DE GRUYTER Adv. Nonlinear Stud. 2016; 16 (1):125–146

Research Article

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# Weighted Gagliardo–Nirenberg Inequalities Involving BMO Norms and Solvability of Strongly Coupled Parabolic Systems

DOI: 10.1515/ans-2015-5006
Received June 11, 2015; accepted September 15, 2015

**Abstract:** New weighted Gagliardo–Nirenberg inequalities are introduced together with applications to the local/global existence of solutions to nonlinear strongly coupled and uniform parabolic systems. Much weaker sufficient conditions than those existing in literature for solvability of these systems will be established.

**Keywords:** Parabolic Systems, Hölder Regularity, $A_p$ Classes, BMO Weak Solutions

**MSC 2010:** 35J70, 35B65, 42B37

**Communicated by:** Shair Ahmad

## 1 Introduction

In [14, 16], for any $p \geq 1$ and $C^2$ function $u$ on $\mathbb{R}^n$, $n \geq 2$, global and local Gagliardo–Nirenberg inequalities of the form

$$\int_{\mathbb{R}^n} |Du|^{2p+2}\,dx \leq C(n,p)\|u\|_{\mathrm{BMO}}^2 \int_{\mathbb{R}^n} |Du|^{2p-2}|D^2u|^2\,dx \tag{1.1}$$

were established and applied to the solvability of *scalar* elliptic equations.

In this paper, we provide global and local versions of the above inequality with the Lebesgue measure $dx$ replaced by $wdx$ where $w$ is some $A_p$ weight. The purpose of such generalization becomes clear when we apply the results to the study of local/global existence of strong solutions to the following nonlinear strongly coupled and *nonregular* but *uniform* parabolic system:

$$\begin{cases} u_t = \mathrm{div}(A(x,t,u,Du)) + \hat{f}(x,t,u,Du), & (x,t) \in Q = \Omega \times (0,T_0), \\ u(x,0) = U_0(x), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T_0). \end{cases} \tag{1.2}$$

Here, and throughout this paper, $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. A typical point in $\mathbb{R}^n$ is denoted by $x$ and a point in $\mathbb{R}^n \times [0,\infty)$ is denoted by $z = (x,t)$. The temporal and $k$-order spatial derivatives of a vector-valued function

$$u(x,t) = (u_1(x,t), \ldots, u_m(x,t))^T, \quad m > 1$$

are denoted by $u_t$ and $D^k u$, respectively. $A(x,t,u,Du)$ is a *full* $m \times n$ matrix, and $\hat{f} : \Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$. The initial data $U_0$ is given in $W^{1,r_0}(\Omega,\mathbb{R}^m)$ for some $r_0 > n$. As usual, $W^{k,p}(\Omega,\mathbb{R}^m)$, where $k$ is an integer and

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$p \geq 1$, denotes the standard Sobolev spaces whose elements are vector-valued functions $u : \Omega \to \mathbb{R}^m$ with finite norm

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R}^m)} = \|u\|_{L^p(\Omega,\mathbb{R}^m)} + \sum_{i=1}^{k} \|D^k u\|_{L^p(\Omega,\mathbb{R}^m)}.$$

By a strong solution of (1.2) we mean a vector-valued function $u$ that solves (1.2) a.e. in $\Omega \times (0, T_0)$ and continuously assumes the initial value $U_0$ at $t = 0$ and boundary data on $\partial\Omega \times (0, T_0)$. Moreover, for some $\alpha > 0$ and all $t \in (0, T_0)$ we have $Du(\cdot, t) \in C^{\alpha,\frac{\alpha}{2}}(\Omega)$ and $D^2 u(\cdot, t) \in L^p(\Omega)$ for a.e. $t \in (0, T_0)$ and all $p > 1$.

The strongly coupled system (1.2) appears in many physical applications, for instance, Maxwell–Stephan systems describing the diffusive transport of multicomponent mixtures, models in reaction and diffusion in electrolysis, flows in porous media, diffusion of polymers, or population dynamics. We refer the reader to the recent work [10] and the references therein for the models and the existence of their *weak* solutions. Besides the question whether a *strong* solution of (1.2) can exist locally near $t = 0$, we face with a fundamental problem in the theory of PDEs to establish that this local solution exists globally. Unlike the well-established theory for scalar parabolic equations (i.e. $m = 1$), where bounded solutions usually exist globally, there are counter examples for systems ($m > 1$) which exhibit solutions that start smoothly and remain bounded but develop singularities in higher norms in finite times (see [8]). Even more, bounded solutions to (1.2) may not be even Hölder continuous everywhere.

We will impose the following structural conditions on (1.2). In this paper, for a vector- or matrix-valued function $f(u, \zeta)$, $u \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^d$, its partial derivatives will be denoted by $f_u, f_\zeta$.

