Research Article

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Minimal Energy Solutions and Infinitely Many Bifurcating Branches for a Class of Saturated Nonlinear Schrödinger Systems

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Abstract: We prove a conjecture recently formulated by Maia, Montefusco and Pellacci saying that minimal energy solutions of the saturated nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^n, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^n \end{cases}$$

are necessarily semitrivial whenever $\alpha, \beta, \lambda_1, \lambda_2 > 0$ and $0 < s < \max\{\alpha/\lambda_1, \beta/\lambda_2\}$ except for the symmetric case $\lambda_1 = \lambda_2$, $\alpha = \beta$. Moreover, it is shown that for most parameter samples $\alpha, \beta, \lambda_1, \lambda_2$, there are infinitely many branches containing seminodal solutions which bifurcate from a semitrivial solution curve parametrized by s.

Keywords: Nonlinear Schrödinger System with Saturation, Bifurcation, Ground States

MSC 2010: 35J47, 35J50, 35Q55

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1 Introduction

In this paper, we intend to continue the study on nonlinear Schrödinger systems for saturated optical materials that was recently initiated by Maia, Montefusco and Pellacci [10]. In their paper, the system of elliptic partial differential equations

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^n, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^n \end{cases} \tag{1.1}$$

was suggested in order to model the interaction of two pulses within the optical material under investigation. Here, the parameters satisfy $\lambda_1, \lambda_2, \alpha, \beta, s > 0$ and $n \in \mathbb{N}$. One way to find classical fully nontrivial solutions of (1.1) is to use variational methods. The Euler functional $I_s : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$ associated to (1.1) is given by

$$\begin{aligned} I_s(u, v) &:= \frac{1}{2}\Big(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 - \frac{\alpha}{s}\|u\|_2^2 - \frac{\beta}{s}\|v\|_2^2\Big) + \frac{1}{2s^2}\int_{\mathbb{R}^n} \ln(1 + s(\alpha u^2 + \beta z^2)) \\ &= \frac{1}{2}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ), \end{aligned} \tag{1.2}$$

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where $Z(x) := \alpha u(x)^2 + \beta v(x)^2$ and $g(z) := z - \ln(1+z)$ for all $z \geq 0$. The symbol $\|\cdot\|_2$ denotes the standard norm on $L^2(\mathbb{R}^n)$ and the norms $\|\cdot\|_{\lambda_1}, \|\cdot\|_{\lambda_2}$ are defined via

$$\|u\|_{\lambda_1} := \Big(\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 u^2\Big)^{1/2}, \quad \|v\|_{\lambda_2} := \Big(\int_{\mathbb{R}^n} |\nabla v|^2 + \lambda_2 v^2\Big)^{1/2}.$$

Since we are interested in minimal energy solutions (that is, ground states) for (1.1), the ground states u_s, v_s of the scalar problems associated to (1.1) turn out to be of particular importance. These are positive radially symmetric and radially decreasing smooth functions satisfying

$$\begin{cases} -\Delta u_s + \lambda_1 u_s = \dfrac{\alpha^2 u_s^3}{1 + s\alpha u_s^2} & \text{in } \mathbb{R}^n, \\ -\Delta v_s + \lambda_2 v_s = \dfrac{\beta^2 v_s^3}{1 + s\beta v_s^2} & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

Since we will encounter these solutions many times, let us recall some facts from the literature. The existence of positive finite energy solutions u_s, v_s of (1.3) for parameters $0 < s < \alpha/\lambda_1$ and $0 < s < \beta/\lambda_2$ can be deduced from [19, Theorem 2.2] for $n \geq 3$ or from [5, Theorem 1 (i)] for $n \geq 2$, respectively. In the case $n = 1$, the positive functions u_s, v_s are given by $u_s(x) = u_s(-x)$, $v_s(x) = v_s(-x)$ for all $x \in \mathbb{R}$ and

$$u_s|_{[0,+\infty)}^{-1}(z) = \int_z^{u_s(0)} \Big(\frac{1}{\lambda_1 x^2 - s^{-2} g(s\alpha x^2)}\Big)^{1/2} dx \quad \text{for } z \in (0, u_s(0)],$$

$$v_s|_{[0,+\infty)}^{-1}(z) = \int_z^{v_s(0)} \Big(\frac{1}{\lambda_2 x^2 - s^{-2} g(s\beta x^2)}\Big)^{1/2} dx \quad \text{for } z \in (0, v_s(0)],$$

where $u_s(0), v_s(0) > 0$ are uniquely determined by

$$\lambda_1 u_s(0)^2 - s^{-2} g(s\alpha u_s(0)^2) = \lambda_2 v_s(0)^2 - s^{-2} g(s\beta v_s(0)^2) = 0. \tag{1.4}$$

As in the explicit one-dimensional case, it is known also in the higher-dimensional case that u_s, v_s are radially symmetric, see [6, Theorem 2]. Finally, the uniqueness of u_s, v_s follows from [17, Theorem 1] in the case $n \geq 3$ and from [12, Theorem 1] in the case $n = 2$. The uniqueness result for $n = 1$ is a direct consequence of the existence proof we gave above.

In this paper, we strengthen the results obtained by Maia, Montefusco and Pellacci [10] concerning ground state solutions and (component-wise) positive solutions of (1.1), so let us shortly comment on their achievements. In Theorem 3.7 of their paper, they proved the existence of nonnegative radially symmetric and nonincreasing ground state solutions of (1.1) for all $n \geq 2$ and for parameter values $0 < s < \max\{\alpha/\lambda_1, \beta/\lambda_2\}$, where the upper bound for s is in fact optimal by Lemma 3.2 in the same paper. It was conjectured that each of these ground states is semitrivial except for the special case $\alpha = \beta$, $\lambda_1 = \lambda_2$, where the totality of ground state solutions is known in a somehow explicit way, see [10, Theorem 2.1] or Theorem 1.1 (i) below. In [10], this conjecture was proved for parameters $s \geq \min\{\alpha/\lambda_1, \beta/\lambda_2\}$, see Theorem 3.15 and Theorem 3.17 therein. Our first result shows that the full conjecture is true even in the case $n = 1$, which was not considered in [10].

Theorem 1.1. *Let $n \in \mathbb{N}$, $\alpha, \beta, \lambda_1, \lambda_2 > 0$ and $0 < s < \max\{\alpha/\lambda_1, \beta/\lambda_2\}$. Then, the following holds.*
(i) *In the case $\alpha = \beta$ and $\lambda_1 = \lambda_2$, all ground states of (1.1) are given by $(\cos(\theta)u_s, \sin(\theta)v_s)$ for $\theta \in [0, 2\pi)$.*
(ii) *In the case $\alpha \neq \beta$ or $\lambda_1 \neq \lambda_2$, every ground state solution of (1.1) is semitrivial.*

The proof of this result will we presented in Section 2. Our approach is based on a suitable min-max characterization of the mountain pass level associated to (1.1) involving a fibering map technique as in [11]. This method even allows to give an alternative proof for the existence of a ground state solution of (1.1) which is significantly shorter than the one presented in [10] and which, moreover, incorporates the case $n = 1$, see Proposition 2.1. More importantly, this approach yields the optimal result.

In view of Theorem 1.1, it is natural to ask how the existence of fully nontrivial solutions of (1.1) can be proved. In [10], Maia, Montefusco and Pellacci found necessary conditions and sufficient conditions for the existence of positive solutions of (1.1) which, however, partly contradict each other. For instance, [10, Theorem 3.21] claims that positive solutions exist for parameters $\alpha = \beta$, $\lambda_1 \neq \lambda_2$ and $s > 0$ sufficiently small contradicting the nonexistence result from [10, Theorem 3.10]. The error leading to this contradiction is located on [10, p. 338, l. 13], where the number λ_2/s must be replaced by $\lambda_2 s$, which makes the results from Theorem 3.19 and Theorem 3.21 in that paper break down. Our approach to finding positive solutions and, more generally, seminodal solutions of (1.1) is to apply bifurcation theory to the semitrivial solution branches

$$\mathcal{T}_1 := \Big\{(0, v_s, s) : 0 < s < \frac{\beta}{\lambda_2}\Big\}, \quad \mathcal{T}_2 := \Big\{(u_s, 0, s) : 0 < s < \frac{\alpha}{\lambda_1}\Big\},$$

which was motivated by the papers of Ostrovskaya and Kivshar [13] and Champneys and Yang [3]. In the case $n = 1$ and $\lambda_1 = 1$, $\lambda_2 = \omega^2 \in (0, 1)$, $\alpha = \beta = 1$, they numerically detected a large number of solution branches emanating from $\mathcal{T}_2$ containing seminodal solutions. Moreover, they conjectured that the bifurcation points on $\mathcal{T}_2$ accumulate near $s = 1$, see [3, p. 2184 ff.]. Our results confirm these observations. For simplicity, we will only discuss the bifurcations from $\mathcal{T}_2$ since the corresponding analysis for $\mathcal{T}_1$ is the same up to interchanging the roles of λ_1, λ_2 and α, β. Investigating the linearized problems associated to (1.1) near $(u_s, 0, s)$ for parameters close to the boundary of the parameter interval $(0, \alpha/\lambda_1)$, we prove the existence of infinitely many bifurcating branches containing fully nontrivial solutions of a certain nodal pattern. Despite the fact that the question whether fully nontrivial solutions bifurcate from $\mathcal{T}_1, \mathcal{T}_2$ makes perfect sense for all space dimensions $n \in \mathbb{N}$, our bifurcation result is restricted to $n \in \{1, 2, 3\}$. Later, we will comment on this issue in more detail, see Remark 3.6. In order to formulate our bifurcation result, let us define the positive numbers $\overline{\mu}_k$ to be the k-th eigenvalues of the linear compact self-adjoint operators $\phi \mapsto (-\Delta + \lambda_2)^{-1}(\alpha\beta u_0^2\phi)$ mapping $H_r^1(\mathbb{R}^n)$ to itself, where u_0 denotes the positive ground state solution of the first equation in (1.3) for $s = 0$. By Sturm–Liouville theory, we know that these eigenvalues are simple and that they satisfy

$$\overline{\mu}_0 > \overline{\mu}_1 > \overline{\mu}_2 > \cdots > \overline{\mu}_k \to 0^+ \quad \text{as } k \to +\infty.$$

Deferring some more or less standard notational conventions to a later stage, we come to the statement of our result.

Theorem 1.2. *Let $n \in \{1, 2, 3\}$ and let $\alpha, \beta, \lambda_1, \lambda_2 > 0$ and $k_0 \in \mathbb{N}_0$ satisfy*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} \quad \text{and} \quad \overline{\mu}_{k_0} < 1.$$

Then, there is an increasing sequence $(s_k)_{k \geq k_0}$ of positive numbers converging to α/λ_1 such that continua $\mathcal{C}_k \subset \mathcal{S}$ containing $(0, k)$-nodal solutions of (1.1) emanate from $\mathcal{T}_2$ at $s = s_k$ for all $k \geq k_0$. In the case $k_0 = 0$, we necessarily have $\lambda_1 > \lambda_2$ and there is a $C > 0$ such that all positive solutions $(u, v, s) \in \mathcal{C}_0$ with $s \geq 0$ satisfy

$$\|u\|_{\lambda_1} + \|v\|_{\lambda_2} < C \quad \text{and} \quad s < \frac{\alpha - \beta}{\lambda_1 - \lambda_2} < \frac{\alpha}{\lambda_1}. \tag{1.5}$$

In the case $n \in \{2, 3\}$, we can estimate $\overline{\mu}_0$ from above in order to obtain a sufficient condition for the conclusions of Theorem 1.2 to hold for $k_0 = 0$. This estimate, which leads to Corollary 1.3, is based on the Courant–Fischer min-max principle and Hölder's inequality. In the one-dimensional case, the values of all eigenvalues $\overline{\mu}_k$ are explicitly known, which results in Corollary 1.4.

