DE GRUYTER Adv. Nonlinear Stud. 2016; 16 (1):75–85

Research Article

Tonia Ricciardi* and Gabriella Zecca

On the Blow-Up of Solutions to Liouville-Type Equations

DOI: 10.1515/ans-2015-5015
Received January 28, 2015; accepted May 19, 2015

Abstract: We estimate some complex structures related to perturbed Liouville equations defined on a compact Riemannian 2-manifold. As a byproduct, we obtain a quick proof of the mass quantization and we locate the blow-up points.

Keywords: Liouville Equation, Peak Solutions, Mean Field Equation

MSC 2010: 35J20, 35J60

Communicated by: Donato Fortunato

1 Introduction

In the article [6], Nagasaki and Suzuki considered the Liouville-type problem

$$\begin{cases} -\Delta u = \rho f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $\rho > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$f(t) = e^t + \varphi(t) \quad \text{with } \varphi(t) = o(e^t) \text{ as } t \to +\infty. \tag{1.2}$$

Equations of the form (1.1) are of actual interest in several contexts, including turbulent Euler flows, chemotaxis, and the Nirenberg problem in geometry; see, e.g., [5] and the references therein. A recent example is given by the mean field equation

$$\begin{cases} -\Delta u = \lambda \displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha u}\, \mathcal{P}(d\alpha)}{\iint_{[-1,1]\times\Omega} e^{\alpha u}\, \mathcal{P}(d\alpha) dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which was derived in [7] for turbulent flows with variable intensities, where $\mathcal{P} \in \mathcal{M}([-1, 1])$ is a probability measure related to the vortex intensity distribution. In this case, setting

$$f(t) = \int_{[-1,1]} \alpha e^{\alpha t}\, \mathcal{P}(d\alpha), \quad \rho = \lambda \Bigg(\iint_{[-1,1]\times\Omega} e^{\alpha u}\, \mathcal{P}(d\alpha) dx \Bigg)^{-1},$$

it is readily seen that if $\mathcal{P}(\{1\}) > 0$, then along a blow-up sequence, (1.3) is of the form (1.1). See [10–13] for more details, where the existence of solutions by variational arguments and blow-up analysis are also

***Corresponding author: Tonia Ricciardi:** Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Napoli, Italy, e-mail: tonricci@unina.it
Gabriella Zecca: Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Napoli, Italy, e-mail: g.zecca@unina.it

considered. Blow-up solution sequences for (1.3) have also been recently constructed in [9] following the approach introduced in [4].

In [6], Nagasaki and Suzuki derived a concentration-compactness principle for (1.1), mass quantization, and the location of blow-up points, under some additional technical assumptions for f. More precisely, they assumed that

$$|\varphi(t)-\varphi'(t)|\leq \mathcal{G}(t)\quad \text{for some } \mathcal{G}\in C^1(\mathbb{R},\mathbb{R}) \text{ satisfying } \mathcal{G}(t)+|\mathcal{G}'(t)|\leq Ce^{\gamma t} \text{ with } \gamma<\frac{1}{4} \tag{1.4}$$

and

$$f(t)\geq 0\quad \text{for all } t\geq 0. \tag{1.5}$$

By a complex analysis approach, they established the following result.

Theorem 1.1 ([6]). *Let f satisfy assumptions* (1.2), (1.4), *and* (1.5). *Let u_n be a solution sequence to* (1.1) *with $\rho=\rho_n\to 0$. Suppose u_n converges to some nontrivial function u_0. Then,*

$$u_0(x)=8\pi\sum_{j=1}^{m}G_\Omega(x,p_j)$$

for some $p_1,\dots,p_m\in\Omega$, $m\in\mathbb{N}$, where G_Ω denotes the Green's function for the Dirichlet problem on Ω. Furthermore, at each blow-up point p_j, $j=1,\dots,m$, there holds that

$$\nabla\Big[G_\Omega(x,p_j)+\frac{1}{2\pi}\log|x-p_j|\Big]\Big|_{x=p_j}+\nabla\Big[\sum_{i\neq j}G_\Omega(p_j,p_i)\Big]=0.$$

The original estimates in [6] are involved and require the technical assumption $\gamma\in(0,\frac{1}{4})$. It should be mentioned that this assumption was later weakened to the natural assumption $\gamma\in(0,1)$ in [14], by taking a different viewpoint on the line of [1].

Here, we are interested in revisiting the complex analysis framework introduced in [6]. In particular, we study the effect of the lower-order terms which naturally appear when the equation is considered on a compact Riemannian 2-manifold. We observe that, although the very elaborate key L^∞-estimate obtained in [6], namely, Proposition 1.2 below, may be extended in a straightforward manner to the case of manifolds (see Appendix A for the details), the lower-order terms are naturally estimated only in L^1. Therefore, we are led to consider an L^1-framework, which turns out to be significantly simpler and which holds under the weaker assumption $\gamma\in(0,\frac{1}{2})$. As a byproduct, we obtain a quick proof of mass quantization and blow-up point location for the case $\gamma\in(0,\frac{1}{2})$.

