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Research Article

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Closed Geodesics on Positively Curved Finsler 3-Spheres

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Abstract: In [33], Wang proved that for every Finsler three-dimensional sphere (S^3, F) with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, there exist at least three distinct closed geodesics. In this paper, we prove that for every Finsler three-dimensional sphere (S^3, F) with reversibility λ and flag curvature K satisfying $(9/4)(\lambda/(1+\lambda))^2 < K \leq 1$ with $\lambda < 2$, if there exist exactly three prime closed geodesics, then two of them are irrationally elliptic and the third one is infinitely degenerate.

Keywords: Closed Geodesics, Multiplicity, Stability, Finsler 3-Sphere, Mean Index Identity

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1 Introduction and Main Results

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here, the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbb{R}$. Note that unlike on Riemannian manifolds, the inverse curve c^{-1} of a closed geodesic c on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. On a reversible Finsler (or Riemannian) manifold, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., if their image sets in M are distinct. We shall omit the word *distinct* when we talk about more than one prime closed geodesic.

For a closed geodesic c on an n-dimensional manifold (M, F), denote by P_c the linearized Poincaré map of c. Then, $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane $\mathbb{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. A closed geodesic c is called *elliptic* if $e(P_c) = 2(n-1)$, i.e., if all the eigenvalues of P_c are located on $\mathbb{U}$, *irrationally elliptic* if it is elliptic and P_c is suitably homotopic to the $\diamond$-product of $n-1$ rotation (2×2) matrices with rotation angles being irrational multiples of π, *hyperbolic* if $e(P_c) = 0$, i.e., all the eigenvalues of P_c are located away from $\mathbb{U}$, *infinitely degenerate* if 1 is an eigenvalue of P_{c^m} for infinitely many $m \in \mathbb{N}$, and, finally, *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler manifold (M, F) is called *bumpy* if all the closed geodesics on it are non-degenerate.

There is a famous conjecture in Riemannian geometry which claims that there exist infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved except for compact rank-one symmetric spaces. The results of Franks [15] and Bangert [4] imply that this conjecture is true for any

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Riemannian 2-sphere (cf. [17, 18]). But once one moves to the Finsler case, the conjecture becomes false. It was quite surprising when Katok [19] found some irreversible Finsler metrics on spheres with only finitely many closed geodesics and all closed geodesics being non-degenerate and elliptic (cf. [37]).

Recently, index iteration theory of closed geodesics (cf. [6, 22–24]) has been applied to study the closed geodesic problem on Finsler manifolds. For example, Bangert and Long show in [5] that there exist at least two closed geodesics on every (S^2, F). After that, a great number of multiplicity and stability results has appeared (cf. [10–14, 25, 26, 31–36] and the references therein).

In [30], Rademacher has introduced the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold as

$$\lambda = \max\{F(-X) \mid X \in TM, F(X) = 1\} \geq 1.$$

Then, in [31], he obtained some results on the multiplicity and the length of closed geodesics and their stability properties. For example, letting F be a Finsler metric on S^n with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, there exist at least $n/2 - 1$ closed geodesics with length $< 2n\pi$. If $(9/4)(\lambda/(1+\lambda))^2 < K \leq 1$ with $\lambda < 2$, then there exists an elliptic-parabolic closed geodesic, i.e., its linearized Poincaré map is split into two-dimensional rotations and a part whose eigenvalues are ± 1. Some similar results in the Riemannian case are obtained in [2, 3].

Recently, Wang proved in [33] that for every Finsler n-dimensional sphere S^n with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exists one elliptic closed geodesic whose linearized Poincaré map has at least one eigenvalue which is of the form $\exp(\sqrt{-1}\pi\mu)$ with an irrational μ. The same author proved in [36] that for every Finsler n-dimensional sphere S^n for $n \geq 6$ with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, either there exist infinitely many prime closed geodesics or there exist $[n/2] - 2$ closed geodesics possessing irrational mean indices. Furthermore, assuming that the metric F is bumpy, he showed in [35] that there exist $2[(n+1)/2]$ closed geodesics on (S^n, F). Also, in [35], he showed that for every bumpy Finsler metric F on S^n satisfying $(9/4)(\lambda/(1+\lambda))^2 < K \leq 1$, there exist two prime elliptic closed geodesics provided the number of closed geodesics on (S^n, F) is finite.

Very recently, Duan proved in [9] that for every Finsler n-dimensional sphere (S^n, F), $n \geq 2$, with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, either there exist infinitely many closed geodesics or there exist at least two elliptic closed geodesics and each linearized Poincaré map has at least one eigenvalue of the form $\exp(\theta\sqrt{-1})$ with θ being an irrational multiple of π. Furthermore, in [8], he proved that for every Finsler metric F on the n-dimensional sphere S^n, $n \geq 3$, with reversibility λ and flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$, either there exist infinitely many closed geodesics or there exist always three prime closed geodesics and at least two of them are elliptic; when $n \geq 6$, these three distinct closed geodesics are non-hyperbolic. If the metric is bumpy, Duan and Long proved in [11] that on every bumpy Finsler three-dimensional sphere (S^3, F), either there exist two non-hyperbolic prime closed geodesics or there exist at least three prime closed geodesics.

Note that Wang proved in [33, Theorem 1.5] that there exist at least three distinct closed geodesics on (S^3, F) with flag curvature K satisfying $(\lambda/(1+\lambda))^2 < K \leq 1$. Motivated by the results mentioned above, in this paper, we prove the following theorem.

Theorem 1.1. *For every Finsler metric F on the three-dimensional sphere S^3 with reversibility λ and flag curvature K satisfying $(9/4)(\lambda/(1+\lambda))^2 < K \leq 1$ with $\lambda < 2$, if there exist exactly three prime closed geodesics, then two of them are irrationally elliptic and the third one is infinitely degenerate.*

Remark 1.2. Note that Anosov conjectured in [1] that the lower bound of the number of distinct closed geodesics on a Finsler three-dimensional sphere (S^3, F) is four, where Katok's examples in [19] show that this lower bound can be attained. However, Ziller in [37, pp. 155–156] conjectured that the lower bound of the number of distinct closed geodesics on a Finsler three-dimensional sphere (S^3, F) is three. To our knowledge, it is not clear whether there exist some irreversible Finsler metrics on S^3 with exactly three distinct closed geodesics. This is an interesting problem.

