Research Article

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On a Generalization of a Global Implicit Function Theorem

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Abstract: In this paper, we derive generalized versions of the results on the existence, uniqueness and continuous differentiability of a global implicit function obtained in [5] and we give some examples.

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1 Introduction

The aim of the present paper is to weaken the assumptions of a global implicit function theorem which was obtained in [5] and to show that such changes are essential.

Using the same method of proof as in [6] (cf. also [5]), based on the mountain pass theorem, we derive a generalized version of a global implicit function theorem obtained in [5] for the equation

$$F(x, y) = 0,$$

where $F : X \times Y \to H$ with X, Y real Banach spaces and H a real Hilbert space. More precisely, the global implicit function theorem obtained in [5] has been split in the present paper into three parts: the existence of a global implicit function, its uniqueness, and its continuous differentiability. We show that the conditions for the existence of a global implicit function can be weakened in relation to the assumptions given in [5] (the assumptions in [5] are simply assumptions guarantying the existence, uniqueness, and continuous differentiability of a global implicit function). Namely, we assume the following:

- The Gâteaux differentiability of F in one variable x instead of the continuous Gâteaux differentiability in both variables (x, y).
- The weak (PS) condition and the boundedness of a minimizing sequence instead of the (PS) condition.
- A "non-orthogonality" condition $F(x, y) \notin (\operatorname{Im} F'_x(x, y))^{\perp}$ instead of the condition $F(x, y) \in \operatorname{Im} F'_x(x, y)$ for points $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$, see [5, Remark 1]; we omit the bijectivity of $F'_x(x, y)$ for points $(x, y) \in X \times Y$ such that $F(x, y) = 0$.

In Section 2.1, we give an example of an operator that satisfies the new assumptions and does not satisfy the assumptions from [5] (including the assumptions given in [5, Remark 1]).

Similarly, we show that the conditions for the uniqueness of a global implicit function can be weakened in relation to the assumptions given in [5]. Namely, we assume the following:

- The continuous Gâteaux differentiability of F in one variable x instead of the continuous Gâteaux differentiability in both variables (x, y).
- The non-orthogonality condition instead of the condition that $F(x, y) \in \operatorname{Im} F'_x(x, y)$ for points $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$.

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The new theorem on the continuous differentiability of a global implicit function is a simple consequence of the above two theorems and the classical local implicit function theorem. In this new theorem, we assume:

- The non-orthogonality condition instead of the condition that $F(x, y) \in \operatorname{Im} F'_x(x, y)$ for points $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$.

In Section 4.1, we give an example of a two-dimensional problem showing that the new assumptions of the global implicit function theorem are satisfied whereas the assumptions of such a theorem from [5] (including the assumptions given in [5, Remark 1]) are not satisfied.

In Remark 2.5, we discuss the non-orthogonality condition $F(x, y) \notin (\operatorname{Im} F'_x(x, y))^{\perp}$ and give some tools to check it in concrete situations. In a particular case, it can be replaced by the condition $F(x, y) \notin \operatorname{Ker} F'_x(x, y)$.

The method of proof of the theorem on existence is the same as in [5] (and [6]) but we use [10, Corollary 3.4] instead of [10, Corollary 2.5] because of the weak (PS) condition. Similarly, the proof of the uniqueness of a global implicit function is the same as in [5] (and [6]) (we give here this proof to show that it allows us to weaken the assumptions in relation to [5]).

2 Existence of a Global Implicit Function

Let X be a real Banach space and $I : X \to \mathbb{R}$ a functional differentiable in the Gâteaux sense. We say that I satisfies the *Palais–Smale condition* (shortly, (PS) condition) if any sequence (x_m) satisfying the conditions

- $|I(x_m)| \leq M$ for all $m \in \mathbb{N}$ and some $M > 0$,
- $I'(x_m) \to 0$,

admits a convergent subsequence (here, $I'(x_m)$ denotes the Gâteaux differential of I at x_m). A sequence (x_m) satisfying the above conditions is called a (PS) sequence. We say that I satisfies the *weak* (PS) *condition* if any bounded (PS) sequence (x_m) admits a convergent subsequence.

