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Research Article

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Estimates for Eigenvalues of Poly-Harmonic Operators

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Abstract: In this paper, we study the eigenvalues of the poly-Laplacian with arbitrary order on a bounded domain in an n-dimensional Euclidean space and we obtain a lower bound which generalizes the results due to Cheng and Wei [5] and gives an improvement of the results due to Cheng, Qi and Wei [3].

Keywords: Eigenvalue Problem, Lower Bound for Eigenvalues, Poly-Laplacian with Arbitrary Order

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1 Introduction

Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbb{R}^n$. Let λ_i be the i-th eigenvalue of the Dirichlet eigenvalue problem of the poly-Laplacian with arbitrary order

$$\begin{cases} (-\Delta)^l u = \lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where l is a positive integer, Δ is the Laplacian in $\mathbb{R}^n$, and ν denotes the outward unit normal vector field of the boundary $\partial\Omega$. It is well known that the spectrum of this eigenvalue problem is real and discrete, that is,

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty,$$

where each λ_i has finite multiplicity and is repeated according to its multiplicity. Let $V(\Omega)$ denote the volume of Ω and let B_n denote the volume of the unit ball in $\mathbb{R}^n$.

When $l = 1$, the eigenvalue problem (1.1) is called a fixed membrane problem. In this case, one has Weyl's asymptotic formula

$$\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to +\infty.$$

From the above asymptotic formula, one can derive the formula

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to +\infty. \tag{1.2}$$

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Pólya [11] proved that

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots,$$

if Ω is a tiling domain in $\mathbb{R}^n$. Furthermore, he proposed the following conjecture.

Conjecture 1.1 (Pólya). *If Ω is a bounded domain in $\mathbb{R}^n$, then the k-th eigenvalue λ_k of the fixed membrane problem satisfies*

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots .$$

Berezin [2] and Lieb [8] gave a partial solution for the conjecture of Pólya. In particular, Li and Yau [7] proved that

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots . \tag{1.3}$$

The formula (1.2) shows that the result of Li and Yau is sharp in the sense of average. From (1.3) one can infer that

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots,$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$. Recently, Melas [9] has improved (1.3) to the estimate

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{1}{24(n+2)} \frac{V(\Omega)}{I(\Omega)}, \quad k = 1, 2, \ldots,$$

where

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x-a|^2 \, dx$$

is called *the moment of inertia* of Ω.

When $l = 2$, the eigenvalue problem (1.1) is called a clamped plate problem. For the eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [10] obtained

$$\lambda_k \sim \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to +\infty. \tag{1.4}$$

From (1.4) one can obtain

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \to +\infty. \tag{1.5}$$

Furthermore, Levine and Protter [6] proved that the eigenvalues of the clamped plate problem satisfy the inequality

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}. \tag{1.6}$$

The formula (1.5) shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible constant. By adding to its right-hand side two terms of lower order in k, Cheng and Wei [4] obtained the estimate

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq{}& \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} + \left(\frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)}\right) \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} \frac{n}{n+2} \frac{V(\Omega)}{I(\Omega)} k^{\frac{2}{n}} \\ &+ \left(\frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)}\right)\left(\frac{V(\Omega)}{I(\Omega)}\right)^2,\end{aligned} \tag{1.7}$$

which is an improvement of (1.6). Very recently, Cheng and Wei [5] have improved the estimate (1.7) to the estimate

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} + \frac{n+2}{12n(n+4)} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} \frac{n}{n+2} \frac{V(\Omega)}{I(\Omega)} k^{\frac{2}{n}} + \frac{(n+2)^2}{1152n(n+4)^2}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2.$$

When l is arbitrary, Levine and Protter [6] proved the estimate

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2l}\frac{\pi^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}}, \quad k=1,2,\dots, \tag{1.8}$$

which implies that

$$\lambda_k \geq \frac{n}{n+2l}\frac{\pi^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}}, \quad k=1,2,\dots.$$

By adding l terms of lower order of $k^{\frac{2l}{n}}$ to its right-hand side, Cheng, Qi and Wei [3] obtained a sharper result than (1.8), that is,

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{n}{(n+2l)}\sum_{p=1}^{l}\frac{l+1-p}{(24)^p n\cdots(n+2p-2)}\frac{(2\pi)^{2(l-p)}}{(B_n V(\Omega))^{\frac{2(l-p)}{n}}}\left(\frac{V(\Omega)}{I(\Omega)}\right)^p k^{\frac{2(l-p)}{n}}. \tag{1.9}$$

In this paper, we study the eigenvalues of the Dirichlet eigenvalue problem (1.1) of a Laplacian with arbitrary order and prove the following result.

Theorem 1.2. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that $l \geq 2$ and λ_i is the i-th eigenvalue of the eigenvalue problem* (1.1). *Then, the eigenvalues satisfy*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq{}& \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{l}{24(n+2l)}\frac{(2\pi)^{2(l-1)}}{(B_n V(\Omega))^{\frac{2(l-1)}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2(l-1)}{n}}\\ &+ \frac{l(n+2(l-1))^2}{2304n(n+2l)^2}\frac{(2\pi)^{2(l-2)}}{(B_n V(\Omega))^{\frac{2(l-2)}{n}}}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2 k^{\frac{2(l-2)}{n}}.\end{aligned} \tag{1.10}$$

Remark 1.3. When $l = 2$, Theorem 1.2 reduces to the result of Cheng and Wei [5].

Remark 1.4. When $l \geq 2$, we give an important improvement of the result in (1.9) due to Cheng, Qi and Wei [3], since the inequality (1.10) is sharper than (1.9). We will give a proof of this fact in Section 2.