In order to state our results, for a function $u\in C^2(\Omega)$, we define the quantity

$$S(u)=\frac{u_z^2}{2}-u_{zz}, \tag{1.6}$$

where

$$\partial_z=\frac{\partial_x-i\partial_y}{2},\qquad \partial_{\bar z}=\frac{\partial_x+i\partial_y}{2}.$$

Then, if u is a solution to (1.1), we have

$$\partial_{\bar z}[S(u)]=-\frac{\rho}{4}u_z[f(u)-f'(u)]=\frac{\rho}{4}u_z[\varphi(u)-\varphi'(u)].$$

In particular, in the Liouville case $f(u)=e^u$, the function $S(u)$ is holomorphic. Therefore, the complex derivative $\partial_{\bar z}[S(u)]$ may be viewed as an estimate of the "distance" between the equation in (1.1) and the standard Liouville equation.

We recall that the main technical estimate in [6] is given by the following proposition.

Proposition 1.2 ([6]). *Let u_ρ be a blow-up sequence for* (1.1). *Assume* (1.2), (1.4), *and* (1.5). *Then,*

$$\|\partial_{\bar z}S(u)\|_{L^\infty(\Omega)}=\frac{\rho}{4}\|\nabla u_\rho(f'(u_\rho)-f(u_\rho))\|_{L^\infty(\Omega)}\to 0.$$

It is natural to expect that corresponding results should hold on a compact Riemannian 2-manifold (M, g) without boundary. We show that, in fact, the L^∞-convergence as stated in Proposition 1.2 still holds true on M (see Proposition A.1 in Appendix A). However, a modified point of view is needed in order to suitably locally define a function S corresponding to (1.6), such that the lower-order terms may be controlled, as well as to prove its convergence to a holomorphic function in some suitable norm, so that the mass quantization and the location of the blow-up points may be derived. As we shall see, our point of view holds under the weaker assumption $\gamma \in (0, \frac{1}{2})$ and is significantly simpler than the original L^∞-framework.

More precisely, on a compact Riemannian 2-manifold without boundary (M, g), we consider the problem

$$\begin{cases} -\Delta_g u = \rho f(u) - c_\rho & \text{in } M, \\ \int_M u\, dx = 0, \end{cases} \tag{1.7}$$

where $c_\rho = \rho |M|^{-1} \int_M f(u)\, dx \in \mathbb{R}$, dx denotes the volume element on M, and Δ_g denotes the Laplace–Beltrami operator. We assume that $f(t) = e^t + \varphi(t)$ satisfies (1.2) and, moreover, that

$$|\varphi(t) - \varphi'(t)| \le \mathcal{G}(t) \quad \text{for some } \mathcal{G} \in C^1(\mathbb{R}, \mathbb{R}) \text{ satisfying } \mathcal{G}(t) + |\mathcal{G}'(t)| \le Ce^{\gamma t} \text{ with } \gamma < \frac{1}{2} \tag{1.8}$$

and

$$f(t) \ge -C \quad \text{for all } t \ge 0. \tag{1.9}$$

In the spirit of [3], we assume that along a blow-up sequence we have

$$\rho \int_M f(u)\, dx \le C. \tag{1.10}$$

In particular, without loss of generality, we may assume that

$$c_\rho \to c_0 \quad \text{as } \rho \to 0^+. \tag{1.11}$$

We note that (1.9) implies that $u \ge -C$. We now define the modified quantity corresponding to $S(u)$. Let $\mathcal{S} = \{p_1, \dots, p_m\}$ denote the blow-up set. Let $p \in \mathcal{S}$ and denote $X = (x_1, x_2)$. We consider a local isothermal chart $(\Psi, \mathcal{U})$ such that $\mathcal{B}_\varepsilon(p) \subset \mathcal{U}$, $\Psi(p) = 0$, $\mathcal{B}_\varepsilon(p) \cap \mathcal{S} = \varnothing$, $g(X) = e^{\xi(X)}(dx_1^2 + dx_2^2)$, and $\xi(0) = 0$. For the sake of simplicity, we identify here functions on M with their pullback functions to $B = B(0, r) = \Psi(\mathcal{B}_\varepsilon(p))$. We denote by $G_B(X, Y)$ the Green's function of $\Delta_X = \partial_{x_1}^2 + \partial_{x_2}^2$ on B. We set

$$K(X) = -\int_B G_B(X, Y) e^{\xi(Y)}\, dY + c_1 z \tag{1.12}$$

with $c_1 \in \mathbb{C}$ defined by

$$\partial_z[\xi(z, \bar z) + c_0 K(z, \bar z)]|_{z=0} = 0, \tag{1.13}$$

where c_0 is defined in (1.11). Let u denote a solution sequence to (1.7). We define $w(z) = u - c_\rho K$, so that $-\Delta w = e^\xi \rho f(u)$ in B. Finally, consider $S(w)$, where S is defined in (1.6). Our main estimate is given in the following theorem.