Our proof of Theorem 1.1 in Section 3 contains mainly three ingredients: the common index jump theorem of [27], Morse theory, and some new symmetric information about the index jump. In addition, we also follow some ideas from our recent preprints [8, 9, 13].

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers, respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. For an S^1-space X, we denote by $\overline{X}$ the quotient space X/S^1. We define the functions

$$[a] = \max\{k \in \mathbb{Z} \mid k \le a\}, \quad E(a) = \min\{k \in \mathbb{Z} \mid k \ge a\}, \quad \varphi(a) = E(a) - [a], \quad \{a\} = a - [a]. \tag{1.1}$$

In particular, we have $\varphi(a) = 0$ if $a \in \mathbb{Z}$ and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$.

2 Morse Theory and Morse Index of Closed Geodesics

2.1 Morse Theory for Closed Geodesics

Let $M = (M, F)$ be a compact Finsler manifold. Then, the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries (cf. [20]). This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 \, dt \tag{2.1}$$

and is $C^{1,1}$ and invariant under the S^1-action. The critical points of E of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional E is well-defined along any closed geodesic c on M, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c) - 1$ the Morse index and the nullity of E at c, respectively. In the following, we denote

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \le \kappa\}, \quad \Lambda^{\kappa-} = \{d \in \Lambda \mid E(d) < \kappa\}$$

for all $\kappa \ge 0$. For a closed geodesic c, we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

Recall that the mean index $\hat{\imath}(c)$ and the S^1-critical modules of c^m are defined by

$$\hat{\imath}(c) = \lim_{m\to\infty} \frac{i(c^m)}{m}, \quad \overline{C}_*(E, c^m) = H_*\big((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1; \mathbb{Q}\big),$$

respectively.

We say that a closed geodesic satisfies the isolation condition if

$$\text{the orbit } S^1 \cdot c^m \text{ is an isolated critical orbit of } E \text{ for all } m \in \mathbb{N}. \tag{Iso}$$

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all closed geodesics satisfy (Iso).

If c has multiplicity m, then the subgroup $\mathbb{Z}_m = \{n/m \mid 0 \le n < m\}$ of S^1 acts on $\overline{C}_*(E, c)$. As studied in [29, p. 59], for all $m \in \mathbb{N}$, let $H_*(X, A)^{\pm\mathbb{Z}_m} = \{[\xi] \in H_*(X, A) \mid T_*[\xi] = \pm[\xi]\}$, where T is a generator of the $\mathbb{Z}_m$-action. On S^1-critical modules of c^m, the following lemma holds (cf. [29, Satz 6.11], [5]).

Lemma 2.1. *Suppose c is a prime closed geodesic on a Finsler manifold M satisfying* (Iso). *Then, there exist U_{c^m} and N_{c^m}, the so-called local negative disk and the local characteristic manifold at c^m, respectively, such that $\nu(c^m) = \dim N_{c^m}$ and*

$$\begin{aligned}\overline{C}_q(E, c^m) &\equiv H_q\big((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\big)\\ &= \big(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})\big)^{+\mathbb{Z}_m},\end{aligned}$$

where $U^-_{c^m} = U_{c^m} \cap \Lambda(c^m)$, $N^-_{c^m} = N_{c^m} \cap \Lambda(c^m)$.

(i) *When* $\nu(c^m) = 0$, *there holds*

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbb{Q} & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \text{ and } q = i(c^m), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *When* $\nu(c^m) > 0$, *there holds*

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\epsilon(c^m)\mathbb{Z}_m},$$

where $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$.

Define

$$k_j(c^m) \equiv \dim H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m}), \quad k_j^{\pm 1}(c^m) \equiv \dim H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\pm\mathbb{Z}_m}.$$

Then, we have the following lemma (cf. [26, 29, 33]).

Lemma 2.2. *Let c be a prime closed geodesic on a Finsler manifold* (M, F). *Then, we have the following.*
(i) *For any* $m \in \mathbb{N}$, *there holds* $k_j(c^m) = 0$ *for* $j \neq [0, \nu(c^m)]$.
(ii) *For any* $m \in \mathbb{N}$, *there holds* $k_0(c^m) + k_{\nu(c^m)}(c^m) \leq 1$ *and if* $k_0(c^m) + k_{\nu(c^m)}(c^m) = 1$, *then there holds* $k_j(c^m) = 0$ *for* $j \in (0, \nu(c^m))$.
(iii) *For any* $m \in \mathbb{N}$, *there holds* $k_0^{+1}(c^m) = k_0(c^m)$ *and* $k_0^{-1}(c^m) = 0$. *In particular, if* c^m *is non-degenerate, then there holds* $k_0^{+1}(c^m) = k_0(c^m) = 1$ *and* $k_0^{-1}(c^m) = k_j^{\pm 1}(c^m) = 0$ *for all* $j \neq 0$.
(iv) *Suppose that the nullities satisfy* $\nu(c^m) = \nu(c^n)$ *for some integer* $m = np \geq 2$ *with* $n, p \in \mathbb{N}$. *Then, there holds* $k_j(c^m) = k_j(c^n)$ *and* $k_j^{\pm 1}(c^m) = k_j^{\pm 1}(c^n)$ *for any integer j.*

Let (M, F) be a compact simply connected Finsler manifold with finitely many closed geodesics. Denote those prime closed geodesics on (M, F) with positive mean indices by $\{c_j\}_{1 \leq j \leq k}$. Rademacher established in [28, 29] a celebrated mean index identity relating all c_j with the global homology of M (cf. [29, Section 7], especially Satz 7.9 therein) for compact simply connected Finsler manifolds. Here, we give a brief review on this identity (cf. [29, Satz 7.9] and also [12, 26, 33]).