Corollary 1.3. *Let $n \in \{2, 3\}$. Then, the conclusions of Theorem 1.2 are true for $k_0 = 0$ if*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} < \Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{4-n}{4}}. \tag{1.6}$$

Corollary 1.4. *Let $n = 1$. Then, the conclusions of Theorem 1.2 are true in the case*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} < \frac{1}{2}\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k_0\Big)\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k_0 + 1\Big). \tag{1.7}$$

Remark 1.5. As we mentioned above, one can find sufficient criteria for the existence of $(k, 0)$-nodal solutions bifurcating from $\mathcal{T}_1$ by reversing the roles of λ_1, λ_2 and α, β in the statement of Theorem 1.2 as well as in its corollaries.

Theorem 1.2 gives rise to many questions which would be interesting to solve in the future. A list of open problems is provided in Section 5. Before going on with the proof of our results, let us clarify the notation which we used in Theorem 1.2. The set $\mathcal{S} \subset X \times \mathbb{R}$ is the closure of all solutions of (1.1) which do not belong to $\mathcal{T}_2$ and a subset of $\mathcal{S}$ is called a continuum if it is a maximal connected set within $\mathcal{S}$. Finally, a fully nontrivial solution (u, v) of (1.1) is called (k, l)-nodal if both component functions are radially symmetric and u has precisely $k + 1$ nodal annuli and v has precisely $l + 1$ nodal annuli. In other words, since double zeros cannot occur, (u, v) is (k, l)-nodal if the radial profiles of u, respectively v, have precisely k, respectively l, zeros.

2 Proof of Theorem 1.1

According to the assumptions of Theorem 1.1, we will assume throughout this section that the numbers $\lambda_1, \lambda_2, \alpha, \beta$ are positive, that s lies between 0 and $\max\{\alpha/\lambda_1, \beta/\lambda_2\} =: s^*$, and that the space dimension is an arbitrary natural number. Furthermore, we define the energy levels

$$
\begin{aligned}
c_s &= \inf\{I_s(u, v) : (u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \text{ solves (1.1)}, \ (u, v) \neq (0, 0)\}, \\
c_s^* &= \inf\{I_s(u, v) : (u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \text{ solves (1.1)}, \ u = 0, v \neq 0 \text{ or } u \neq 0, v = 0\}.
\end{aligned}
$$

The first step towards the proof of Theorem 1.1 is a more suitable min-max characterization of the least energy level c_s of (1.1) which, as in [11], gives rise to a simple proof for the existence of a ground state. To this end, we introduce the Nehari manifold

$$
c_{\mathcal{N}_s} := \inf_{\mathcal{N}_s} I_s, \quad \mathcal{N}_s := \{(u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : (u, v) \neq (0, 0) \text{ and } I_s'(u, v)[(u, v)] = 0\}.
$$

Proposition 2.1. *The value*

$$
c_s = c_{\mathcal{N}_s} = \inf_{(u,v)\neq(0,0)} \sup_{r>0} I_s(\sqrt{r}u, \sqrt{r}v). \tag{2.1}
$$

is attained at a radially symmetric and radially nonincreasing ground state of (1.1).

Proof. From [10, (3.15), (3.52)] we get $c_s = c_{\mathcal{N}_s}$, so let us prove the second equation in (2.1). For every fixed $u, v \in H^1(\mathbb{R}^n)$ satisfying $(u, v) \neq (0, 0)$, we set

$$
\beta(r) := I_s(\sqrt{r}u, \sqrt{r}v) = \frac{r}{2}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \frac{1}{2s^2} \int_{\mathbb{R}^n} g(rsZ),
$$

so that $(\sqrt{r}u, \sqrt{r}v) \in \mathcal{N}_s$ holds for $r > 0$ if and only if $\beta'(r) = 0$. Since β is smooth and strictly concave with $\beta'(0) > 0$, a critical point of β is uniquely determined and it is a maximizer (whenever it exists). Since the supremum of β is $+\infty$, when there is no maximizer of β we obtain

$$
c_{\mathcal{N}_s} = \inf_{\mathcal{N}_s} I_s = \inf_{(u,v)\neq(0,0)} \sup_{r>0} I_s(\sqrt{r}u, \sqrt{r}v),
$$

which proves (2.1).

Due to $0 < s < \max\{\alpha/\lambda_1, \beta/\lambda_2\}$, we can find a semitrivial function $(u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying

$$
\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 < \frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2,
$$

which implies that $c_s < +\infty$ according to (2.1). So, let (u_k, v_k) be a minimizing sequence in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying $\sup_{r>0} I_s(\sqrt{r}u_k, \sqrt{r}v_k) \to c_s$ as $k \to +\infty$. Using the classical Polya–Szegő inequality and the extended Hardy–Littlewood inequality

$$
\int_{\mathbb{R}^n} \ln(1 + rs(\alpha u_k^2 + \beta v_k^2)) \geq \int_{\mathbb{R}^n} \ln(1 + rs(\alpha u_k^{*2} + \beta v_k^{*2})) \quad \text{for all } r > 0,
$$

for the spherical rearrangement taken from [1, Theorem 2.2], we may assume u_k, v_k to be radially symmetric and radially decreasing. Since the function $g(z) = z - \ln(1+z)$ strictly increases on $(0, +\infty)$ from 0 to $+\infty$, we may moreover assume that (u_k, v_k) are rescaled in such a way that the equality

$$\frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ_k) = 1$$

holds for $Z_k := \alpha u_k^2 + \beta v_k^2$. The inequality

$$c_s + o(1) = \lim_{k\to+\infty}\sup_{r>0} I_s(\sqrt{r}u_k, \sqrt{r}v_k) \geq \limsup_{k\to+\infty} I_s(u_k, v_k) = \frac{1}{2}\limsup_{k\to+\infty}(\|u_k\|_{\lambda_1}^2 + \|v_k\|_{\lambda_2}^2) - 1$$

implies that the sequence (u_k, v_k) is bounded in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. Using the uniform decay rate and the resulting compactness properties of radially decreasing functions bounded in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ (apply, for instance, [18, Compactness Lemma 2]), we may take a subsequence, again denoted by (u_k, v_k), such that $(u_k, v_k) \rightharpoonup (u, v)$ in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ pointwise always everywhere and

$$\frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ) = \lim_{k\to+\infty}\frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ_k) \quad \text{for all } r > 0.$$

From this we infer that

$$\frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ) = 1$$

and, thus, $(u, v) \neq (0, 0)$. Hence, for all $r > 0$, we obtain

$$\begin{aligned} c_s &= \lim_{k\to+\infty}\sup_{\rho>0} I_s(\sqrt{\rho}u_k, \sqrt{\rho}v_k) \\ &\geq \limsup_{k\to+\infty}\Big(\frac{r}{2}(\|u_k\|_{\lambda_1}^2 + \|v_k\|_{\lambda_2}^2) - \frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ_k)\Big) \\ &\geq \frac{r}{2}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ) \\ &= I_s(\sqrt{r}u, \sqrt{r}v), \end{aligned}$$

so that (u, v) is a nontrivial radially symmetric and radially decreasing minimizer. Taking for r the maximizer of the map $r \mapsto I_s(\sqrt{r}u, \sqrt{r}v)$, we obtain the ground state solution $(\overline{u}, \overline{v}) := (\sqrt{r}u, \sqrt{r}v)$ having the properties we claimed to hold. Indeed, the Nehari manifold may be rewritten as

$$\mathcal{N}_s = \{(u, v) \in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n) : (u, v) \neq (0, 0),\ H(u, v) = 0\}$$

for

$$H(u, v) := I_s'(u, v)[(u, v)] = \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 - \int_{\mathbb{R}^n}\frac{Z^2}{1+sZ},$$

so that the Lagrange multiplier rule applies due to

$$H'(u, v)[(u, v)] = 2(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \int_{\mathbb{R}^n}\frac{4Z^2 + 2sZ^3}{(1+sZ)^2} = -\int_{\mathbb{R}^n}\frac{2Z^2}{1+sZ} < 0$$

for all $(u, v) \in \mathcal{N}_s$. □

Let us note that c_s equals $c = m_{\mathcal{N}} = m_{\mathcal{P}}$ from [10, Lemma 3.6] and, therefore, corresponds to the mountain pass level of I_s. Given Proposition 2.1, we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Part (i) was proved in [10, Lemma 3.2], so let us prove (ii). First, we show that the ground state energy level c_s equals c_s^*. Since we have $c_s \le c_s^*$ by definition, we have to show that

$$\sup_{r>0} I_s(\sqrt{r}u, \sqrt{r}v) \ge c_s^* \quad \text{for all } u, v \in H^1(\mathbb{R}^n) \text{ with } u, v \neq 0. \tag{2.2}$$

From (1.2) we deduce that if $\|u\|_{\lambda_1}^2 \ge (\alpha/s)\|u\|_2^2$, then we have $I_s(\sqrt{r}u, \sqrt{r}v) \ge I_s(0, \sqrt{r}v)$ for all $v \neq 0$ and $r > 0$, which implies (2.2). In the same way, one proves (2.2) in the case $\|v\|_{\lambda_2}^2 \ge (\beta/s)\|v\|_2^2$, so it remains to prove (2.2) for functions (u, v) satisfying

$$\|u\|_{\lambda_1}^2 < \frac{\alpha}{s}\|u\|_2^2, \quad \text{and} \quad \|v\|_{\lambda_2}^2 < \frac{\beta}{s}\|v\|_2^2. \tag{2.3}$$

To this end, let $r > 0$ be arbitrary but fixed. From (2.3) we infer that the numbers

$$t(u, v) := \frac{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2}{\frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2 - \|u\|_{\lambda_1}^2 - \|v\|_{\lambda_2}^2}, \qquad r(u, v) := r\Big(\frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2 - \|u\|_{\lambda_1}^2 - \|v\|_{\lambda_2}^2\Big)$$

satisfy $t(u, v) \in (0, 1)$ and $r(u, v) > 0$ as well as

$$I_s(\sqrt{r}u, \sqrt{r}v) = -\frac{r(u, v)}{2} + \frac{1}{2s^2}\int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s(\alpha u^2 + \beta v^2)}{\frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2 - \|u\|_{\lambda_1}^2 - \|v\|_{\lambda_2}^2}\Big). \tag{2.4}$$

The concavity of the logarithm yields

$$\begin{aligned}
\int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s(\alpha u^2 + \beta v^2)}{\frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2 - \|u\|_{\lambda_1}^2 - \|v\|_{\lambda_2}^2}\Big) &= \int_{\mathbb{R}^n} \ln\Big(t(u, v)\Big(1 + \frac{r(u, v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2}\Big) + (1 - t(u, v))\Big(1 + \frac{r(u, v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2 - \|v\|_{\lambda_2}^2}\Big)\Big) \\
&\ge t(u, v)\int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2}\Big) + (1 - t(u, v))\int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2 - \|v\|_{\lambda_2}^2}\Big) \\
&\ge \min\Big\{\int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2}\Big), \int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2 - \|v\|_{\lambda_2}^2}\Big)\Big\}.
\end{aligned}$$

Combining this inequality with (2.4) gives

$$I_s(\sqrt{r}u, \sqrt{r}v) \ge \min\Big\{-\frac{r(u, v)}{2} + \int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2}\Big), -\frac{r(u, v)}{2} + \int_{\mathbb{R}^n} \ln\Big(1 + \frac{r(u, v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2 - \|v\|_{\lambda_2}^2}\Big)\Big\}.$$

Taking the supremum with respect to $r > 0$, gives (2.2) and, therefore, $c_s \ge c_s^*$, which is what we had to show.

