Research Article

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Multiple Solutions to (p, q)-Laplacian Problems with Resonant Concave Nonlinearity

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Abstract: The existence of multiple solutions to a Dirichlet problem involving the (p, q)-Laplacian is investigated via variational methods, truncation-comparison techniques, and Morse theory. The involved reaction term is resonant at infinity with respect to the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ and exhibits a concave behavior near zero.

Keywords: (p, q)-Laplacian, Resonance from the Left (Right), Concave Nonlinearity, Multiple Solutions

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, let $1 < q \le p < +\infty$, and let $\mu \ge 0$. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_r, $r > 1$, denotes the r-Laplacian, namely,

$$\Delta_r u := \operatorname{div}(\|\nabla u\|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega),$$

the reaction term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions, while, as usual, $p = q$ if and only if $\mu = 0$. Elliptic equations involving differential operators of the form

$$Au := -\Delta_p u - \Delta_q u,$$

often called (p, q)-Laplacian, occur in many important concrete situations. For instance, this happens when one seeks stationary solutions to the reaction-diffusion system

$$u_t + Au = c(x, u),$$

which exhibits a wide range of applications in physics and related sciences such as biophysics, quantum and plasma physics, and chemical reaction design; see [3, 6]. Consequently, they have been the subject of numerous investigations, both in bounded domains and on the whole space, mainly concerning the multiplicity of solutions or bifurcation-type results.

This paper falls within the first framework. We show that if, roughly speaking, f has a subcritical growth and, moreover,

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(i) $\lim_{|t|\to+\infty}\frac{p}{|t|^p}\int_0^t f(x,\xi)\,d\xi=\lambda_{1,p}$ uniformly in $x\in\Omega$, where $\lambda_{1,p}$ denotes the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$,

(ii) $\lim_{|t|\to+\infty}\left[f(x,t)t-p\int_0^t f(x,\xi)\,d\xi\right]=+\infty$ uniformly in $x\in\Omega$,

(iii) $c|t|^\theta\le f(x,t)t\le\theta\int_0^t f(x,\xi)\,d\xi$ for all $(x,t)\in\Omega\times[-\delta,\delta]$, where $c>0$, $\theta\in(1,q)$, while $\delta>0$,

then (1.1) possesses at least three nontrivial solutions in $C_0^1(\overline{\Omega})$, one greatest negative v_-, another smallest positive u_+, and a third nodal u_0, such that $v_-\le u_0\le u_+$; see Theorem 3.9 below.

Assumptions (i)–(ii) directly give

$$\lim_{|t|\to+\infty}\left[\lambda_{1,p}|t|^p-p\int_0^t f(x,\xi)\,d\xi\right]=+\infty\quad\text{uniformly in } x\in\Omega;\tag{1.2}$$

see the proof of Lemma 3.1. Hence, resonance with respect to $\lambda_{1,p}$ from the left occurs and, a fortiori, the energy functional φ associated with (1.1) is coercive.

Now, the question of investigating what happens if there is resonance from the right of $\lambda_{1,p}$, i.e., the limit in (1.2) equals $-\infty$, naturally arises. Accordingly, φ turns out to be indefinite and direct methods no longer work. However, via linking arguments and, in place of (ii), via the hypothesis that

(iv) either $\mu>0$ and

$$\liminf_{|t|\to+\infty}\frac{1}{|t|^\eta}\left[p\int_0^t f(x,\xi)\,d\xi-f(x,t)t\right]\ge C>0\quad\text{uniformly in } x\in\Omega,$$

where $\eta\in(q,p]$, or $\mu=0$ and

$$\lim_{|t|\to+\infty}\left[p\int_0^t f(x,\xi)\,d\xi-f(x,t)t\right]=+\infty\quad\text{uniformly in } x\in\Omega,$$

we still obtain a nontrivial solution $u_0\in C_0^1(\overline{\Omega})$ of (1.1); cf. Theorem 4.5 below.

It should be also noted that, in both settings, due to (ii), the nonlinearity $f(x,\cdot)$ exhibits a *concave* behavior at the origin. Such a type of growth rate has been widely studied, also combined with further conditions, provided $p=2$ and $\mu=0$, i.e., the equation is semilinear. As an example, besides the seminal paper [2], let us mention [8, 16, 21, 22]. A similar comment holds true also when $p\neq 2$ but $\mu=0$, in which case the literature looks to be daily increasing; see for instance the very recent papers [12, 14, 18, 19] and, concerning the nonsmooth framework, [13, 17].

Another meaningful feature of (1.1) is the following. If $\mu>0$, then the differential operator $u\mapsto-\Delta_p u-\mu\Delta_q u$ turns out to be nonhomogeneous. Hence, standard results for the p-Laplacian not always extend in a simple way to it.

Our approach is variational, based on critical point theory, together with appropriate truncation-comparison arguments and results from Morse theory.

2 Mathematical Background

Let $(X,\|\cdot\|)$ be a real Banach space. Given a set $V\subseteq X$, write $\overline{V}$ for the closure of V and ∂V for the boundary of V. If $x\in X$, $\delta>0$, then $B_\delta(x):=\{z\in X:\|z-x\|<\delta\}$, while $B_\delta:=B_\delta(0)$. The symbol X^* denotes the dual space of X, $\langle\cdot,\cdot\rangle$ indicates the duality brackets for the pair (X^*,X), and $x_n\to x$ (respectively, $x_n\rightharpoonup x$) in X means that 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'. An operator $A:X\to X^*$ is called of type $(S)_+$ provided

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0 \quad \text{imply} \quad x_n \to x \quad \text{in } X.$$

Let $\varphi \in C^1(X)$ and let $c \in \mathbb{R}$. Put

$$\varphi^c := \{x \in X : \varphi(x) \le c\}, \quad \varphi_c := \{x \in X : \varphi(x) \ge c\}, \quad K_\varphi := \{x \in X : \varphi'(x) = 0\}, \quad K_\varphi^c := \{x \in K_\varphi : \varphi(x) = c\}.$$

We say that φ satisfies the Cerami condition when

(C) every sequence $\{x_n\} \subseteq X$ such that $\{\varphi(x_n)\}$ is bounded and

$$\lim_{n\to+\infty}(1 + \|x_n\|)\varphi'(x_n) = 0 \quad \text{in } X^*$$

admits a strongly convergent subsequence.

This compactness-type assumption turns out to be weaker than the usual Palais–Smale condition. Nevertheless, it suffices to prove a deformation theorem, from which the minimax theory for the critical values of φ follows. In such a framework, the topological notion of linking sets plays a key role.

Definition 2.1. Suppose Q_0, Q, E are three nonempty closed subsets of a Hausdorff topological space Y with $Q_0 \subseteq Q$. The pair (Q_0, Q) links E in Y if $Q_0 \cap E = \emptyset$ and, for every $\gamma \in C^0(Q, Y)$ such that $\gamma|_{Q_0} = \mathrm{id}|_{Q_0}$, one has $\gamma(Q) \cap E \neq \emptyset$.

The following general minimax principle is well known; see, e.g., [10, Theorem 5.2.5].