Theorem 2.3. *Assume that there exist finitely many closed geodesics on* (S^3, F) *and denote the prime closed geodesics with positive mean indices by* $\{c_j\}_{1 \leq j \leq k}$ *for some* $k \in \mathbb{N}$. *Then, we have the identity*

$$\sum_{j=1}^{k} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = 1, \tag{2.2}$$

where

$$\hat{\chi}(c_j) = \frac{1}{n(c_j)} \sum_{\substack{1 \leq m \leq n(c_j) \\ 0 \leq l \leq 2(n-1)}} \chi(c_j^m) = \frac{1}{n(c_j)} \sum_{\substack{1 \leq m \leq n(c_j) \\ 0 \leq l \leq 4}} (-1)^{i(c_j^m)+l} k_l^{\epsilon(c_j^m)}(c_j^m) \in \mathbb{Q} \tag{2.3}$$

and the analytical period $n(c_j)$ *of* c_j *is defined by (cf.* [26]*)*

$$n(c_j) = \min\Big\{ l \in \mathbb{N} \;\Big|\; \nu(c_j^l) = \max_{m \geq 1} \nu(c_j^m) \text{ with } i(c_j^{m+l}) - i(c_j^m) \in 2\mathbb{Z} \text{ for all } m \in \mathbb{N} \Big\}. \tag{2.4}$$

Set

$$\overline{\Lambda}^0 = \overline{\Lambda}^0 S^3 = \{\text{constant point curves in } S^3\} \cong S^3.$$

Let (X, Y) be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbb{Q})$ are finite for all $i \in \mathbb{Z}$. As usual, the *Poincaré series* of (X, Y) is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following well-known version of results on Betti numbers and the Morse inequality. For Lemma 2.4 below, see [28, Theorem 2.4 and Remark 2.5], [16], and also [12, Lemma 2.5]), and for Theorem 2.5, see [7, Theorem I.4.3].

Lemma 2.4. *Let* (S^3, F) *be a three-dimensional Finsler sphere. Then, the Betti numbers are given by*

$$b_j = \operatorname{rank} H_j(\Lambda S^3/S^1, \Lambda^0 S^3/S^1; \mathbb{Q}) = \begin{cases} 2 & \text{if } j = 2k \geq 4, \\ 1 & \text{if } j = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Theorem 2.5. *Let (M, F) be a Finsler manifold with finitely many closed geodesics, denoted by $\{c_j\}_{1\le j\le k}$. Set*

$$M_q = \sum_{\substack{1\le j\le k\\ m\ge 1}} \dim \overline{C}_q(E, c_j^m), \quad q \in \mathbb{Z}.$$

Then, for every integer $q \ge 0$, there holds

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \ge b_q - b_{q-1} + \cdots + (-1)^q b_0 \tag{2.6}$$

$$M_q \ge b_q. \tag{2.7}$$

2.2 Index Iteration Theory of Closed Geodesics

In [22], Long established the basic normal form decomposition of symplectic matrices. Based on this result, he further established the precise iteration formulae of indices of symplectic paths in [23]. Note that this index iteration formulae works for Morse indices of iterated closed geodesics (cf. [21] and [24, Chapter 12]). Since every closed geodesic on a sphere must be orientable, then, by [21, Theorem 1.1], the initial Morse index of a closed geodesic on a Finsler S^n coincides with the index of a corresponding symplectic path.

As in [23], we denote

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1,\ b \in \mathbb{R},$$

$$H(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R} \setminus \{0, \pm 1\},$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$

and

$$N_2(\exp(\theta\sqrt{-1}), B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$

where

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad b_j \in \mathbb{R},\ b_2 \ne b_3.$$

Here, $N_2(\exp(\theta\sqrt{-1}), B)$ is non-trivial if $(b_2 - b_3)\sin\theta < 0$ and trivial if $(b_2 - b_3)\sin\theta > 0$.

As in [23], the $\diamond$-sum (direct sum) of any two real matrices is defined by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i} \diamond \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

For every $M \in \mathrm{Sp}(2n)$, the homotopy set $\Omega(M)$ of M in $\mathrm{Sp}(2n)$ is defined by

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U} \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M) \text{ for all } \omega \in \Gamma\},$$

where $\sigma(M)$ denotes the spectrum of M, $\nu_\omega(M) \equiv \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - \omega I)$ for $\omega \in \mathbb{U}$. The component $\Omega^0(M)$ of P in $\mathrm{Sp}(2n)$ is defined by the path-connected component of $\Omega(M)$ containing M.

For Theorem 2.6 below, cf. [22, Theorem 7.8], [23, Theorems 1.2 and 1.3] and also [24, Theorem 1.8.10, Lemma 2.3.5, and Theorem 8.3.1].

Theorem 2.6. *For every $P \in \mathrm{Sp}(2n-2)$, there exists a continuous path $f \in \Omega^0(P)$ such that $f(0) = P$ and*

$$\begin{aligned} f(1) = {} & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond N_2(\exp(\alpha_1\sqrt{-1}), A_1) \diamond \cdots \diamond N_2(\exp(\alpha_{r_*}\sqrt{-1}), A_{r_*}) \\ & \diamond N_2(\exp(\beta_1\sqrt{-1}), B_1) \diamond \cdots \diamond N_2(\exp(\beta_{r_0}\sqrt{-1}), B_{r_0}) \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_{r'}) \diamond R(\theta_{r'+1}) \diamond \cdots \diamond R(\theta_r) \diamond H(\pm 2)^{\diamond h}, \end{aligned} \tag{2.8}$$

where $\theta_j/2\pi \in \mathbb{Q} \cap (0,1)$ *for* $1 \le j \le r'$ *and* $\theta_j/2\pi \notin \mathbb{Q} \cap (0,1)$ *for* $r'+1 \le j \le r$. *The terms* $N_2(\exp(\alpha_j\sqrt{-1}), A_j)$ *are non-trivial and* $N_2(\exp(\beta_j\sqrt{-1}), B_j)$ *are trivial, and the non-negative integers* $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, r_0, h$ *satisfy the equality*

$$p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_* + 2r_0 + h = n - 1. \tag{2.9}$$