It remains to prove that every ground state is semitrivial unless $\lambda_1 = \lambda_2$, $\alpha = \beta$. To this end, assume that (u, v) is a fully nontrivial ground state solution of (1.1), so that in particular $I_s(u, v) = c_s$ holds. Then, $c_s = c_s^*$ implies that the inequalities above are equalities for some $r > 0$. In particular, since the logarithm is strictly concave and $t(u, v) \in (0, 1)$, we get

$$1 + \frac{r(u, v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2 - \|u\|_{\lambda_1}^2} = k\Big(1 + \frac{r(u, v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2 - \|v\|_{\lambda_2}^2}\Big) \quad \text{a.e. on } \mathbb{R}^n$$

for some $k > 0$. This implies that $k = 1$, so that u, v have to be positive multiples of each other. From the Euler–Lagrange equation (1.1) we deduce that $\lambda_1 = \lambda_2$ and $\alpha = \beta$, which finishes the proof. □

3 Proof of Theorem 1.2

In this section, we assume $\lambda_1, \lambda_2, \alpha, \beta > 0$ as before but the space dimension n is supposed to be 1, 2 or 3. In Remark 3.6, we will comment on the reason for this restriction. Let us first provide the functional analytic framework we will be working in. In the case $n \geq 2$, we set $X := H^1_r(\mathbb{R}^n) \times H^1_r(\mathbb{R}^n)$ to be the product of the radially symmetric functions in $H^1(\mathbb{R}^n)$ and define $F : X \times (0, +\infty) \to X$ by

$$F(u, v, s) := \begin{pmatrix} u - (-\Delta + \lambda_1)^{-1}(\alpha u Z(1 + sZ)^{-1}) \\ v - (-\Delta + \lambda_2)^{-1}(\beta v Z(1 + sZ)^{-1}) \end{pmatrix}, \quad \text{where } Z := \alpha u^2 + \beta v^2. \tag{3.1}$$

Hence, finding solutions of (1.1) is equivalent to finding zeros of F. Using the compactness of the embeddings $H^1_r(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $n \geq 2$ and $2 < q < 2n/(n-2)$, one can check that the function $F(\cdot\,, s)$ is a smooth compact perturbation of the identity in X for all s, so that the Krasnosel'skii–Rabinowitz global bifurcation theorem [9, 15] is applicable. In the case $n = 1$, however, this structural property is not satisfied, which motivates a different choice for X. In Appendix A, we show that one can define a suitable Hilbert space X of exponentially decreasing functions such that $F(\cdot\,, s) : X \to X$ is again a smooth compact perturbation of the identity in X. Except for this technical inconvenience, the case $n = 1$ can be treated in a similar way to the case $n \in \{2, 3\}$, so we carry out the proofs for the latter case only. Furthermore, we always assume that $\lambda_2/\lambda_1 < \beta/\alpha$ according to the assumption of Theorem 1.2.

The first step in our bifurcation analysis is to investigate the linearized problems associated to the equation $F(u, v, s) = 0$ around the elements of the semitrivial solution branch $\mathcal{T}_2$. While doing this, we make use of a nondegeneracy result for ground states of semilinear problems which is due to Bates and Shi [2]. Amongst other things, it tells us that u_s is a nondegenerate solution of the first equation in (1.3), that is, we have the following result.

Proposition 3.1. *The linear problem*

$$-\Delta\phi + \lambda_1\phi = \frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}\phi, \quad \phi \in H^1_r(\mathbb{R}^n),\ 0 < s < \frac{\alpha}{\lambda_1},$$

only admits the trivial solution $\phi = 0$.

Proof. In order to apply [2, Theorem 5.4 (6)], we set

$$g(z) := -\lambda_1 z + \frac{\alpha^2 z^3}{1 + s\alpha z^2}, \quad z \in \mathbb{R},$$

so that u_s is the ground state solution of $-\Delta u = g(u)$ in $\mathbb{R}^n$ which is centered at the origin. In the notation of [2], one can check that g is of class (A). Indeed, the properties (g1), (g2), (g3A), (g4A), (g5A) from [2, p. 258] are satisfied for

$$b = \left(\frac{\lambda_1}{\alpha^2 - \alpha\lambda_1 s}\right)^{1/2}, \quad K_\infty = 1$$

and the unique positive number $\theta > b$ satisfying

$$\left(\frac{\alpha}{s} - \lambda_1\right)\theta^2 - \frac{1}{s^2}\ln(1 + s\alpha\theta^2) = 0.$$

Notice that (g4A), (g5A) follow from the fact that $K_g(z) := zg'(z)/g(z)$ decreases from 1 to $-\infty$ on the interval $(0, b)$ and that it decreases from $+\infty$ to $K_\infty = 1$ on $(b, +\infty)$. Having checked the assumptions of [2, Theorem 5.4 (6)], we obtain that the space of solutions of $-\Delta\phi - g'(u_s)\phi = 0$ in $\mathbb{R}^n$ is spanned by $\partial_1 u_s, \dots, \partial_n u_s$, implying that the linear problem only has the trivial solution in $H^1_r(\mathbb{R}^n)$. Due to

$$g'(u_s) = -\lambda_1 + \frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}, \tag{3.2}$$

this proves the claim. □

Using this preliminary result, we can characterize all possible bifurcation points on $\mathcal{T}_2$ which are, due to the implicit function theorem, the points where the kernel of the linearized operator is nontrivial. For notational purposes, we introduce the linear compact self-adjoint operator $L(s) : H^1_r(\mathbb{R}^n) \to H^1_r(\mathbb{R}^n)$ for parameters $0 < s < \alpha/\lambda_1$ by setting

$$L(s)\phi := (-\Delta + \lambda_2)^{-1}(W_s\phi), \quad W_s(x) := \frac{\alpha\beta u_s(x)^2}{1 + s\alpha u_s(x)^2}, \quad 0 < s < \frac{\alpha}{\lambda_1},$$

for $\phi \in H^1_r(\mathbb{R}^n)$. Denoting by $(\mu_k(s))_{k\in\mathbb{N}_0}$ the decreasing null sequence of eigenvalues of $L(s)$, we will observe that finding bifurcation points on $\mathcal{T}_2$ amounts to solving $\mu_k(s) = 1$ for $s \in (0, \alpha/\lambda_1)$ and $k \in \mathbb{N}_0$. In fact, we have the following result.

Proposition 3.2. *We have*

$$\ker(\partial_X F(u_s, 0, s)) = \{0\} \times \ker(\mathrm{Id} - L(s)) \quad \text{for } 0 < s < \frac{\alpha}{\lambda_1}.$$

Proof. For $(u, v), (\phi_1, \phi_2) \in X$, we have

$$\partial_X F_1(u, v, s)[\phi_1, \phi_2] = \phi_1 - (-\Delta + \lambda_1)^{-1}\Big(\frac{s\alpha Z^2 + 3\alpha^2 u^2 + \alpha\beta v^2}{(1 + sZ)^2}\phi_1 + \frac{2\alpha\beta uv}{(1 + sZ)^2}\phi_2\Big),$$

$$\partial_X F_2(u, v, s)[\phi_1, \phi_2] = \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{s\beta Z^2 + 3\beta^2 v^2 + \alpha\beta u^2}{(1 + sZ)^2}\phi_2 + \frac{2\alpha\beta uv}{(1 + sZ)^2}\phi_1\Big).$$

Plugging in $u = u_s$, $v = 0$ and $Z = \alpha u^2 + \beta v^2 = \alpha u_s^2$ gives

$$\partial_X F_1(u_s, 0, s)[\phi_1, \phi_2] = \phi_1 - (-\Delta + \lambda_1)^{-1}\Big(\frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}\phi_1\Big),$$

$$\begin{aligned}\partial_X F_2(u_s, 0, s)[\phi_1, \phi_2] &= \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{s\beta\alpha^2 u_s^4 + \alpha\beta u_s^2}{(1 + s\alpha u_s^2)^2}\phi_2\Big)\\ &= \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{\alpha\beta u_s^2}{1 + s\alpha u_s^2}\phi_2\Big)\\ &= \phi_2 - (-\Delta + \lambda_2)^{-1}(W_s\phi_2)\\ &= \phi_2 - L(s)\phi_2.\end{aligned}$$

From these formulas and Proposition 3.1, we deduce the claim. □

Given this result, our aim is to find sufficient conditions for the equation $\mu_k(s) = 1$ to be solvable. Since there is only few information available for any given $s > 0$, our approach consists of proving the continuity of μ_k and calculating the limits of $\mu_k(s)$ as s approaches the boundary of $(0, \alpha/\lambda_1)$. It will turn out that the limits at both sides of the interval exist and that they lie on opposite sides of the value 1 provided our sufficient conditions from Theorem 1.2 are satisfied. As a consequence, these conditions and the intermediate value theorem imply the solvability of $\mu_k(s) = 1$ and it remains to add some technical arguments in order to apply the Krasnosel'skii–Rabinowitz global bifurcation theorem to prove Theorem 1.2. Calculating the limits of μ_k at the ends of $(0, \alpha/\lambda_1)$ requires Proposition 3.3 and Proposition 3.4.

Proposition 3.3. *We have*

$$u_s \to u_0 \quad \text{and} \quad W_s \to \alpha\beta u_0^2 \quad \text{as } s \to 0,$$

where the convergence is uniform in $\mathbb{R}^n$.

Proof. As in Lemma A.1 in Appendix A, one shows that on every interval $[0, s_0]$ with $0 < s_0 < \alpha/\lambda_1$, there is an exponentially decreasing function which bounds each of the functions u_s with $s \in [0, s_0]$ from above. In particular, the Arzelà–Ascoli theorem shows that $u_s \to u_0$ and $W_s \to \alpha\beta u_0^2$ as $s \to 0$ locally uniformly in $\mathbb{R}^n$, so that the uniform exponential decay gives $u_s \to u_0$ and $W_s \to \alpha\beta u_0^2$ uniformly in $\mathbb{R}^n$. □

Proposition 3.4. *We have*

$$u_s \to +\infty \quad \text{and} \quad W_s \to \frac{\beta\lambda_1}{\alpha} \quad \text{as } s \to \frac{\alpha}{\lambda_1},$$

where the convergence is uniform on bounded sets in $\mathbb{R}^n$.