(A) $A(x, t, u, \zeta)$ is $C^1$ in $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^{mn}$. Moreover, there are constants $C_*, C, \lambda_0 > 0$ and a scalar $C^1$ function $\lambda(x, t, u)$ such that, for any $(x, t) \in \mathbb{R}^{n+1}$, $u \in \mathbb{R}^m$ and $\zeta, \xi \in \mathbb{R}^{nm}$,

$$\lambda(x, t, u)|\zeta|^2 \leq \langle A_\zeta(x, t, u, \zeta)\xi, \xi\rangle \quad \text{and} \quad |A_\zeta(x, t, u, \zeta)| \leq C_* \lambda(x, t, u). \tag{1.3}$$

We also assume $\lambda(x, t, u) \geq \lambda_0$ and $|A_u(x, t, u, \zeta)| \leq C|\lambda_u||\zeta|$.

If $\lambda(x, t, u)$ is also bounded from above by a constant, we say that $A$ is regular elliptic. Otherwise, $A$ is uniformly elliptic. The constant $C_*$ in (1.3) concerns the ratio between the largest and smallest eigenvalues of $A_\zeta$. We assume that these constants are not too far apart in the following sense.

(SG) (The spectral gap condition) $(n - 2)/n < C_*^{-1}$.

We note that if this condition is somewhat violated then examples of blowing up in finite time can occur (see [1]).

Concerning $\hat{f}$, we will assume the following.

(F) There exist a constant $C$ and a function $f(x, t, u)$ which is $C^1$ in $x, u$ such that, for any $C^1$ functions $u : \Omega \to \mathbb{R}^m$ and $p : \Omega \to \mathbb{R}^{mn}$,

$$|\hat{f}(x, t, u, p)| \leq C\lambda^{\frac{1}{2}}(x, t, u)|p| + f(x, t, u),$$

$$|D\hat{f}(x, t, u, p)| \leq C\lambda^{\frac{1}{2}}(x, t, u)|Dp| + C\frac{|\lambda_u(x, t, u)|}{\lambda^{\frac{1}{2}}(x, t, u)}|Du||p| + |f_u(x, t, u)||Du|, \tag{1.4}$$

$$|f_u(x, t, u)| \leq C\lambda(x, t, u). \tag{1.5}$$

For simplicity in our statements and proof, as the presence of $x, t$ can be treated similarly, we will mostly assume that $A, \hat{f}$ are independent of $x, t$ in this paper.

In the last decades, papers concerning strongly coupled parabolic systems like (1.2), with $A(x, t, u, Du)$ being *linear* in $Du$, i.e. $A(x, t, u, Du) = A(x, t, u)Du$, usually relied on the results of Amann [2, 3] who showed that a solution to (4.1) exists globally if its $W^{1,r_0}(\Omega)$ norm for some $r_0 > n$, where $n$ is the dimension of $\Omega$, does not blow up in finite time. This requires the existence of a continuous function $\mathcal{C}$ on $(0, \infty)$ such that

$$\|u(\cdot, t)\|_{W^{1,r_0}(\Omega,\mathbb{R}^m)} \leq \mathcal{C}(t) \quad \text{for all } t \in (0, T_0) \text{ and some } r_0 > n. \tag{1.6}$$

The verification of (1.6) is very difficult and equivalently requires Hölder continuity of the solution $u$. This is a very hard problem in the theory of PDEs as known techniques for the regularity of solutions to scalar equations could not be extended to systems, and counterexamples were available. Maximum or comparison principles for systems generally do not hold so that the boundedness of solutions to (4.1) is unknown. Even if the solutions are bounded, only *partial* regularity results are known (see [6]).

Furthermore, the assumption (1.6) gives the boundedness of $u$ so that the ellipticity constants for the matrix $A(x, t, u)$ are bounded. Thus $A$ is regular elliptic. Without this assumption, one has to consider the case $A(x, t, u)$ being uniformly elliptic when the smallest and largest eigenvalues of $A(x, t, u)$ can be unbounded but comparable as in (1.3).

In this paper, we will replace (1.6) by a much weaker condition. Namely, we will show that it suffices to control the BMO norm of $u$ and the uniform continuity of this norm in small balls. Roughly speaking, we will replace condition (1.6) by the following:

$$\text{for any } \varepsilon > 0 \text{ there is } R_\varepsilon > 0 \text{ such that } \sup_{B_{R_\varepsilon} \subset \Omega} \|u(\cdot, t)\|_{\mathrm{BMO}(B_{R_\varepsilon})} \le \varepsilon \text{ for all } t \in (0, T_0). \tag{1.7}$$

By the Poincaré–Sobolev inequality, it is clear that (1.6) implies (1.7), even when $r_0 = n$.