Let

$$\gamma \in \mathcal{P}_\tau(2n-2) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n-2)) \mid \gamma(0) = I\}$$

and denote the basic normal form decomposition of $P \equiv \gamma(\tau)$ *by* (2.8). *Then, we have*

$$\begin{aligned} i(\gamma^m) &= m(i(\gamma) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E\Big(\frac{m\theta_j}{2\pi}\Big) - r \\ &\quad - p_- - p_0 - \frac{1+(-1)^m}{2}(q_0 + q_+) + 2\sum_{j=1}^{r_*} \varphi\Big(\frac{m\alpha_j}{2\pi}\Big) - 2r_*, \end{aligned} \tag{2.10}$$

$$\nu(\gamma^m) = \nu(\gamma) + \frac{1+(-1)^m}{2}(q_- + 2q_0 + q_+) + 2\varsigma(m, \gamma(\tau)), \tag{2.11}$$

where

$$\varsigma(m,\gamma(\tau)) = r - \sum_{j=1}^{r} \varphi\Big(\frac{m\theta_j}{2\pi}\Big) + r_* - \sum_{j=1}^{r_*} \varphi\Big(\frac{m\alpha_j}{2\pi}\Big) + r_0 - \sum_{j=1}^{r_0} \varphi\Big(\frac{m\beta_j}{2\pi}\Big).$$

We have that $i(\gamma,1)$ *is odd if* $f(1) = N_1(1,1)$, I_2, $N_1(-1,1)$, $-I_2$, $N_1(-1,-1)$ *and* $R(\theta)$; $i(\gamma,1)$ *is even if* $f(1) = N_1(1,-1)$ *and* $N_2(\omega,b)$; $i(\gamma,1)$ *can be any integer if* $\sigma(f(1)) \cap \mathbb{U} = \emptyset$.

The following is the common index jump theorem of Long and Zhu [27] (cf. [27, Theorems 4.1–4.3] and [24]).

Theorem 2.7. *Let* γ_k, $k = 1, \dots, q$, *be a finite collection of symplectic paths and* $M_k = \gamma_k(\tau_k) \in Sp(2n-2)$. *Suppose* $\hat{i}(\gamma_k, 1) > 0$ *for all* $k = 1, \dots, q$. *Then, for every* $k = 1, \dots, q$, *there exist infinitely many* $(N, m_1, \dots, m_q) \in \mathbb{N}^{q+1}$ *such that*

$$\begin{aligned} \nu(\gamma_k, 2m_k - 1) &= \nu(\gamma_k, 1), \\ \nu(\gamma_k, 2m_k + 1) &= \nu(\gamma_k, 1), \\ i(\gamma_k, 2m_k - 1) + \nu(\gamma_k, 2m_k - 1) &= 2N - (i(\gamma_k, 1) + 2S^+_{M_k}(1) - \nu(\gamma_k, 1)), \\ i(\gamma_k, 2m_k + 1) &= 2N + i(\gamma_k, 1), \\ i(\gamma_k, 2m_k) &\ge 2N - \frac{e(M_k)}{2}, \\ i(\gamma_k, 2m_k) + \nu(\gamma_k, 2m_k) &\le 2N + \frac{e(M_k)}{2}, \end{aligned}$$

where $S^+_{M_k}(1)$ *is the splitting number of* M_k.

More precisely, by [27, (4.10) and (4.40)] *, we have*

$$m_k = \Big(\Big[\frac{N}{M\hat{i}(\gamma_k, 1)}\Big] + \chi_k\Big)M, \quad 1 \le k \le q, \tag{2.12}$$

where $\chi_k = 0$ *or* 1 *for* $1 \le k \le q$ *and* $m_k\theta/\pi \in \mathbb{Z}$ *whenever* $\exp(\theta\sqrt{-1}) \in \sigma(M_k)$ *and* $\theta/\pi \in \mathbb{Q}$ *for some* $1 \le k \le q$. *Furthermore, given* $M_0 \in \mathbb{N}$, *by the proof of* [27, Theorem 4.1], *we may further require* $M_0|N$ *(since the closure of the set* $\{\{N\nu\} \mid N \in \mathbb{N}, M_0|N\}$ *is still a closed additive subgroup of* $\mathbb{T}^h$ *for some* $h \in \mathbb{N}$, *where we use notation as in* [27, (4.21)]. *Then, we can use the proof of* [27, Theorem 4.1, Step 2] *to get* N*).*

We also have the following properties in the index iteration theory (cf. [27, Theorem 2.2] or [24, Theorem 10.2.3]).

Theorem 2.8. *Let* $\gamma \in \mathcal{P}_\tau(2n)$. *Then, for any* $m \in \mathbb{N}$, *there holds*

$$\nu(\gamma, m) - \frac{e(M)}{2} \le i(\gamma, m+1) - i(\gamma, m) - i(\gamma, 1) \le \nu(\gamma, 1) - \nu(\gamma, m+1) + \frac{e(M)}{2},$$

where $e(M)$ *is the elliptic height defined in Section 1.*

3 Proof of Theorem 1.1

In this section, we prove our main theorem by using the mean index equality in Theorem 2.3, the Morse inequality in Theorem 2.5, and the index iteration theory developed by Long and his coworkers, especially a new observation on a symmetric property for closed geodesics in the common index jump intervals, i.e., Lemma 3.2.

First, we make the assumption that

there exist only finitely many closed geodesics c_k, $k = 1, \dots, q$, on (S^3, F) with reversibility λ and flag curvature K satisfying $(9/4)(\lambda/(1+\lambda))^2 < K \le 1$ with $\lambda < 2$. (FCG)

Then, we have an estimate on the index and on the mean index of c_k.

Lemma 3.1. *We have $i(c_k) \ge 2$ and $\hat{i}(c_k) > 3$ for $k = 1, \dots, q$.*

Proof. By assumption, since the flag curvature K satisfies $(9/4)(\lambda/(1+\lambda))^2 < K \le 1$, we can choose a δ in [31, Lemma 2] to satisfy

$$\delta > \frac{9}{4}\Big(\frac{\lambda}{1+\lambda}\Big)^2$$

and

$$\hat{i}(c_k) \ge 2\sqrt{\delta}\frac{\lambda+1}{\lambda} > 3.$$

The claim $i(c_k) \ge 2$ follows from [30, Theorem 3 and Lemma 3]. □

Combining Lemma 3.1 with Theorem 2.8, it follows that

$$i(c_k^{m+1}) - i(c_k^m) - \nu(c_k^m) \ge i(c_k) - \frac{e(P_{c_k})}{2} \ge 0 \tag{3.1}$$

for all $m \in \mathbb{N}$. Here, the last inequality holds by the fact that $e(P_{c_k}) \le 4$.