2 Proofs of Theorem 1.2 and Remark 1.4

In this section, we give the proof of Theorem 1.2. At first, we need the following key lemma which will play an important role in the proof of Theorem 1.2. Its proof will be given in Appendix A.

Lemma 2.1. *Let $b \geq 2$ be a positive real number and $\mu > 0$. If $\psi : [0, +\infty) \to [0, +\infty)$ is a decreasing function such that*

$$-\mu \leq \psi'(s) \leq 0$$

and

$$A := \int_0^\infty s^{b-1}\psi(s)\,ds > 0,$$

then, for any positive integer $l \geq 2$, we have

$$\begin{aligned}\int_0^\infty s^{b+2l-1}\psi(s)\,ds \geq{}& \frac{1}{b+2l}(bA)^{\frac{b+2l}{b}}\psi(0)^{-\frac{2l}{b}} + \frac{l}{6b(b+2l)\mu^2}(bA)^{\frac{b+2(l-1)}{b}}\psi(0)^{\frac{2b-2l+2}{b}}\\ &+ \frac{l(b+2(l-1))^2}{144b^2(b+2l)^2\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}}.\end{aligned}$$

Proof of Theorem 1.2. In this proof, we will use the same notation as in [3]. Let $\widehat{\varphi}_j(z)$ be the Fourier transform of the trial function $\varphi_j(x)$ defined as

$$\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $u_j(x)$ is an orthonormal eigenfunction corresponding to the eigenvalue λ_j, $f(z) := \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2$, and f^* is the symmetric decreasing rearrangement of f. We assume that $A = \frac{k}{nB_n}$, $b = n$, and $\psi(s) = \phi(s)$, where $\phi : [0, +\infty) \to [0, (2\pi)^{-n} V(\Omega)]$ is a non-increasing function of $|x|$ and $\phi(x)$ is defined by $\phi(|x|) := f^*(x)$. Then, from Lemma 2.1 we can obtain that

$$\begin{aligned}\sum_{j=1}^k \lambda_j &\geq nB_n \int_0^\infty s^{n+2l-1}\phi(s)\,ds \\ &\geq \frac{nB_n(\frac{k}{B_n})^{\frac{n+2l}{n}}}{n+2l}\phi(0)^{-\frac{2l}{n}} + \frac{lB_n(\frac{k}{B_n})^{\frac{n+2(l-1)}{n}}}{6(n+2l)\mu^2}\phi(0)^{\frac{2n-2l+2}{n}} + \frac{l(n+2(l-1))^2B_n(\frac{k}{B_n})^{\frac{n+2l-4}{n}}}{144n(n+2l)^2\mu^4}\phi(0)^{\frac{4n-2l+4}{n}}. \end{aligned} \tag{2.1}$$

Now, we define a function $\xi(t)$ as

$$\xi(t) = \frac{nB_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{2l}{n}} + \frac{lB_n}{6(n+2l)\mu^2}\Big(\frac{k}{B_n}\Big)^{\frac{n+2(l-1)}{n}} t^{\frac{2n-2l+2}{n}} + \frac{l(n+2(l-1))^2B_n}{144n(n+2l)^2\mu^4}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l-4}{n}} t^{\frac{4n-2l+4}{n}}. \tag{2.2}$$

Here, we assume that $l \leq n+1$. The other cases (i.e., $n+1 < l < 2(n+1)$, $l \geq 2(n+1)$) can be discussed by using a similar method. After differentiating (2.2) with respect to the variable t, we derive

$$\begin{aligned}\xi'(t) = \frac{B_n}{n+2l}\Big(\frac{k}{B_n}\Big)^{\frac{n+2l}{n}} t^{-\frac{2l}{n}-1}\bigg[&-2l + \frac{l(2n-2l+2)}{6n\mu^2}\Big(\frac{k}{B_n}\Big)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} \\ &+ \frac{l(4n-2l+4)(n+2(l-1))^2}{144n^2(n+2l)\mu^4}\Big(\frac{k}{B_n}\Big)^{-\frac{4}{n}} t^{\frac{4n+4}{n}}\bigg].\end{aligned} \tag{2.3}$$

Putting

$$\zeta(t) = \xi'(t)\frac{n+2l}{B_n}\Big(\frac{k}{B_n}\Big)^{-\frac{n+2l}{n}} t^{\frac{2l}{n}+1}$$

and noticing that $\mu \geq (2\pi)^{-n}B_n^{-\frac{1}{n}}V(\Omega)^{\frac{n+1}{n}}$, from (2.3) we have

$$\begin{aligned}\zeta(t) &= -2l + \frac{l(2n-2l+2)}{6n\mu^2}\Big(\frac{k}{B_n}\Big)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} + \frac{l(4n-2l+4)(n+2(l-1))^2}{144n^2(n+2l)\mu^4}\Big(\frac{k}{B_n}\Big)^{-\frac{4}{n}} t^{\frac{4n+4}{n}} \\ &\leq -2l + \frac{l(2n-2l+2)}{6n(2\pi)^{-2n}B_n^{-\frac{2}{n}}V(\Omega)^{\frac{2(n+1)}{n}}}\Big(\frac{k}{B_n}\Big)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} + \frac{l(4n-2l+4)(n+2(l-1))^2}{144n^2(n+2l)(2\pi)^{-4n}B_n^{-\frac{4}{n}}V(\Omega)^{\frac{4(n+1)}{n}}}\Big(\frac{k}{B_n}\Big)^{-\frac{4}{n}} t^{\frac{4n+4}{n}}.\end{aligned} \tag{2.4}$$