Proof. First, we show that

$$u_s(0) = \max_{\mathbb{R}^n} u_s \to +\infty \quad \text{as } s \to s^* := \frac{\alpha}{\lambda_1}. \tag{3.3}$$

Otherwise, we would observe that $u_s(0) \to a$ for some subsequence, where $a \geq 0$. In the case $a > 0$, a combination of elliptic regularity theory for (1.3) and the Arzelà–Ascoli theorem would imply that u_s converges locally uniformly to a nontrivial radially symmetric function $u \in C^1(\mathbb{R}^n)$ satisfying

$$-\Delta u + \lambda_2 u = \frac{\alpha^2 u^3}{1 + s^* \alpha u^2} \quad \text{in } \mathbb{R}^n$$

in the weak sense and $u(0) = \|u\|_\infty = a$. As in Lemma A.1, we conclude that the functions u_s are uniformly exponentially decaying, so that u even lies in $H^1_r(\mathbb{R}^n)$. Hence, we may test the differential equation with u and obtain

$$\lambda_1 \int_{\mathbb{R}^n} u^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 u^2 = \int_{\mathbb{R}^n} \frac{\alpha^2 u^4}{1 + s^* \alpha u^2} < \frac{\alpha}{s^*} \int_{\mathbb{R}^n} u^2 = \lambda_1 \int_{\mathbb{R}^n} u^2,$$

which is impossible. It therefore remains to exclude the case $a = 0$. In this case, the functions u_s would converge uniformly in $\mathbb{R}^n$ to the trivial solution, implying that $u_s/u_s(0)$ would converge to a nonnegative bounded function $\phi \in C^1(\mathbb{R}^n)$ satisfying $-\Delta\phi + \lambda_1\phi = 0$ in $\mathbb{R}^n$ and $\phi(0) = \|\phi\|_\infty = 1$. Hence, ϕ is smooth, so that Liouville's theorem applied to the function $(x, y) \mapsto \phi(x)\cos(\sqrt{\lambda_1} y)$ defined on $\mathbb{R}^{n+1}$ implies that ϕ is constant and, thus, $\phi \equiv 0$, contradicting $\phi(0) = 1$. This proves (3.3).

Now, set $\phi_s := u_s/u_s(0)$. Using

$$-\Delta\phi_s + \lambda_1\phi_s = \alpha\phi_s \frac{\alpha u_s^2}{1 + s\alpha u_s^2} \quad \text{in } \mathbb{R}^n$$

and the fact that $\alpha u_s^2/(1 + s\alpha u_s^2)$ remains bounded as $s \to s^*$, we get that the functions ϕ_s converge locally uniformly as $s \to s^*$ to some nonnegative radially nonincreasing function $\phi \in C^1(\mathbb{R}^n)$ satisfying $\phi(0) = \|\phi\|_\infty = 1$. In order to prove our claim, it is sufficient to show that $\phi \equiv 1$, since this implies $u_s = u_s(0)\phi_s \to +\infty$ locally uniformly and, in particular, $W_s \to \beta\lambda_1/\alpha$ locally uniformly.

First, we show that $\phi > 0$. If this were not true, then there would exist a number $\rho \in (0, +\infty)$ such that $\phi|_{B_\rho} > 0$ and $\phi|_{\partial B_r} = 0$ for all $r \in [\rho, +\infty)$. In B_ρ, we have $u_s \to +\infty$ and $\alpha^2 u_s^2/(1 + s\alpha u_s^2) \to \lambda_1$ implies $-\Delta\phi + \lambda_1\phi = \lambda_1\phi$ in B_ρ and $\phi|_{\partial B_\rho} = 0$, in contradiction to the maximum principle. Hence, we must have $\phi > 0$ in $\mathbb{R}^n$. Repeating the above argument, we find $-\Delta\phi + \lambda_1\phi = \lambda_1\phi$ in $\mathbb{R}^n$ and $\phi(0) = \|\phi\|_\infty = 1$, so that Liouville's theorem implies $\phi \equiv \phi(0) = 1$. □

The previous propositions enable us to calculate the limits of the eigenvalue functions $\mu_k(s)$ as s approaches the boundary of $(0, \alpha/\lambda_1)$.

Proposition 3.5. *For all $k \in \mathbb{N}_0$, the functions μ_k are positive and continuous on $(0, \alpha/\lambda_1)$. Moreover, we have*

$$\mu_k(s) \to \overline{\mu}_k \quad \text{as } s \to 0, \qquad \mu_k(s) \to \frac{\beta\lambda_1}{\alpha\lambda_2} \quad \text{as } s \to \frac{\alpha}{\lambda_1}.$$

Proof. As in Proposition 3.3, the uniform exponential decay of the functions u_s for $s \in [0, s^*)$ for $s^* := \alpha/\lambda_1$ implies that $u_s \to u_{s_0}$, $W_s \to W_{s_0}$ uniformly in $\mathbb{R}^n$ whenever $s_0 \in [0, s^*]$. Hence, the Courant–Fischer min-max characterization for the eigenvalues $\mu_k(s)$ implies the continuity of μ_k as well as $\mu_k(s) \to \overline{\mu}_k$ as $s \to 0$.

In order to evaluate $\mu_k(s)$ for $s \to s^*$, we apply Lemma C.1 from Appendix C. The conditions (i) and (ii) of the lemma are satisfied since we have $\|W_s\|_\infty = W_s(0) \to \beta\lambda_1/\alpha$ and $W_s \to \beta\lambda_1/\alpha$ locally uniformly as $s \to s^*$ by Proposition 3.4. From the lemma we get $\mu_k(s) \to \beta\lambda_1/\alpha\lambda_2$ as $s \to s^*$, which is all we had to show. □

Remark 3.6. When $n \geq 4$, the statement of Proposition 3.3 is not meaningful since u_0 does not exist in this case by Pohožaev's identity. So, it is natural to ask how u_s, W_s and μ_k behave when s approaches zero and $n \geq 4$. Having found an answer to this question, it might be possible to modify our reasoning in order to prove sufficient conditions for the existence of bifurcation points from $\mathcal{T}_2$ in the case $n \geq 4$.

The above propositions are sufficient for proving the mere existence of the continua $\mathcal{C}_k$ from Theorem 1.2. So, it remains to show that positive solutions lie to the left of the threshold value $(\alpha - \beta)/(\lambda_1 - \lambda_2)$ and that they are equibounded in X. The latter result will be proved in Lemma A.1 whereas the first claim follows from the following nonexistence result which slightly improves [10, Theorem 3.10 and Theorem 3.11].

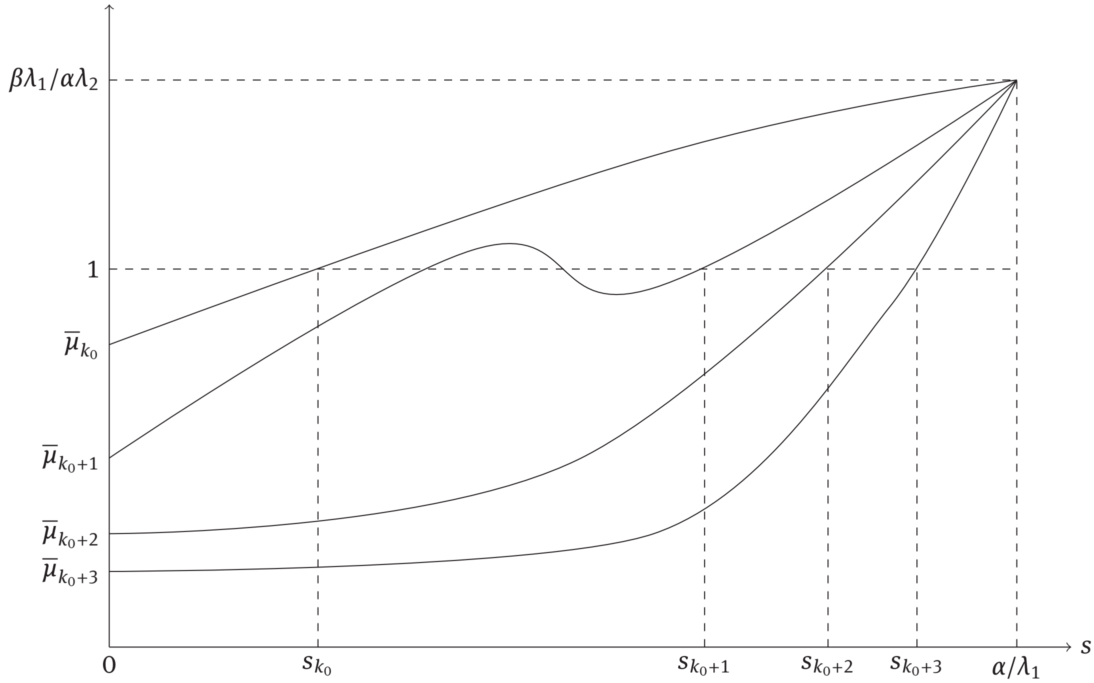

Figure 1. The eigenvalue functions $\mu_{k_0}, \dots, \mu_{k_0+3}$ on $(0, \alpha/\lambda_1)$.

Proposition 3.7. *If positive solutions of* (1.1) *exist, then we either have*

$$\text{(i)}\ \lambda_1 = \lambda_2,\ \alpha = \beta \quad \textit{or} \quad \text{(ii)}\ s < \frac{\alpha - \beta}{\lambda_1 - \lambda_2} < \min\Big\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\Big\}.$$

Proof. Assume there is a positive solution (u, v) of (1.1). Testing (1.1) with (v, u) leads to

$$\int_{\mathbb{R}^n} uv\Big(\lambda_1 - \lambda_2 - (\alpha - \beta)\frac{Z}{1 + sZ}\Big) = 0.$$

Hence, the function $\lambda_1 - \lambda_2 - (\alpha - \beta)Z/(1 + sZ)$ vanishes identically or it changes sign in $\mathbb{R}^n$. In the first case, we get (i), so let us assume that the function changes sign. Then, we have $\lambda_1 \neq \lambda_2$ and $\alpha \neq \beta$, so that [10, Theorem 3.11 and Remark 3.18] imply that

$$0 < \frac{\alpha - \beta}{\lambda_1 - \lambda_2} < \min\Big\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\Big\}.$$

Moreover, $s \geq (\alpha - \beta)/(\lambda_1 - \lambda_2)$ would imply that

$$\Big|\lambda_1 - \lambda_2 - (\alpha - \beta)\frac{Z}{1 + sZ}\Big| > |\lambda_1 - \lambda_2| - \frac{|\alpha - \beta|}{s} \geq 0 \quad \text{in } \mathbb{R}^n,$$

contradicting the assumption that $\lambda_1 - \lambda_2 - (\alpha - \beta)Z/(1 + sZ)$ changes sign. Hence, we have $s < (\alpha - \beta)/(\lambda_1 - \lambda_2)$, which concludes the proof. □

Proof of Theorem 1.2. The main ingredient of our proof is the Krasnosel'skii–Rabinowitz global bifurcation theorem (cf. [9, 15] or [8, Theorem II.3.3]) which, roughly speaking, says that a change of the Leray–Schauder index along a given solution curve over some parameter interval implies the existence of a bifurcating continuum emanating from the solution curve within this parameter interval. In our application, the solution curve is $\mathcal{T}_2$ and the first task is to identify parameter intervals within $(0, \alpha/\lambda_1)$ where the index changes. For notational purposes, we set $s^* := \alpha/\lambda_1$.