It follows from Lemma 3.1 and Theorem 2.7 that there exist infinitely many $(q+1)$-tuples of the form $(N, m_1, \dots, m_q) \in \mathbb{N}^{q+1}$ such that, for any $1 \le k \le q$, there holds

$$i(c_k^{2m_k-1}) + \nu(c_k^{2m_k-1}) = 2N - (i(c_k) + 2S^+_{M_k}(1) - \nu(c_k)), \tag{3.2}$$

$$i(c_k^{2m_k}) \ge 2N - \frac{e(P_{c_k})}{2}, \tag{3.3}$$

$$i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \le 2N + \frac{e(P_{c_k})}{2}, \tag{3.4}$$

$$i(c_k^{2m_k+1}) = 2N + i(c_k). \tag{3.5}$$

Note that by [24, List 9.1.12] and the fact that $\nu(c_k) = p_{k_-} + 2p_{k_0} + p_{k_+}$ we obtain

$$2S^+_{M_k}(1) - \nu(c_k) = 2(p_{k_-} + p_{k_0}) - (p_{k_-} + 2p_{k_0} + p_{k_+}) = p_{k_-} - p_{k_+}. \tag{3.6}$$

So, by (3.1)–(3.6) and the fact that $e(P_{c_k}) \le 4$, we have

$$i(c_k^m) + \nu(c_k^m) \le 2N - i(c_k) - p_{k_-} + p_{k_+}, \quad \text{for all } 1 \le m < 2m_k, \tag{3.7}$$

$$i(c_k^{2m_k}) + \nu(c_k^{2m_k}) \le 2N + \frac{e(P_{c_k})}{2} \le 2N + 2, \tag{3.8}$$

$$2N + 2 \le i(c_k^m), \qquad \text{for all } m > 2m_k. \tag{3.9}$$

In addition, the precise formulae of $i(c_k^{2m_k})$ and $i(c_k^{2m_k}) + \nu(c_k^{2m_k})$ for $k = 1, \dots, q$ can be computed as follows (cf. [9, (3.16) and (3.21)] for the details):

$$i(c_k^{2m_k}) = 2N - S^+_{M_k}(1) - C(M_k) + 2\Delta_k, \tag{3.10}$$

$$\begin{aligned} i(c_k^{2m_k}) + \nu(c_k^{2m_k}) = 2N &+ p_{k_0} + p_{k_+} + q_{k_-} + q_{k_0} \\ &+ 2r'_{k_0} - 2(r_{k_*} - r'_{k_*}) + 2r'_k - r_k + 2\Delta_k \end{aligned} \tag{3.11}$$

where r_k, r_{k_*}, and r_{k_0} denote the number of normal forms $R(\theta)$, $N_2(\exp(\alpha\sqrt{-1}), A)$, and $N_2(\exp(\beta\sqrt{-1}), B)$ in (2.8) of Theorem 2.6 with $P = P_{c_k}$, $k = 1, 2$, respectively, and r'_k, r'_{k_*}, and r'_{k_0} denote the number of normal forms $R(\theta)$, $N_2(\exp(\alpha\sqrt{-1}), A)$, and $N_2(\exp(\beta\sqrt{-1}), B)$ with θ, α, β being the rational multiples of π in (2.8) of Theorem 2.6 with $P = P_{c_k}$, $k = 1, 2$, respectively, and

$$\Delta_k \equiv \sum_{0<\{m_k\theta/\pi\}<\delta} S^-_{M_k}(\exp(\theta\sqrt{-1})) \le r_k - r'_k + r_{k_*} - r'_{k_*}, \quad C(M_k) \equiv \sum_{\theta\in(0,2\pi)} S^-_{M_k}(\exp(\theta\sqrt{-1})), \tag{3.12}$$

where $\delta > 0$ is a small enough number (cf. [27, (4.43)]) and the estimate of Δ_k follows from the inequality [9, (3.18)].

Under the assumption (FCG), using [9, Theorem 1.1], we have that there exist at least two elliptic closed geodesics c_1 and c_2 on (S^3, F) whose flag curvature satisfies $(\lambda/(1+\lambda))^2 < K \le 1$. The next lemma (cf. [9, Section 3]) lists some properties of these two closed geodesics which will be useful in the proof of Theorem 1.1.

Lemma 3.2. *Under the assumption* (FCG), *there exist at least two elliptic closed geodesics c_1 and c_2 on (S^3, F) whose flag curvature satisfies $(\lambda/(1+\lambda))^2 < K \le 1$. Moreover, there exist infinitely many pairs of $(q+1)$-tuples of the form $(N, m_1, m_2, \dots, m_q) \in \mathbb{N}^{q+1}$ and $(N', m'_1, m'_2, \dots, m'_q) \in \mathbb{N}^{q+1}$ such that*

$$i(c_1^{2m_1}) + \nu(c_1^{2m_1}) = 2N + 2, \quad \overline{C}_{2N+2}(E, c_1^{2m_1}) = \mathbb{Q}, \tag{3.13}$$

$$i(c_2^{2m'_2}) + \nu(c_2^{2m'_2}) = 2N' + 2, \quad \overline{C}_{2N'+2}(E, c_2^{2m'_2}) = \mathbb{Q}, \tag{3.14}$$

and

$$p_{k_-} = q_{k_+} = r_{k_*} = r_{k_0} - r'_{k_0} = h_k = 0, \quad k = 1, 2, \tag{3.15}$$

$$r_1 - r'_1 = \Delta_1 \ge 1, \quad r_2 - r'_2 = \Delta'_2 \ge 1, \tag{3.16}$$

$$\Delta_k + \Delta'_k = r_k - r'_k, \quad k = 1, 2, \tag{3.17}$$

where we can require $2|N$ or $2|N'$ as remarked in Theorem 2.7 and

$$\Delta'_k \equiv \sum_{0<\{m'_k\theta/\pi\}<\delta} S^-_{M_k}(\exp(\theta\sqrt{-1})), \quad k = 1, 2. \tag{3.18}$$

