Research Article

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On the Profile of Globally and Locally Minimizing Solutions of the Spatially Inhomogeneous Allen–Cahn and Fisher–KPP Equations

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Abstract: We show that the spatially inhomogeneous Allen–Cahn equation $-\varepsilon^2\Delta u = u(u - a(x))(1 - u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $u = 0$ on $\partial\Omega$, with $0 < a(\cdot) < 1$ continuous and $\varepsilon > 0$ a small parameter, cannot have globally minimizing solutions with transition layers in a smooth subdomain of Ω whereon $a - \frac{1}{2}$ does not change sign and $a - \frac{1}{2} \neq 0$ on that subdomain's boundary. Under the assumption of radial symmetry, this property was shown by Dancer and Yan in [5]. Our approach may also be used to simplify some parts of the latter and related references. In particular, for this model, we can give a streamlined new proof of the existence of locally minimizing transition layered solutions with nonsmooth interfaces, considered originally by del Pino in [6] using different techniques. Besides of its simplicity, the main advantage of our proof is that it allows one to deal with more degenerate situations. We also establish analogous results for a class of problems that includes the spatially inhomogeneous Fisher–KPP equation $-\varepsilon^2\Delta u = \rho(x)u(1 - u)$ with ρ sign-changing.

Keywords: Singular Perturbations, Variational Methods, Elliptic Equation, Transition Layer, Spatial Inhomogeneity, Allen–Cahn Equation, Fife–Greenlee Problem, Fisher Equation

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1 Introduction and Main Results

Consider the well-studied elliptic problem

$$\begin{cases} -\varepsilon^2\Delta u = u(u - a(x))(1 - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a(\cdot)$ is a continuous function satisfying $0 < a(x) < 1$ for $x \in \bar{\Omega}$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary, and $\varepsilon > 0$ is a small number. In [15], this problem was referred to as the spatially inhomogeneous Allen–Cahn equation, while in [7] as the Fife–Greenlee problem.

For the physical motivation behind this problem as well as for the extensive mathematical studies that have been carried out over the last decades, we refer the interested reader to the recent articles [7, 15] and the references therein.

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The functional corresponding to (1.1) is

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_\Omega |Du|^2\,dx - \int_\Omega F(x,u)\,dx, \quad u \in H_0^1(\Omega),$$

where

$$F(x,t) = \int_0^t s(s-a(x))(1-s)\,ds. \tag{1.2}$$

In this paper, we will study the behavior of global and local minimizers of the above functional as $\varepsilon \to 0$. Using the same techniques, we will also study the globally minimizing solutions of the spatially inhomogeneous Fisher–KPP type equation. In the appendixes, we state two variational lemmas that we will use throughout this paper.

1.1 Global Minimizers of the Spatially Inhomogeneous Allen–Cahn Equation

It is easy to see that the minimization problem

$$\inf\{I_\varepsilon(u) : u \in H_0^1(\Omega)\}$$

has a minimizer. Minimizers furnish classical solutions of (1.1) (at least when a is Hölder continuous) with values in $[0,1]$ and, more precisely, in $(0,1)$, provided that ε is sufficiently small (see [5, Lemma 2.2]). Let

$$A = \Big\{x : x \in \Omega,\ a(x) < \frac{1}{2}\Big\} \quad \text{and} \quad B = \Big\{x : x \in \Omega,\ a(x) > \frac{1}{2}\Big\}.$$

In [5, Theorem 1.1], Dancer and Yan show that any global minimizer u_ε of I_ε in $H_0^1(\Omega)$ satisfies

$$u_\varepsilon \to \begin{cases} 1 & \text{uniformly on any compact subset of } A,\\ 0 & \text{uniformly on any compact subset of } B,\end{cases} \tag{1.3}$$

as $\varepsilon \to 0$. However, this result provides no information about the global minimizers near the set $S = \{x \in \Omega : a(x) = \frac{1}{2}\}$. Their proof uses a comparison argument (see Lemma B.1 below) together with a result from [3] (see also Lemma A.1 herein) that the minimizer of the problem

$$\inf\Big\{\frac{\varepsilon^2}{2}\int_{B_\tau(x_0)} |Du|^2\,dx - \int_{B_\tau(x_0)} F_b(u)\,dx : u - \varphi \in H_0^1(B_\tau(x_0))\Big\} \tag{1.4}$$