Theorem 1.3. *Assume that $f(t) = e^t + \varphi(t)$ satisfies* (1.2), (1.8), *and* (1.9). *Let u_ρ be a blow-up solution sequence for* (1.7). *Then,*

(i) *for every $1 \le s < (\gamma + \frac{1}{2})^{-1}$,*

$$\rho \|\nabla u_\rho (f'(u_\rho) - f(u_\rho))\|_{L^s(M)} \to 0 \quad \text{as } \rho \to 0^+;$$

(ii) *for every blow-up point $p \in \mathcal{S}$, the function $S(w) \to S_0$ in $L^1(B)$ as $\rho \to 0^+$, where S_0 is holomorphic in B.*

Consequently, we derive the following corollary.

Corollary 1.4. *Assume that $f(t) = e^t + \varphi(t)$ satisfies* (1.2), (1.8), *and* (1.9). *Suppose u_n converges to some nontrivial function u_0. Then,*

$$u_0(x) = 8\pi \sum_{j=1}^{m} G_M(x, p_j). \tag{1.14}$$

Moreover, for all $p \in \mathcal{S}$, we have the relation

$$\left[\nabla_X\left(\sum_{q\in\mathcal{S}\setminus\{p\}} G_M(\Psi^{-1}(X), q) + G_M(\Psi^{-1}(X), p) + \frac{1}{2\pi}\log|X| + \frac{1}{8\pi}\xi(X)\right)\right]\Big|_{X=0} = 0. \tag{1.15}$$

We provide the proofs of Theorem 1.3 and Corollary 1.4 in Section 2. For the sake of completeness and in order to readily allow a comparison with the L^∞-framework employed in [6], in Appendix A we extend Proposition 1.2 to the case of Riemannian 2-manifolds without boundary.

Throughout this note, we denote by $C > 0$ a constant whose actual value may vary from line to line.

2 Proof of Theorem 1.3

We begin by establishing the following result.

Lemma 2.1. *Let u be a solution to* (1.7). *For every $r > 0$, we have*

$$r\int_M e^{-ru}|\nabla u|^2\,dx \le C, \tag{2.1}$$

where $C = C(r, M, \varphi, c_0)$.

Proof. We multiply the equation $-\Delta_g u = \rho f(u) - c_\rho$ by e^{-ru}. Integrating, we have

$$\begin{aligned} r\int_M e^{-ru}|\nabla u|^2\,dx &= \int_M e^{-ru}\Delta_g u\,dx \\ &= -\rho\int_M e^{-ru} f(u)\,dx + c_\rho\int_M e^{-ru}\,dx \\ &\le \rho\int_M e^{-ru}|\varphi(u)|\,dx + c_\rho\int_M e^{rC}\,dx \\ &\le \rho\int_M e^{-ru}|\varphi(u)|\,dx + c_\rho e^{rC}|M|, \end{aligned}$$

since $u \ge -C$. Using the assumptions on φ, there exists $t_0 > 0$ such that $|g(u)| < e^u$ for $u > t_0$, so that

$$r\int_M e^{-ru}|\nabla u|^2\,dx \le C + \rho\left(\int_{\{u>t_0\}} e^{(1-r)u}\,dx + \int_{\{u\le t_0\}} e^{-ru}|\varphi(u)|\,dx\right) \le C + \rho\left(\int_M e^u\,dx + \int_{\{u\le t_0\}} e^{-ru}|\varphi(u)|\,dx\right),$$

and the claim follows using again the fact that $u \ge -C$. □

The following proposition proves Theorem 1.3 (i).

Proposition 2.2. *Let u be a solution to* (1.7). *Then, for every $1 \le s < (\gamma + \frac{1}{2})^{-1}$ and for every $\varepsilon > 0$, we have*

$$\|\nabla u(f'(u) - f(u))\|_{L^s(M)} \le C\rho^{-\gamma-\varepsilon}$$

for $0 < \rho < 1$.

Proof. In view of (1.8), we have

$$0 \le |f(u) - f'(u)| \le Ce^{\gamma u}.$$

Hence,

$$\|(f(u) - f'(u))\nabla u\|_{L^s} \le C\|e^{\gamma u}\nabla u\|_{L^s}. \tag{2.2}$$

Moreover, (1.10) implies that

$$\int_M e^u \, dx \le c\rho^{-1}.$$

Then, for every $1 \le q < \gamma^{-1}$, using Hölder's inequality we have

$$\|e^{\gamma u}\|_{L^q(M)} \le C|M|^{\frac{1}{q}-\gamma}\rho^{-\gamma}. \tag{2.3}$$