Since the right-hand side of (2.4) is an increasing function of t, if the right-hand side of (2.4) is not larger than 0 at $t = (2\pi)^{-n}V(\Omega)$, that is,

$$\zeta(t) \leq -2l + \frac{l(2n-2l+2)}{6n}k^{-\frac{2}{n}}\frac{B_n^{\frac{4}{n}}}{(2\pi)^2} + \frac{l(4n-2l+4)(n+2(l-1))^2}{144n^2(n+2l)}k^{-\frac{4}{n}}\frac{B_n^{\frac{8}{n}}}{(2\pi)^4} \leq 0, \tag{2.5}$$

we can claim from (2.5) that $\xi'(t) \leq 0$ on $(0, (2\pi)^{-n}V(\Omega)]$. If $\xi'(t) \leq 0$, then $\xi(t)$ is a decreasing function on $(0, (2\pi)^{-n}V(\Omega)]$. In fact, by a direct calculation, we can obtain

$$\zeta(t) \leq -2l + \frac{l(2n-2l+2)}{6n} + \frac{l(4n-2l+4)(n+2(l-1))^2}{144n^2(n+2l)} \leq 0,$$

since

$$\frac{B_n^{\frac{4}{n}}}{(2\pi)^2} < 1.$$

On the other hand, since $0 < \phi(0) \le (2\pi)^{-n}V(\Omega)$ and the right-hand side of (2.1) is $\xi(\phi(0))$, which is a decreasing function of $\phi(0)$ on $(0, (2\pi)^{-n}V(\Omega)]$, we can replace $\phi(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2.1), which gives the inequality

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\lambda_j \ge \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{l}{24(n+2l)}\frac{(2\pi)^{2(l-1)}}{(B_nV(\Omega))^{\frac{2(l-1)}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2(l-1)}{n}} \\ + \frac{l(n+2(l-1))^2}{2304n(n+2l)^2}\frac{(2\pi)^{2(l-2)}}{(B_nV(\Omega))^{\frac{2(l-2)}{n}}}\Big(\frac{V(\Omega)}{I(\Omega)}\Big)^2 k^{\frac{2(l-2)}{n}}.\end{aligned}$$

This completes the proof of the theorem. □

Next, we will prove that the inequality (1.10) is sharper than (1.9).

Proof of Remark 1.4. Under the same assumptions as in Lemma 2.1, letting $b = n$ and $A = \frac{k}{nB_n}$, from

$$\mu \ge (2\pi)^{-n}B_n^{-\frac{1}{n}}V(\Omega)^{\frac{n+1}{n}}$$

we obtain that

$$\frac{(bA)^{-\frac{2}{b}}\psi(0)^{2+\frac{2}{b}}}{\mu^2} \le \frac{(2\pi)^{-2n-2}V(\Omega)^{2+\frac{2}{n}}}{(2\pi)^{-2n}B_n^{-\frac{2}{n}}V(\Omega)^{\frac{2(n+1)}{n}}}\Big(\frac{k}{B_n}\Big)^{-\frac{2}{n}} = \frac{(2\pi)^{-2}}{(B_n)^{-\frac{4}{n}}}k^{-\frac{2}{n}} < \frac{(2\pi)^{-2}}{(B_n)^{-\frac{4}{n}}} < 1$$

and then we have

$$\begin{aligned}&\frac{1}{b+2l}\sum_{p=2}^{l}\frac{(l+1-p)}{6^pb\cdots(b+2p-2)\mu^{2p}}(bA)^{\frac{b+2(l-p)}{b}}\psi(0)^{\frac{2pb-2(l-p)}{b}} \\ &< \frac{1}{b+2l}\sum_{p=2}^{l}\frac{(l+1-p)}{6^pb\cdots(b+2p-2)\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}} \\ &< \frac{l-1}{36(b+2l)b(b+2)\mu^4}\sum_{p=0}^{\infty}\frac{1}{6^p(b+2)^p}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}} \\ &= \frac{l-1}{6b(b+2l)(6(b+2)-1)\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}}.\end{aligned} \tag{2.6}$$

By a direct calculation, we derive

$$l(6(b+2)-1)(b+2(l-1))^2 > 24b(b+2l)(l-1) > 0.$$

In fact,

$$\begin{aligned}l(6(b+2)-1)(b+2(l-1))^2 - 24b(b+2l)(l-1) &= 4b(l-1)(6b(l-1)-l) + l(6b+11)(b^2+4(l-1)^2) \\ &> 24b^2(l-1)^2 + 4bl(l-1)(6(l-1)-1) > 0,\end{aligned}$$

that is,

$$\frac{24b(b+2l)(l-1)}{l(6(b+2)-1)(b+2(l-1))^2} < 1. \tag{2.7}$$

Therefore, from (2.6) and (2.7) we get

$$\begin{aligned}\frac{1}{b+2l}\sum_{p=2}^{l}\frac{(l+1-p)}{6^pb\cdots(b+2p-2)\mu^{2p}}(bA)^{\frac{b+2(l-p)}{b}}\psi(0)^{\frac{2pb-2(l-p)}{b}} &< \frac{l-1}{6b(b+2l)(6(b+2)-1)\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}} \\ &= \frac{24b(b+2l)(l-1)}{l(6(b+2)-1)(b+2(l-1))^2} \\ &\quad\times\frac{l(b+2(l-1))^2}{144b^2(b+2l)^2\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}} \\ &< \frac{l(b+2(l-1))^2}{144b^2(b+2l)^2\mu^4}(bA)^{\frac{b+2l-4}{b}}\psi(0)^{\frac{4b-2l+4}{b}}.\end{aligned} \tag{2.8}$$