with $F_b(t) = \int_0^t s(s-b)(1-s)\,ds$ tends to 1 (or 0) uniformly on $B_{\frac{\tau}{2}}(x_0)$ if $b < \frac{1}{2}$ (or $b > \frac{1}{2}$), as $\varepsilon \to 0$, for any φ with $0 \le \varphi \le 1$; here, $B_\tau(x_0) = \{x : x \in \mathbb{R}^N, |x - x_0| < \tau\}$. There is no similar result for the case $b = \frac{1}{2}$. Actually, in the latter case, the minimizer may have an interior transition layer for some φ with $0 \le \varphi \le 1$ (see [2] and the references therein). On the other hand, if Ω is a ball centered at the origin and $a(\cdot)$ is radially symmetric, then so is every global minimizer u_ε of I_ε in $H_0^1(\Omega)$ (see [5, Proposition 2.6]). Moreover, [5, Theorem 1.3 (i)–(ii)] tells us that for any $0 < r_1 < r_2 \le r_3 < r_4$ with $a(r_i) = \frac{1}{2}$, $i = 1,2,3,4$, such that $a(r) < \frac{1}{2}$ (or $> \frac{1}{2}$) for $r \in (r_1, r_2) \cup (r_3, r_4)$ and $a(r) = \frac{1}{2}$ for $r \in [r_2, r_3]$, we have that $u_\varepsilon \to 1$ (or 0) uniformly on any compact subset of (r_1, r_4), as $\varepsilon \to 0$. The proof of this result relies heavily on the radial symmetry of u_ε making use of a blow-up argument together with results stemming from the proof of De Giorgi's conjecture in low dimensions and an energy comparison argument (using the same approach, with a few modifications, a more general radially symmetric problem was treated in [16]). As was pointed out in [5], the nonsymmetric case is far from understood. Nevertheless, in the current paper, we are able to verify the validity of the corresponding nonradial version of the above result as follows.

Theorem 1.1. *Assume that $a(x) \le \frac{1}{2}$ (or $\ge \frac{1}{2}$) in a smooth domain A_1 (or B_1) such that $\bar{A}_1 \subset \Omega$ (or $\bar{B}_1 \subset \Omega$) and $a(x) < \frac{1}{2}$ (or $> \frac{1}{2}$) on ∂A_1 (or ∂B_1). Then, any global minimizer u_ε of I_ε in $H_0^1(\Omega)$ satisfies $u_\varepsilon \to 1$ (or $u_\varepsilon \to 0$) uniformly on $\bar{A}_1$ (or $\bar{B}_1$), as $\varepsilon \to 0$.*

Proof. We will only consider the case A, since the case B is identical. Let $\eta > 0$ be any number such that

$$2\eta < \min_{x\in\bar{\Omega}}(1 - a(x)). \tag{1.5}$$

For small $\delta > 0$, we have $a(x) < \frac{1}{2}$ if $\text{dist}(x, \partial A_1) \leq \delta$. Therefore, by (1.3), we deduce that $u_\varepsilon \to 1$ uniformly on the compact subset of A that is described by $\{x \in \Omega : \text{dist}(x, \partial A_1) \leq \frac{\delta}{2}\}$, as $\varepsilon \to 0$. Consider the subset of Ω that is defined by $A_2 = A_1 \cup \{x \in \Omega : \text{dist}(x, \partial A_1) < \frac{\delta}{2}\}$. We fix a small δ such that $A_2 \supset A_1$ is smooth and $\bar{A}_2 \subset \Omega$. Since any global minimizer satisfies $0 < u_\varepsilon < 1$ if ε is small, we have that

$$1 - u_\varepsilon(x) \leq \eta, \quad x \in \partial A_2. \tag{1.6}$$

We claim that $1 - u_\varepsilon(x) \leq \eta$, $x \in \bar{A}_2$, which clearly implies the validity of the assertion of the theorem. Suppose that the claim is false. Then, for some sequence of small ε's, there exists an $x_\varepsilon \in A_2$ such that

$$1 - u_\varepsilon(x_\varepsilon) = \max_{x\in\bar{A}_2}(1 - u_\varepsilon(x)) > \eta. \tag{1.7}$$