On the other hand, since we will consider nonregular parabolic systems with $A$ being nonlinear in $Du$, Amann's results are not applicable here to give the solvability of (1.2). We will then provide an alternative approach to establish local/global existence results for (1.2) via Leray–Schauder fixed point theories. The existence results will be proven under a set of general and practical structural conditions on $A, \hat{f}$. Roughly speaking, we will embed (1.2) in a family of nonlinear systems which satisfy the same set of assumptions for (1.2). The strong solutions of these systems are fixed points of a family of compact vector fields in some appropriate Banach space. The key step in the argument is the establishment of a uniform bound for such solutions. We obtain the desired bound by using the local weighted Gagliardo–Nirenberg inequalities in Section 2 to deduce a decay estimate for local norms of the solutions so that an iteration argument can apply.

Though (1.7) will be required to hold uniformly for all strong solutions of the systems in the family, but, as these systems assume the same hypotheses for (4.1), we practically need only verify (1.7) for (1.2).

The techniques in this paper also give higher regularity of solutions to other systems where (1.7) yields that $u$ is Hölder continuous. We thus devote Section 3 to the a-priori estimates and the regularity of solutions to the following system:

$$u_t = \mathrm{div}(A(x, t, W, Du)) + \hat{f}(x, t, W, Du),$$

where $W, u$ are related in some way. Later, the case $W = \sigma u$ for some $\sigma \in (0, 1]$ will be used in our fixed point argument to obtain local and global existence results.

We conclude this paper with Section 4 where we apply the estimates in Section 3 to study the solvability of (1.2) under the assumption (1.7).

## 2 Weighted Gagliardo–Nirenberg Inequalities

In this section we will establish global and local weighted Gagliardo–Nirenberg interpolation inequalities which allow us to control the $L^p$ norm of the derivatives of the solutions in the proof of our main theorems. These inequalities generalize those in [14, 16], where no weight versions were proved (see Remark 2.2).

Here and throughout this paper, we write $B_R(x)$ for a ball centered at $x$ with radius $R$ and will omit $x$ if no ambiguity can arise. In our statements and proofs, we use $C, C_1, \ldots$ to denote various constants which can change from line to line but depend only on the parameters of the hypotheses in an obvious way. We will write $C(a, b, \ldots)$ when the dependence of a constant $C$ on its parameters $a, b, \ldots$ is needed to emphasize that $C$ is bounded in terms of its parameters.

For any measurable subset $A$ of $\Omega$ and any locally integrable function $U : \Omega \to \mathbb{R}^m$ we denote by $|A|$ the Lebesgue measure of $A$ and by $U_A$ the average of $U$ over $A$. That is,

$$U_A = \fint_A U(x)\, dx = \frac{1}{|A|} \int_A U(x)\, dx.$$

In order to state the assumption for this type of inequalities, we recall some well-known notions from Harmonic Analysis.

For $\gamma \in (1,\infty)$ we say that a nonnegative locally integrable function $w$ belongs to the class $A_\gamma$ or $w$ is an $A_\gamma$ weight if the quantity

$$[w]_\gamma := \sup_{B_R(y)\subset\Omega} \Big( \fint_{B_R(y)} w\,dx \Big)\Big( \fint_{B_R(y)} w^{1-\gamma'}\,dx \Big)^{\gamma-1}$$

is finite. Here, $\gamma' = \gamma/(\gamma-1)$. The $A_\infty$ and $A_1$ classes are defined by $A_\infty = \bigcup_{\gamma>1} A_\gamma$ and $A_1 = \bigcap_{\gamma>1} A_\gamma$. For more details on these classes we refer the reader to [4, 13, 15]. Clearly, the above implies

$$\Big( \fint_{B_R(y)} w\,dx \Big)\Big( \fint_{B_R(y)} w^{-\frac{1}{\mu}}\,dx \Big)^{\mu} \le [w]_{\mu+1} \quad \text{for all } \mu > 0.$$

A locally integrable function $U : \Omega \to \mathbb{R}^m$ is said to be BMO if the quantity

$$[U]_* := \sup_{B_R(y)\subset\Omega} \fint_{B_R(y)} |U - U_{B_R(y)}|\,dx$$

is finite.

The Banach space $\mathrm{BMO}(\Omega,\mathbb{R}^m)$ consists of functions with finite norm

$$\|U\|_{\mathrm{BMO}(\Omega,\mathbb{R}^m)} := [U]_* + \|U\|_{L^1(\Omega,\mathbb{R}^m)}.$$

When no ambiguity can arise, we simply say $U$ is BMO and omit $\Omega$ or $\mathbb{R}^m$ from the above notations.

We first have the following global weighted Gagliardo–Nirenberg inequality.