Taking

$$b = n, \quad A = \frac{k}{nB_n}, \quad \psi(0) = (2\pi)^{-n} V(\Omega), \quad \mu = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)} \tag{2.9}$$

and substituting (2.9) into (2.8), we have

$$\begin{aligned} &\frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{l}{24(n+2l)}\frac{(2\pi)^{2(l-1)}}{(B_nV(\Omega))^{\frac{2(l-1)}{n}}}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2(l-1)}{n}} + \frac{l(n+2(l-1))^2}{2304n(n+2l)^2}\frac{(2\pi)^{2(l-2)}}{(B_nV(\Omega))^{\frac{2(l-2)}{n}}}\Big(\frac{V(\Omega)}{I(\Omega)}\Big)^2 k^{\frac{2(l-2)}{n}} \\ &\quad > \frac{n}{n+2l}\frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}k^{\frac{2l}{n}} + \frac{n}{(n+2l)}\sum_{p=1}^{l}\frac{l+1-p}{(24)^p n\cdots(n+2p-2)}\frac{(2\pi)^{2(l-p)}}{(B_nV(\Omega))^{\frac{2(l-p)}{n}}}\Big(\frac{V(\Omega)}{I(\Omega)}\Big)^p k^{\frac{2(l-p)}{n}}. \end{aligned}$$

This completes the proof of the remark. □

A Proof of Lemma 2.1

In this section, we give the proof of Lemma 2.1.

Proof of Lemma 2.1. Letting

$$\varrho(t) = \frac{\psi(\frac{\psi(0)}{\mu}t)}{\psi(0)},$$

we have $\varrho(0) = 1$ and $-1 \le \varrho'(t) \le 0$. Without loss of generality, we can assume that

$$\psi(0) = 1 \quad \text{and} \quad \mu = 1.$$

Define

$$D_l := \int_0^\infty s^{b+2l-1}\psi(s)\,ds.$$

One can assume that $D_l < \infty$, otherwise there is nothing to prove. Since $D_l < \infty$, we can conclude that

$$\lim_{s\to\infty} s^{b+2l-1}\psi(s) = 0.$$

Putting $h(s) = -\psi'(s)$ for $s \ge 0$, we get

$$0 \le h(s) \le 1 \quad \text{and} \quad \int_0^\infty h(s)\,ds = \psi(0) = 1.$$

By making use of integration by parts, one has

$$\int_0^\infty s^b h(s)\,ds = b\int_0^\infty s^{b-1}\psi(s)\,ds = bA$$

and

$$\int_0^\infty s^{b+2l}h(s)\,ds \le (b+2l)D_l,$$

since $\psi(s) > 0$. By the same assertion as in [9], one can infer that there exists an $\epsilon \ge 0$ such that

$$\int_\epsilon^{\epsilon+1} s^b\,ds = \int_0^\infty s^b h(s)\,ds = bA \tag{A.1}$$

and

$$\int_\epsilon^{\epsilon+1} s^{b+2l}\,ds \le \int_0^\infty s^{b+2l}h(s)\,ds \le (b+2l)D_l. \tag{A.2}$$

Letting

$$\Theta(s) = bs^{b+2l} - (b+2l)\tau^{2l}s^b + 2l\tau^{b+2l} - 2l\tau^{b+2(l-1)}(s-\tau)^2,$$

we can prove that $\Theta(s) \geq 0$. By integrating the function $\Theta(s)$ from ϵ to $\epsilon + 1$, from (A.1) and (A.2) we deduce, for any $\tau > 0$, that

$$b(b+2l)D_l - (b+2l)\tau^{2l}bA + 2l\tau^{b+2l} \geq \frac{l}{6}\tau^{b+2(l-1)}. \tag{A.3}$$

Defining

$$f(\tau) := (b+2l)\tau^{2l}bA - 2l\tau^{b+2l} + \frac{l}{6}\tau^{b+2(l-1)},$$

for any $\tau > 0$, we can obtain from (A.3) that

$$D_l = \int_0^\infty s^{b+2l-1}\psi(s)\,ds \geq \frac{f(\tau)}{b(b+2l)}.$$

Then, taking

$$\tau = (bA)^{\frac{1}{b}}\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{1}{b}},$$

we have

$$\begin{aligned} f(\tau) = (bA)^{\frac{b+2l}{b}}\Big(b - \frac{l(b+2(l-1))}{6(b+2l)}(bA)^{-\frac{2}{b}}\Big)\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{2l}{b}} \\ + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{b+2(l-1)}{b}}. \end{aligned}$$

Next, we consider the following four cases.

Case 1. $b \geq 2l$. For $t > 0$, from Taylor's formula we have

$$(1+t)^{\frac{2l}{b}} \geq 1 + \frac{2l}{b}t + \frac{2l(2l-b)}{2b^2}t^2 + \frac{2l(2l-b)(2l-2b)}{6b^3}t^3 + \frac{2l(2l-b)(2l-2b)(2l-3b)}{24b^4}t^4$$

and

$$(1+t)^{\frac{b+2(l-1)}{b}} \geq 1 + \frac{2(l-1)+b}{b}t + \frac{(2(l-1)+b)(l-1)}{b^2}t^2 + \frac{(2(l-1)-b)(l-1)(2(l-1)+b)}{3b^3}t^3.$$