We will first exclude the possibility that

$$1 - u_\varepsilon(x) \leq 2\eta, \quad x \in \bar{A}_2. \tag{1.8}$$

To this end, we will argue by contradiction. Let

$$\tilde{u}_\varepsilon(x) = \begin{cases} \max\{u_\varepsilon(x), 2 - 2\eta - u_\varepsilon(x)\}, & x \in A_2, \\ u_\varepsilon(x), & x \in \Omega \setminus A_2. \end{cases}$$

Since $\max\{u_\varepsilon, 2 - 2\eta - u_\varepsilon\}$ is the composition of a Lipschitz function with an $H^1(A_2)$ function, it follows from [8] that $\tilde{u}_\varepsilon \in H^1(A_2)$. Furthermore, from (1.6) and the Lipschitz regularity of A_2 we obtain that $\tilde{u}_\varepsilon \in H_0^1(\Omega)$, see again [8]. Note that $\tilde{u}_\varepsilon \in C(\bar{\Omega})$. On the other hand, (1.8) implies that

$$1 - 2\eta \leq u_\varepsilon(x) \leq \tilde{u}_\varepsilon(x) \leq 1, \quad x \in \bar{A}_2.$$

In turn, recalling (1.2) and (1.5), this implies that

$$F(x, u_\varepsilon(x)) \leq F(x, \tilde{u}_\varepsilon(x)), \quad x \in \bar{A}_2. \tag{1.9}$$

To see this, observe that

$$\text{for each } x \in \bar{\Omega} \text{ the function } F(x, t) \text{ is increasing with respect to } t \in [1 - 2\eta, 1], \tag{1.10}$$

since $F_t(x, t) = t(t - a(x))(1 - t)$. (Note that $t \mapsto F(t, x)$ changes monotonicity in $(0, 1)$ only at $t = a(x)$). From (1.7), which implies that $u_\varepsilon(x_\varepsilon) < \tilde{u}_\varepsilon(x_\varepsilon)$, it follows that $F(x, u_\varepsilon(x)) < F(x, \tilde{u}_\varepsilon(x))$ on an open subset of A_2 containing x_ε. Hence,

$$\int_\Omega F(x, u_\varepsilon(x))\, dx < \int_\Omega F(x, \tilde{u}_\varepsilon(x))\, dx. \tag{1.11}$$

Moreover, it holds that

$$\int_\Omega |D\tilde{u}_\varepsilon|^2\, dx \leq \int_\Omega |Du_\varepsilon|^2\, dx, \tag{1.12}$$

see [14, p. 93]. The above two relations yield that $I_\varepsilon(\tilde{u}_\varepsilon) < I_\varepsilon(u_\varepsilon)$, contradicting the fact that u_ε is a global minimizer of I_ε in $H_0^1(\Omega)$. Consequently, we have that

$$0 < 1 - u_\varepsilon(x_\varepsilon) < 1 - 2\eta. \tag{1.13}$$

Now, let

$$\hat{u}_\varepsilon(x) = \begin{cases} \min\{1, \max\{u_\varepsilon(x), 2 - 2\eta - u_\varepsilon(x)\}\}, & x \in A_2, \\ u_\varepsilon(x), & x \in \Omega \setminus A_2, \end{cases}$$

see also [11]. As before, it is easy to see that $\hat{u}_\varepsilon \in H_0^1(\Omega)$. Since $a(x) \le \frac{1}{2}$, $x \in \bar{A}_2$, it follows readily that

$$F(x,t) < F(x,1) \quad \text{for all } t \in (0,1),\ x \in \bar{A}_2.$$

Hence, as before, making use of (1.10), (1.13), and the above relation, we get (1.11), (1.12) with $\hat{u}_\varepsilon$ in place of $\tilde{u}_\varepsilon$, which again contradict the minimality of u_ε. □

Remark 1.2. In the radially symmetric case, if $0 < r_1 < r_2 \le r_3 < r_4$ satisfy $a(r_i) = \frac{1}{2}$, $i = 1, 2, 3, 4$, and $a(r) < \frac{1}{2}$ (or $> \frac{1}{2}$) for $r \in (r_1, r_2)$, $a(r) > \frac{1}{2}$ (or $< \frac{1}{2}$) for $r \in (r_3, r_4)$, and $a(r) = \frac{1}{2}$ for $r \in [r_2, r_3]$, incorporating our approach into the proof of [5, Theorem 1.3 (iii)–(iv)] can lead to a simpler proof of the fact that global minimizers have only one transition layer in (r_1, r_4), see also [1], which for $N \ge 2$ takes place near r_2 (or r_2).