A point $x^* \in X$ is called a *critical point* of I if $I'(x^*) = 0$. In such a case, $I(x^*)$ is called a *critical value* of I. By $\inf I$ we denote the value $\inf_{x \in X} I(x)$ and by a minimizing sequence we mean any sequence (x_n) such that $I(x_n) \to \inf I$. In [8, Corollary 3.4], the following result is deduced.

Lemma 2.1. *Let X be a real Banach space. If $I : X \to \mathbb{R}$ has a bounded minimizing sequence and it is a lower semi-continuous, bounded from below, and differentiable in the Gâteaux sense functional satisfying the weak* (PS) *condition, then* $\inf I$ *is the critical value of I.*

Remark 2.2. Let us observe that if $I : X \to \mathbb{R}$ is a lower semi-continuous, bounded from below, and differentiable in the Gâteaux sense functional that satisfies the (PS) condition, then it has a bounded minimizing sequence and satisfies the weak (PS) condition. Indeed, let (x_n) be a minimizing sequence. Then, from the theorem on almost critical points[1] it follows that there exists a minimizing sequence (y_n) which is a (PS) sequence for I. Since I satisfies the (PS) condition, (y_n) has a convergent and, consequently, bounded subsequence. The fact that I satisfies the weak (PS) condition is obvious.

Using the above lemma, we obtain the following theorem.

Theorem 2.3. *Let X be a real Banach space and H a real Hilbert space. If $G : X \to H$ is differentiable in the Gâteaux sense and*

- *the functional*
$$\varphi : X \ni x \mapsto \frac{1}{2}\|G(x)\|^2 \in \mathbb{R} \tag{2.1}$$
is lower semi-continuous, has a bounded minimizing sequence, and satisfies the weak (PS) *condition,*
- $G(x) \notin (\operatorname{Im} G'(x))^{\perp}$ *for any $x \in X$ such that $G(x) \neq 0$,*

then there exists $x \in X$ such that $G(x) = 0$.

1 In [8, Corollary 3.2], the following theorem is proved: If X is a real Banach space and $I : X \to \mathbb{R}$ is a lower semi-continuous, bounded from below, and differentiable in the Gâteaux sense functional, then, for any minimizing sequence (x_n), there exists a minimizing sequence (y_n) such that $I(y_n) \leq I(x_n)$ for $n \in \mathbb{N}$, $\|y_n - x_n\| \to 0$, and $\|I'(y_n)\| \to 0$.

Proof. The functional φ, being a superposition of the mapping $\frac{1}{2}\|\cdot\|^2$, which is differentiable in the Fréchet sense on H, and the mapping G, which is differentiable in the Gâteaux sense on X, is differentiable in the Gâteaux sense on X and its Gâteaux differential $\varphi'(x)$ at $x \in X$ is given by

$$\varphi'(x)h = \langle G(x), G'(x)h\rangle$$

for $h \in X$. Moreover, φ is bounded from below and, by assumption, lower semi-continuous, has a bounded minimizing sequence, and satisfies the weak (PS) condition. So, by Lemma 2.1, there exists a point $x \in X$ such that $\varphi(x) = \inf \varphi$ and

$$\langle G(x), G'(x)h\rangle = 0$$

for $h \in X$, i.e., $G(x) \in (\operatorname{Im} G'(x))^{\perp}$. So, $G(x) = 0$. □

Remark 2.4. The assumption on the lower semi-continuity of φ can be replaced by a more restrictive one but concerning directly G, namely, the continuity of G.