Proof. In fact, all these properties have already been obtained in [9, Section 3] and here we only list references. More precisely, (3.13) follows from [9, Claim 1] and the arguments between [9, (3.25) and (3.26)], (3.14) follows from [9, Claim 3] and similar arguments as those for c_1 between [9, (3.25) and (3.26)], (3.15) and (3.16) follow from [9, (3.25), Claim 2, and Claim 3], and, finally, (3.17) follows from [9, (3.31)] and (3.15). In one word, the properties of c_1 and c_2 are symmetric. □

Lemma 3.3. *Under the assumption* (FCG), *for the two elliptic closed geodesics c_1, c_2 found in Lemma 3.2, there holds*

$$k^{\epsilon(c_k^{n(c_k)})}_{\nu(c_k^{n(c_k)})}(c_k^{n(c_k)}) = 1, \quad k^{\epsilon(c_k^{n(c_k)})}_j(c_k^{n(c_k)}) = 0 \tag{3.19}$$

for all $0 \le j < \nu(c_k^{n(c_k)})$, $k = 1, 2$, and then $\hat{\chi}(c_k) \le 1$ for $k = 1, 2$.

Proof. We only give the proof for c_1. The proof for c_2 is identical.

First, by (3.13) and Lemma 2.1, we have

$$\begin{aligned} 1 &= \dim \overline{C}_{2N+2}(E, c_1^{2m_1}) \\ &= \dim H_{2N+2-i(c_1^{2m_1})}(N_{c_1^{2m_1}} \cup \{c_1^{2m_1}\}, N_{c_1^{2m_1}})^{\epsilon(c_1^{2m_1})\mathbb{Z}_{2m_1}} \\ &= \dim H_{\nu(c_1^{2m_1})}(N_{c_1^{2m_1}} \cup \{c_1^{2m_1}\}, N_{c_1^{2m_1}})^{\epsilon(c_1^{2m_1})\mathbb{Z}_{2m_1}} \\ &= k^{\epsilon(c_1^{2m_1})}_{\nu(c_1^{2m_1})}(c_1^{2m_1}), \end{aligned}$$

which implies that

$$k_j^{\epsilon(c_1^{2m_1})}(c_1^{2m_1}) = 0$$

for any $0 \le j < \nu(c_1^{2m_1})$ by Lemma 2.2 (ii). In addition, note that since $n(c_1)|2m_1$ and $\nu(c_1^{2m_1}) = \nu(c_1^{n(c_1)})$ by (2.4) and (2.12), there holds

$$k_j^{\epsilon(c_1^{2m_1})}(c_1^{2m_1}) = k_j^{\epsilon(c_1^{n(c_1)})}(c_1^{n(c_1)})$$

for any $0 \le j \le \nu(c_1^{2m_1})$ by Lemma 2.2 (iv). Thus, (3.19) holds.

Note that by (3.16), the linearized Poincaré map P_{c_k} of the elliptic closed geodesic c_k is conjugate to $R(\theta_1) \diamond R(\theta_2)$ or $R(\theta_1) \diamond N_1(\lambda, b)$ for some $\theta_1/2\pi \in (0,1)\backslash\mathbb{Q}$, $\lambda = \pm 1$, and $b = 0, 1$. Then,

$$\nu(c_k^m) = 0 \tag{3.20}$$

for all $m < n(c_k)$. In fact, when P_{c_k} is conjugate to $R(\theta_1) \diamond N_1(1, b)$, we have $n(c_k) = 1$. By (2.4), (3.20) holds. When P_{c_k} is conjugate to $R(\theta_1) \diamond N_1(-1, b)$, we have $n(c_k) = 2$. By (2.4), (3.20) also holds. When P_{c_k} is conjugate to $R(\theta_1) \diamond R(\theta_2)$, (3.20) holds by (2.4).

Then, (3.20) yields

$$k_0^{\epsilon(c_k^m)}(c_k^m) = 1, \quad k_j^{\epsilon(c_k^m)}(c_k^m) = 0$$

for all $0 < j \le 4$ and for $1 \le m < n(c_k)$, which together with (2.3) and (3.19) gives

$$\hat{\chi}(c_k) = \frac{1}{n(c_k)}\left((-1)^{i(c_k^{n(c_k)})+\nu(c_k^{n(c_k)})} + \sum_{1\le m<n(c_k)} (-1)^{i(c_k^m)}\right) \le 1. \qquad \square$$

Proof of Theorem 1.1. In order to prove Theorem 1.1, based on [33, Theorem 1.5] (cf. also [8, Theorem 1.1]), we make the assumption that

> there exist exactly two elliptic distinct closed geodesics c_1, c_2 possessing all properties listed in Lemmas 3.2 and 3.3 and a third closed geodesic c_3 on (S^3, F) with reversibility λ and flag curvature K satisfying $(9/4)(\lambda/(1+\lambda))^2 < K \le 1$ with $\lambda < 2$. (TCG)

Claim 1. *c_1^m has no contribution to the Morse-type numbers M_{2N+1}, M_{2N}, and M_{2N-1} for any $m \in \mathbb{N}$, c_2^m has possible contribution to the Morse-type numbers M_{2N+1}, M_{2N}, or M_{2N-1} only when $m = 2m_2$, and this time $c_2^{2m_2}$ has no contribution to M_{2N+1} and M_{2N-1}, but contributes at most one to M_{2N}.*