Step 1. Existence of Solution Continua $\mathcal{C}_k$ Bifurcating from $\mathcal{T}_2$. By the assumptions of Theorem 1.2 and Proposition 3.5, we have

$$\lim_{s\to 0}\mu_k(s) = \overline{\mu}_k < 1 \quad \text{and} \quad \lim_{s\to s^*}\mu_k(s) = \frac{\beta\lambda_1}{\alpha\lambda_2} > 1 \quad \text{for all } k \geq k_0.$$

The continuity of the eigenvalue functions μ_k on $(0, s^*)$ as well as the fact that $\mu_k(s) > \mu_{k+1}(s)$ for all $k \geq k_0$, $s \in (0, s^*)$, therefore implies that $0 < a_{k_0} < a_{k_0+1} < a_{k_0+2} < \cdots < \alpha/\lambda_1$ for the numbers a_k given by

$$a_k := \sup\Big\{0 < s < \frac{\alpha}{\lambda_1} : \mu_k(s) < 1\Big\}, \quad k \geq k_0.$$

By the definition of a_k, we can find $\underline{a}_k < a_k < \overline{a}_k$ such that the following inequalities hold:

$$\begin{aligned} &\text{(i)} && \mu_k(s) < 1 < \mu_{k-1}(\underline{a}_k) && \text{for all } s \leq \underline{a}_k,\ k \geq k_0,\\ &\text{(ii)} && \mu_k(s) > 1 > \mu_{k-1}(\overline{a}_k) && \text{for all } s \geq \overline{a}_k,\ k \geq k_0,\\ &\text{(iii)} && a_k - 1/k < \underline{a}_k < \overline{a}_k < \underline{a}_{k+1} && \text{for all } k \geq k_0. \end{aligned} \tag{3.4}$$

In fact, one first chooses $\overline{a}_k \in (a_k, a_{k+1})$ such that (ii) is satisfied and then $\underline{a}_k < a_k$ sufficiently close to a_k such that (i) and (iii) hold.

Now, let us show that the Leray–Schauder index $\operatorname{ind}(F(\,\cdot\,, s), (u_s, 0))$ changes sign on each of the mutually disjoint intervals $(\underline{a}_k, \overline{a}_k)$. The index of $F(\,\cdot\,, s)$ near $(u_s, 0)$ is computed using the Leray–Schauder formula which involves the algebraic multiplicities of the eigenvalues $\mu > 1$ of the compact linear operator $\operatorname{Id} - \partial_X F(u_s, 0, s)$, see [8, (II.2.11)]. From the formulas appearing in Proposition 3.2 we find that $\mu > 1$ is such an eigenvalue if and only if one of the equations

$$(-\Delta + \lambda_1)^{-1}\Big(\frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}\phi\Big) = \mu\phi \quad \text{in } \mathbb{R}^n, \quad \phi \in H_r^1(\mathbb{R}^n),\ \phi \neq 0,$$

$$L(s)\psi = (-\Delta + \lambda_2)^{-1}(W_s\psi) = \mu\psi \quad \text{in } \mathbb{R}^n, \quad \psi \in H_r^1(\mathbb{R}^n),\ \psi \neq 0,$$

is solvable. If $s = \underline{a}_k$, then the second equation is solvable with $\mu > 1$ if and only if μ is an eigenvalue of $L(\underline{a}_k)$ larger than 1. By (3.4) (i), this is equivalent to $\mu \in \{\mu_0(\underline{a}_k), \dots, \mu_{k-1}(\underline{a}_k)\}$. Due to Sturm–Liouville theory, each of these eigenvalues is simple. The first equation is solvable with $\mu > 1$ if and only if $\Delta + g'(u_s)$ has a negative eigenvalue in $H_r^1(\mathbb{R}^n)$, where g is defined as in (3.2). From [2, Theorem 5.4 (4)–(6)] we infer that there is precisely one such eigenvalue $\mu > 1$ and μ has algebraic multiplicity one. Denoting the $H_r^1(\mathbb{R}^n)$ spectrum with σ, we arrive at the formula

$$\begin{aligned} \operatorname{ind}(F(\,\cdot\,, \underline{a}_k), (0, v_{\underline{a}_k})) &= (-1)^{\#\{\mu\in\sigma(\operatorname{Id} - \partial_X F(0, v_{\underline{a}_k}, \underline{a}_k)) : \mu>1\}}\\ &= (-1)^{k+1}\\ &= -(-1)^{k+2}\\ &= -(-1)^{\#\{\mu\in\sigma(\operatorname{Id} - \partial_X F(0, v_{\overline{a}_k}, \overline{a}_k)) : \mu>1\}}\\ &= -\operatorname{ind}(F(\,\cdot\,, \overline{a}_k), (0, v_{\overline{a}_k})). \end{aligned}$$

The Krasnosel'skii–Rabinowitz theorem implies that the interval $(\underline{a}_k, \overline{a}_k)$ contains at least one bifurcation point $(u_{s_k}, 0, s_k)$, so that the maximal component $\mathcal{C}_k$ in $\mathcal{S}$ satisfying $(u_{s_k}, 0, s_k) \in \mathcal{C}_k$ is nonempty. By Proposition 3.2, this implies $\mu_j(s_k) = 1$ for some $j \in \mathbb{N}_0$ and (3.4) implies $j = k$, that is, $\mu_k(s_k) = 1$. Indeed, property (ii) gives $\mu_{k-1}(s_k) > 1$ and (i) gives $\mu_{k+1}(s_k) < 1$.

Step 2. $s_k \to s^*$ as $k \to +\infty$. If the claim did not hold, then we would have $s_k \to \overline{s}$ from below for some $\overline{s} < s^*$. From $s_k \in (\underline{a}_k, \overline{a}_k)$, the inequality $\underline{a}_k > a_k - 1/k$ and the definition of a_k, we deduce that $\mu_k(t) \geq 1$ whenever $t \geq s_k + 1/k$, $k \geq k_0$, and, thus,

$$\mu_k(t) \geq 1 \quad \text{for all } t \in \Big(\frac{\overline{s} + s^*}{2}, s^*\Big) \text{ and } k \geq k_1$$

for some sufficiently large $k_1 \in \mathbb{N}$. This contradicts $\mu_k(t) \to 0$ as $k \to +\infty$ for all $t \in (0, s^*)$ and the claim is proved.

Step 3. Existence of Seminodal Solutions within $\mathcal{C}_k$. We briefly show that fully nontrivial solutions of (1.1) belonging to a sufficiently small neighbourhood of $(u_{s_k}, 0, s_k)$ are $(0, k)$-nodal. Indeed, if solutions (u^m, v^m, s^m) of (1.1) converge to $(u_{s_k}, 0, s_k)$, then $v^m/v^m(0)$ converges to the eigenfunction ϕ of $L(s_k)$ with $\phi(0) = 1$ which is associated to the eigenvalue 1. Due to the fact that $\mu_k(s_k) = 1$ and Sturm–Liouville theory, ϕ has precisely $k + 1$ nodal annuli, so that the same is true for v^m and sufficiently large $m \in \mathbb{N}$. On the other hand, the convergence $u^m \to u$ implies that u^m must be positive for large m, which proves the claim.

Step 4. Positive Solutions. The claim concerning positive solutions of (1.1) follows directly from Proposition 3.7 and Lemma A.1 from Appendix A. □

4 Proof of Corollary 1.3 and Corollary 1.4

Let $\zeta \in H_r^1(\mathbb{R}^n)$ be the unique positive function which satisfies $-\Delta\zeta + \zeta = \zeta^3$ in $\mathbb{R}^n$, so that u_0, v_0 can be rewritten as

$$u_0(x) = \sqrt{\lambda_1}\alpha^{-1}\zeta(\sqrt{\lambda_1}x), \quad v_0(x) = \sqrt{\lambda_2}\beta^{-1}\zeta(\sqrt{\lambda_2}x).$$

Hence, Corollary 1.3 follows from Theorem 1.2 and the estimate

$$\overline{\mu}_0 = \max_{\phi\neq 0} \frac{\alpha\beta\|u_0\phi\|_2^2}{\|\phi\|_{\lambda_2}^2} \le \max_{\phi\neq 0} \frac{\alpha\beta\|u_0\|_4^2\|\phi\|_4^2}{\|\phi\|_{\lambda_2}^2} = \frac{\alpha\beta\|u_0\|_4^2\|v_0\|_4^2}{\|v_0\|_{\lambda_2}^2} = \frac{\alpha}{\beta}\frac{\|u_0\|_4^2}{\|v_0\|_4^2} = \frac{\beta}{\alpha}\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{4-n}{4}}.$$

In the case $n = 1$, we have $\zeta(x) = \sqrt{2}\,\mathrm{sech}(x)$ and it is known (see, for instance, [4, Lemma 5.1]) that the eigenvalue problem $\mu(-\phi'' + \omega^2\phi) = \zeta^2\phi$ in $\mathbb{R}$ admits nontrivial solutions in $H_r^1(\mathbb{R})$ if and only if $2/\mu = (\omega + 2k)(\omega + 2k + 1)$ for some $k \in \mathbb{N}_0$. This implies that

$$\overline{\mu}_k = \frac{\beta}{\alpha}\frac{2}{\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k\Big)\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k + 1\Big)}, \qquad k \in \mathbb{N}_0,$$

and Corollary 1.4 follows from Theorem 1.2.

5 Open Problems

Let us finally summarize some open problems concerning (1.1) which we were not able to solve and which we believe provide a better understanding of the equation. Especially the open questions concerning global bifurcation scenarios are supposed to be very difficult from the analytical point of view so that numerical indications would be very helpful, too. The following questions might be of interest.

(i) As in the author's work on weakly coupled nonlinear Schrödinger systems [11], one could try to prove the existence of positive solutions by minimizing the Euler functional over the "system Nehari manifold" $\mathcal{M}_s$ consisting of all fully nontrivial functions $(u, v) \in X$ which satisfy $I'(u, v)[(u, 0)] = I'(u, v)[(0, v)] = 0$. For which parameter values $\alpha, \beta, \lambda_1, \lambda_2, s$ are there such minimizers and do they belong to $\mathcal{C}_0$?

(ii) What is the existence theory and the bifurcation scenario when $\alpha\lambda_2 = \beta\lambda_1$ and $\alpha \neq \beta$, $\lambda_1 \neq \lambda_2$?