Putting

$$t = \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}},$$

we have from

$$(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3} > \frac{1}{4}$$

that $t < \frac{1}{3}$ and $b - 2lt > \frac{4l}{3} > 0$ (see also [5]). Then, we obtain

$$\begin{aligned} &\Big(b - \frac{l(b+2(l-1))}{6(b+2l)}(bA)^{-\frac{2}{b}}\Big)\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{2l}{b}} \\ &\quad = (b-2lt)(1+t)^{\frac{2l}{b}} \\ &\quad \geq (b-2lt)\Big(1 + \frac{2l}{b}t + \frac{2l(2l-b)}{2b^2}t^2 + \frac{2l(2l-b)(2l-2b)}{6b^3}t^3 + \frac{2l(2l-b)(2l-2b)(2l-3b)}{24b^4}t^4\Big) \\ &\quad \geq b - \frac{l(2l+b)}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 - \frac{(2l-b)(8l^2+4lb)}{6b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3 \\ &\qquad - \frac{(2l-b)(2l-2b)(12l^2+6lb)}{24b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^4 \end{aligned}$$

and also

$$\begin{aligned}\left(1+\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^{\frac{b+2(l-1)}{b}} &= (1+t)^{\frac{b+2(l-1)}{b}}\\ &\geq 1+\frac{2(l-1)+b}{b}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)\\ &\quad+\frac{(2(l-1)+b)(l-1)}{b^2}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^2\\ &\quad+\frac{(2(l-1)-b)(l-1)(2(l-1)+b)}{3b^3}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^3.\end{aligned}$$

Therefore, we have

$$\begin{aligned}f(\tau) &= (b+2l)\tau^{2l}bA-2l\tau^{b+2l}+\frac{l}{6}\tau^{b+2(l-1)}\\ &\geq (bA)^{\frac{b+2l}{b}}\left[b-\frac{l(2l+b)}{b}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^2-\frac{(2l-b)(8l^2+4lb)}{6b^2}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^3\right.\\ &\quad\left.-\frac{(2l-b)(2l-2b)(12l^2+6lb)}{24b^3}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^4\right]\\ &\quad+\frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}\left[1+\frac{2(l-1)+b}{b}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)\right.\\ &\quad+\frac{(2(l-1)+b)(l-1)}{b^2}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^2\\ &\quad\left.+\frac{(2(l-1)-b)(l-1)(2(l-1)+b)}{3b^3}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^3\right]\\ &= b(bA)^{\frac{b+2l}{b}}+\frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}+\frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}+\eta_1,\end{aligned}$$

where

$$\begin{aligned}\eta_1 &= \frac{2l(l+b-3)(b+2l)}{3b^2}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-6}{b}}\\ &\quad+\frac{l(b+2(l-1))(4(l-1)(2(l-1)-b)-3(2l-b)(l-b))}{72b^3}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}.\end{aligned}$$

Since

$$(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3} > \frac{1}{4}$$

and $b \geq 2l$, we have

$$\begin{aligned}\eta_1 &\geq \frac{2l(l+b-3)(b+2l)}{12b^2}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}\\ &\quad+\frac{l(b+2(l-1))(4(l-1)(2(l-1)-b)-3(2l-b)(l-b))}{72b^3}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}\\ &= \frac{l(9b^3+(35l-26)b^2+(36l^2-90l)b+(4l^3-36l^2+48l-16))}{72b^3}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}\\ &\geq \frac{l(72l^3+(70l^2-52l)b+(36l^2-90l)b-36l^2)}{72b^3}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}\\ &\geq \frac{l((72l^3-36l^2)+(140l-52l)b+(72l-90l)b)}{72b^3}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^3(bA)^{\frac{b+2l-8}{b}}\\ &\geq 0,\end{aligned}$$

which implies

$$f(\tau) \geq b(bA)^{\frac{b+2l}{b}}+\frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}+\frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}.$$

Case 2. $2l-2 \le b < 2l$. By using Taylor's formula, for $t>0$, we obtain the inequalities

$$(1+t)^{\frac{2l}{b}} \ge 1 + \frac{2l}{b}t + \frac{2l(2l-b)}{2b^2}t^2 + \frac{2l(2l-b)(2l-2b)}{6b^3}t^3$$

and

$$(1+t)^{\frac{b+2(l-1)}{b}} \ge 1 + \frac{2(l-1)+b}{b}t + \frac{(2(l-1)+b)(l-1)}{b^2}t^2 + \frac{(2(l-1)+b)(l-1)(2(l-1)-b)}{3b^3}t^3.$$

Putting

$$t = \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}},$$

we have $b-2lt > \frac{l}{3} > 0$,

$$\begin{aligned}
&\Big(b - \frac{l(b+2(l-1))}{6(b+2l)}(bA)^{-\frac{2}{b}}\Big)\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{2l}{b}} \\
&\quad = (b-2lt)(1+t)^{\frac{2l}{b}} \\
&\quad \ge (b-2lt)\Big(1 + \frac{2l}{b}t + \frac{2l(2l-b)}{2b^2}t^2 + \frac{2l(2l-b)(2l-2b)}{6b^3}t^3\Big) \\
&\quad = b - \frac{l(b+2l)}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 - \frac{(2l-b)(8l^2+4lb)}{6b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3 \\
&\qquad - \frac{4l^2(2l-b)(2l-2b)}{6b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^4,
\end{aligned}$$

and

$$\begin{aligned}
\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{b+2(l-1)}{b}} &= (1+t)^{\frac{b+2(l-1)}{b}} \\
&\ge 1 + \frac{2(l-1)+b}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big) \\
&\quad + \frac{(2(l-1)+b)(l-1)}{b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 \\
&\quad + \frac{(2(l-1)+b)(l-1)(2(l-1)-b)}{3b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3.
\end{aligned}$$