Theorem 2.2. *Let X be a Banach space, let Q_0, Q, and E be such that the pair (Q_0, Q) links E in X, and let $\varphi \in C^1(X)$ satisfy condition* (C). *If, moreover,* $\sup_{Q_0}\varphi < \inf_E \varphi$ *and*

$$c := \inf_{\gamma\in\Gamma}\sup_{x\in Q}\varphi(\gamma(x)), \quad \textit{where } \Gamma := \{\gamma \in C^0(Q, X) : \gamma|_{Q_0} = \mathrm{id}|_{Q_0}\},$$

then $c \ge \inf_E \varphi$ and $K_\varphi^c \neq \emptyset$.

Appropriate choices of linking sets in Theorem 2.2 produce meaningful critical point results. For later use, we state here the famous Ambrosetti–Rabinowitz mountain pass theorem.

Theorem 2.3. *If $(X, \|\cdot\|)$ is a Banach space, $\varphi \in C^1(X)$ fulfills* (C), *$x_0, x_1 \in X$, $0 < \rho < \|x_1 - x_0\|$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < m_\rho := \inf_{\partial B_\rho(x_0)}\varphi,$$

and

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[-1,1]}\varphi(\gamma(t)), \quad \textit{where } \Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \ge m_\rho$ and $K_\varphi^c \neq \emptyset$.

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and let k be any nonnegative integer. We denote by $H_k(Y_1, Y_2)$ the k-th relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Given an isolated critical point $x_0 \in K_\varphi^c$,

$$C_k(\varphi, x_0) := H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}), \quad k \in \mathbb{N}_0,$$

is the k-th critical group of φ at x_0. Here, U indicates any neighborhood of x_0 fulfilling $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology ensures that this definition does not depend on the choice of U. The monographs [5, 11] are general references on this subject.

Hereafter, $\|\cdot\|$ stands for the $\mathbb{R}^N$-norm, while $|A|$ denotes the N-dimensional Lebesgue measure of $A \subseteq \mathbb{R}^N$. If $p \in [1, +\infty)$, then p' indicates the conjugate exponent of p and $\|\cdot\|_p$ is the usual norm of the Sobolev space $W_0^{1,p}(\Omega)$, namely, thanks to the Poincaré inequality,

$$\|u\|_p := \|\nabla u\|_{L^p(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $u, v : \Omega \to \mathbb{R}$ and let $t \in \mathbb{R}$. The symbol $u \le v$ means $u(x) \le v(x)$ for almost every $x \in \Omega$, $t^\pm := \max\{\pm t, 0\}$, as well as $u^\pm(\cdot) := u(\cdot)^\pm$. It is known that $u^\pm \in W_0^{1,p}(\Omega)$ provided $u \in W_0^{1,p}(\Omega)$. Next, define

$$C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

With the standard norm of $C^1(\overline{\Omega})$, this set is an ordered Banach space whose positive cone

$$C_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ in } \overline{\Omega}\}$$

has nonempty interior given by

$$\operatorname{int}(C_+) = \left\{u \in C_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$; see [10, Remark 6.2.10]. If

$$p \le r < p^* := \begin{cases} \dfrac{Np}{N-p} & \text{for } p < N, \\ +\infty & \text{otherwise,} \end{cases}$$

then, due to the continuous embedding $W_0^{1,p}(\Omega) \subseteq L^r(\Omega)$ and the Poincaré inequality, one has

$$\|u\|_{L^r(\Omega)} \le c_{r,p}\|u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{2.1}$$

Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$ and let $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative p-Laplacian, i.e.,

$$\langle A_p(u), v\rangle := \int_\Omega \|\nabla u(x)\|^{p-2}\nabla u(x) \cdot \nabla v(x)\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

Denote by $\lambda_{1,p}$ (respectively, $\lambda_{2,p}$) the first (respectively, second) eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. The following properties of $\lambda_{1,p}$, $\lambda_{2,p}$, and A_p can be found in [7, 15]; see also [10, Section 6.2].

(p_1) $0 < \lambda_{1,p} < \lambda_{2,p}$.

(p_2) $\|u\|^p_{L^p(\Omega)} \le \dfrac{1}{\lambda_{1,p}}\|u\|_p^p$ for all $u \in W_0^{1,p}(\Omega)$.

(p_3) There is a unique eigenfunction $u_{1,p}$ corresponding to $\lambda_{1,p}$ such that

$$u_{1,p} \in \operatorname{int}(C_0^1(\overline{\Omega})_+), \quad \|u_{1,p}\|_{L^p(\Omega)} = 1.$$

Any other eigenfunction is a scalar multiple of $u_{1,p}$.

(p_4) If $U := \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ and

$$\Gamma_0 := \{\gamma \in C^0([-1, 1], U) : \gamma(-1) = -u_{1,r} = -\gamma(1)\},$$

then

$$\lambda_{2,p} = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|u\|_p^p.$$

(p_5) The operator A_p is bounded, continuous, strictly monotone, and of type $(S)_+$.

Now, with p, q, μ, and f as in Section 1, suppose that

$$|f(x, t)| \le c(1 + |t|^{p-1}), \quad (x, t) \in \Omega \times \mathbb{R}, \tag{2.2}$$

for appropriate $c > 0$, put

$$F(x, t) := \int_0^t f(x, \xi)\, d\xi, \tag{2.3}$$

and consider the C^1-functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\varphi(u) := \frac{1}{p}\|u\|_p^p + \frac{\mu}{q}\|u\|_q^q - \int_\Omega F(x, u(x))\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next result establishes a relation between local $C_0^1(\overline{\Omega})$-minimizers and local $W_0^{1,p}(\Omega)$-minimizers of φ. Its proof is the same as that of [1, Proposition 2], with the (p, q)-Laplacian instead of the differential operator considered therein. This idea goes back to the pioneering works of Brézis and Nirenberg [4] for $p = 2$ and García Azorero, Manfredi, and Peral Alonso [9] when $p \neq 2$.

Proposition 2.4. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ, then u_0 lies in $C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 turns out to be a local $W_0^{1,p}(\Omega)$-minimizer of φ.*

Finally, we shall write $N_f(u)(\cdot) := f(\cdot, u(\cdot))$ for every $u \in L^p(\Omega)$. The function

$$N_f : L^p(\Omega) \to L^{p'}(\Omega)$$

is often called the Nemytskii operator associated with f. Moreover, given $u : \Omega \to \mathbb{R}$ and $c \in \mathbb{R}$,

$$\Omega(u \le c) := \{x \in \Omega : u(x) \le c\}.$$

The meaning of $\Omega(u > c)$ etc. is analogous.

3 Resonance from the Left

To avoid unnecessary technicalities, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' and the variable x will be omitted when no confusion may arise. Moreover, $p = q$ if and only if $\mu = 0$ and $f(x, 0) \equiv 0$. We will posit the following assumptions, where F is given by (2.3).

(h_1) For appropriate $c > 0$, one has

$$|f(x, t)| \le c(1 + |t|^{p-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

(h_2) $\displaystyle\lim_{|t|\to+\infty} \frac{pF(x, t)}{|t|^p} = \lambda_{1,p}$ uniformly in $x \in \Omega$.