Let $0 < r < 1 - s(\gamma + \frac{1}{2})$. By Lemma 2.1, for

$$q = \frac{s + \frac{r}{\gamma}}{1 - \frac{s}{2}} < \frac{1}{\gamma},$$

using Hölder's inequality again, we have

$$\|e^{\gamma u}\nabla u\|^s_{L^s(M)} = \int_M e^{(s\gamma+r)u}(e^{-ru}|\nabla u|^s)\, dx \le \Big(\int_M e^{\gamma u q}\, dx\Big)^{1-\frac{s}{2}} \Big(\int_M e^{-2ru}|\nabla u|^2 \, dx\Big)^{\frac{s}{2}} \le C\|e^{\gamma u}\|^{s+\frac{r}{\gamma}}_{L^q(M)}. \tag{2.4}$$

Then, by (2.3) and (2.4) we have

$$\|e^{\gamma u}\nabla u\|_{L^s(M)} \le C\rho^{-\gamma-\frac{r}{s}}. \tag{2.5}$$

Combining (2.2) and (2.5), the claim is proved. □

Let $p \in \mathcal{S}$. We denote by $(\Psi, \mathcal{U})$ an isothermal chart satisfying

$$\bar{\mathcal{U}} \cap \mathcal{S} = \{p\}, \quad \Psi(\mathcal{U}) = \mathcal{O} \subset \mathbb{R}^2, \quad \Psi(p) = 0, \quad g(X) = e^{\xi(X)}(dx_1^2 + dx_2^2), \quad \xi(0) = 0,$$

where $X = (x_1, x_2)$ denotes a coordinate system on $\mathcal{O}$. We consider $\varepsilon > 0$ sufficiently small so that $\mathcal{B}(p, \varepsilon) \Subset \mathcal{U}$ and let $B = B(0, r) = \Psi(\mathcal{B}(p, \varepsilon))$. The Laplace–Beltrami operator Δ_g is then mapped to the operator $e^{-\xi(X)}\Delta_X$ on $\mathcal{O}$, where $\Delta_X = \partial^2_{x_1^2} + \partial^2_{x_2^2}$. By $G_B(X, Y)$ we denote the Green's function of Δ_X on B, namely,

$$\begin{cases} -\Delta_X G_B(X, Y) = \delta_Y & \text{in } B, \\ \quad G_B(X, Y) = 0 & \text{on } \partial B. \end{cases}$$

We recall from (1.12) that

$$K(X) = -\int_B G_B(X, Y)e^{\xi(Y)}\, dY + c_1 z$$

with c_1 the constant defined by (1.13), namely,

$$\partial_z[\xi(z, \bar{z}) + c_0 K(z, \bar{z})]|_{z=0} = 0,$$

where $c_0 = \lim_{\rho\to 0} c_\rho$. Then, $K \in C^\infty(B)$ and

$$\Delta_X K = e^{\xi} \quad \text{in } \bar{B}.$$

Let u_ρ be a blow-up solution sequence for (1.7). As $\rho \to 0$, $u \to u_0$ in $C^\infty_{\mathrm{loc}}(M \setminus \mathcal{S})$, $u - u_0 \in W^{1,q}(M)$ for $1 \le q < 2$, and $f(u) \to f(u_0)$ in $C^\infty_{\mathrm{loc}}(M \setminus \mathcal{S})$, we have $\Delta_g u \to \Delta_g u_0$ in $C^\infty_{\mathrm{loc}}(M \setminus \mathcal{S})$, so that

$$\Delta_g u_0 = c_0 \quad \text{in } M \setminus \mathcal{S}.$$

We consider the following functions defined in B:

$$\begin{aligned} \tilde{u} &= u \circ \Psi^{-1}, & \tilde{u}_0 &= u_0 \circ \Psi^{-1}, \\ w(z) &= \tilde{u} - c_\rho K, & w_0(z) &= \tilde{u}_0 - c_0 K, \\ S(w) &= w_{zz} - \frac{1}{2}w_z^2, & S_0 &= w_{0zz} - \frac{1}{2}w_{0z}^2. \end{aligned} \tag{2.6}$$

The following proposition proves Theorem 1.3 (ii).

Proposition 2.3. *The complex function S_0 defined in* (2.6) *is holomorphic in B and $S \to S_0$ in $L^1(B)$.*

Proof. By (2.6) we have

$$-\Delta_X w = \rho f(\tilde{u})e^{\xi} \quad \text{and} \quad w_z = \tilde{u}_z - c_\rho K_z.$$

Then, using $\Delta_X = 4\partial_{z\bar{z}}$ we compute

$$\begin{aligned}\partial_{\bar{z}}[S(w)] &= \frac{1}{4}(\partial_z \Delta_X w - w_z \Delta_X w)\\ &= -\frac{\rho}{4}e^{\xi}(f(\tilde{u})\xi_z + \tilde{u}_z f'(\tilde{u})) + \frac{\rho}{4}e^{\xi}f(\tilde{u})(\tilde{u}_z - c_\rho K_z)\\ &= \frac{\rho}{4}e^{\xi}(f(\tilde{u}) - f'(\tilde{u}))\tilde{u}_z - \frac{\rho}{4}e^{\xi}f(\tilde{u})(\xi_z + c_0 K_z) + (c_0 - c_\rho)\frac{\rho}{4}e^{\xi}f(\tilde{u})K_z. \end{aligned} \tag{2.7}$$