1.2 Local Minimizers of the Spatially Inhomogeneous Allen–Cahn Equation

In the case where there exists a smooth $(n-1)$-dimensional submanifold Γ of Ω that divides Ω in an interior and an exterior subdomain, which we denote by Ω_- and Ω_+, respectively, such that $a = \frac{1}{2}$ and $\frac{\partial a}{\partial \nu} > 0$ (or < 0) on Γ, where ν denotes the outer normal to Γ, it was shown in the pioneering work of Fife and Greenlee [9] that (1.1) has a solution $0 < w_\varepsilon < 1$ such that

$$w_\varepsilon \to \begin{cases} 1 \text{ (or } 0), & \text{uniformly on any compact subset of } \Omega_-, \\ 0 \text{ (or } 1), & \text{uniformly on any compact subset of } \Omega_+, \end{cases} \tag{1.14}$$

as $\varepsilon \to 0$. Their approach was based on matched asymptotics and on bifurcation arguments. Such a solution is said to have a transition layer along the interface $w_\varepsilon = 0$, which collapses in a smooth manner to Γ, as $\varepsilon \to 0$. In fact, they considered more general equations of the form $\varepsilon^2 \Delta u = f(x,u)$ and their proof carries over to the case of finitely many such interfaces. This result was extended by del Pino in [6], via degree-theoretic arguments, for general (even nonsmooth) interfaces. In the following theorem, we present a truly simple proof of the result in [6] for (1.1), which also allows for transition layers between degenerate stable roots of the equation $f(x,\cdot) = 0$ (see also [1, Hypothesis (h)]). In fact, with a little more work in the proof and using some ideas from [21], even more degenerate situations can be allowed.

Theorem 1.3. *Assume the existence of a closed set $\Gamma \subset \Omega$ and of open disjoint subsets Ω_+ and Ω_- of Ω such that*

$$\Omega = \Omega_+ \cup \Gamma \cup \Omega_-.$$

Assume also the existence of an open neighborhood $\mathcal{N}$ of Γ such that

$$a(x) < \frac{1}{2} \left(\text{or } > \frac{1}{2}\right) \quad \text{for } x \in \mathcal{N} \cap \Omega_-, \quad a(x) > \frac{1}{2} \left(\text{or } < \frac{1}{2}\right) \quad \text{for } x \in \mathcal{N} \cap \Omega_+.$$

Then, there exists a solution $0 < w_\varepsilon < 1$ of (1.1) that satisfies (1.14). Moreover, w_ε is a local minimizer of I_ε in $H_0^1(\Omega)$.

Proof. We will only consider the first scenario, since the one depicted in parentheses can be handled identically. Let η, δ be any positive numbers such that

$$4\eta < \min_{x\in\bar{\Omega}} a(x) + \min_{x\in\bar{\Omega}} (1 - a(x)) \quad \text{and} \quad \{x : \operatorname{dist}(x,\Gamma) \le \delta\} \subset \mathcal{N}.$$

For convenience purposes, we will assume that $\partial\Omega$ is a part of $\partial\Omega_+$ (otherwise, the solution would also have a boundary layer along $\partial\Omega$). Let

$$\Omega_\pm^\delta = \{x \in \Omega_\pm : \operatorname{dist}(x,\Gamma) > \delta\}$$

and

$$C = \{u \in H_0^1(\Omega) : u \le 2\eta \text{ a.e. on } \bar{\Omega}_+^\delta,\ 1 - u \le 2\eta \text{ a.e. on } \bar{\Omega}_-^\delta\}.$$