Furthermore, using the same method as in Case 1, we deduce that

$$\begin{aligned}
f(\tau) &= (b+2l)\tau^{2l}bA - 2l\tau^{b+2l} + \frac{l}{6}\tau^{b+2(l-1)} \\
&\ge (bA)^{\frac{b+2l}{b}}\Big[b - \frac{l(b+2l)}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 - \frac{(2l-b)(8l^2+4lb)}{6b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3 \\
&\qquad - \frac{4l^2(2l-b)(2l-2b)}{6b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^4\Big] \\
&\qquad + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}\Big[1 + \frac{2(l-1)+b}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big) \\
&\qquad + \frac{(2(l-1)+b)(l-1)}{b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 \\
&\qquad + \frac{(2(l-1)+b)(l-1)(2(l-1)-b)}{3b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3\Big] \\
&= b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}} + \eta_2,
\end{aligned}$$

where

$$\begin{aligned}
\eta_2 &= \frac{2l(l+b-3)(b+2l)}{3b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-6}{b}} \\
&\quad + \frac{l(2(l-1)+b)((l-1)(2(l-1)-b)(b+2l)-l(2l-b)(2l-2b))}{18b^3(b+2l)}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} \\
&\geq \frac{2l(l+b-3)(b+2l)}{9b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} + \frac{l(2(l-1)+b)(l-1)(2(l-1)-b)}{18b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} \\
&= \Big(\frac{4bl(l+b-3)(b+2l)}{18b^3} + \frac{l(2(l-1)+b)(l-1)(2(l-1)-b)}{18b^3}\Big)\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} \\
&\geq \Big(\frac{4bl(l+b-3)(b+2l)}{18b^3} + \frac{lb(b+2l)(2(l-1)-b)}{18b^3}\Big)\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} \\
&\geq \frac{bl(b+2l)(6l+3b-14)}{18b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-8}{b}} \\
&\geq 0,
\end{aligned}$$

since

$$(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3}.$$

Therefore, we have

$$f(\tau) \geq b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}.$$

Case 3. $l \leq b < 2l-2$. By using Taylor's formula, for $t > 0$, we have

$$(1+t)^{\frac{2l}{b}} \geq 1 + \frac{2l}{b}t + \frac{l(2l-b)}{b^2}t^2 + \frac{l(2l-b)(2l-2b)}{3b^3}t^3$$

and

$$(1+t)^{\frac{b+2(l-1)}{b}} \geq 1 + \frac{2(l-1)+b}{b}t + \frac{(2(l-1)+b)(l-1)}{b^2}t^2.$$

Putting

$$t = \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}} > 0,$$

we have $b - 2lt > 0$,

$$\begin{aligned}
&\Big(b - \frac{l(b+2(l-1))}{6(b+2l)}(bA)^{-\frac{2}{b}}\Big)\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{2l}{b}} \\
&\quad = (b-2lt)(1+t)^{\frac{2l}{b}} \\
&\quad \geq (b-2lt)\Big(1 + \frac{2l}{b}t + \frac{l(2l-b)}{b^2}t^2 + \frac{l(2l-b)(2l-2b)}{3b^3}t^3\Big) \\
&\quad = b - \frac{l(b+2l)}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 - \frac{(2l-b)(4l^2+2lb)}{3b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3 \\
&\qquad - \frac{4l^2(2l-b)(l-b)}{3b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^4,
\end{aligned}$$

and

$$\begin{aligned}
\Big(1 + \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^{\frac{b+2(l-1)}{b}} &= (1+t)^{\frac{b+2(l-1)}{b}} \\
&\geq 1 + \frac{2(l-1)+b}{b}\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}} \\
&\quad + \frac{(2(l-1)+b)(l-1)}{b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2.
\end{aligned}$$

By the same argument as in Case 2, we can deduce that

$$\begin{aligned}
f(\tau) &= (b+2l)\tau^{2l}bA - 2l\tau^{b+2l} + \frac{l}{6}\tau^{b+2(l-1)} \\
&\geq (bA)^{\frac{b+2l}{b}}\Bigg[b - \frac{l(b+2l)}{b}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2 - \frac{(2l-b)(4l^2+2lb)}{3b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^3 \\
&\quad - \frac{4l^2(2l-b)(l-b)}{3b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^4\Bigg] + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}\Bigg[1 + \frac{2(l-1)+b}{b}\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}} \\
&\quad + \frac{(2(l-1)+b)(l-1)}{b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\Big)^2\Bigg] \\
&= b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}} + \eta_3,
\end{aligned}$$

where

$$\eta_3 = \frac{2l(l+b-3)(b+2l)}{3b^2}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^3(bA)^{\frac{b+2l-6}{b}} - \frac{4l^2(2l-b)(l-b)}{3b^3}\Big(\frac{b+2(l-1)}{12(b+2l)}\Big)^4(bA)^{\frac{b+2l-8}{b}} \geq 0.$$

Therefore, we have

$$f(\tau) \geq b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}.$$

Case 4. $2 \leq b < l$. Since $2 \leq b < l$, there exists a positive integer k such that $2 \leq k-1 \leq \frac{2l}{b} < k$ and, for $t > 0$, we have

$$\begin{aligned}
(1+t)^{\frac{2l}{b}} &\geq 1 + \frac{2l}{b}t + \frac{1}{2!}\frac{2l}{b}\Big(\frac{2l}{b}-1\Big)t^2 + \frac{1}{3!}\frac{2l}{b}\Big(\frac{2l}{b}-1\Big)\Big(\frac{2l}{b}-2\Big)t^3 + \cdots + \frac{1}{(k+1)!}\frac{2l}{b}\Big(\frac{2l}{b}-1\Big)\cdots\Big(\frac{2l}{b}-k\Big)t^{k+1} \\
&= 1 + \sum_{p=0}^{k}\Big\{\frac{1}{(p+1)!}\prod_{q=0}^{p}\Big(\frac{2l}{b}-q\Big)\Big\}t^{p+1}
\end{aligned}$$