Using (2.7) we derive that

$$\partial_{\bar{z}} S \to 0 \quad \text{in } L^1(B). \tag{2.8}$$

Indeed, this follows by Proposition 2.2, (1.13), and by the fact that $|\rho f(\tilde{u})| \overset{*}{\to} a\delta_0(dx)$ for some $a > 0$. On the other hand, by (2.6), since $u \to u_0$ in $C^\infty_{\mathrm{loc}}(M \setminus \mathcal{S})$, we have

$$w \to w_0 \quad \text{in } C^\infty_{\mathrm{loc}}(\bar{B} \setminus \{0\})$$

and then

$$S \to S_0 \quad \text{in } C^\infty_{\mathrm{loc}}(\bar{B} \setminus \{0\}). \tag{2.9}$$

At this point, we set $\Xi = (\xi_1, \xi_2)$ and $\zeta = \xi_1 + i\xi_2$ and we observe that by the Cauchy integral formula we may write

$$[S(w)](\zeta) = \frac{1}{\pi}\int_B \frac{\partial_{\bar{z}}S(z)}{\zeta - z}\,dX + \frac{i}{2\pi}\int_{+\partial B}\frac{[S(w)](z)}{\zeta - z}\,dz = g(\zeta) + h(\zeta). \tag{2.10}$$

We have

$$h(\zeta) \to h_0(\zeta) = \frac{i}{2\pi}\int_{+\partial B}\frac{S_0(z)}{\zeta - z}\,dz \quad \text{in } C^0_{\mathrm{loc}}(B) \tag{2.11}$$

and h_0 is holomorphic in B. On the other hand, we have

$$g \to 0 \quad \text{in } L^1(B). \tag{2.12}$$

To prove (2.12), it is sufficient to observe that for every $z \in B = B(0, r)$, we have $B \subset B(z, 2r)$ and then

$$\|g\|_{L^1(B)} \le \iint_{B\times B} |\partial_{\bar{z}}S(z)|\frac{1}{|\zeta - z|}\,dX\,d\Xi \le \int_B |\partial_{\bar{z}}S(z)|\bigg(\int_{B(z,2r)}\frac{1}{|\zeta - z|}\,d\Xi\bigg)\,dX = 4\pi r\int_B |\partial_{\bar{z}}S(z)|\,dX,$$

which tends to zero by (2.8). Combining (2.10), (2.11), and (2.12), we have

$$S \to h_0 \quad \text{in } L^1(B) \text{ as } \rho \to 0,$$

and hence, up to subsequences,

$$S \to h_0 \quad \text{a.e. in } B \text{ as } \rho \to 0,$$

so that by (2.9),

$$S_0(\zeta) = h_0(\zeta) \quad \text{for all } \zeta \in B \setminus \{0\}.$$

This completes our proof. ☐

Finally, we use the following result from [2].

Proposition 2.4 ([2]). *For $B = B(0, 1) \subset \mathbb{R}^n$, $n \ge 2$, the conditions $v \in W^{1,p}(B)$, $1 < p < \infty$, and $\Delta v = 0$ in $B \setminus \{0\}$ imply that $H = v - \ell E$ is harmonic in B, where ℓ is some constant and*

$$E(x) = \begin{cases} |x|^{2-n} & \text{if } n > 2,\\ \log|x| & \text{if } n = 2.\end{cases}$$

Now, we are ready to prove Corollary 1.4. By G_M we denote the Green's function on the manifold M, defined by

$$\begin{cases} -\Delta_g G_M(x,y) = \delta_y - \dfrac{1}{|M|} \\ \displaystyle\int_M G_M(x,y)dx = 0. \end{cases}$$

Proof of Corollary 1.4. Assume that $p \in \mathcal{S}$. Let us start by observing that w_0 in (2.6) is harmonic in $B \setminus \{0\}$ by definition and that $w_0 \in W^{1,q}(B)$ for all $1 < q < 2$. Hence, also by using Proposition 2.4, we have

$$w_0(z) = \ell \log \frac{1}{|z|} + H(z),$$

where H is harmonic in B and $\ell \neq 0$. Then, using the fact that

$$\partial_z \log|z| = \frac{1}{2}\partial_z \log(z\bar{z}) = (2z)^{-1},$$

we compute

$$w_{0z} = -\frac{\ell}{2z} + H_z, \qquad w_{0zz} = \frac{\ell}{2z^2} + H_{zz}.$$