Remark 2.5. It is known that in the case of a densely defined linear (not necessarily continuous) operator $A : D(A) \subset H \to H$ in a Hilbert space H, the orthogonal decomposition

$$\overline{\operatorname{Im} A} \oplus \operatorname{Ker} A^* = H$$

of this space holds true, where $A^* : D(A^*) \to H$ is the adjoint operator of A defined on

$$D(A^*) = \{x \in H \mid \text{there exists } z \in H \text{ such that for all } y \in D(A) \text{ there holds } \langle Ay, x\rangle = \langle y, z\rangle\}$$

by $A^*x = z$. The above decomposition implies the equality

$$(\overline{\operatorname{Im} A})^{\perp} = \operatorname{Ker} A^*$$

and, consequently,

$$(\operatorname{Im} A)^{\perp} = \operatorname{Ker} A^*.$$

So, if X is a dense linear subspace of H, then the non-orthogonality condition $G(x) \notin (\operatorname{Im} G'(x))^{\perp}$ in Theorem 2.3 can be replaced by the condition

$$G(x) \notin \operatorname{Ker} G'(x)^*.$$

When, for some $x \in X$, $G'(x)$ is self-adjoint, then the non-orthogonality condition takes the form

$$G(x) \notin \operatorname{Ker} G'(x). \tag{2.2}$$

The non-orthogonality condition takes the above form also in the case of $G'(x) : X \subset H \to H$ being a densely defined closed linear operator with closed range $\operatorname{Im} G'(x)$ and such that

$$\operatorname{Im} G'(x) = (\operatorname{Ker} G'(x))^{\perp}.$$

Examples of elliptic, parabolic, and hyperbolic operators A with the property $\operatorname{Im} A = (\operatorname{Ker} A)^{\perp}$ can be found in [3].

In [7], the following lemma is proved: If U, V are Banach spaces and $\Lambda : U \to V$ is a linear continuous operator such that $\operatorname{Im} \Lambda = V$, then $(\operatorname{Ker} \Lambda)^{\dashv} = \operatorname{Im} \Lambda^{\circledast}$, where $(\operatorname{Ker} \Lambda)^{\dashv} = \{f \in U^* \mid f(u) = 0 \text{ for } u \in \operatorname{Ker} \Lambda\}$ and $\Lambda^{\circledast} : V^* \to U^*$ is given by $(\Lambda^{\circledast} g)(u) = g(\Lambda u)$ for $g \in V^*$, $u \in U$. If $U = V = H$, then the operators $\Lambda^* : H \to H$ and $\Lambda^{\circledast} : H^* \to H^*$ are connected in the following way:

$$\Lambda^* h = h_{\Lambda^{\circledast} h^*},$$

where $h^* : H \ni z \mapsto \langle h, z\rangle \in \mathbb{R}$ and $h_{\Lambda^{\circledast} h^*} \in H$ is such that $(\Lambda^{\circledast} h^*)z = \langle h_{\Lambda^{\circledast} h^*}, z\rangle$ (cf. [4, Examples VI.2.3.3]). It is easy to see that in the case of a Hilbert space $U = V = H$ and a linear continuous operator $\Lambda : H \to H$ such that $\operatorname{Im} \Lambda = H$, the equality $(\operatorname{Ker} \Lambda)^{\dashv} = \operatorname{Im} \Lambda^{\circledast}$ implies that $(\operatorname{Ker} \Lambda)^{\perp} = \operatorname{Im} \Lambda^*$ and, consequently, $\operatorname{Ker} \Lambda = (\operatorname{Im} \Lambda^*)^{\perp}$.

Thus, if in Theorem 2.3, for some $x \in H$ such that $G(x) \neq 0$, the operator $G'(x) : H \to H$ is the adjoint of a linear continuous operator $\Lambda : H \to H$ with $\operatorname{Im} \Lambda = H$, then the "non-orthogonality" condition $G(x) \notin (\operatorname{Im} G'(x))^{\perp}$ can be replaced by

$$G(x) \notin \operatorname{Ker} \Lambda.$$

Denoting the above operator Λ by ${}^{*}G'(x)$, we can write the above condition in the form

$$G(x) \notin \operatorname{Ker} {}^{*}G'(x).$$

If Λ is self-adjoint, then $G'(x)$ has the same property and the above condition reduces to (2.2).