It is easy to verify that the constrained minimization problem

$$\inf\{I_\varepsilon(u) : u \in C\}$$

has a minimizer $w_\varepsilon \in C$ such that $0 \le w_\varepsilon \le 1$ (see the related paper [11]). Our goal is to show that w_ε does not realize (touch) the constraints if $\varepsilon > 0$ is sufficiently small. Naturally, this will imply that w_ε is a local minimizer of $I_\varepsilon(u)$ in $H_0^1(\Omega)$ and thus a classical solution of (1.1) satisfying the desired assertions of the theorem. The minimizer w_ε of the constrained problem is a classical solution of the equation (1.1) in $\{x : \operatorname{dist}(x, \Gamma) < \delta\}$, and in fact a global minimizer in the sense that $I_\varepsilon(w_\varepsilon) \le I_\varepsilon(w_\varepsilon + \phi)$ for every ϕ that is compactly supported in this region. Furthermore, by the strong maximum principle (see, for example, [12, Lemma 3.4]), we deduce that $0 < w_\varepsilon < 1$ in the same region. As in [5], making use of Lemma B.1 in Appendix B, we can bound w_ε from below by the minimizer of (1.4), with $b = \max\{a(x), x \in \overline{\mathcal{N} \cap \Omega_-}\} < \frac{1}{2}$ and $\varphi \equiv 0$, over every ball that is contained in $\Omega_- \cap \{x : \operatorname{dist}(x, \Gamma) < \delta\}$. From the result of [3] which we mentioned in the introduction (see also Lemma A.1 herein), we obtain that $w_\varepsilon \to 1$, uniformly on $\Omega_- \cap \{x : \operatorname{dist}(x, \Gamma) \in [\frac{\delta}{4}, \frac{\delta}{2}]\}$, as $\varepsilon \to 0$. In particular, for small $\varepsilon > 0$, we have

$$0 < 1 - w_\varepsilon(x) \le \eta \quad \text{if } x \in \Omega_- \text{ such that } \operatorname{dist}(x, \Gamma) = \frac{\delta}{2}.$$

As in the part of the proof of Theorem 1.1 that is below (1.6), it follows that the above relation holds for all $x \in \Omega_-$ such that $\operatorname{dist}(x, \Gamma) \ge \frac{\delta}{2}$. We point out that here the function w_ε may not be continuous in the vicinity of the constraints, but it is as long as it does not touch them, since there it is a classical solution of (1.1), which suffices for our purposes. Analogous relations hold in Ω_+. Consequently, w_ε stays away from the constraints for small $\varepsilon > 0$ and is therefore a local minimizer of I_ε in $H_0^1(\Omega)$ with the desired asymptotic behavior (1.14), since $\eta, \delta > 0$ can be chosen arbitrarily small. □

1.3 Global Minimizers of the Spatially Inhomogeneous Fisher–KPP Equation

Using the same approach, we can treat the elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta u = \rho(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

where Ω is as before, $g \in C^1$ such that

$$g(0) = g(1) = 0, \quad g(t) > 0 \text{ for } t \in (0, 1), \quad g(t) < 0 \text{ for } t \in \mathbb{R} \setminus (0, 1),$$

$\rho \in C(\bar{\Omega})$, and $\varepsilon > 0$ is a small number. Note that this includes the important Fisher–KPP equation, where $g(t) = t(1 - t)$, arising in population genetics (see [10]).

The functional corresponding to (1.15) is

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega \rho(x) G(u) \, dx, \quad u \in H_0^1(\Omega),$$

where

$$G(t) = \int_0^t g(s) \, ds. \tag{1.16}$$

It is easy to see that the minimization problem

$$\inf\{J_\varepsilon(u) : u \in H_0^1(\Omega)\}$$

has a minimizer. Minimizers furnish classical solutions of (1.15) (at least when ρ is Hölder continuous) with values in $[0, 1]$ and, more precisely, in $(0, 1)$, provided that ε is sufficiently small. Let

$$A = \{x : x \in \Omega, \rho(x) > 0\} \quad \text{and} \quad B = \{x : x \in \Omega, \rho(x) < 0\}.$$

Similarly to [5, Theorem 1.1], using Lemmas A.1 and B.1 below, we can show that any global minimizer u_ε of $J_\varepsilon(u)$ satisfies (1.3) (related results can be found in [4] and in [13, Chapter 10]).