(h_3) $\displaystyle\lim_{|t|\to+\infty} [f(x, t)t - pF(x, t)] = +\infty$ uniformly in $x \in \Omega$.

(h_4) There exist $\theta \in (1, q)$ and $\delta_0, c_0 > 0$ such that

$$c_0|t|^\theta \le f(x, t)t \le \theta F(x, t), \quad (x, t) \in \Omega \times [-\delta_0, \delta_0].$$

The energy functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ stemming from (1.1) is defined by

$$\varphi(u) := \frac{1}{p}\|u\|_p^p + \frac{\mu}{q}\|u\|_q^q - \int_\Omega F(x, u(x))\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\varphi \in C^1(W_0^{1,p}(\Omega))$. Moreover, once

$$f_+(x, t) := f(x, t^+), \quad f_-(x, t) := f(x, -t^-), \quad F_\pm(x, t) := \int_0^t f_\pm(x, \xi)\,d\xi,$$

one has $F_+(x, t) = F(x, t^+)$, $F_-(x, t) = F(x, -t^-)$, while the associated truncated functionals

$$\varphi_\pm(u) := \frac{1}{p}\|u\|_p^p + \frac{\mu}{q}\|u\|_q^q - \int_\Omega F_\pm(x, u(x))\,dx, \quad u \in W_0^{1,p}(\Omega),$$

turn out to be C^1 as well.

Lemma 3.1. *If* (h_1)–(h_3) *hold true, then* φ, φ_+, *and* φ_- *are coercive and weakly sequentially lower semicontinuous.*

Proof. We will verify the conclusion for φ_+, the other cases being similar. The space $W_0^{1,p}(\Omega)$ compactly embeds in $L^p(\Omega)$ while the Nemytskii operator N_{f_+} turns out to be continuous on $L^p(\Omega)$. Thus, a standard argument ensures that φ_+ is weakly sequentially lower semicontinuous. In view of (h_3), given any $K > 0$, there exists $\delta > 0$ such that

$$f_+(x,t)t - pF_+(x,t) \geq K \quad \text{for all } (x,t) \in \Omega \times [\delta, +\infty),$$

which clearly means that

$$\frac{d}{dt}\frac{F_+(x,t)}{t^p} \geq \frac{K}{t^{p+1}}.$$

After integration, we obtain

$$\frac{F_+(x,s)}{s^p} - \frac{F_+(x,t)}{t^p} \geq -\frac{K}{p}\Big(\frac{1}{s^p} - \frac{1}{t^p}\Big) \quad \text{provided } s \geq t \geq \delta. \tag{3.1}$$

Thanks to (h_2), letting $s \to +\infty$ in (3.1) yields

$$\frac{\lambda_{1,p}}{p}t^p - F_+(x,t) \geq \frac{K}{p}, \quad (x,t) \in \Omega \times [\delta, +\infty).$$

Therefore,

$$\lim_{t\to+\infty}\Big[\frac{\lambda_{1,p}}{p}t^p - F_+(x,t)\Big] = +\infty \quad \text{uniformly with respect to } x \in \Omega. \tag{3.2}$$

Now, suppose by contradiction that there exists a sequence $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ such that

$$\lim_{n\to+\infty}\|u_n\|_p = +\infty \quad \text{but} \quad \varphi_+(u_n) \leq C < +\infty \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Write $v_n := u_n^+/\|u_n^+\|_p$. Since $\|v_n\|_p \equiv 1$, passing to a subsequence when necessary, one has

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega), \quad v_n \to v \quad \text{in } L^p(\Omega), \quad v_n \to v \geq 0 \quad \text{a.e. in } \Omega.$$

Fix any $\varepsilon > 0$ and, through (h_2), choose $\delta > 0$ fulfilling

$$F_+(x,t) \leq \frac{\lambda_{1,p}+\varepsilon}{p}t^p, \quad (x,t) \in \Omega \times [\delta, +\infty).$$

Moreover, set $M := \sup_{\Omega\times[0,\delta]} F_+$. From (3.3) it evidently follows that

$$\varphi_+(u_n^+) \leq C \quad \text{for all } n \in \mathbb{N}, \tag{3.4}$$

because $F_+(x, u_n(x)) = 0$ as soon as $u_n(x) \leq 0$, while $\|u_n^+\|_r \leq \|u_n\|_r$.

We claim that $\{u_n^+\}$ is bounded in $W_0^{1,p}(\Omega)$. In fact, if the assertion were false, then, up to subsequences, $\|u_n^+\|_p \to +\infty$. Dividing (3.4) by $\|u_n^+\|_p^p$ gives

$$\begin{aligned}
\frac{1}{p}\|v_n\|_p^p + \frac{\mu}{q\|u_n^+\|^{p-q}}\|v_n\|_q^q &\leq \frac{C}{\|u_n^+\|_p^p} + \int_\Omega \frac{F(x,u_n^+(x))}{\|u_n^+\|_p^p}\,dx \\
&\leq \frac{C}{\|u_n^+\|_p^p} + \int_{\Omega(u_n^+\leq\delta)} \frac{F(x,u_n^+(x))}{\|u_n^+\|_p^p}\,dx + \int_{\Omega(u_n^+>\delta)} \frac{F(x,u_n^+(x))}{\|u_n^+\|_p^p}\,dx \\
&\leq \frac{C}{\|u_n^+\|_p^p} + M\frac{|\Omega|}{\|u_n^+\|_p^p} + \frac{\lambda_{1,p}+\varepsilon}{p}\int_\Omega \frac{|u_n^+(x)|^p}{\|u_n^+\|_p^p}\,dx.
\end{aligned} \tag{3.5}$$

Recall next that $p \geq q$, but $p = q$ only when $\mu = 0$. As $n \to +\infty$ and $\varepsilon \to 0^+$, we get

$$\|v\|_p^p \leq \lambda_{1,p}\|v\|_{L^p(\Omega)}^p.$$

On account of (p_3), this implies that $v = \xi u_{1,p}$ for some $\xi \ge 0$. If $\xi = 0$, then $v_n \to 0$ in $L^p(\Omega)$. Thus, by (3.5), $v_n \to 0$ in $W_0^{1,p}(\Omega)$, which contradicts $\|v_n\|_p \equiv 1$. So, suppose $\xi > 0$, whence $u_n^+(x) \to +\infty$ for every $x \in \Omega$. Through (p_2), Fatou's lemma, and (3.2), one gets

$$\frac{1}{p}\|u_n^+\|_p^p - \int_\Omega F_+(x, u_n^+(x))\,dx \ge \int_\Omega \Big(\frac{\lambda_{1,p}}{p}|u_n^+(x)|^p - F(x, u_n^+(x))\Big)\,dx \to +\infty,$$

against (3.4). Consequently, the claim holds true.