and

$$\begin{aligned}
(1+t)^{\frac{b+2l}{b}} &\leq 1 + \frac{b+2l}{b}t + \frac{1}{2!}\frac{b+2l}{b}\frac{2l}{b}t^2 + \frac{1}{3!}\frac{b+2l}{b}\frac{2l}{b}\Big(\frac{2l}{b}-1\Big)t^3 + \cdots \\
&\quad + \frac{1}{(k+1)!}\frac{b+2l}{b}\frac{2l}{b}\Big(\frac{2l}{b}-1\Big)\cdots\Big(\frac{2l}{b}-(k-1)\Big)t^{k+1} \\
&= 1 + \sum_{p=0}^{k}\Big\{\frac{1}{(p+1)!}\prod_{q=0}^{p}\Big(\frac{2l}{b}-q+1\Big)\Big\}t^{p+1},
\end{aligned}$$

and also

$$\begin{aligned}
(1+t)^{\frac{b+2(l-1)}{b}} &\geq 1 + \frac{2(l-1)+b}{b}t + \frac{1}{2!}\frac{(2(l-1)+b)}{b}\frac{2(l-1)}{b}t^2 + \frac{1}{3!}\frac{(2(l-1)+b)}{b}\frac{2(l-1)}{b}\Big(\frac{2(l-1)}{b}-1\Big)t^3 + \cdots \\
&\quad + \frac{1}{k!}\frac{(2(l-1)+b)}{b}\frac{2(l-1)}{b}\cdots\Big(\frac{2(l-1)}{b}-(k-2)\Big)t^k \\
&\quad - \Big|\frac{1}{(k+1)!}\frac{(2(l-1)+b)}{b}\frac{2(l-1)}{b}\cdots\Big(\frac{2(l-1)}{b}-(k-1)\Big)\Big|t^{k+1} \\
&= 1 + \sum_{p=0}^{k-1}\Big\{\frac{1}{(p+1)!}\prod_{q=0}^{p}\Big(\frac{2(l-1)}{b}-q+1\Big)\Big\}t^{p+1} - \Big|\frac{1}{(k+1)!}\prod_{q=0}^{k}\Big(\frac{2(l-1)}{b}-q+1\Big)\Big|t^{k+1}.
\end{aligned}$$

Putting

$$t = \frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}$$

and $f(\tau) = (bA)^{\frac{b+2l}{b}}h(\tau)$, where

$$h(\tau) = (b+2l)(1+t)^{\frac{2l}{b}} - 2l(1+t)^{\frac{b+2l}{b}} + \frac{1}{6}(bA)^{-\frac{2}{b}}(1+t)^{\frac{b+2(l-1)}{b}},$$

then, for $2 \le b < l$, we have

$$
\begin{aligned}
h(\tau) &\ge (b+2l)\left\{1+\sum_{p=0}^{k}\left[\frac{1}{(p+1)!}\prod_{q=0}^{p}\left(\frac{2l}{b}-q\right)\right]t^{p+1}\right\} - 2l\left\{1+\sum_{p=0}^{k}\left[\frac{1}{(p+1)!}\prod_{q=0}^{p}\left(\frac{2l}{b}-q+1\right)\right]t^{p+1}\right\}\\
&\quad + \frac{l}{6}(bA)^{-\frac{2}{b}}\left\{1+\sum_{p=0}^{k-1}\left[\frac{1}{(p+1)!}\prod_{q=0}^{p}\left(\frac{2(l-1)}{b}-q+1\right)\right]t^{p+1} - \left|\frac{1}{(k+1)!}\prod_{q=0}^{k}\left(\frac{2(l-1)}{b}-q+1\right)\right|t^{k+1}\right\}\\
&= b+\frac{l}{6}(bA)^{-\frac{2}{b}} + \sum_{p=1}^{k}\left\{\frac{b+2l}{(p+1)!}\frac{2l}{b}\left[\prod_{q=1}^{p}\left(\frac{2l}{b}-q\right)-\prod_{q=1}^{p}\left(\frac{2l}{b}-q+1\right)\right]\right\}t^{p+1}\\
&\quad + \sum_{p=0}^{k-1}\left\{\frac{l(bA)^{-\frac{2}{b}}}{6(p+1)!}\prod_{q=0}^{p}\left(\frac{2(l-1)}{b}-q+1\right)\right\}t^{p+1} - \left|\frac{l(bA)^{-\frac{2}{b}}}{6(k+1)!}\prod_{q=0}^{k}\left(\frac{2(l-1)}{b}-q+1\right)\right|t^{k+1}\\
&= b+\frac{l}{6}(bA)^{-\frac{2}{b}} - \sum_{p=1}^{k}\left\{\frac{p2l(b+2l)}{b(p+1)!}\prod_{q=1}^{p-1}\left(\frac{2l}{b}-q\right)\right\}t^{p+1} + \sum_{p=1}^{k}\left\{\frac{l(bA)^{-\frac{2}{b}}}{6p!}\prod_{q=0}^{p-1}\left(\frac{2(l-1)}{b}-q+1\right)\right\}t^{p}\\
&\quad - \left|\frac{l(bA)^{-\frac{2}{b}}}{6(k+1)!}\prod_{q=0}^{k}\left(\frac{2(l-1)}{b}-q+1\right)\right|t^{k+1}\\
&= b+\frac{l}{6}(bA)^{-\frac{2}{b}} - \sum_{p=1}^{k}\left\{\frac{p}{b^p(p+1)!}\prod_{q=0}^{p}(2l-(q-1)b)\right\}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^{p+1}\\
&\quad + \sum_{p=1}^{k}\left\{\frac{l(bA)^{-\frac{2}{b}}}{6b^p p!}\prod_{q=0}^{p-1}(2(l-1)-(q-1)b)\right\}\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^{p}\\
&\quad - \left|\frac{l(bA)^{-\frac{2}{b}}}{6b^{k+1}(k+1)!}\prod_{q=0}^{k}(2(l-1)-(q-1)b)\right|\left(\frac{b+2(l-1)}{12(b+2l)}(bA)^{-\frac{2}{b}}\right)^{k+1}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
f(\tau) &\ge b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}\\
&\quad - \sum_{p=2}^{k}\left\{\frac{p}{b^p(p+1)!}\prod_{q=0}^{p}(2l-(q-1)b)\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{p+1}(bA)^{\frac{b+2l-2p-2}{b}}\\
&\quad + \sum_{p=2}^{k}\left\{\frac{l}{6b^p p!}\prod_{q=0}^{p-1}(2(l-1)-(q-1)b)\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{p}(bA)^{\frac{b+2l-2p-2}{b}}\\
&\quad - \left|\frac{l}{6b^{k+1}(k+1)!}\prod_{q=0}^{k}(2(l-1)-(q-1)b)\right|\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{k+1}(bA)^{\frac{b+2l-2k-4}{b}}\\
&= b(bA)^{\frac{b+2l}{b}} + \frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}} + \frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}} + \eta_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_4 &= \sum_{p=2}^{k}\left\{\frac{(b+2(l-1))2l}{12b^p p!}\left[\prod_{q=1}^{p-1}(2(l-1)-(q-1)b) - \frac{p}{p+1}\prod_{q=1}^{p-1}(2l-qb)\right]\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{p}(bA)^{\frac{b+2l-2p-2}{b}}\\
&\quad - \left|\frac{l}{6b^{k+1}(k+1)!}\prod_{q=0}^{k}(2(l-1)-(q-1)b)\right|\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{k+1}(bA)^{\frac{b+2l-2k-4}{b}}.
\end{aligned}
$$