From Theorem 2.3 we obtain the following corollary.

Corollary 2.6. *Let X be a real Banach space, Y a non-empty set, and H a real Hilbert space. If $F : X \times Y \to H$ is differentiable with respect to $x \in X$ in the Gâteaux sense and*

- *for any $y \in Y$, the functional*
$$\varphi_y : X \ni x \mapsto \frac{1}{2}\|F(x, y)\|^2 \in \mathbb{R} \tag{2.3}$$
is lower semi-continuous, has a bounded minimizing sequence, and satisfies the weak (PS) condition,
- *$F(x, y) \notin (\operatorname{Im} F'_x(x, y))^{\perp}$ for any $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$ (here, $F'_x(x, y)$ denotes the Gâteaux differential of F at (x, y) with respect to x),*

then, for any $y \in Y$, there exists $x_y \in X$ such that $F(x_y, y) = 0$.

2.1 Example

Let us consider the operator

$$F : W \times S^{\perp} \ni (x, g) \mapsto x'' + x - g \in L^2,$$

where

$$W = \{x : [0, \pi] \to \mathbb{R} \mid x, x' \text{ are absolutely continuous},\ x'' \in L^2,\ x(0) = x(\pi) = 0\}$$

and $S = \{a \sin(\cdot) \mid a \in \mathbb{R}\}$ (here, $S^{\perp}$ denotes the orthogonal subspace to S in $L^2 = L^2([0, \pi], \mathbb{R})$). We consider the space W with the scalar product

$$\langle x, y\rangle = \langle x, y\rangle_{L^2} + \langle x'', y''\rangle_{L^2}, \quad x, y \in W.$$

The norm

$$\|x\|^2 = \|x\|_{L^2}^2 + \|x''\|_{L^2}^2 \tag{2.4}$$

generated by this product is equivalent to the norm (see [2, Part VIII.2])

$$\|x\|_W^2 = \|x\|_{L^2}^2 + \|x'\|_{L^2}^2 + \|x''\|_{L^2}^2. \tag{2.5}$$

Of course, the space W with the norm $\|\cdot\|$ is complete.

We shall show that F satisfies the assumptions of Corollary 2.6 and that it does not satisfy the assumptions of the global implicit function theorem proved in [5, Theorem 4.1 and Remark 1], i.e., it does not satisfy the (PS) condition and its differential is not bijective for all $(x, g) \in W \times S^{\perp}$ satisfying $F(x, g) = 0$.

It is well known (see [8]) that the range $\operatorname{Im} F'_x(x, g)$ of the operator

$$F'_x(x, g) : W \ni h \mapsto h'' + h \in L^2$$

is equal to $S^{\perp}$ for any $(x, g) \in W \times S^{\perp}$. So, the assumption on the bijectivity of $F'_x(x, g)$ is not satisfied but it is clear that

$$F(x, g) = x'' + x - g \in \operatorname{Im} F'_x(x, g)$$

for any $(x, g) \in W \times S^{\perp}$ and, consequently, $F(x, g) \notin (\operatorname{Im} F'_x(x, g))^{\perp}$ for any $(x, g) \in W \times S^{\perp}$ such that $F(x, g) \neq 0$. Alternatively, using Remark 2.5, one can show that $F(x, g) \notin \operatorname{Ker} F'_x(x, g)$ for any $(x, g) \in W \times S^{\perp}$ such that $F(x, g) \neq 0$. It follows from the equality $\operatorname{Ker} F'_x(x, g) = S$.