Finally, also the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, because $F_+(x, -u_n^-(x)) \equiv 0$ and $\varphi_+(u_n) \le C$ for all $n \in \mathbb{N}$. This completes the proof. □

Lemma 3.2. *Let* (h_1)–(h_4) *be satisfied. Then,* (1.1) *has at least two nontrivial constant-sign solutions* $u_0 \in \operatorname{int}(C_+)$, $v_0 \in -\operatorname{int}(C_+)$, *both local minimizers of* φ.

Proof. By Lemma 3.1, the functional φ_+ possesses a global minimizer $u_0 \in W_0^{1,p}(\Omega)$. If θ, δ_0, c_0 come from (h_4), $w \in \operatorname{int}(C_+)$, and $\|w\|_{L^\infty(\Omega)} \le 1$, then

$$\varphi_+(tw) \le \frac{t^p}{p}\|w\|_p^p + \mu\frac{t^q}{q}\|w\|_q^q - \frac{c_0}{\theta}t^\theta\|w\|_{L^\theta(\Omega)}^\theta \quad \text{for all } t \in (0, \delta_0].$$

Since $\theta < q \le p$ but $q = p$ if and only if $\mu = 0$, for sufficiently small $t > 0$, the right-hand side in the above inequality turns out to be negative, which evidently forces $\varphi_+(u_0) < 0$, namely, $u_0 \ne 0$. Proceeding as in [20, Theorem 4.1] then gives $u_0 \in \operatorname{int}(C_+)$. Moreover, u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ, because $\varphi|_{C_+} = \varphi_+|_{C_+}$. Now, the conclusion follows from Proposition 2.4. A similar argument yields a function v_0 with the asserted properties. □

To establish the existence of a third nodal solution, we will first show that there exist two extremal constant-sign solutions, i.e., a smallest positive one and a biggest negative one. In fact, through (h_1) and (h_4) one has

$$f(x, t)t \ge c_0|t|^\theta - c_1|t|^p \quad \text{in } \Omega \times \mathbb{R}, \tag{3.6}$$

where $c_1 > 0$. Thus, it is quite natural to compare solutions of (1.1) with those of the auxiliary problem

$$-\Delta_p u - \mu\Delta_q u = c_0|u|^{\theta-2}u - c_1|u|^{p-2}u, \tag{3.7}$$

which, by [20, Lemma 2.2], possesses a unique positive solution $\overline{u} \in \operatorname{int}(C_+)$ and a unique negative solution $\overline{v} = -\overline{u}$. Reasoning as in the proof of [20, Lemma 2.2] yields the next result.

Lemma 3.3. *Under* (h_1)–(h_4), *any positive (respectively, negative) solution* u *to* (1.1) *fulfills* $u \ge \bar{u}$ *(respectively,* $u \le -\bar{u}$*).*

Proof. Let u be a positive solution of (1.1). For every $(x, t) \in \Omega \times \mathbb{R}$, define the functions

$$j(x, t) := \begin{cases} 0 & \text{if } t \le 0, \\ c_0 t^{\theta-1} - c_1 t^{p-1} & \text{if } 0 < t \le u(x), \\ c_0 u(x)^{\theta-1} - c_1 u(x)^{p-1} & \text{otherwise,} \end{cases} \tag{3.8}$$

$J(x, t) := \int_0^t j(x, \xi)\,d\xi$, as well as

$$\eta(w) := \frac{1}{p}\|w\|_p^p + \frac{\mu}{q}\|w\|_q^q - \int_\Omega J(x, w(x))\,dx, \quad w \in W_0^{1,p}(\Omega).$$

Obviously, the functional η belongs to $C^1(W_0^{1,p}(\Omega))$, is coercive, and weakly sequentially lower semicontinuous. So, there exists $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\eta(\tilde{u}) = \min_{w \in W_0^{1,p}(\Omega)} \eta(w). \tag{3.9}$$

As in the above proof, for sufficiently small $t > 0$, we have $\eta(tu) < 0$, whence $\eta(\tilde{u}) < 0$ and, a fortiori, $\tilde{u} \neq 0$. Now, from (3.9) it follows that

$$\langle A_p(\tilde{u}), w\rangle + \mu\langle A_q(\tilde{u}), w\rangle = \int_\Omega j(x, \tilde{u}(x))w(x)\, dx \quad \text{for all } w \in W_0^{1,p}(\Omega). \tag{3.10}$$

Setting $w := -\tilde{u}^-$ in (3.10), one obtains $\tilde{u}^- = 0$, i.e., $\tilde{u} \geq 0$. Likewise, if $w := (\tilde{u} - u)^+$, then, on account of (3.10), (3.8), (3.6), and the properties of u, one gets

$$\begin{aligned}\langle A_p(\tilde{u}), (\tilde{u} - u)^+\rangle + \mu\langle A_q(\tilde{u}), (\tilde{u} - u)^+\rangle &= \int_\Omega (c_0 u^{\theta-1} - c_1 u^{p-1})(\tilde{u} - u)^+ dx \\ &\leq \int_\Omega f(x, u)(\tilde{u} - u)^+\, dx \\ &= \langle A_p(u), (\tilde{u} - u)^+\rangle + \mu\langle A_q(u), (\tilde{u} - u)^+\rangle.\end{aligned}$$

Therefore,

$$\langle A_p(\tilde{u}) - A_p(u), (\tilde{u} - u)^+\rangle + \mu\langle A_q(\tilde{u}) - A_q(u), (\tilde{u} - u)^+\rangle \leq 0.$$

By (p_5), this evidently forces $u \geq \tilde{u}$. Through (3.10) and (3.8) we thus see that the function $\tilde{u}$ is a nonnegative nontrivial solution of (3.7). Since, due to [23, Theorem 5.4.1 and Theorem 5.5.1], $\tilde{u} \in \mathrm{int}(C_+)$, while (3.7) possesses a unique positive solution, we get $\tilde{u} = \bar{u}$, and the desired inequality follows. A similar argument works for the other conclusion. □

Remark 3.4. Weaker versions of (h_4) allow to achieve the last two lemmas, namely,

$$\text{there exists } \theta \in (1, q) \text{ such that } \liminf_{t\to 0} \frac{F(x, t)}{|t|^\theta} > 0 \text{ uniformly in } x \in \Omega$$

for Lemma 3.2 and (3.6) for Lemma 3.3. So, instead of any comparison between $F(x, t)$ and $f(x, t)t$, only the behavior of $t \mapsto f(x, t)$ and $t \mapsto F(x, t)$ for t close to zero needs to be prescribed.

From now on, Σ will denote the set of all solutions to (1.1), while

$$\Sigma_+ := \Sigma \cap \mathrm{int}(C_+), \quad \Sigma_- := \Sigma \cap (-\mathrm{int}(C_+)).$$

Proceeding exactly as in the proof of [20, Lemma 4.2], one obtains the next result.

Lemma 3.5. *If* (h_1)–(h_4) *hold true, then* (1.1) *has a smallest positive solution* $u_+ \in \mathrm{int}(C_+)$ *and a greatest negative solution* $v_- \in -\mathrm{int}(C_+)$.

A mountain pass procedure can now provide a third solution, but in order to exclude that it is the trivial one, we need further information on the critical groups of φ at zero, which will be achieved as in [21]. This is the point where (h_4) plays a crucial role.