Therefore,

$$S_0 = w_{0zz} - \frac{1}{2}w_{0z}^2 = \frac{\ell}{2z^2} + H_{zz} - \frac{1}{2}\Big(\frac{\ell}{2z} - H_z\Big)^2 = \frac{\ell(4-\ell)}{8z^2} + \frac{\ell}{2z}H_z + H_{zz} - \frac{1}{2}H_z^2.$$

By Proposition 2.3, we know that S_0 is holomorphic. Hence, we can conclude that $\ell = 4$ and $H_z(0) = 0$. Since

$$H = w_0 - 4\log\frac{1}{|z|} \tag{2.13}$$

is harmonic in B, we have

$$\Delta_X\Big(\tilde{u}_0 - 4\log\frac{1}{|z|}\Big) = c_0 e^{\xi} \quad \text{in } B(0,r)$$

and, therefore,

$$\Delta_g(u_0(x) - 8\pi G_M(x,p)) = c_0 - \frac{8\pi}{|M|} + h_p \quad \text{in } \mathcal{B}(p,\varepsilon)$$

for some harmonic function h_p. Arguing similarly for each $p \in \mathcal{S} = \{p_1, p_2, \dots, p_m\}$, we conclude that

$$\Delta_g\Big(u_0(x) - 8\pi\sum_{j=1}^m G_M(x,p_j)\Big) = c_0 - \frac{8\pi m}{|M|} \quad \text{in } M.$$

In particular, we obtain

$$u_0(x) - 8\pi\sum_{j=1}^m G_M(x,p_j) = \text{constant} \quad \text{in } M.$$

Observing that $\int_M u_0 = 0$, this completes the proof of (1.14). To obtain (1.15) it is sufficient to observe that, in view of (2.13) and (1.13), we have

$$\begin{aligned} 0 &= \frac{1}{8\pi}\partial_z H(X)|_{X=0} \\ &= \partial_z\Big[\sum_{q\in\mathcal{S}} G_M(\Psi^{-1}(X),q) + \frac{1}{2\pi}\log|X|\Big]\Big|_{X=0} - \Big[\frac{m}{|M|}\partial_z K(X)\Big]\Big|_{X=0} \\ &= \partial_z\Big[\sum_{q\in\mathcal{S}} G_M(\Psi^{-1}(X),q) + \frac{1}{2\pi}\log|X| - \frac{1}{8\pi}\xi(X)\Big]\Big|_{X=0}. \end{aligned}$$

Now, Corollary 1.4 is completely established. □

A The L^∞-estimate on M

In this appendix, for the sake of completeness and in order to outline the original arguments in [6], so that the simplification of our L^1-approach may be seen, we check that Proposition 1.2 may be actually extended to (1.7) on a compact Riemannian 2-manifold (M, g) without boundary with minor modifications. We consider a solution sequence for (1.7). We assume that f satisfies (1.2), (1.4), and (1.5). Moreover, we assume (1.10), so that $c_\rho \to c_0$ as $\rho \to 0^+$. We show the following proposition.

Proposition A.1. *Let u be a solution to* (1.7). *Then,*

$$\rho\|\nabla u(f'(u) - f(u))\|_{L^\infty(M)} \to 0 \quad \text{as } \rho \to 0.$$

The proof relies on the following relation, due to Obata.

Lemma A.2 ([8]). *Let $w = w(x) > 0$ be a solution to*

$$\Delta w = \frac{|\nabla w|^2}{w} + F(w) \quad \text{on } M, \tag{A.1}$$

where F is a C^1-function. Then, there holds the identity

$$\operatorname{div} V = J + \frac{1}{2}|\nabla w|^2 w^{-2}(F(w) + wF'(w)), \tag{A.2}$$

where, in local coordinates,

$$V_j = w^{-1}\Big\{\nabla\Big(\frac{\partial w}{\partial x_i}\Big)\cdot\nabla w - \frac{1}{2}\frac{\partial w}{\partial x_i}\Delta w\Big\}, \quad j = 1, 2,$$

and

$$J = w^{-1}\Big\{\sum_{i,j=1}^{2}\Big(\frac{\partial^2 w}{\partial x_i \partial x_j}\Big)^2 - \frac{1}{2}(\Delta w)^2\Big\} \geq 0.$$

Lemma A.3. *Let u be a solution to* (1.7). *Then, for every $r > 0$, there holds*

$$\rho\int_M e^{-ru}|\nabla u|^2(2rf(u) - f'(u)) \leq 2rc_\rho \int_M e^{-ru}|\nabla u|^2. \tag{A.3}$$

Proof. Let u be a solution to (1.7). Denoting $w = e^{-ru}$, it is easy to see that Obata's assumption (A.1) is satisfied by the function w with

$$F(w) = re^{-ru}(\rho f(u) - c_\rho).$$

On the other hand, we have

$$F(w) + wF'(w) = \rho e^{-ru}(2rf(u) - f'(u)) - 2re^{-ru}c_\rho.$$

In view of Obata's identity (A.2), we conclude that

$$\int_M \frac{|\nabla w|^2}{w^2}(F(w) + wF(w)) \leq 2\int_M \operatorname{div} V = 0.$$