Now, let us fix $g \in S^{\perp}$ and consider the functional

$$\varphi_g : W \ni x \mapsto \frac{1}{2}\|x'' + x - g\|^2 \in \mathbb{R}.$$

Of course, the differential $\varphi_g'(x) : W \to \mathbb{R}$ is given by

$$\varphi_g'(x)h = \int_0^{\pi} (x''(t) + x(t) - g(t))(h''(t) + h(t))\,dt.$$

It is clear that (x_n), where $x_n(t) = n \sin t$, is a (PS) sequence for φ_0 (in fact, $\varphi_0(x_n) = \frac{1}{2}\|0\|_{L^2}^2 = 0$ and $\varphi_0'(x_n) = 0$ for $n \in \mathbb{N}$) and it does not contain any subsequence which is converging in L^2. So, φ_0 does not satisfy the (PS) condition (we see that φ_0 is not coercive, too).

But, for any $g \in S^{\perp}$, we have $\inf_{x\in W} \varphi_g(x) = 0$ and a constant sequence (x_n), where $x_n \in W$ is a fixed solution to the equation

$$x'' + x = g$$

for any $n \in \mathbb{N}$, is minimizing for φ_g, and, of course, it is bounded.

Now, we shall show that φ_g satisfies the weak (PS) condition. Let (x_n) be a bounded (PS) sequence for φ_g. Since W is a Hilbert space, we may assume, without loss of generality, that there exists $x_0 \in W$ such that $x_n \rightharpoonup x_0$ weakly in W. Using the Arzelà–Ascoli theorem and the equivalence of the norms (2.4) and (2.5), one can show that there exists a subsequence (x_{n_k}) such that $x_{n_k} \rightrightarrows x_0$ uniformly on $[0, \pi]$ and $x_{n_k}'' \rightharpoonup x_0''$ weakly in L^2. Consequently, from the identity

$$(\varphi_g'(x_{n_k}) - \varphi_g'(x_0))(x_{n_k} - x_0) = \|x_{n_k} - x_0\|_W^2 + 2\langle x_{n_k}'' - x_0'', x_{n_k} - x_0\rangle_{L^2}$$

it follows that $x_{n_k} \to x_0$ in W (with respect to the norm).

3 Uniqueness of a Global Implicit Function

We have the following celebrated result due to Ambrosetti and Rabinowitz (see [1, 9]).

Theorem 3.1 (Mountain Pass Theorem). *Let $I : X \to \mathbb{R}$ be a functional which is continuously differentiable in the Gâteaux sense (equivalently, in the Fréchet sense), satisfies the* (PS) *condition, and $I(0) = 0$. If there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B(0,\rho)} \geq \alpha$ and $I(e) \leq 0$ for some $e \in X \setminus \overline{B(0,\rho)}$, then*

$$b := \sup_{U\in W_e} \inf_{u\in\partial U} I(u)$$

is the critical value of I and $b \geq \alpha$, where

$$W_e = \{U \subset X \mid U \text{ is open}, 0 \in U, \text{and } e \notin \overline{U}\}.$$

Using the above theorem and the method of proof applied in [5, 6], we can prove the following theorem (we give here the proof to show that the method allows us to weaken the assumptions with relation to [5]).

Theorem 3.2. *Let X be a real Banach space and H a real Hilbert space. If $G : X \to H$ is continuously differentiable and*

- *the functional*
$$\varphi : X \ni x \mapsto \frac{1}{2}\|G(x)\|^2 \in \mathbb{R}$$
satisfies the (PS) *condition,*
- *$G'(x) : X \to H$ is bijective for any $x \in X$ such that $G(x) = 0$ and $G(x) \notin (\operatorname{Im} G'(x))^{\perp}$ for any $x \in X$ such that $G(x) \neq 0$,*

then there exists a unique point $x \in X$ such that $G(x) = 0$.