Theorem 3.6. *Let* (h_1), (h_4) *be satisfied, let* $\varphi(u) \geq 0$ *for some* $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, *and let zero be an isolated critical point of* φ. *Then,* $C_k(\varphi, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. Observe that

$$\begin{aligned}\frac{d}{d\tau}\varphi(\tau u)\Big|_{\tau=1} &= \langle \varphi'(u), u\rangle \\ &= \|u\|_p^p + \mu\|u\|_q^q - \int_\Omega f(x, u(x))u(x)\, dx \\ &\geq \left(1 - \frac{\theta}{p}\right)\|u\|_p^p + \mu\left(1 - \frac{\theta}{p}\right)\|u\|_q^q + \int_\Omega [\theta F(x, u(x)) - f(x, u(x))u(x)]\, dx.\end{aligned}$$

By (h_1), (h_4), and (2.1), one has

$$\int_\Omega |\theta F(x,u) - f(x,u)u|\,dx \le c_2 \int_{\Omega(|u|\ge\delta_0)} |u(x)|^p\,dx \le c_3 \int_\Omega |u(x)|^r\,dx \le c_4\|u\|_p^r,$$

where $c_i > 0$, $i = 2, 3, 4$, are suitable constants, while $p < r < p^*$. Consequently,

$$\frac{d}{d\tau}\varphi(\tau u)\Big|_{\tau=1} \ge \Big(1 - \frac{\theta}{p}\Big)\|u\|_p^p - c_4\|u\|_p^r > 0$$

whenever $\|u\|_p$ is sufficiently small, say $u \in \bar{B}_{2\rho} \setminus \{0\}$ for some $\rho > 0$. Thus, in particular, if $\tau_0 > 0$, $\tau_0 u \in \bar{B}_{2\rho} \setminus \{0\}$, and $\varphi(\tau_0 u) \ge 0$, then

$$\frac{d}{d\tau}\varphi(\tau u)\Big|_{\tau=\tau_0} = \frac{1}{\tau_0}\frac{d}{d\tau}\varphi(\tau\tau_0 u)\Big|_{\tau=1} > 0.$$

This means that the C^1-function $\tau \mapsto \varphi(\tau u)$, $\tau \in (0, +\infty)$, turns out to be increasing at the point τ provided $\tau u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \varphi_0$. So, it vanishes at most once in the open interval $(0, 2\rho/\|u\|_p)$. On the other hand, (h_1) and (h_4) force

$$F(x,t) \ge \frac{c_0}{\theta}|t|^\theta - c_5|t|^p, \quad (x,t) \in \Omega \times \mathbb{R},$$

with appropriate $c_5 > 0$. Hence,

$$\int_\Omega F(x, \tau u(x))\,dx \ge \frac{c_0}{\theta}\tau^\theta\|u\|^\theta_{L^\theta(\Omega)} - c_5\tau^p\|u\|^p_{L^p(\Omega)} \quad \text{for all } \tau > 0.$$

Since $\theta < q \le p$, we get

$$\lim_{\tau\to 0^+}\frac{\varphi(\tau u)}{\tau^\theta} \le -\frac{c_0}{\theta}\|u\|^\theta_{L^\theta(\Omega)} < 0, \tag{3.11}$$

i.e., $\varphi(\tau u) < 0$ for $\tau > 0$ small enough. Summing up, given any $u \in \bar{B}_{2\rho} \setminus \{0\}$, either $\varphi(\tau u) < 0$ as soon as $\tau u \in \bar{B}_{2\rho}$ or

$$\text{there exists a unique } \bar{\tau}(u) > 0 \text{ such that } \bar{\tau}(u)u \in \bar{B}_{2\rho} \setminus \{0\},\ \varphi(\bar{\tau}(u)u) = 0. \tag{3.12}$$

Moreover, if $u \in (\bar{B}_{2\rho} \setminus \{0\}) \cap \varphi_0$, then $0 < \bar{\tau}(u) \le 1$ and

$$\varphi(\tau u) < 0 \text{ for all } \tau \in (0, \bar{\tau}(u)), \quad \varphi(\tau u) > 0 \text{ for all } \tau > \bar{\tau}(u) \quad \text{with } \tau u \in \bar{B}_{2\rho}. \tag{3.13}$$

Let $\tau : \bar{B}_\rho \setminus \{0\} \to (0, +\infty)$ be defined by

$$\tau(u) := \begin{cases} 1 & \text{when } u \in (\bar{B}_\rho \setminus \{0\}) \cap \varphi^0, \\ \bar{\tau}(u) & \text{when } u \in (\bar{B}_\rho \setminus \{0\}) \cap \varphi_0. \end{cases}$$

We claim that the function $\tau(u)$ is continuous. This immediately follows once one knows that $\bar{\tau}(u)$ turns out to be continuous on $(\bar{B}_\rho \setminus \{0\}) \cap \varphi_0$, because, by uniqueness, $u \in \bar{B}_\rho \setminus \{0\}$ and $\varphi(u) = 0$ evidently imply $\bar{\tau}(u) = 1$; cf. (3.12). Pick $\hat{u} \in (\bar{B}_\rho \setminus \{0\}) \cap \varphi_0$. The function $\phi(t,u) := \varphi(tu)$ belongs to $C^1(\mathbb{R} \times W_0^{1,p}(\Omega))$ and, on account of (3.13), we have

$$\phi(\bar{\tau}(\hat{u}), \hat{u}) = 0, \quad \frac{\partial\phi}{\partial u}(\bar{\tau}(\hat{u}), \hat{u}) = \bar{\tau}(\hat{u})\varphi'(\bar{\tau}(\hat{u})\hat{u}).$$

Since zero turns out to be an isolated critical point for φ, there is no loss of generality in assuming that $K_\varphi \cap \bar{B}_\rho = \{0\}$. So, the implicit function theorem furnishes $\sigma \in C^1(B_\varepsilon(\hat{u}))$, $\varepsilon > 0$, such that

$$\phi(\sigma(u), u) = 0 \text{ for all } u \in B_\varepsilon(\hat{u}), \quad \sigma(\hat{u}) = \bar{\tau}(\hat{u}).$$

Through $0 < \bar{\tau}(\hat{u}) \le 1$, we thus get $0 < \sigma(u) < 2$ for all $u \in U$, where $U \subseteq B_\varepsilon(\hat{u})$ denotes a convenient neighborhood of $\hat{u}$. Consequently,

$$\sigma(u)u \in \bar{B}_{2\rho} \setminus \{0\} \quad \text{and} \quad \varphi(\sigma(u)u) = 0 \quad \text{provided } u \in (\bar{B}_\rho \setminus \{0\}) \cap \varphi_0 \cap U.$$

By (3.12), this results in $\sigma(u) = \bar{\tau}(u)$, from which the continuity of $\bar{\tau}(u)$ at $\hat{u}$ follows. As $\hat{u}$ was arbitrary, the function $\bar{\tau}(u)$ turns out to be continuous on $(\bar{B}_\rho \setminus \{0\}) \cap \varphi_0$.