(iii) In the case $\alpha = \beta$, $\lambda_1 = \lambda_2$, the points on $\mathcal{T}_1, \mathcal{T}_2$ are connected by a smooth curve and the same is true for every semitrivial solution. Do these connections break up when the parameters of the equation are perturbed? This is related to the question whether the continuum $\mathcal{C}_0$ contains $\mathcal{T}_1$.

(iv) It would be interesting to know if the eigenvalue functions μ_k are strictly monotone. The monotonicity of μ_k would imply that s_k are the only solutions of $\mu_k(s) = 1$ so that the totality of bifurcation points is given by $(s_k)_{k\ge k_0}$.

(v) We expect that $\mathcal{T}_1, \mathcal{T}_2$ extend to semitrivial solution branches $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ containing also negative parameter values s. A bifurcation analysis for such branches remains open. Let us shortly comment on why we expect an interesting outcome from such a study. In the model case $n = 1$ and $\beta = \lambda_2 = 1$, one obtains from (1.4) the existence of u_s for all $s < 0$ as well as the a priori information $u_s(0)^2 \in (1/(|s|+1), 1/|s|)$. Using this, one successively proves that $su_s(0)^2 \to -1$ and $s(1+su_s(0)^2) \to 0$ as $s \to -\infty$. This implies that $W_s(0) = u_s(0)^2/(1+su_s(0)^2) \to +\infty$ as $s \to -\infty$, so that one expects that $\mu_k(s) \to +\infty$ as $s \to -\infty$ for all $k \in \mathbb{N}_0$. In view of $\overline{\mu}_{k_0} < 1$, this leads to the natural conjecture that there are also infinitely many bifurcating branches $(\tilde{\mathcal{C}}_k)_{k \geq k_0}$ in the parameter range $s < 0$.

(vi) Our paper does not contain any existence result for fully nontrivial solutions when $n \geq 4$ and $\lambda_1 \neq \lambda_2$ or $\alpha \neq \beta$. It would be interesting to know whether there is such a nonexistence result.

A A Priori Bounds

In our proof of the a priori bounds for positive solutions (u, v) of (1.1), we will use the notation $s^* := \min\{\alpha/\lambda_1, \beta/\lambda_2\}$ and $u(x) = \hat{u}(|x|)$, $v(x) = \hat{v}(|x|)$, so that $\hat{u}, \hat{v}$ denote the radial profiles of u, v. Notice that all nonnegative solutions are radially symmetric and radially decreasing by [10, Lemma 3.8]. We want to highlight the fact that the main ideas leading to Lemma A.1 are taken from [7, Section 2].

Lemma A.1. *Let $n \in \{1,2,3\}$. For all $\varepsilon > 0$, there are $c_\varepsilon, C_\varepsilon > 0$ such that all nonnegative solutions (u, v) of (1.1) for $\lambda_1, \lambda_2, \alpha, \beta \in [\varepsilon, \varepsilon^{-1}]$ and $s \in [0, \min\{\alpha/\lambda_1, \beta/\lambda_2\} - \varepsilon]$ satisfy*

$$\|u\|_{\lambda_1} + \|v\|_{\lambda_2} < C_\varepsilon \quad \text{and} \quad u(x) + v(x) \leq C_\varepsilon e^{-c_\varepsilon |x|} \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. We will break the proof into three steps.

Step 1. Boundedness in $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. Assume that there is an sequence (u_k, v_k) of nonnegative solutions of (1.1) for parameters $(\lambda_1)_k, (\lambda_2)_k, \alpha_k, \beta_k \in [\varepsilon, \varepsilon^{-1}]$ and $s_k \in [0, s^* - \varepsilon]$ which is unbounded in $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. As always, we write $Z_k(x) := \alpha_k u_k(x)^2 + \beta_k v_k(x)^2$. Passing to a subsequence, we may assume that $Z_k(0) = \max_{\mathbb{R}^n} Z_k \to +\infty$ and $((\lambda_1)_k, (\lambda_2)_k, \alpha_k, \beta_k, s_k) \to (\lambda_1, \lambda_2, \alpha, \beta, s)$ for some $s \in [0, s^* - \varepsilon]$ and $\lambda_1, \lambda_2, \alpha, \beta \in [\varepsilon, \varepsilon^{-1}]$. Let us distinguish the cases $s > 0$ and $s = 0$ to lead this assumption to a contradiction.

For the case $s > 0$, the functions

$$\phi_k := u_k Z_k(0)^{-1/2}, \quad \psi_k := v_k Z_k(0)^{-1/2}$$

are bounded in $L^\infty(\mathbb{R}^n)$ and satisfy $\alpha_k \phi_k(0)^2 + \beta_k \psi_k(0)^2 = 1$ as well as

$$-\Delta \phi_k + (\lambda_1)_k \phi_k = \alpha_k \phi_k \frac{Z_k}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n,$$

$$-\Delta \psi_k + (\lambda_2)_k \phi_k = \beta_k \psi_k \frac{Z_k}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n.$$

Using the fact that $Z_k/(1+s_k Z_k) \leq s_k^{-1} = s^{-1} + o(1)$ and De Giorgi–Nash–Moser estimates, we obtain from the Arzelà–Ascoli theorem that there are bounded nonnegative radially symmetric limit functions $\phi, \psi \in C^1(\mathbb{R}^n)$ satisfying $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ and

$$-\Delta \phi + \lambda_1 \phi = \frac{\alpha}{s} \phi \quad \text{in } \mathbb{R}^n,$$

$$-\Delta \psi + \lambda_2 \psi = \frac{\beta}{s} \psi \quad \text{in } \mathbb{R}^n.$$

From $\lambda_1 < \alpha/s$ and $\lambda_2 < \beta/s$ we obtain

$$\phi(r) = \kappa_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(\left(\frac{\alpha}{s} - \lambda_1\right)^{1/2} r\right) \quad \text{and} \quad \psi(r) = \kappa_2 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(\left(\frac{\beta}{s} - \lambda_2\right)^{1/2} r\right) \quad \text{for } r \geq 0$$

and for some $\kappa_1, \kappa_2 \in \mathbb{R}$. Since the functions ϕ, ψ are nonnegative, this is only possible in the case $\kappa_1 = \kappa_2 = 0$, which contradicts $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$. Hence, the case $s > 0$ does not occur.

For the case $s = 0$, we first show that $s_k Z_k \to 0$ uniformly on $\mathbb{R}^n$ which, due to the fact that $Z_k(0) = \max_{\mathbb{R}^n} Z_k$, is equivalent to proving that $s_k Z_k(0) \to 0$. So, let κ be an arbitrary accumulation point of the sequence $(s_k Z_k(0))_{k\in\mathbb{N}}$ and without loss of generality we assume that $s_k Z_k(0) \to \kappa \in [0, +\infty]$, so that we are left to show that $\kappa = 0$. To this end, set

$$\phi_k(x) := u_k(\sqrt{s_k}x)Z_k(0)^{-1/2}, \quad \psi_k(x) := v_k(\sqrt{s_k}x)Z_k(0)^{-1/2}.$$

The functions ϕ_k, ψ_k satisfy $\alpha_k\phi_k(0)^2 + \beta_k\psi_k(0)^2 = 1$ as well as

$$-\Delta\phi_k + s_k(\lambda_1)_k\phi_k = \alpha_k\phi_k\frac{s_k Z_k}{1 + s_k Z_k} = \alpha_k\phi_k\frac{s_k Z_k(0)(\alpha_k\phi_k^2 + \beta_k\psi_k^2)}{1 + s_k Z_k(0)(\alpha_k\phi_k^2 + \beta_k\psi_k^2)} \quad \text{in } \mathbb{R}^n,$$

$$-\Delta\psi_k + s_k(\lambda_2)_k\psi_k = \beta_k\psi_k\frac{s_k Z_k}{1 + s_k Z_k} = \beta_k\psi_k\frac{s_k Z_k(0)(\alpha_k\phi_k^2 + \beta_k\psi_k^2)}{1 + s_k Z_k(0)(\alpha_k\phi_k^2 + \beta_k\psi_k^2)} \quad \text{in } \mathbb{R}^n.$$

The Arzelà–Ascoli theorem implies that a subsequence (ϕ_k), (ψ_k) converges locally uniformly to nonnegative functions $\phi, \psi \in C^1(\mathbb{R}^n)$ satisfying $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ and

$$-\Delta\phi = \alpha\phi\frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \quad \text{in } \mathbb{R}^n,$$

$$-\Delta\psi = \beta\psi\frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \quad \text{in } \mathbb{R}^n.$$

For the case $\kappa = +\infty$, we arrive at a contradiction as in the case $s > 0$, so let us assume that $\kappa < +\infty$. Then, $z := \phi + \psi$ is nonnegative, nontrivial and the inequality $\alpha\phi^2 + \beta\psi^2 \le \alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ implies that

$$\begin{aligned}
-\Delta z &= (\alpha\phi + \beta\psi)\frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \\
&\ge \min\{\alpha, \beta\}(\phi + \psi)\frac{\kappa}{1 + \kappa}(\alpha\phi^2 + \beta\psi^2) \\
&\ge c(\kappa)(\phi + \psi)^3 \\
&= c(\kappa)z^3,
\end{aligned}$$

where $c(\kappa) = \min\{\alpha, \beta\}^2\kappa/(2(1 + \kappa))$. From [14, Theorem 8.4] we infer that $c(\kappa) = 0$ and, thus, $\kappa = 0$. Hence, every accumulation point of the sequence $(s_k Z_k(0))$ is zero, so that $s_k Z_k$ converges to the trivial function uniformly on $\mathbb{R}^n$.

With this result at hand, one can use the classical blow-up technique by considering

$$\tilde{\phi}_k(x) := u_k(Z_k(0)^{-1/2}x)Z_k(0)^{-1/2}, \quad \tilde{\psi}_k(x) := v_k(Z_k(0)^{-1/2}x)Z_k(0)^{-1/2}.$$

These functions satisfy $\alpha_k\tilde{\phi}_k(0)^2 + \beta_k\tilde{\psi}_k(0)^2 = 1$ as well as

$$-\Delta\tilde{\phi}_k + Z_k(0)^{-1}(\lambda_1)_k\tilde{\phi}_k = \alpha_k\tilde{\phi}_k\frac{Z_k Z_k(0)^{-1}}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n,$$

$$-\Delta\tilde{\psi}_k + Z_k(0)^{-1}(\lambda_2)_k\tilde{\psi}_k = \beta_k\tilde{\psi}_k\frac{Z_k Z_k(0)^{-1}}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n.$$

Then, we have $s_k Z_k \to 0$ uniformly in $\mathbb{R}^n$ and similar arguments as the ones used above lead to a bounded nonnegative nontrivial solution ϕ, ψ of

$$-\Delta\phi = \alpha\phi(\alpha\phi^2 + \beta\psi^2) \quad \text{in } \mathbb{R}^n,$$

$$-\Delta\psi = \beta\psi(\alpha\phi^2 + \beta\psi^2) \quad \text{in } \mathbb{R}^n,$$

which we may lead to a contradiction as above. This finally shows that $Z_k(0) \to +\infty$ is also impossible in the case $s = 0$, so that the nonnegative solutions (u, v) of (1.1) are pointwise bounded by some constant depending on ε.