In fact, by (3.16), for $k = 1, 2$, the linearized Poincaré map P_{c_k} of the elliptic closed geodesic c_k is conjugate to $R(\theta_1) \diamond R(\theta_2)$ or $R(\theta_1) \diamond N_1(\lambda, b)$ for some $\theta_1/2\pi \in (0,1)\backslash\mathbb{Q}$, $\lambda = \pm 1$, and $b = 0, 1$. Combining this fact with Lemma 3.1 and (3.7), we have

$$i(c_k^m) + \nu(c_k^m) \le 2N - i(c_k) - p_{k_-} + p_{k_+} \le 2N - 1 \tag{3.21}$$

for $m < 2m_k$, $k = 1, 2$, where the equality in (3.21) holds if and only if P_{c_k} is conjugate to $R(\theta_1) \diamond N_1(1, -1)$ and $i(c_k) = 2$, but $i(c_k) \in 2\mathbb{N} - 1$ when P_{c_k} is conjugate to $R(\theta_1) \diamond N_1(1, -1)$, thus the equality in (3.21) does not hold. Then,

$$i(c_k^m) + \nu(c_k^m) \le 2N - 2 \tag{3.22}$$

for $m < 2m_k$, $k = 1, 2$. Combining Lemma 2.2 (i) with (3.9) and (3.22), we know that c_k^m has no contribution to the Morse-type numbers M_{2N+1}, M_{2N}, and M_{2N-1} for $m \ne 2m_k$, where $k = 1, 2$. Note that by (3.13) and (3.19), $c_1^{2m_1}$ has also no contribution to M_{2N+1}, M_{2N}, and M_{2N-1}.

On one hand, there holds

$$\nu(c_2^{2m_2}) = \nu(c_2^{2m_2'})$$

by the choices of m_2 and m_2' in (2.12) of Theorem 2.7. On the other hand, it yields

$$i(c_2^{2m_2'}) = i(c_2^{2m_2}) \pmod 2$$

by (2.10) of Theorem 2.6. So, $i(c_2^{2m_2}) + \nu(c_2^{2m_2})$ is even since $i(c_2^{2m_2'}) + \nu(c_2^{2m_2'})$ is even by (3.14) of Lemma 3.2, and then $c_2^{2m_2}$ has no contribution to M_{2N+1} and M_{2N-1} by (3.19). If $c_2^{2m_2}$ has contribution to M_{2N}, then $c_2^{2m_2}$ contributes exactly one to M_{2N} by (3.19). Hence, Claim 1 holds.

Claim 2. *c_3^m has no contribution to the Morse-type numbers M_{2N+1}, M_{2N}, and M_{2N-1} for any $m \neq 2m_3$.*

First, by (3.9) and Lemma 2.2 (i), we know that c_3^m has no contribution to the Morse-type numbers M_{2N+1}, M_{2N}, and M_{2N-1} for $m > 2m_3$.

On the other hand, from Lemma 3.1 and (3.1)–(3.2) along with the fact that $\nu(c_3^{2m_3-1}) = \nu(c_3)$, we have

$$i(c_3^m) + \nu(c_3^m) \leq i(c_3^{2m_3-1}) = 2N - (i(c_3) + 2S^+_{M_3}(1)) \leq 2N - 2$$

for all $1 \leq m < 2m_3 - 1$, which, together with Lemma 2.2 (i), implies that c_3^m has no contribution to the Morse-type numbers M_{2N+1}, M_{2N}, and M_{2N-1} for any $m < 2m_3 - 1$.

Now, we prove Claim 2 by contradiction. We can assume that $c_3^{2m_3-1}$ has contribution to the Morse-type numbers M_{2N+1}, M_{2N}, or M_{2N-1}, i.e.,

$$\sum_{q=2N-1}^{2N+1} \dim \overline{C}_q(E, c^{2m_3-1}) \geq 1. \tag{3.23}$$

Note that, by (3.2) and (3.6), we have

$$i(c_3^{2m_3-1}) + \nu(c_3^{2m_3-1}) = 2N - i(c_3) - p_{3_-} + p_{3_+} \leq 2N - 2 + 2 = 2N, \tag{3.24}$$

which, together with Lemma 2.2 (i) and the assumption (3.23), gives $i(c_3^{2m_3-1}) + \nu(c_3^{2m_3-1}) = 2N$ or $2N - 1$.

We continue the proof by distinguishing two cases.

Case 1. $i(c_3^{2m_3-1}) + \nu(c_3^{2m_3-1}) = 2N$. In this case, by (3.24), P_{c_3} is conjugate to $N_1(1, -1)^{\diamond 2}$ and $i(c_3) = 2$, $\nu(c_3^m) = 2$ for all $m \geq 1$. Then, using Theorem 2.6, we obtain

$$i(c_3^m) + \nu(c_3^m) - 2 = i(c_3^m) = mi(c_3) = m(i(c_3) + \nu(c_3) - 2) = 2m \tag{3.25}$$

for all $m \geq 1$.

Now, by (2.5) and (2.7), we obtain $M_{2N} \geq b_{2N} = 2$, which, together with Claim 1, implies that c_3^m must have contribution to M_{2N} for some $m \in \mathbb{N}$, i.e.,

$$\sum_{m \geq 1} \dim \overline{C}_{2N}(E, c_3^m) \geq 1. \tag{3.26}$$

Thus, $c_3^{2m_3-1}$ has contribution to M_{2N} and $k_{\nu(c_3)}(c_3) = 1$, since otherwise $c_3^{2m_3-1}$ contributes to M_{2N-1} and $k_1(c_3) \neq 0$, and then c_3^m has no contribution to M_{2N} for any $m \in \mathbb{N}$ by (3.25), which contradicts (3.26). Now, $k_{\nu(c_3)}(c_3) = 1$ and (3.25) imply that c_3 satisfies the condition of Hingston's result (cf. [17, Proposition 1] and [33, Theorem 4.2]), which yields the existence of infinitely many closed geodesics which contradicts the assumption (TCG).

Case 2. $i(c_3^{2m_3-1}) + \nu(c_3^{2m_3-1}) = 2N - 1$. In this case, by (3.24), one of the following cases may happen.
(i) $i(c_3) = 3$ and $p_{3_+} = 2$.
(ii) $i(c_3) = 2$ and $p_{3_+} = 1$.

For (i), we have that P_{c_3} is conjugate to $N_1(1, -1)^{\diamond 2}$, which implies that $i(c_3)$ is even, thus case (i) cannot happen.