Next, observe that $\tau u \in \bar{B}_\rho \cap \varphi^0$ for all $\tau \in [0, 1]$, $u \in \bar{B}_\rho \cap \varphi^0$. Hence, if

$$h(t, u) := (1 - t)u, \quad (t, u) \in [0, 1] \times (\bar{B}_\rho \cap \varphi^0),$$

then $h([0, 1] \times (\bar{B}_\rho \cap \varphi^0)) \subseteq \bar{B}_\rho \cap \varphi^0$, namely, $\bar{B}_\rho \cap \varphi^0$ is contractible in itself. Moreover, the function

$$g(u) := \tau(u)u \quad \text{for all } u \in \bar{B}_\rho \setminus \{0\}$$

is continuous and one has $g(\bar{B}_\rho \setminus \{0\}) \subseteq (\bar{B}_\rho \cap \varphi^0) \setminus \{0\}$. Since

$$g|_{(\bar{B}_\rho \cap \varphi^0) \setminus \{0\}} = \mathrm{id}|_{(\bar{B}_\rho \cap \varphi^0) \setminus \{0\}},$$

the set $(\bar{B}_\rho \cap \varphi^0) \setminus \{0\}$ turns out to be a retract of $\bar{B}_\rho \setminus \{0\}$. Being $\bar{B}_\rho \setminus \{0\}$ contractible in itself, because $W_0^{1,p}(\Omega)$ is infinite dimensional, we get (see, e.g., [11, p. 389])

$$C_k(\varphi, 0) := H_k(\bar{B}_\rho \cap \varphi^0, (\bar{B}_\rho \cap \varphi^0) \setminus \{0\}) = 0, \quad k \in \mathbb{N}_0.$$

This completes the proof. □

Remark 3.7. A careful inspection of the above argument shows that the second inequality in (h_4) can be weakened to achieve the same conclusion, requiring instead

$$f(x, t)t - \theta F(x, t) \leq c_6 |t|^r \quad \text{in } \Omega \times [-\delta_0, \delta_0]$$

for suitable $\theta < p < r$ and $c_6, \delta_0 > 0$.

We are now ready to find a nodal solution of (1.1). Write, provided u, v lie in $W_0^{1,p}(\Omega)$ and $v \leq u$,

$$[v, u] = \{w \in W_0^{1,p}(\Omega) : v \leq w \leq u\}.$$

Theorem 3.8. *If* (h_1)–(h_4) *hold true, then* (1.1) *admits a sign-changing solution* $u_0 \in C_0^1(\overline{\Omega}) \cap [v_-, u_+]$.

Proof. For every $(x, t) \in \Omega \times \mathbb{R}$, define the function

$$\hat{f}(x, t) := \begin{cases} f(x, v_-(x)) & \text{if } t < v_-(x), \\ f(x, t) & \text{if } v_-(x) \leq t \leq u_+(x), \\ f(x, u_+(x)) & \text{if } u_+(x) < t, \end{cases} \tag{3.14}$$

as well as

$$\hat{f}_+(x, t) := \hat{f}(x, t^+), \quad \hat{f}_-(x, t) := \hat{f}(x, -t^-).$$

Moreover, provided $u \in W_0^{1,p}(\Omega)$, set

$$\hat{\varphi}(u) := \frac{1}{p}\|u\|_p^p + \frac{\mu}{q}\|u\|_q^q - \int_\Omega \hat{F}(x, u(x))\, dx, \quad \hat{\varphi}_\pm(u) := \frac{1}{p}\|u\|^p + \frac{\mu}{q}\|u\|_q^q - \int_\Omega \hat{F}_\pm(x, u(x))\, dx,$$

where

$$\hat{F}(x, t) := \int_0^t \hat{f}(x, \xi)\, d\xi, \quad \hat{F}_\pm(x, t) := \int_0^t \hat{f}_\pm(x, \xi)\, d\xi.$$

The same reasoning as in the proof of [20, Theorem 4.3] guarantees here that

$$K_{\hat{\varphi}} \subseteq [v_-, u_+], \quad K_{\hat{\varphi}_-} \subseteq \{v_-, 0\}, \quad K_{\hat{\varphi}_+} \subseteq \{0, u_+\}, \tag{3.15}$$

besides

$$u_+ \in \mathrm{int}(C_+) \quad \text{and} \quad v_- \in -\mathrm{int}(C_+) \quad \text{are local } W_0^{1,p}(\Omega)\text{-minimizers for } \hat{\varphi}. \tag{3.16}$$

Since, by (3.15), one has $K_{\hat{\varphi}} = K_\varphi \cap [v_-, u_+]$, it suffices to find a nontrivial critical point of $\hat{\varphi}$. Suppose that $\hat{\varphi}(v_-) \leq \hat{\varphi}(u_+)$ (the opposite case is analogous). Due to (3.16) there exists $\rho \in (0, 1)$ such that

$$\|v_- - u_+\|_p > \rho, \quad \hat{\varphi}(u_+) < m_\rho := \inf_{\partial B_\rho(u_+)} \hat{\varphi}. \tag{3.17}$$

Furthermore, the functional $\hat{\varphi}$ fulfills condition (C), because it is coercive by construction; cf. (3.14). Hence, Theorem 2.3 applies and we obtain a point $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}}, \quad m_\rho \leq \hat{\varphi}(u_0).$$

The strict inequality in (3.17) and (3.15) forces $u_0 \in [v_-, u_+] \setminus \{v_-, u_+\}$. Now, if $K_{\hat{\varphi}}$ possesses infinitely many elements, then the conclusion follows at once. Otherwise, $C_1(\hat{\varphi}, u_0) \neq 0$, because u_0 is a critical point of mountain pass type; see [5, p. 89]. Through $u_+ \in \operatorname{int}(C_+)$, $v_- \in -\operatorname{int}(C_+)$, and $\hat{\varphi}|_{[v_-,u_+]} = \varphi|_{[v_-,u_+]}$, we infer that

$$C_k(\hat{\varphi}|_{C_0^1(\overline{\Omega})}, 0) = C_k(\varphi|_{C_0^1(\overline{\Omega})}, 0), \quad k \in \mathbb{N}_0.$$

Moreover, recalling that $C_0^1(\overline{\Omega})$ turns out to be dense in $W_0^{1,p}(\Omega)$,

$$C_k(\hat{\varphi}|_{C_0^1(\overline{\Omega})}, 0) = C_k(\hat{\varphi}, 0), \quad C_k(\varphi|_{C_0^1(\overline{\Omega})}, 0) = C_k(\varphi, 0).$$

So, thanks to Theorem 3.6, $C_k(\hat{\varphi}, 0) = 0$ for all $k \in \mathbb{N}_0$, whence $u_0 \neq 0$. The solution u_0 is nodal by the extremality of v_- and u_+, while standard nonlinear regularity results yield $u_0 \in C_0^1(\overline{\Omega})$. □

Combining Lemma 3.5 with Theorem 3.8 directly produces the next result.