From $k-2 \le \frac{2(l-1)}{b} < k$ we have

$$
\frac{k-2-i}{k+1-i} \le \frac{\frac{2(l-1)}{b}-i}{k+1-i} < \frac{k-i}{k+1-i}. \tag{A.4}
$$

Then, from (A.4) we have

$$
\left|\frac{\frac{2(l-1)}{b}-i}{k+1-i}\right| \le 1, \quad i = 0, 1, 2, \dots, k-1.
$$

Note that

$$2(l-1)-(q-1)b \geq 2l-qb \geq 0, \quad p=2,3,\dots,k,$$

and

$$\prod_{q=1}^{p-1}(2(l-1)-(q-1)b)-\frac{p}{p+1}\prod_{q=1}^{p-1}(2l-qb) \geq \prod_{q=1}^{p-1}(2(l-1)-(q-1)b)-\prod_{q=1}^{p-1}(2l-qb) \geq 0.$$

Therefore, we obtain

$$\begin{aligned}\sum_{p=2}^{k}&\left\{\frac{(b+2(l-1))2l}{12b^p p!}\left[\prod_{q=1}^{p-1}(2(l-1)-(q-1)b)-\frac{p}{p+1}\prod_{q=1}^{p-1}(2l-qb)\right]\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^p (bA)^{\frac{b+2l-2p-2}{b}}\\ &\geq \frac{(b+2(l-1))2l}{24b^2}\left(2(l-1)-\frac{2(2l-b)}{3}\right)\left(\frac{b+2(l-1)}{12(b+2l)}\right)^2 (bA)^{\frac{b+2l-6}{b}}.\end{aligned}$$

From

$$(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3}$$

we have

$$\begin{aligned}\eta_4 &\geq \frac{(b+2(l-1))2l}{24b^2}\left(2(l-1)-\frac{2(2l-b)}{3}\right)\left(\frac{b+2(l-1)}{12(b+2l)}\right)^2 (bA)^{\frac{b+2l-6}{b}}\\ &\quad -\left|\frac{l}{6b^{k+1}(k+1)!}\prod_{q=0}^{k}(2(l-1)-(q-1)b)\right|\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{k+1}(bA)^{\frac{b+2l-2k-4}{b}}\\ &= \frac{l(b+2(l-1))}{12b^2}\left\{\left(2(l-1)-\frac{2(2l-b)}{3}\right)\right.\\ &\quad \left.-2b\left|\frac{1}{b^k(k+1)!}\prod_{q=1}^{k}(2(l-1)-(q-1)b)\right|\left(\frac{b+2(l-1)}{12(b+2l)}\right)^{k-1}(bA)^{\frac{-2k+2}{b}}\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^2(bA)^{\frac{b+2l-6}{b}}\\ &\geq \frac{l(b+2(l-1))}{12b^2}\left\{\frac{2l+2b-6}{3}-2b\left|\prod_{q=0}^{k-1}\left(\frac{\frac{2(l-1)}{b}-q}{k+1-q}\right)\right|\left(\frac{1}{4}\right)^{k-1}\right\}\left(\frac{b+2(l-1)}{12(b+2l)}\right)^2(bA)^{\frac{b+2l-6}{b}}\\ &\geq \frac{l(b+2(l-1))}{12b^2}\left(\frac{2l+2b-6}{3}-\frac{b}{8}\right)\left(\frac{b+2(l-1)}{12(b+2l)}\right)^2(bA)^{\frac{b+2l-6}{b}}\\ &\geq 0,\end{aligned}$$

which implies that

$$f(\tau) \geq b(bA)^{\frac{b+2l}{b}}+\frac{l}{6}(bA)^{\frac{b+2(l-1)}{b}}+\frac{l(b+2(l-1))^2}{144b(b+2l)}(bA)^{\frac{b+2l-4}{b}}.$$

This completes the proof of the lemma. □

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