Noticing that $i(c_3) = 2$ is even in case (ii), we have that P_{c_3} is conjugate to $N_1(1, -1) \diamond H(2)$. So, by Theorem 2.6, we have

$$i(c_3^m) + \nu(c_3^m) = mi(c_3) + \nu(c_3^m) = 2m + 1 \tag{3.27}$$

for $m \geq 1$. Now, in this case it follows from (3.23) that $c_3^{2m_3-1}$ has contribution to M_{2N-1} and then $k_{\nu(c_3)}(c_3) = 1$, which together with (3.27) implies that c_3^m has no contribution to M_{2N} for any $m \in \mathbb{N}$, which in turn contradicts (3.26). This completes the proof of Claim 2.

Claim 3. *$c_2^{2m_2}$ has no contribution to M_{2N}.*

In fact, $c_2^{2m_2}$ contributes otherwise exactly one to M_{2N} by Claim 1. By (2.5) and (2.7), $M_{2N} \geq b_{2N} = 2$, and then $c_3^{2m_3}$ must have contribution to M_{2N} by Claims 1 and 2. Thus, $c_3^{2m_3}$ has no contribution to M_{2N+2} and M_{2N-2} by (3.3)–(3.4) and Lemma 2.2 (ii). So, we obtain that

$$-M_{2N+1} + M_{2N} - M_{2N-1} = \sum_{0\leq l\leq 4} (-1)^{i(c_3^{2m_3})+l} k_l^{\epsilon(c_3^{2m_3})}(c_3^{2m_3}) + 1. \tag{3.28}$$

On the other hand, by (2.6) and Lemma 2.4, we have

$$M_{2N+1} - M_{2N} + M_{2N-1} \geq b_{2N+1} - b_{2N} + b_{2N-1} = -2. \tag{3.29}$$

Combining (3.28) and (3.29), we get

$$\chi(c_3^{2m_3}) = \sum_{0\leq l\leq 4} (-1)^{i(c_3^{2m_3})+l} k_l^{\epsilon(c_3^{2m_3})}(c_3^{2m_3}) \leq 1. \tag{3.30}$$

Note that since $n(c_3)|2m_3$ and $\nu(c_3^{2m_3}) = \nu(c_3^{n(c_3)})$ by (2.4) and (2.12), there holds

$$k_j^{\epsilon(c_3^{2m_3})}(c_3^{2m_3}) = k_j^{\epsilon(c_3^{n(c_3)})}(c_3^{n(c_3)})$$

for any $0 \leq j \leq \nu(c_3^{2m_3})$ by Lemma 2.2 (iv). Then, it follows from (2.4) and (3.30) that

$$\chi(c_3^{n(c_3)}) = \chi(c_3^{2m_3}) \leq 1. \tag{3.31}$$

Now, we can obtain that

$$\chi(c_3^{m}) \leq 1 \tag{3.32}$$

for all $1 \leq m < n(c_3)$.

In fact, if c_3 is totally degenerate, i.e., if 1 is the unique eigenvalue of P_{c_3}, then $n(c_3) = 1$ and (3.32) holds by (3.31).

If c_3 is not totally degenerate, by (2.4), either $\nu(c_3^m) < 2$ for $1 \leq m < n(c_3)$ or $\nu(c_3^{m_0}) = 2$ for $1 \leq m_0 < n(c_3)$ with $P_{c_3^{m_0}}$ conjugating to $I \diamond R(\theta)$ for some $\theta/2\pi \in \mathbb{Q}$ and $i(c_3^{m_0}) \in 2\mathbb{N}$. In any case, (3.32) follows from Lemma 2.2 (ii).

Now, we combine (3.31) and (3.32) to get $\hat{\chi}(c_3) \leq 1$, which, together with Lemma 3.1 and Lemma 3.3, implies that

$$\sum_{j=1}^{3} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1,$$

which contradicts the identity (2.2) in Theorem 2.3. Hence, Claim 3 holds.

Claim 4. *c_1 and c_2 are irrationally elliptic.*

By (3.16) and (3.17), there holds $\Delta_2 = 0$. Then, together with the fact that $r_{2_*} = 0$ from (3.15), it follows from (3.16) and (3.11) that

$$\begin{aligned} 2N &\geq i(c_2^{2m_2}) + \nu(c_2^{2m_2}) && (3.33)\\ &= 2N + (p_{2_0} + p_{2+} + q_{2_-} + q_{2_0} + 2r'_{2_0} + r'_2) - (r_2 - r'_2) \\ &\geq 2N - 2, && (3.34) \end{aligned}$$

where (3.33) holds by the fact that $p_{2_0} + p_{2+} + q_{2_-} + q_{2_0} + 2r'_{2_0} + r'_2 \leq 1$ from (2.9) and (3.16) and $r_2 - r'_2 \geq 1$ from (3.16), and the equality in (3.34) holds if and only if $r_2 - r'_2 = 2$. On the other hand, by Claim 3, we have $i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \neq 2N$ and by (3.14), we have that

$$i(c_2^{2m_2}) + \nu(c_2^{2m_2})$$

is even since it has the same parity with $i(c_2^{2m_2'}) + \nu(c_2^{2m_2'})$. Thus, by (3.33), we obtain $i(c_2^{2m_2}) + \nu(c_2^{2m_2}) \leq 2N - 2$, which together with (3.34) implies $r_2 - r_2' = 2$, i.e., c_2 is irrationally elliptic. By the symmetry of c_1 and c_2, we also obtain that c_1 is irrationally elliptic. Thus, Claim 4 is true.

To conclude with the proof of Theorem 1.1, first note that if 1 is an eigenvalue of $P_{c_3^{m_0}}$ for some $m_0 \in \mathbb{N}$, then 1 must be an eigenvalue of $P_{c_3^{2lm_0}}$ for any $l \in \mathbb{N}$ by (2.11) of Theorem 2.6. So, if c_3 is not infinitely degenerate, then all iterates c_3^m of c_3 with $m \in \mathbb{N}$ are non-degenerate and then all closed geodesics c_k, $k = 1, 2, 3$, and their iterates are non-degenerate by Claim 4. Using [35, Theorem 1.2], we get four prime closed geodesics, which contradicts the assumption (TCG). Hence, c_3 is infinitely degenerate. □

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