In particular, since

$$\frac{\nabla w}{w} = -r\nabla u,$$

by the last inequality we obtain

$$\int_M r^2|\nabla u|^2(F(w) + wF'(w)) = r^2\rho\int_M e^{-ru}|\nabla u|^2(2rf(u) - f'(u)) - 2r^3 c_\rho \int_M e^{-ru}|\nabla u|^2 \leq 0. \qquad \square$$

We note that combining (A.3) and (2.1), for $\frac{1}{2} < r < 1$, we obtain

$$\rho \int_M e^{-ru} |\nabla u|^2 f(u) \le C\Big(1 + \rho \int_M e^{-(r-\gamma)u} |\nabla u|^2\Big). \tag{A.4}$$

Since $\gamma < \frac{1}{4}$, combining (2.1) and (A.4) we obtain

$$\rho \int_M e^{-ru} |\nabla u|^2 f(u)\,dx \le C$$

and then, since $u \ge -C$, using (2.1) again we have

$$\rho \int_M e^{-ru} |\nabla u|^2 |f(u)|\,dx \le C \quad \text{if } \frac{1}{2} < r < 1. \tag{A.5}$$

For $r > 0$, we define

$$G_r(t) = \int_0^t e^{-\frac{r}{2}s} \sqrt{|f(s)|}\,ds.$$

Then, (A.5) may be written in the form

$$\|\nabla G_r(u)\|_{L^2(M)} \le \frac{C}{\sqrt{\rho}}. \tag{A.6}$$

Lemma A.4. *There holds*

$$\|G_r(u)\|_{L^1(M)} \le \frac{C}{\sqrt{\rho}}. \tag{A.7}$$

Proof. The proof can be easily obtained as in Lemma 2.1. Let us observe that in our assumption, for every $\frac{1}{2} < r < 1$, we have

$$\int_{\{x\in M: u(x)\ge 0\}} G_r(u)\,dx \le \frac{2}{r} \int_{\{u\ge 0\}} \sqrt{|f(u)|}\,dx \le C\Big(\int_M |f(u)|\,dx\Big)^{\frac{1}{2}} \le \frac{C}{\sqrt{\rho}}. \tag{A.8}$$

On the other hand, since $-u \le C$, we have

$$\int_{\{x\in M: u(x)\le 0\}} |G_r(u)|\,dx \le C \int_{\{u\le 0\}} dx \int_u^0 e^{\frac{Cr}{2}} \le Ce^{\frac{Cr}{2}} |M| \le C. \tag{A.9}$$

Combining (A.8) and (A.9), we conclude the proof of (A.7). □

Reducing (A.6) to

$$\|\nabla G_r(u)\|_{L^p(M)} \le \frac{C}{\sqrt{\rho}} \quad \text{for } 1 < p < 2,$$

and using (A.7) and the Sobolev embedding, we obtain

$$\|G_r(u)\|_{L^{p^*}(M)} \le \frac{C}{\sqrt{\rho}}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}.$$

Moreover, we have

$$|f(t)|^{\frac{1}{2\sigma}} \le C(|G_r(t)| + 1)$$

for $\sigma = \frac{1}{1-r}$ (> 2). We choose $\frac{1}{2} < r < 1$ such that

$$\Big(\gamma + \frac{1}{2}\Big)\sigma < \frac{3}{2}. \tag{A.10}$$

Arguing as in [6], for every $\varepsilon > 0$, we obtain

$$\|f(u)\|_{L^p(M)} \le C\rho^{-\sigma + \frac{\sigma-1}{p} - \varepsilon}, \quad 1 < p < \infty, \tag{A.11}$$

and, for $q > 2$,

$$\|\nabla u\|_{L^q(M)} \le C\rho^{(-\frac{1}{2}+\frac{1}{q})(\sigma-1)-\varepsilon}. \tag{A.12}$$

Now, we conclude the proof of Proposition A.1.

Proof of Proposition A.1. There holds

$$\|(f'(u) - f(u))\nabla u\|_{L^\infty(M)} \le C\|e^{\gamma u}\nabla u\|_{L^\infty(M)} = \frac{C}{\gamma}\|\nabla e^{\gamma u}\|_{L^\infty(M)}. \tag{A.13}$$

Moreover, by (1.7) we have

$$-\Delta_g e^{\gamma u} = -\gamma^2 e^{\gamma u}|\nabla u|^2 + \rho\gamma e^{\gamma u} f(u) - c_\rho \gamma e^{\gamma u} \quad \text{in } M.$$

Hence, for $p > 2$, we have

$$\|\nabla e^{\gamma u}\|_{L^\infty(M)} \le C(\|\Delta_g e^{\gamma u}\|_{L^p(M)} + \|e^{\gamma u}\|_{L^1(M)}) \le C(\|e^{\gamma u}|\nabla u|^2\|_{L^p(M)} + \rho\|e^{\gamma u} f(u)\|_{L^p(M)} + \|c_\rho e^{\gamma u}\|_{L^p(M)}).$$