Theorem 3.9. *Let* (h_1)–(h_4) *be satisfied. Then,* (1.1) *admits a smallest positive solution* $u_+ \in \operatorname{int}(C_+)$*, a greatest negative solution* $v_- \in -\operatorname{int}(C_+)$*, and a nodal solution* $u_0 \in C_0^1(\overline{\Omega})$ *such that* $v_- \leq u_0 \leq u_+$.

4 Resonance from the Right

The notation in this section is the same as in Section 3. Conditions (h_2) and (h_3) furnish that

$$\lim_{|t|\to+\infty} [\lambda_{1,p}|t|^p - pF(x, t)] = +\infty \quad \text{uniformly in } x \in \Omega; \tag{4.1}$$

cf. (3.2). So, under these hypotheses, resonance with respect to $\lambda_{1,p}$ from the left occurs and, a fortiori, the energy functional φ turns out to be coercive (Lemma 3.1). Now, the question of investigating what happens when there is resonance from the right of $\lambda_{1,p}$, i.e., the limit in (4.1) equals $-\infty$, naturally arises. In this case, φ turns out to be indefinite and direct methods no longer work. However, the linking structure of suitably defined sets still fits our purpose.

The following assumption will take the place of (h_3).

(h_3') If $\mu > 0$, then there exist $\eta \in (q, p]$ and $\alpha_0 > 0$ such that

$$\liminf_{|t|\to+\infty} \frac{pF(x, t) - f(x, t)t}{|t|^\eta} \geq \alpha_0 > 0 \quad \text{uniformly in } x \in \Omega.$$

If $\mu = 0$ then,

$$\liminf_{|t|\to+\infty} [pF(x, t) - f(x, t)t] = +\infty \quad \text{uniformly in } x \in \Omega.$$

Lemma 4.1. *Suppose* (h_1)–(h_3') *hold true. Then,* φ *satisfies condition* (C).

Proof. Since $W_0^{1,p}(\Omega)$ compactly embeds in $L^p(\Omega)$, the Nemytskii operator N_f is continuous on $L^p(\Omega)$, and A_p enjoys property (p_5), it suffices to show that every sequence $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ fulfilling

$$|\varphi(u_n)| \leq C \quad \text{for all } n \in \mathbb{N} \tag{4.2}$$

and

$$\lim_{n\to+\infty}(1+\|u_n\|_p)\varphi'(u_n)=0 \tag{4.3}$$

turns out to be bounded. If the assertion were false, then, along a subsequence when necessary, $\|u_n\|_p \to +\infty$. Let $v_n := u_n/\|u_n\|_p$. We may evidently assume

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega), \qquad v_n \to v \quad \text{in } L^p(\Omega), \qquad v_n(x) \to v(x) \quad \text{for every } x \in \Omega,$$

because $\|v_n\|_p \equiv 1$. Inequality (4.2) gives

$$\frac{1}{p}\|v_n\|_p^p + \frac{\mu}{q}\frac{1}{\|u_n\|_p^{p-q}}\|v_n\|_q^q \le \frac{C}{\|u_n\|_p^p} + \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_p^p}\,dx.$$

Proceeding exactly as in the proof of Lemma 3.1, one obtains $\|v\|_p^p \le \lambda_{1,p}\|v\|_{L^p(\Omega)}^p$, which forces $v = \xi u_{1,p}$ for appropriate $\xi \in \mathbb{R}\setminus\{0\}$. Therefore, $|v| > 0$ and thus

$$|u_n| \to +\infty \quad \text{a.e. in } \Omega. \tag{4.4}$$

Through (4.3), we easily have $\langle \varphi'(u_n), u_n\rangle \to 0$, whence

$$\|u_n\|_p^p + \mu\|u_n\|_q^q - \int_\Omega f(x,u_n(x))u_n(x)\,dx \le \varepsilon_n, \tag{4.5}$$

where $\varepsilon_n \to 0^+$. From (4.2) it follows that

$$-\|u_n\|_p^p - \frac{\mu p}{q}\|u_n\|_q^q + \int_\Omega pF(x,u_n(x))\,dx \le C. \tag{4.6}$$

Combining (4.5) and (4.6) leads to

$$\int_\Omega [pF(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx \le C + \varepsilon_n + \mu\Big(\frac{p}{q}-1\Big)\|u_n\|_q^q, \tag{4.7}$$

i.e., after an elementary calculation,

$$\int_\Omega [pF(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx \le c_7(1+\|u_n\|_p^q) \tag{4.8}$$

for all $n \in \mathbb{N}$. If $\mu > 0$, then, because of (h_3'), Fatou's lemma, and (4.4),

$$\liminf_{n\to+\infty}\frac{1}{\|u_n\|_p^\eta}\int_\Omega [pF(x,u_n)-f(x,u_n)u_n]\,dx = \liminf_{n\to+\infty}\int_\Omega \frac{pF(x,u_n)-f(x,u_n)u_n}{|u_n|^\eta}|v_n|^\eta\,dx \ge \alpha_0\|v\|_{L^\eta(\Omega)}^\eta > 0. \tag{4.9}$$

However, since $\eta > q$, dividing (4.8) by $\|u_n\|_p^\eta$ and letting $n \to +\infty$ produces

$$\limsup_{n\to+\infty}\frac{1}{\|u_n\|_p^\eta}\int_\Omega [pF(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx \le 0,$$

against (4.9). So, suppose $\mu = 0$. Thanks to (h_3'), one has

$$\lim_{n\to+\infty}\int_\Omega [pF(x,u_n(x)) - f(x,u_n(x))u_n(x)]\,dx = +\infty,$$

which contradicts (4.7). Therefore, the sequence $\{u_n\}$ turns out to be bounded in $W_0^{1,p}(\Omega)$, as required. □

Lemma 4.2. *Let* (h_1)*–*(h_3') *be satisfied. Then,* $\lim_{t\to\pm\infty}\varphi(tu_{1,p})=-\infty$.

Proof. Consider first the case $\mu>0$. Without loss of generality, we may suppose $\eta<p$ in (h_3'). Thus, there exist $\alpha_1,\delta_1>0$ such that

$$\alpha_1|t|^\eta\le pF(x,t)-f(x,t)t\quad\text{for every } x\in\Omega,\ |t|\ge\delta_1.$$

Consequently,

$$\frac{d}{dt}\frac{F(x,t)}{t^p}=\frac{f(x,t)t-pF(x,t)}{t^{p+1}}\le-\alpha_1t^{\eta-p-1},\quad (x,t)\in\Omega\times[\delta_1,+\infty).$$

After integration, this results in

$$\frac{F(x,t)}{t^p}-\frac{F(x,s)}{s^p}\le\frac{\alpha_1}{p-\eta}\left(t^{\eta-p}-s^{\eta-p}\right)\quad\text{provided } t\ge s\ge\delta_1.$$

Letting $t\to+\infty$, on account of (h_2) we have

$$\frac{\lambda_{1,p}}{p}s^p-F(x,s)\le-\frac{\alpha_1}{p-\eta}s^\eta\quad\text{in }\Omega\times[\delta_1,+\infty),$$

which clearly implies that

$$\frac{\lambda_{1,p}}{p}s^p-F(x,s)\le-\frac{\alpha_1}{p-\eta}s^\eta+c_8\quad\text{for all }(x,s)\in\Omega\times[0,+\infty).$$

Hence, for any $t>0$,

$$\varphi(tu_{1,p})=\frac{t^p}{p}\lambda_{1,p}\|u_{1,p}\|^p_{L^p(\Omega)}+\frac{\mu t^q}{q}\|u_{1,p}\|_q^q-\int_\Omega F(x,tu_{1,p}(x))\,dx\le-\frac{\alpha_1}{p-\eta}t^\eta\|u_{1,p}\|^\eta_{L^\eta(\Omega)}+\frac{\mu t^q}{q}\|u_{1,p}\|_q^q+c_8|\Omega|,$$

namely, $\varphi(tu_{1,p})\to-\infty$ as $t\to+\infty$. The proof for $t\to-\infty$ is analogous.