Proof. From Theorem 2.3 and Remark 2.2 it follows that there exists a point $x \in X$ such that $G(x) = 0$. Let us suppose that there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $G(x_1) = G(x_2) = 0$. Put $e = x_2 - x_1$ and

$$g(x) = G(x + x_1)$$

for $x \in X$. Of course,

$$g(x) = g'(0)x + o(x) = G'(x_1)x + o(x)$$

for $x \in X$, where

$$\frac{o(x)}{\|x\|_X} \to 0$$

in H when $x \to 0$ in X. So,

$$\beta\|x\|_X \leq \|G'(x_1)x\|_H \leq \|g(x)\|_H + \|o(x)\|_H \leq \|g(x)\|_H + \frac{1}{2}\beta\|x\|_X$$

for sufficiently small $\|x\|_X$ and some $\beta > 0$ (the existence of such a β follows from the bijectivity of $G'(x_1)$). Thus, there exists $\rho > 0$ such that

$$\frac{1}{2}\beta\|x\|_X \leq \|g(x)\|_H$$

for $x \in \overline{B(0,\rho)}$. Without loss of generality one may assume that $\rho < \|e\|_X$. Put

$$\psi(x) = \frac{1}{2}\|g(x)\|_H^2 = \frac{1}{2}\|G(x + x_1)\|_H^2 = \varphi(x + x_1)$$

for $x \in X$. Of course, ψ is continuously differentiable on X and

$$\psi'(x) = \varphi'(x + x_1).$$

Consequently, since φ satisfies the (PS) condition, ψ has this property, too. Moreover, $\psi(0) = \psi(e) = 0$, $e \notin \overline{B(0,\rho)}$, and $\psi(x) \geq \alpha$ for $x \in \partial B(0,\rho)$ with $\alpha = \frac{1}{8}\beta^2\rho^2 > 0$.

Thus, the mountain pass theorem implies that $b = \sup_{U \in W_e} \inf_{x \in \partial U} \psi(x)$ is a critical value of ψ and $b \geq \alpha$, i.e., there exists a point $x^* \in X$ such that $\psi(x^*) = b > 0$ and

$$0 = \psi'(x^*)h = \varphi'(x^* + x_1)h = \langle G(x^* + x_1), G'(x^* + x_1)h\rangle$$

for $h \in X$. The first condition means that $G(x^* + x_1) \neq 0$ and the second condition means that $G(x^* + x_1) \in (\operatorname{Im} G'(x^* + x_1))^{\perp}$. The obtained contradiction completes the proof. □

As we remarked in [5], if $X = \mathbb{R}^n$, then the (PS) condition imposed on φ can be replaced by the coercivity of φ.

Corollary 3.3. *Let X be a real Banach space, Y a non-empty set, and H a real Hilbert space. If $F : X \times Y \to H$ is continuously differentiable with respect to $x \in X$ and*

- *for any $y \in Y$, the functional*
$$\varphi_y : X \ni x \mapsto \frac{1}{2}\|F(x,y)\|^2 \in \mathbb{R}$$
satisfies the (PS) *condition,*
- *$F'_x(x,y) : X \to Y$ is bijective for any $(x,y) \in X \times Y$ such that $F(x,y) = 0$ and $F(x,y) \notin (\operatorname{Im} F'_x(x,y))^{\perp}$ for any $(x,y) \in X \times Y$ such that $F(x,y) \neq 0$,*

then, for any $y \in Y$, there exists a unique $x_y \in X$ such that $F(x_y, y) = 0$.

4 A Global Implicit Function Theorem

From Corollary 3.3 and the classical local implicit function theorem we immediately obtain the following global implicit function theorem.