Now, observing that $e^u \le C(f(u) + 1)$, by (A.11) we obtain

$$\rho\|e^{\gamma u} f(u)\|_{L^p(M)} \le C\rho\|e^{(\gamma+1)u}\|_{L^p(M)} = C\rho\|e^u\|^{\gamma+1}_{L^{p(\gamma+1)}(M)} \le C\rho^{\tau-\varepsilon} \tag{A.14}$$

for every $\varepsilon > 0$ with

$$\tau = 1 + (\gamma+1)\Big(\frac{\sigma-1}{p(\gamma+1)} - \sigma\Big) = 1 + \frac{\sigma-1}{p} - \sigma(\gamma+1). \tag{A.15}$$

Hence, as $p \downarrow 2$, we have

$$\tau \uparrow 1 + \frac{1}{2}(\sigma-1) - \sigma(\gamma+1) > -1 \tag{A.16}$$

by (A.10). On the other hand, by (2.3), for $1 \le p < \frac{1}{\gamma}$, we have

$$\|c_\rho e^{\gamma u}\|_{L^p(M)} \le C\rho^{-\gamma}.$$

Moreover, if $q > \frac{1}{2\gamma}$ (> 2), then

$$\|e^{\gamma u}|\nabla u|^2\|_{L^p(M)} \le \|e^{\gamma u}\|_{L^{pq}(M)}\|\nabla u|\|^2_{L^{2pq'}(M)},$$

where $qq' = q + q'$. By (A.12), for every $\varepsilon > 0$ and since $2pq' > 2$, we have

$$\|\nabla u\|^2_{L^{2pq'}(M)} \le C\rho^{(-1+\frac{1}{pq'})(\sigma-1)-\varepsilon}.$$

Using again (A.11), for every $\varepsilon > 0$, we have

$$\|e^{\gamma u}\|_{L^{pq}(M)} \le C\|e^u\|^\gamma_{L^{pq\gamma}(M)} \le C\rho^{-\gamma\sigma+\frac{\sigma-1}{pq}-\varepsilon}.$$

Then, for every $\varepsilon > 0$, we have

$$\|e^{\gamma u}|\nabla u|^2\|_{L^p(M)} \le C\rho^{\tau-\varepsilon} \tag{A.17}$$

with τ defined by (A.15). Combining (A.13)–(A.14) and (A.16)–(A.17), we complete the proof. □

Funding: The first author acknowledges the support of FP7-MC-2009-IRSES-247486 “MaNEqui”.

References

[1] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg–Landau Vortices*, Progr. Nonlinear Differential Equations Appl. 13, Birkhäuser, Boston, 1994.

[2] H. Brézis and P. L. Lions, A note on isolated singularities for linear elliptic equations, in: *Mathematical Analysis and Applications, Part A*, Adv. in Math. Suppl. Stud. 7A, Academic Press, New York (1981), 263–266.

[3] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations* **16** (1991), no. 8–9, 1223–1253.

[4] P. Esposito, M. Grossi and A. Pistoia, On the existence of blowing-up solutions for a mean field equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), no. 2, 227–257.

[5] C.-S. Lin, An expository survey on the recent development of mean field equations, *Discrete Contin. Dyn. Syst.* **19** (2007), no. 2, 387–410.

[6] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially-dominated nonlinearities, *Asymptot. Anal.* **3** (1990), no. 2, 173–188.

[7] C. Neri, Statistical mechanics of the *N*-point vortex system with random intensities on a bounded domain, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004), no. 3, 381–399.

[8] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geom.* **6** (1971), no. 2, 247–258.

[9] A. Pistoia and T. Ricciardi, Concentrating solutions for a Liouville type equation with variable intensities in 2D-turbulence, preprint (2015), http://arxiv.org/abs/1505.05304.

[10] T. Ricciardi and T. Suzuki, Duality and best constant for a Trudinger–Moser inequality involving probability measures, *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 7, 1327–1348.

[11] T. Ricciardi and G. Zecca, Blow-up analysis for some mean field equations involving probability measures from statistical hydrodynamics, *Differential Integral Equations* **25** (2012), no. 3–4, 201–222.

[12] T. Ricciardi and G. Zecca, Mean field equations with probability measure in 2D-turbulence, *Ric. Mat.* **63** (2014), no. 1, suppl., S255–S264.

[13] T. Ricciardi and G. Zecca, Mass quantization and minimax solutions for Neri's mean field equation in 2D-turbulence, *J. Differential Equations* **260** (2016), no. 1, 339–369.

[14] D. Ye, Une remarque sur le comportement asymptotique des solutions de $-\Delta u = \lambda f(u)$, C. R. Acad. Sci. Paris Sér I Math. **325** (1997), no. 12, 1279–1282.