In the nondegenerate case, where Γ is a finite union of smooth $(n-1)$-dimensional submanifolds of Ω such that $\rho = 0$ and $\frac{\partial \rho}{\partial \nu} \neq 0$ on Γ, where ν denotes the outer normal to Γ, it can be shown that the width of the transition region of w_ε is of order $\varepsilon^{\frac{2}{3}}$ (see [18]). On the other side, in the corresponding nondegenerate case of (1.1) considered in [9], the width of the transition region is of order ε. This difference can be traced back to the fact that the one-dimensional version of (1.1) falls in the framework of standard geometric singular perturbation theory, see [20] ($u = 0$, $u = 1$ are asymptotically stable roots of $f(x, u) = 0$, with respect to the dynamics of $\dot{u} = f(x, u)$, for all $x \in \bar{\Omega}$), whereas the corresponding version of (1.15) is not (here, the roots $u = 0$, $u = 1$ of $g(u) = 0$ exchange stability as x crosses Γ) and one has to use a blow-up transformation (see [17]).

A Minimizers of a Homogeneous Problem over Balls

The following lemma can be found in [19] and generalizes the result of [3] that we mentioned in relation to (1.4).

Lemma A.1. *Suppose that $W \in C^2$ satisfies $0 = W(\mu) < W(t)$, $t \in [0, \mu)$, $W(t) \geq 0$, $t \in \mathbb{R}$, $W(-t) \geq W(t)$, $t \in [0, \mu]$, or $W'(t) < 0$, $t < 0$, for some $\mu > 0$. Let $x_0 \in \mathbb{R}^N$, $\tau > 0$, $\eta \in (0, \mu)$, and $D > D'$, where D' is determined from the relation $\boldsymbol{U}(D') = \mu - \eta$, where in turn $\boldsymbol{U}$ is the only function in $C^2[0, \infty)$ that satisfies*

$$\boldsymbol{U}'' = W'(\boldsymbol{U}) \ \text{for} \ s > 0, \quad \boldsymbol{U}(0) = 0, \quad \lim_{s \to \infty} \boldsymbol{U}(s) = \mu,$$

(keep in mind that $\boldsymbol{U}' > 0$). There exists a positive constant ε_0, depending only on τ, η, D, W, and n, such that there exists a global minimizer u_ε of the energy functional

$$E(v) = \frac{\varepsilon^2}{2} \int_{B_\tau(x_0)} |Dv|^2 \, dx + \int_{B_\tau(x_0)} W(v) \, dx, \quad v \in H_0^1(B_\tau(x_0)),$$

which satisfies $0 < u_\varepsilon(x) < \mu$, $x \in B_\tau(x_0)$, and

$$\mu - \eta \leq u_\varepsilon(x), \quad x \in \bar{B}_{(\tau - D\varepsilon)}(x_0),$$

provided that $\varepsilon < \varepsilon_0$.

B A Comparison Lemma from [5]

The following result is [5, Lemma 2.3].

Lemma B.1. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $g_1(x, t)$, $g_2(x, t)$ be locally Lipschitz functions with respect to t, measurable functions with respect to x, and for any bounded interval I, there exists a constant C such that $\sup_{x \in \mathcal{D}, t \in I} |g_i(x, t)| \leq C$, $i = 1, 2$, holds. Let*

$$G_i(x, t) = \int_0^t g_i(x, s) \, ds, \quad i = 1, 2.$$

For $\varphi_i \in W^{1,2}(\mathcal{D}) = H^1(\mathcal{D})$, $i = 1, 2$, consider the minimization problem

$$\inf\{J_i(u; \mathcal{D}) : u - \varphi_i \in W_0^{1,2}(\mathcal{D}) = H_0^1(\mathcal{D})\},$$

where

$$J_i(u; \mathcal{D}) = \int_{\mathcal{D}} \left\{ \frac{1}{2} |\nabla u|^2 - G_i(x, u) \right\} dx.$$

Let $u_i \in W^{1,2}(\mathcal{D})$, $i = 1, 2$, *be minimizers to the minimization problems above. Assume that there exist constants* $m < M$ *such that*

- $m \leq u_i(x) \leq M$ *a.e. for* $i = 1, 2$, $x \in \mathcal{D}$,
- $g_1(x, t) \geq g_2(x, t)$ *a.e. for* $x \in \mathcal{D}$, $t \in [m, M]$,
- $M \geq \varphi_1(x) \geq \varphi_2(x) \geq m$ *a.e. for* $x \in \mathcal{D}$.

Suppose further that $\varphi_i \in W^{2,p}(\mathcal{D})$ *for any* $p > 1$ *and that they are* not identically equal *on* $\partial\mathcal{D}$. *Then, we have*

$$u_1(x) \geq u_2(x), \quad x \in \mathcal{D}.$$

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