\lambda^1 \equiv \max_{\tau\in[0,\tau_1]} \Gamma_\lambda(U_\tau),$$

there exist $\lambda_2 > 0$ and $\mathcal{C}_1 > 0$ such that, for any $0 < \lambda < \lambda_2$,

$$\varnothing \neq \Upsilon_\lambda = \{\gamma \in C([0, \tau_1], H_r^s(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(\tau_1) = U_{\tau_1},\ \|\gamma(\tau)\|_s \leq \mathcal{C}_1 + 1,\ \tau \in [0, \tau_1]\}.$$

Then, for any $\lambda \in (0, \lambda_1)$, we define a min-max value C_λ^1 as

$$C_\lambda^1 = \inf_{\gamma\in\Upsilon_\lambda} \max_{\tau\in[0,\tau_1]} \Gamma_\lambda(\gamma(\tau)).$$

Similarly to Section 3, we have the following proposition.

Proposition 4.5. *We have* $\lim_{\lambda\to 0^+} C_\lambda^1 = \lim_{\lambda\to 0^+} D_\lambda^1 = m$, *where* m *is the least energy of* (1.2).

Now for $\alpha, d > 0$, define

$$\Gamma_\lambda^\alpha := \{u \in H_r^s(\mathbb{R}^3) : \Gamma_\lambda(u) \le \alpha\}$$

and

$$S_1^d = \Big\{u \in H_r^s(\mathbb{R}^3) : \inf_{v\in S_1} \|u - v\|_s \le d\Big\}.$$

Similarly to Section 3, for small $\lambda > 0$ and some $0 < d < 1$, we will find a solution $u \in S_1^d$ of (2.2) in the critical case. Also, similarly to [39], we can get the following compactness result, which can yield the gradient estimate of Γ_λ.

Proposition 4.6. *Let* $\{\lambda_i\}_{i=1}^\infty$ *be such that* $\lim_{i\to\infty} \lambda_i = 0$ *and* $\{u_{\lambda_i}\} \subset S_1^d$ *with*

$$\lim_{i\to\infty} \Gamma_{\lambda_i}(u_{\lambda_i}) \le m \quad \text{and} \quad \lim_{i\to\infty} \Gamma'_{\lambda_i}(u_{\lambda_i}) = 0.$$

Then, for d small enough, there is $u_1 \in S_1$, *up to a subsequence, such that* $u_{\lambda_i} \to u_1$ *in* $H_r^s(\mathbb{R}^3)$.

Proof. For convenience, we write λ for λ_i. Since $u_\lambda \in S_1^d$ and S_1 is compact, we know that $\{u_\lambda\}$ is bounded in $H_r^s(\mathbb{R}^3)$. Moreover, up to a subsequence, there exists $u_1 \in S_1^d$ such that $u_\lambda \to u_1$ weakly in $H^s(\mathbb{R}^3)$, a.e. in $\mathbb{R}^3$, and $\|u_\lambda - u_1\|_s \le 3d$ for i large. Then, by Lemma 2.4, we see that

$$\lim_{i\to\infty} I(u_\lambda) \le m \quad \text{and} \quad \lim_{i\to\infty} I'(u_\lambda) = 0.$$

Then $I'(u_1) = 0$. Obviously, $u_0 \not\equiv 0$ if d small. So, $I(u_1) \ge m$. Meanwhile, thanks to Lemma 3.2, we have

$$I(u_\lambda) = I(u_1) + I(u_\lambda - u_1) + o(1)$$

and

$$I(u_\lambda - u_1) = \frac{1}{2}\|u_\lambda - u_1\|_s^2 - \frac{1}{2_s^*}\|u_\lambda - u_1\|_{2_s^*}^{2_s^*} + o(1) \le o(1).$$

Then, by Lemma 2.2, for d small enough, $u_\lambda \to u_1$ strongly in $H_r^s(\mathbb{R}^3)$. □

By Proposition 4.6, for small $d \in (0, 1)$, there exist $\omega_1 > 0$, $\lambda_2 \in (0, \lambda_1)$ such that

$$\|\Gamma'_\lambda(u)\|_s \ge \omega_1, \quad u \in \Gamma_\lambda^{D_\lambda^1} \cap (S_1^d \setminus S_1^{d/2}),\ \lambda \in (0, \lambda_2). \tag{4.5}$$

Similarly to Section 3, we have the following proposition.

Proposition 4.7. *There exists* $\alpha_1 > 0$ *such that, for small* $\lambda > 0$,

$$\Gamma_\lambda(\gamma(\tau)) \ge C_\lambda^1 - \alpha_1 \quad \textit{implies that} \quad \gamma(\tau) \in S_1^{d/2},$$

where $\gamma(\tau) = U(\cdot/\tau)$ *for* $\tau \in (0, \tau_1]$.

Proof of Theorem 1.4. With the help of (4.5) and Proposition 4.7, similarly to [39], for $\lambda > 0$ small enough, there exists $\{u_n\}_n \subset \Gamma_\lambda^{D_\lambda^1} \cap S_1^d$ such that $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$. As above, there exists $u_\lambda \in S_1^d$ with $u_\lambda \not\equiv 0$ for small $d > 0$. Moreover, up to a subsequence, $u_n \to u_\lambda$ weakly in $H_r^s(\mathbb{R}^3)$, a.e. in $\mathbb{R}^3$, and $\|u_n - u_\lambda\|_s \le 3d$ for n large. Furthermore, $\Gamma'_\lambda(u_\lambda) = 0$. By Lemma 2.4, we have

$$\Gamma_\lambda(u_n) = \Gamma_\lambda(u_\lambda) + \Gamma_\lambda(u_n - u_\lambda) + o(1).$$

By (H2)′–(H3)′, for some $C > 0$, we have

$$\Gamma_\lambda(u_n - u_\lambda) \ge \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{s/2}(u_n - u_\lambda)|^2 + \frac{1}{2}|u_n - u_\lambda|^2)\,dx - C\int_{\mathbb{R}^3} |u_n - u_\lambda|^{2_s^*}\,dx.$$

Then, by Lemma 2.2, $\liminf_{n\to\infty} \Gamma_\lambda(u_n - u_\lambda) \ge 0$ for small $d > 0$. So, $u_\lambda \in \Gamma_\lambda^{D_\lambda^1} \cap S_1^d$ with $\Gamma'_\lambda(u_\lambda) = 0$. Thus, u_λ is a nontrivial solution of (2.2). The asymptotic behavior of u_λ follows from Proposition 4.6. □

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