Step 2. Uniform Exponential Decay. Let us assume for contradiction that there is a sequence (u_k, v_k, s_k) of positive solutions of (1.1) satisfying

$$\hat{u}_k(r_k) + \hat{v}_k(r_k) \geq ke^{-r_k/k} \quad \text{for all } k \in \mathbb{N} \text{ and some } r_k > 0. \tag{A.1}$$

Due to the L^∞-bounds for (u_k, v_k) which we proved in the first step, we can use De Giorgi–Nash–Moser estimates and the Arzelà–Ascoli theorem to obtain a smooth bounded radially symmetric limit function (u, v) of a suitable subsequence of (u_k, v_k). As a limit of positive radially decreasing functions, u, v are also nonnegative and radially nonincreasing. In particular, we may define

$$u_\infty := \lim_{r\to+\infty} \hat{u}(r) \geq 0, \quad v_\infty := \lim_{r\to+\infty} \hat{v}(r) \geq 0.$$

Our first aim is to show that $u_\infty = v_\infty = 0$. Since $(\hat{u}, \hat{v})$ decreases to some limit at infinity, we have $\hat{u}'(r), \hat{v}'(r), \hat{u}''(r), \hat{v}''(r) \to 0$ as $r \to +\infty$, so that (1.1) implies that

$$\lambda_1 u_\infty = \frac{\alpha u_\infty Z_\infty}{1 + sZ_\infty}, \quad \lambda_2 v_\infty = \frac{\beta v_\infty Z_\infty}{1 + sZ_\infty}, \quad \text{where } Z_\infty = \alpha u_\infty^2 + \beta v_\infty^2. \tag{A.2}$$

Now, define

$$\begin{aligned} E_k(r) &:= \hat{u}_k'(r)^2 + \hat{v}_k'(r)^2 - \lambda_1 \hat{u}_k(r)^2 - \lambda_2 \hat{v}_k(r)^2 + s^{-2} g(sZ_k(r)), \\ E(r) &:= \hat{u}'(r)^2 + \hat{v}'(r)^2 - \lambda_1 \hat{u}(r)^2 - \lambda_2 \hat{v}(r)^2 + s^{-2} g(sZ(r)). \end{aligned}$$

The differential equation implies that

$$E_k'(r) = -\frac{2(n-1)}{r}(\hat{u}_k'(r)^2 + \hat{v}_k'(r)^2) \leq 0,$$

so that E_k decreases to some limit at infinity. The monotonicity of $\hat{u}_k, \hat{v}_k$ and the fact that $\hat{u}_k(r), \hat{v}_k(r) \to 0$ as $r \to +\infty$ imply that this limit must be 0. In particular, we obtain that $E_k \geq 0$ and the pointwise convergence $E_k \to E$ implies that E is a nonnegative nonincreasing function. From this we obtain that

$$0 \leq \lim_{r\to+\infty} E(r) = -\lambda_1 u_\infty^2 - \lambda_2 v_\infty^2 + s^{-2} g(sZ_\infty) \overset{\text{(A.2)}}{=} -\frac{Z_\infty^2}{1 + sZ_\infty} + s^{-2} g(sZ_\infty) = \frac{1}{s^2}\Big(\frac{sZ_\infty}{1 + sZ_\infty} - \ln(1 + sZ_\infty)\Big).$$

This equation implies that $Z_\infty = 0$ and, hence, $u_\infty = v_\infty = 0$.

Now, let μ satisfy $0 < \mu < \sqrt{\min\{\lambda_1, \lambda_2\}}$ and choose $\delta > 0$. Due to the fact that $u_\infty = v_\infty = 0$, we may choose $r_0 > 0$ such that $\hat{u}(r_0) + \hat{v}(r_0) < \delta/2$ holds. From $\hat{u}_k(r_0) \to \hat{u}(r_0), \hat{v}_k(r_0) \to \hat{v}(r_0)$ and the fact that $\hat{u}_k, \hat{v}_k$ are decreasing, we obtain that $\hat{u}_k(r) + \hat{v}_k(r) \leq \delta$ for all $r \geq r_0$ and all $k \geq k_0$ for some sufficiently large $k_0 \in \mathbb{N}$. Having chosen $\delta > 0$ sufficiently small, the inequality $\hat{u}_k', \hat{v}_k' \leq 0$ implies that

$$-(\hat{u}_k + \hat{v}_k)'' + \mu^2(\hat{u}_k + \hat{v}_k) \leq 0 \quad \text{on } [r_0, +\infty) \text{ for all } k \geq k_0.$$

Hence, the maximum principle implies that for any given $R > r_0$, the function $w_R(r) := e^{-\mu(r-r_0)} + e^{-\mu(R-r)}$ satisfies $\hat{u}_k + \hat{v}_k \leq w_R$ on (r_0, R). Indeed, w_R dominates $\hat{u}_k + \hat{v}_k$ on the boundary of (r_0, R) due to the fact that

$$w_R(r_0) = w_R(R) \geq 1 \geq \delta \geq (\hat{u}_k + \hat{v}_k)(r_0) = \max\{(\hat{u}_k + \hat{v}_k)(r_0), (\hat{u}_k + \hat{v}_k)(R)\}.$$

Sending R to infinity, we obtain that

$$(\hat{u}_k + \hat{v}_k)(r) \leq e^{-\mu(r-r_0)} \quad \text{for all } r \geq r_0,$$

which, together with the a priori bounds from the first step, yields a contradiction to the assumption (A.1). This proves the uniform exponential decay.

Step 3. Conclusion. Given the uniform exponential decay of (u, v), we obtain a uniform bound on $\|u\|_{L^4(\mathbb{R}^n)}$, $\|v\|_{L^4(\mathbb{R}^n)}$ which, using the differential equation (1.1), gives a uniform bound on $\|u\|_{\lambda_1}, \|v\|_{\lambda_2}$. This finishes the proof. □

Let us mention that in view of Proposition 3.4, the a priori bounds from the above lemma cannot be extended to the interval $s \in [0, \min\{\alpha/\lambda_1, \beta/\lambda_2\}]$. Furthermore, positive solutions of (1.1) are not uniformly bounded for parameters s belonging to neighbourhoods of 0 when $n \geq 4$, see Remark 3.6. Notice that the assumption $n \in \{1, 2, 3\}$ in the proof of the above lemma only becomes important when we apply [14, Theorem 8.4].

B A Functional Analytic Setting for $n = 1$

In this section, we show that in the one-dimensional case, the function $F(\cdot, s) : X \to X$ given by (3.1) is a compact perturbation of the identity for an appropriately chosen Banach space X such that $\mathcal{T}_1, \mathcal{T}_2$ are continuous curves in $X \times (0, +\infty)$. Let $\sigma \in (0, 1)$ be fixed and set $(X, \langle \cdot, \cdot \rangle_X)$ to be the Hilbert space given by

$$X := \{(u, v) \in H_r^1(\mathbb{R}) \times H_r^1(\mathbb{R}) : \langle (u, v), (u, v) \rangle_X < +\infty\}$$

with

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_X := \int_0^{+\infty} e^{2\sigma\mu_1 x}(u'\tilde{u}' + \mu_1^2 u\tilde{u})\, dx + \int_0^{+\infty} e^{2\sigma\mu_2 x}(v'\tilde{v}' + \mu_2^2 v\tilde{v})\, dx,$$

where $\mu_1 := \sqrt{\lambda_1}$ and $\mu_2 := \sqrt{\lambda_2}$. One may check that $(X, \langle \cdot, \cdot \rangle_X)$ is a Hilbert space and the subspace $C_{0,r}^\infty(\mathbb{R}) \times C_{0,r}^\infty(\mathbb{R})$ consisting of smooth even functions having compact support is dense in X. We will use the formula

$$((-\Delta + \mu^2)^{-1} f)(x) = \frac{\mu}{2} \int_{\mathbb{R}} e^{-\mu|x-y|} f(y)\, dy = \int_0^{+\infty} \mu\Gamma(\mu x, \mu y) f(y)\, dy \tag{B.1}$$

for all $f \in C_{0,r}^\infty(\mathbb{R})$ and $\mu > 0$, where

$$\Gamma(x, y) = \frac{1}{2}(e^{-|x-y|} + e^{-|x+y|}).$$

Proof of Well-Definedness. First, let us prove for all $(u, v) \in X$ the estimate

$$\sqrt{\mu_1}|u(r)| \le \|(u, v)\|_X e^{-\sigma\mu_1 r} \quad \text{and} \quad \sqrt{\mu_2}|v(r)| \le \|(u, v)\|_X e^{-\sigma\mu_2 r} \quad \text{for } r \ge 0. \tag{B.2}$$

It suffices to prove these inequalities for $u, v \in C_{0,r}^\infty(\mathbb{R})$. For such functions, we have

$$\mu_1 u(r)^2 \le 2\mu_1 \int_r^{+\infty} |uu'|\, dx \le e^{-2\sigma\mu_1 r} \int_r^{\infty} e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\, dx \le \|(u, v)\|_X^2 e^{-2\sigma\mu_1 r},$$

$$\mu_2 v(r)^2 \le 2\mu_2 \int_r^{+\infty} |vv'|\, dx \le e^{-2\sigma\mu_2 r} \int_r^{\infty} e^{2\sigma\mu_2 x}(v'^2 + \mu_2^2 v^2)\, dx \le \|(u, v)\|_X^2 e^{-2\sigma\mu_2 r}.$$

Next, using that $u'(0) = v'(0) = 0$ and the fact that u, v have compact support, we obtain

$$\begin{aligned}
\int_0^{+\infty} e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\, dx &= \int_0^{+\infty} (e^{2\sigma\mu_1 x} uu')' - 2\sigma\mu_1 e^{2\sigma\mu_1 x} uu' + e^{2\sigma\mu_1 x} u(-u'' + \mu_1^2 u)\, dx \\
&= -2\sigma\mu_1 \int_0^{+\infty} e^{2\sigma\mu_1 x} uu'\, dx + \int_0^{+\infty} e^{2\sigma\mu_1 x} u(-u'' + \mu_1^2 u)\, dx \\
&\le \sigma \int_0^{+\infty} e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\, dx + \int_0^{+\infty} e^{2\sigma\mu_1 x} u(-u'' + \mu_1^2 u)\, dx.
\end{aligned}$$

Then, performing the analogous rearrangements for v, yields for all $u, v \in C_{0,r}^\infty(\mathbb{R})$ that

$$\|(u, v)\|_X^2 \le \frac{1}{1-\sigma} \int_0^{+\infty} e^{2\sigma\mu_1 x} u(-u'' + \mu_1^2 u)\, dx + \frac{1}{1-\sigma} \int_0^{+\infty} e^{2\sigma\mu_2 x} v(-v'' + \mu_2^2 v)\, dx. \tag{B.3}$$