Theorem 4.1. *Let X, Y be real Banach spaces and H a real Hilbert space. If $F : X \times Y \to H$ is continuously differentiable with respect to $(x, y) \in X \times Y$ and*

- *for any $y \in Y$, the functional*
$$\varphi_y : X \ni x \mapsto \frac{1}{2}\|F(x, y)\|^2 \in \mathbb{R}$$
satisfies the (PS) *condition*
- *$F'_x(x, y) : X \to H$ is bijective for any $(x, y) \in X \times Y$ such that $F(x, y) = 0$ and $F(x, y) \notin (\operatorname{Im} F'_x(x, y))^\perp$ for any $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$,*

then there exists a unique function $\lambda : Y \to X$ such that $F(\lambda(y), y) = 0$ for any $y \in Y$ and this function is continuously differentiable with differential $\lambda'(y)$ at y given by

$$\lambda'(y) = -[F'_x(\lambda(y), y)]^{-1} \circ F'_y(\lambda(y), y). \tag{4.1}$$

Proof. Of course, it is sufficient to put $\lambda(y) = x_y$, where x_y is a solution to $F(x, y) = 0$, as described in Corollary 3.3. □

Remark 4.2. The assumption that $F'_x(x, y) : X \to H$ is bijective for any $(x, y) \in X \times Y$ such that $F(x, y) = 0$ can be replaced by the equivalent assumption that $F(x, y) \neq 0$ for any $(x, y) \in X \times Y$ such that $F'_x(x, y) : X \to H$ is not bijective, which may be more convenient in practice.

4.1 Example

Now, we shall give an example of a function which satisfies the assumptions of Theorem 4.1 and does not satisfy the assumptions of the global implicit function theorem derived in [5]. Namely, let us consider the function

$$F : \mathbb{R}^2 \times \mathbb{R} \ni (x_1, x_2, y) \mapsto (x_1 - y + \sin(x_2 - y), x_2 - y + \cos(x_1 - y)) \in \mathbb{R}^2.$$

The Jacobi matrix $[F'_x(x_1, x_2, y)]$ of F with respect to $x = (x_1, x_2) \in \mathbb{R}^2$ has the form

$$[F'_x(x_1, x_2, y)] = \begin{bmatrix} 1 & \cos(x_2 - y) \\ -\sin(x_1 - y) & 1 \end{bmatrix}.$$

So, $F'_x(x_1, x_2, y) : \mathbb{R}^2 \to \mathbb{R}^2$ is not bijective if and only if

$$(x_1, x_2, y) \in \left\{\left(\frac{\pi}{2}+y+2k\pi, \pi+y+2l\pi, y\right) \,\middle|\, k, l \in \mathbb{Z},\ y \in \mathbb{R}\right\} \cup \left\{\left(-\frac{\pi}{2}+y+2k\pi, 0+y+2l\pi, y\right) \,\middle|\, k, l \in \mathbb{Z},\ y \in \mathbb{R}\right\},$$

where $\mathbb{Z}$ is the set of all integers. Thus, it is clear (cf. Remark 4.2) that the first part of the second assumption of Theorem 4.1 is satisfied.

To check the second part of the second assumption, we first observe that (in general) if $F(x_1, x_2, y) \neq 0$ and $F'_x(x_1, x_2, y)$ is bijective, then $F(x_1, x_2, y) \notin (\operatorname{Im} F'_x(x_1, x_2, y))^\perp$ (in such a case, $(\operatorname{Im} F'_x(x_1, x_2, y))^\perp = \{0\}$). So, it remains to show that if a point (x_1, x_2, y) is such that $F(x_1, x_2, y) \neq 0$ and $F'_x(x_1, x_2, y)$ is not bijective, then

$$\langle F(x_1, x_2, y), F'_x(x_1, x_2, y)(h_1, h_2)\rangle \neq 0$$

for some $(h_1, h_2) \in \mathbb{R}^2$. Indeed, for points $(x_1, x_2, y) = (\frac{\pi}{2} + y + 2k\pi, \pi + y + 2l\pi, y)$, we have