Now, let $\mu=0$. By (h_3') again, to every $K>0$ corresponds $\delta>0$ such that

$$pF(x,t)-f(x,t)t>K,\quad (x,t)\in\Omega\times[\delta,+\infty).$$

The same argument as before yields here

$$\frac{\lambda_{1,p}}{p}s^p-F(x,s)\le-\frac{K}{p}\quad\text{in }\Omega\times[\delta,+\infty).$$

Define

$$M:=\sup_{\Omega\times[0,\delta]}\left|\frac{\lambda_{1,p}}{p}s^p-F(x,s)\right|$$

and observe that

$$\begin{aligned}\varphi(tu_{1,p})&=\frac{t^p}{p}\lambda_{1,p}\|u_{1,p}\|^p_{L^p(\Omega)}-\int_\Omega F(x,tu_{1,p}(x))\,dx\\&\le\int_{\Omega\left(u_{1,p}\le\frac{\delta}{t}\right)}\left(\frac{\lambda_{1,p}}{p}t^pu_{1,p}^p-F(x,tu_{1,p})\right)dx+\int_{\Omega\left(u_{1,p}>\frac{\delta}{t}\right)}\left(\frac{\lambda_{1,p}}{p}t^pu_{1,p}^p-F(x,tu_{1,p})\right)dx\\&\le M\left|\Omega\left(u_{1,p}\le\frac{\delta}{t}\right)\right|-\frac{K}{p}|\Omega|\end{aligned}$$

provided $t>0$. Since $u_{1,p}>0$, letting $t\to+\infty$ leads to

$$\limsup_{t\to+\infty}\varphi(tu_{1,p})\le-\frac{K}{p}|\Omega|.$$

As $K>0$ was arbitrary, we actually have $\lim_{t\to+\infty}\varphi(tu_{1,p})=-\infty$. The case $t\to-\infty$ is quite similar. □

Next, write

$$E := \{u \in W_0^{1,p}(\Omega) : \|u\|_p^p = \lambda_{2,p}\|u\|_{L^p(\Omega)}^p\}.$$

Obviously, E turns out to be nonempty and closed.

Lemma 4.3. *If* (h$_1$)–(h$_2$) *hold true, then* $\varphi|_E$ *is coercive.*

Proof. Pick $\xi \in (\lambda_{1,p}, \lambda_{2,p})$. The hypotheses give $K > 0$ such that

$$F(x,t) \le \frac{\xi}{p}|t|^p + K \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Consequently, for any $u \in E$,

$$\varphi(u) \ge \frac{1}{p}\|u\|_p^p + \frac{\mu}{q}\|u\|_q^q - \frac{\xi}{p}\|u\|_{L^p(\Omega)}^p - K|\Omega| \ge \frac{1}{p}\Big(1 - \frac{\xi}{\lambda_{2,p}}\Big)\|u\|_p^p - K|\Omega|.$$

Since $\xi < \lambda_{2,p}$, the assertion follows. □

Lemma 4.3 basically ensures that $\inf_E \varphi > -\infty$. Thanks to Lemma 4.2, we can find $\tau > 0$ fulfilling

$$\varphi(\pm\tau u_{1,p}) < m_E := \inf_E \varphi. \tag{4.10}$$

Define

$$Q_0 := \{\pm\tau u_{1,p}\}, \quad Q := \{tu_{1,p} : t \in [-\tau,\tau]\}.$$

Lemma 4.4. *The pair* (Q_0, Q) *links* E *in* $W_0^{1,p}(\Omega)$.

Proof. One evidently has $Q_0 \cap E = \varnothing$. Moreover, if

$$U := \{u \in W_0^{1,p}(\Omega) : \|u\|_p^p < \lambda_{2,p}\|u\|_{L^p(\Omega)}^p\},$$

then $Q_0 \subseteq U$, because $\lambda_{1,p} < \lambda_{2,p}$. Let us verify that $-\tau u_{1,p}$ and $\tau u_{1,p}$ lie in different pathwise connected components of U. Arguing by contradiction, there exists $\sigma \in C^0([-1,1], U)$ such that $\sigma(-1) = -\tau u_{1,p} = -\sigma(1)$. On the other hand, (p$_4$) forces

$$\lambda_{2,p} \le \max_{t\in[-1,1]} \frac{\|\sigma(t)\|_p^p}{\|\sigma(t)\|_{L^p(\Omega)}^p},$$

which leads to $\sigma(t_0) \notin U$ for some $t_0 \in (0,1)$. However, this is impossible. Hence, any $\gamma \in C^0(Q, W_0^{1,p}(\Omega))$ such that $\gamma|_{Q_0} = \mathrm{id}|_{Q_0}$ must satisfy the condition $\gamma(Q) \cap \partial U \ne \varnothing$. Since $\partial U \subseteq E$, the proof is complete. □

We are now in a position to treat the existence of solutions to (1.1) when resonance from the right of $\lambda_{1,p}$ occurs. To the best of our knowledge, *multiplicity* is still an open question.

Theorem 4.5. *Under assumptions* (h$_1$)–(h$_3'$) *and* (h$_4$), *the problem* (1.1) *has at least one nontrivial solution* $u_0 \in C_0^1(\overline{\Omega})$.

Proof. By Lemma 4.1, Lemma 4.4, and (4.10), one can apply Theorem 2.2. Thus, we get a point $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi'(u_0) = 0, \quad \inf_E \varphi \le \varphi(u_0) = \inf_{\gamma\in\Gamma}\sup_{u\in Q}\varphi(\gamma(u)),$$

where

$$\Gamma := \{\gamma \in C^0(Q, W_0^{1,p}(\Omega)) : \gamma|_{Q_0} = \mathrm{id}|_{Q_0}\}.$$

Moreover, $C_1(\varphi, u_0) \ne 0$, because u_0 is a critical point of mountain pass type; see [5, p. 89]. On the other hand, due to Lemma 4.3 and Theorem 3.6, one has $C_1(\varphi, 0) = 0$. Therefore, $u_0 \ne 0$. Standard results from nonlinear regularity theory then ensure that $u_0 \in C_0^1(\overline{\Omega})$. □

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