Applying this inequality to $(u,v) = ((-\Delta+\mu_1^2)^{-1}(f)\chi_R, (-\Delta+\mu_2^2)^{-1}(g)\chi_R)$ for $f, g \in C^\infty_{0,r}(\mathbb{R})$ and a suitable family $(\chi_R)_{R>0}$ of cut-off functions converging to 1, we obtain

$$\begin{aligned}
\|((-\Delta+\mu_1^2)^{-1}(f), (-\Delta+\mu_2^2)^{-1}(g))\|_X^2 &\overset{(B.3)}{\leq} \frac{1}{1-\sigma}\int_0^{+\infty} e^{2\sigma\mu_1 x}(-\Delta+\mu_1^2)^{-1}(f)(x)f(x)\,dx \\
&\quad + \frac{1}{1-\sigma}\int_0^{+\infty} e^{2\sigma\mu_2 x}(-\Delta+\mu_2^2)^{-1}(g)(x)g(x)\,dx \\
&\overset{(B.1)}{=} \frac{\mu_1}{1-\sigma}\int_0^{+\infty}\int_0^{+\infty} e^{2\sigma\mu_1 x}\Gamma(\mu_1 x, \mu_1 y)f(x)f(y)\,dx\,dy \\
&\quad + \frac{\mu_2}{1-\sigma}\int_0^{+\infty}\int_0^{+\infty} e^{2\sigma\mu_2 x}\Gamma(\mu_2 x, \mu_2 y)g(x)g(y)\,dx\,dy \\
&\leq \frac{\mu_1}{1-\sigma}\int_0^{+\infty}\int_0^{+\infty} e^{\sigma\mu_1 x}e^{\sigma\mu_1 y}|f(x)||f(y)|\,dx\,dy \\
&\quad + \frac{\mu_2}{1-\sigma}\int_0^{+\infty}\int_0^{+\infty} e^{\sigma\mu_2 x}e^{\sigma\mu_2 y}|g(x)||g(y)|\,dx\,dy \\
&= \frac{\mu_1}{1-\sigma}\bigg(\int_0^{+\infty} e^{\sigma\mu_1 x}|f(x)|\,dx\bigg)^2 + \frac{\mu_2}{1-\sigma}\bigg(\int_0^{+\infty} e^{\sigma\mu_2 x}|g(x)|\,dx\bigg)^2.
\end{aligned}$$

Plugging in

$$f := f_{u,v} := \frac{\alpha u Z}{1+sZ} \leq \alpha u(\alpha u^2+\beta v^2), \quad g := g_{u,v} := \frac{\beta v Z}{1+sZ} \leq \beta v(\alpha u^2+\beta v^2)$$

and using the estimate (B.2), we find that there is a positive number C depending on $\sigma, \mu_1, \mu_2, \alpha, \beta$ but not on u, v such that

$$\|((-\Delta+\mu_1^2)^{-1}(f_{u,v}), (-\Delta+\mu_2^2)^{-1}(g_{u,v}))\|_X \leq C\|(u,v)\|_X^3. \tag{B.4}$$

By the density of $C^\infty_{0,r}(\mathbb{R}) \times C^\infty_{0,r}(\mathbb{R})$ in X, this inequality also holds for $(u,v) \in X$. If now (u_k, v_k) is a sequence in $C^\infty_{0,r}(\mathbb{R}) \times C^\infty_{0,r}(\mathbb{R})$ converging to $(u,v) \in X$, then similar estimates based on (B.2) show that

$$\|((-\Delta+\mu_1^2)^{-1}(f_{u_k,v_k}-f_{u_m,v_m}), (-\Delta+\mu_2^2)^{-1}(g_{u_k,v_k}-g_{u_m,v_m}))\|_X \leq C\|(u_k-u_m, v_k-u_m)\|_X(\|(u_k,v_k)\|_X+\|(u_m,v_m)\|_X)^2$$

for some $C > 0$, implying that $F : X \times (0,+\infty) \to X$ is well defined and that (B.4) also holds for $(u,v) \in X$.

Proof of Compactness of $\mathrm{Id}-F$. Let now (u_m, v_m) be a bounded sequence in X. Then, without loss of generality, we can assume that $(u_m, v_m) \rightharpoonup (u,v) \in X$ and $(u_m, v_m) \to (u,v)$ pointwise almost everywhere. We set

$$f_m := \frac{\alpha u_m Z_m}{1+sZ_m}, \quad g_m := \frac{\beta v_m Z_m}{1+sZ_m}, \quad f := \frac{\alpha u Z}{1+sZ}, \quad g := \frac{\beta v Z}{1+sZ},$$

where $Z_m := \alpha u_m^2 + \beta v_m^2$ and $Z := \alpha u^2 + \beta v^2$. Then, we have $f_m \to f$ and $g_m \to g$ pointwise almost everywhere and the estimate (B.2) implies that

$$|f_m(r)| + |f(r)| \leq \alpha(|u_m(r)|Z_m(r) + |u(r)|Z(r)) \leq C(e^{-3\sigma\mu_1 r} + e^{-\sigma(\mu_1+2\mu_2)r}), \tag{B.5}$$

$$|g_m(r)| + |g(r)| \leq \beta(|v_m(r)|Z_m(r) + |v(r)|Z(r)) \leq C(e^{-3\sigma\mu_2 r} + e^{-\sigma(\mu_2+2\mu_1)r}) \tag{B.6}$$

for some positive number $C > 0$. Using the estimate from above, we therefore obtain that

$$\begin{aligned}
\|(\mathrm{Id}-F)(u_m,v_m) - (\mathrm{Id}-F)(u,v)\|_X^2 &= \|((-\Delta+\mu_1^2)^{-1}(f_m-f), (-\Delta+\mu_2^2)^{-1}(g_m-g))\|_X^2 \\
&\leq \frac{\mu_1}{1-\sigma}\bigg(\int_0^{+\infty} e^{\sigma\mu_1 x}|f_m(x)-f(x)|\,dx\bigg)^2 + \frac{\mu_2}{1-\sigma}\bigg(\int_0^{+\infty} e^{\sigma\mu_2 x}|g_m(x)-g(x)|\,dx\bigg)^2.
\end{aligned}$$

Using (B.5), (B.6) and the dominated convergence theorem, we finally get that

$$\|(\mathrm{Id} - F)(u_m, v_m) - (\mathrm{Id} - F)(u, v)\|_X \to 0 \quad \text{as } m \to +\infty,$$

which is all we had to show.

C A Spectral Theoretic Result

Finally, we prove a spectral theoretical result which we used in the proof of Proposition 3.5 and for which we could not find a reference in the literature. The key ingredient of this result is the min-max principle for eigenvalues of semibounded self-adjoint Schrödinger operators, see, for instance, [16, Theorem XIII.2]. As in Proposition 3.5, we denote by $\mu_k(s)$, $k \in \mathbb{N}_0$, the k-th eigenvalue of the compact self-adjoint operator

$$L_s : H_r^1(\mathbb{R}^n) \to H_r^1(\mathbb{R}^n) \quad \text{with} \quad L_s\phi := (-\Delta + \lambda)^{-1}(W_s\phi) \tag{C.1}$$

for potentials W_s vanishing at infinity, that is, $W_s(x) \to 0$ as $|x| \to +\infty$.

Lemma C.1. *Let $n \in \mathbb{N}$, $\kappa, \lambda > 0$, $a < b$ and let $(W_s)_{s\in(a,b)}$ be a family of radially symmetric potentials $W_s : \mathbb{R}^n \to [0, +\infty)$ vanishing at infinity and satisfying*

$$\text{(i)} \ \limsup_{s\to b} \|W_s\|_\infty = \kappa \quad \textit{and} \quad \text{(ii)} \ W_s \to \kappa \textit{ locally uniformly as } s \to b.$$

Then, we have $\mu_k(s) \to \kappa/\lambda$ as $s \to b$ for all $k \in \mathbb{N}_0$.

Proof. The min-max principle and (i) imply that

$$\limsup_{s\to b} \mu_k(s) \le \limsup_{s\to b} \frac{\|W_s\|_\infty}{\lambda} = \frac{\kappa}{\lambda}.$$

So, it remains to show the corresponding estimate from below. Given the assumptions $W_s \ge 0$ and (ii), we find that it is sufficient to show that $\mu_k^\varepsilon \to \kappa/\lambda$ as $\varepsilon \to 0$, where μ_k^ε denotes the k-th eigenvalue of the compact self-adjoint operator $M_\varepsilon : H_r^1(\mathbb{R}^n) \to H_r^1(\mathbb{R}^n)$ defined by $M_\varepsilon\phi = (-\Delta + \lambda)^{-1}((\kappa - \varepsilon)1_{B_{1/\varepsilon}}\phi)$. Here, $1_{B_{1/\varepsilon}}$ denotes the indicator function of the ball in $\mathbb{R}^n$ centered at the origin with radius $1/\varepsilon$. Since $\varepsilon \to M_\varepsilon$ is continuous on $(0, +\infty)$ with respect to the operator norm, the min-max characterization of the eigenvalues implies that the mapping

$$\varepsilon \mapsto \omega_k^\varepsilon$$

is continuous on $(0, +\infty)$, where

$$\omega_k^\varepsilon := \frac{\kappa - \varepsilon}{\mu_k^\varepsilon} - \lambda.$$

By the definition of μ_k^ε, ω_k^ε, the boundary value problem

$$\begin{cases} -\phi''(r) - \dfrac{n-1}{r}\phi'(r) = \omega_k^\varepsilon\phi(r) & \text{for } 0 \le r \le \varepsilon^{-1}, \\ -\phi''(r) - \dfrac{n-1}{r}\phi'(r) = -\lambda\phi(r) & \text{for } r \ge \varepsilon^{-1} \end{cases}$$

with

$$\phi'(0) = 0 \quad \text{and} \quad \phi(r) \to 0 \text{ as } r \to +\infty$$

for $\phi \in C^1([0, +\infty))$ has a nontrivial solution. Testing the differential equation on $[0, \varepsilon^{-1}]$ with ϕ, we obtain that $\omega_k^\varepsilon > 0$. Hence, ϕ is given by

$$\phi(r) = \alpha \begin{cases} c r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon} r) & \text{if } r \le \varepsilon^{-1}, \\ r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(\sqrt{\lambda} r) & \text{if } r \ge \varepsilon^{-1} \end{cases}$$

for some $\alpha \neq 0$. Here, K denotes the modified Bessel function of the second kind and J represents the Bessel function of the first kind. From $\phi \in C^1([0,+\infty))$ we get the conditions

$$K_{\frac{n-2}{2}}(\sqrt{\lambda}\varepsilon^{-1}) = cJ_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon}\varepsilon^{-1}), \qquad \sqrt{\lambda}K'_{\frac{n-2}{2}}(\sqrt{\lambda}\varepsilon^{-1}) = \sqrt{\omega_k^\varepsilon}cJ'_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon}\varepsilon^{-1})$$

on c and ω_k^ε. Due to the continuity of $\varepsilon \to \omega_k^\varepsilon$ on $(0,+\infty)$ and due to the fact that K is positive whereas J has infinitely many zeros going off to infinity, we infer that $\sqrt{\omega_k^\varepsilon}\varepsilon^{-1}$ is bounded on $(0,+\infty)$. In particular, this gives that $\omega_k^\varepsilon \to 0$ and, thus, $\mu_k^\varepsilon \to \kappa/\lambda$ as $\varepsilon \to 0$, which is all we had to show. □

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