$$\begin{aligned}
&\langle F(x_1, x_2, y), F'_x(x_1, x_2, y)(h_1, h_2)\rangle \\
&= \left\langle \left(\frac{\pi}{2} + 2k\pi + \sin(\pi + 2l\pi), \pi + 2l\pi + \cos\left(\frac{\pi}{2} + 2k\pi\right)\right), \begin{bmatrix} 1 & \cos(\pi + 2l\pi) \\ -\sin(\frac{\pi}{2} + 2k\pi) & 1 \end{bmatrix} (h_1, h_2) \right\rangle \\
&= \left\langle \left(\frac{\pi}{2} + 2k\pi, \pi + 2l\pi\right), \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (h_1, h_2) \right\rangle \\
&= \left(\frac{\pi}{2} + 2k\pi\right)(h_1 - h_2) + (\pi + 2l\pi)(h_2 - h_1) \\
&= \left(-\frac{\pi}{2} + 2(k - l)\pi\right)(h_1 - h_2).
\end{aligned}$$

Of course, the last value is different from zero for $h_1 - h_2 \neq 0$. Similarly, for points $(x_1, x_2, y) = (-\frac{\pi}{2} + y + 2k\pi, 0 + y + 2l\pi, y)$, we have

$$
\begin{aligned}
&\langle F(x_1, x_2, y), F'_x(x_1, x_2, y)(h_1, h_2)\rangle \\
&\quad = \left\langle \left(-\frac{\pi}{2} + 2k\pi + \sin(2l\pi), 2l\pi + \cos\left(-\frac{\pi}{2} + 2k\pi\right)\right), \begin{bmatrix} 1 & \cos(2l\pi) \\ -\sin(-\frac{\pi}{2} + 2k\pi) & 1 \end{bmatrix}(h_1, h_2)\right\rangle \\
&\quad = \left\langle \left(-\frac{\pi}{2} + 2k\pi, 2l\pi\right), \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(h_1, h_2)\right\rangle \\
&\quad = \left(-\frac{\pi}{2} + 2k\pi\right)(h_1 + h_2) + 2l\pi(h_1 + h_2) \\
&\quad = \left(-\frac{\pi}{2} + 2(k + l)\pi\right)(h_1 + h_2)
\end{aligned}
$$

and this value is different from zero for $h_1 + h_2 \neq 0$. Let us point out that we did not use the condition $F(x_1, x_2, y) \neq 0$.

To check the first assumption of Theorem 4.1, we observe that

$$
\begin{aligned}
\varphi(x_1, x_2, y) &= \frac{1}{2}\|F(x_1, x_2, y)\|^2 \\
&= \frac{1}{2}((x_1 - y)^2 + 2(x_1 - y)\sin(x_2 - y) + \sin^2(x_2 - y) + (x_2 - y)^2 + 2(x_2 - y)\cos(x_1 - y) + \cos^2(x_1 - y))
\end{aligned}
$$

and this value converges to $+\infty$ whereas $\|(x_1, x_2)\| = (x_1^2 + x_2^2)^{\frac{1}{2}} \to +\infty$. So, φ is coercive with respect to (x_1, x_2) and, consequently, satisfies the (PS) condition.

It is easy to see that F does not satisfy the assumptions of the global implicit function theorem derived in [5] because $F'_x(x_1, x_2, y)$ is not bijective for all points $(x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}$. Moreover, F does not satisfy the assumptions given in [5, Remark 1], namely, that the differential $F'_x(x, y) : X \to H$ is bijective for any $(x, y) \in X \times Y$ such that $F(x, y) = 0$ and $F(x, y) \in F'_x(x, y)X$ for any $(x, y) \in X \times Y$ such that $F(x, y) \neq 0$. Indeed, e.g., the point $(\frac{\pi}{2}, \pi, 0)$ is such that $F(\frac{\pi}{2}, \pi, 0) = (\frac{\pi}{2}, \pi) \neq (0, 0)$ and, of course, there is no point (h_1, h_2) such that

$$
\left(\frac{\pi}{2}, \pi\right) = F'_x\left(\frac{\pi}{2}, \pi, 0\right)(h_1, h_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(h_1, h_2) = (h_1 - h_2, -h_1 + h_2).
$$

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