**Lemma 2.1.** *Let $u, U : \Omega \to \mathbb{R}^m$ be vector-valued functions with $u \in C^1(\Omega)$, $U \in C^2(\Omega)$ and $\Phi : \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ function. Suppose that either $U$ or $\Phi^2(u)\frac{\partial U}{\partial \nu}$ vanish on the boundary $\partial\Omega$ of $\Omega$. We set*

$$I_1 := \int_\Omega \Phi^2(u)|DU|^{2p+2}\,dx, \quad \hat{I}_1 := \int_\Omega \Phi^2(u)|Du|^{2p+2}\,dx, \tag{2.1}$$

$$\bar{I}_1 := \int_\Omega |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2})\,dx,$$

$$I_2 := \int_\Omega \Phi^2(u)|DU|^{2p-2}|D^2U|^2\,dx. \tag{2.2}$$

*Suppose that*

(GN) $\Phi(u)^{\frac{2}{p+2}}$ *belongs to the $A_{p/(p+2)+1}$ class.*

*Then for any $\varepsilon > 0$ there is a constant $C_{\varepsilon,\Phi}$ depending on $\varepsilon$ and $[\Phi^{\frac{2}{p+2}}(u)]_{p/(p+2)+1}$ for which*

$$I_1 \le \varepsilon \hat{I}_1 + C_{\varepsilon,\Phi}\|U\|^2_{\mathrm{BMO}(\Omega)}[\bar{I}_1 + I_2]. \tag{2.3}$$

In the proof of this lemma we will make use of the following well-known facts from Harmonic Analysis. We first recall the definition of the *centered* and *uncentered* Hardy–Littlewood maximal operators acting on function $F \in L^1_{\mathrm{loc}}(\Omega)$:

$$M(F)(y) = \sup_{\varepsilon}\Big\{ \fint_{B_\varepsilon(y)} F(x)\,dx : \varepsilon > 0 \text{ and } B_\varepsilon(y) \subset \Omega \Big\},$$

$$M^*(F)(z) = \sup_{z\in B_\varepsilon(y),\varepsilon}\Big\{ \fint_{B_\varepsilon(y)} F(x)\,dx : \varepsilon > 0 \text{ and } B_\varepsilon(y) \subset \Omega \Big\}.$$

We also note here the Hardy–Littlewood theorem: for any $F \in L^q(\Omega)$ we have

$$\int_\Omega M(F)^q\,dx \le C(q)\int_\Omega F^q\,dx, \quad q > 1. \tag{2.4}$$

More generally, the Muckenhoupt theorem [12] states that if $w$ is an $A_q$ weight then, for any $F \in L^q(\Omega)$,

$$\int_\Omega M(F)^q w\,dx \le \int_\Omega M^*(F)^q w\,dx \le C([w]_q)\int_\Omega F^q w\,dx. \tag{2.5}$$

We also make use of Hardy spaces $\mathcal{H}^1$. For any $y \in \Omega$ and $\varepsilon > 0$, let $\phi$ be any function in $C_0^\infty(B_1(y))$ with $|D\phi| \le C_1$. Let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ (then $|D\phi_\varepsilon| \le C_1\varepsilon^{-1-n}$). From [15], a function $g$ is in $\mathcal{H}^1(\Omega)$ if

$$\sup_{\varepsilon>0} g * \phi_\varepsilon \in L^1(\Omega) \quad \text{and} \quad \|g\|_{\mathcal{H}^1} = \|g\|_{L^1(\Omega)} + \|\sup_{\varepsilon>0} g * \phi_\varepsilon\|_{L^1(\Omega)}.$$

We are now ready to give the proof of Lemma 2.1.

*Proof.* We can assume that $m = 1$ because the proof for the vectorial case is similar. Integrating by parts, we have

$$I_1 = \int_\Omega \Phi^2(u)|DU|^{2p+2}\,dx = -\int_\Omega U \operatorname{div}(\Phi^2(u)|DU|^{2p}DU)\,dx. \tag{2.6}$$

We will show that $g = \operatorname{div}(\Phi^2(u)|DU|^{2p}DU)$ belongs to the Hardy space $\mathcal{H}^1$ and

$$\|g\|_{\mathcal{H}^1} = \int_\Omega \sup_\varepsilon |g * \phi_\varepsilon|\,dx \le C([\Phi^{\frac{2}{p+2}}(u)]_{p/(p+2)+1})\Big[\bar{I}_1^{\frac12}(I_1^{\frac12} + \hat{I}_1^{\frac12}) + I_1^{\frac12} I_2^{\frac12}\Big]. \tag{2.7}$$

Once this is established, (2.6) and the Fefferman–Stein theorem on the duality of the BMO and Hardy spaces yield

$$I_1 \le \|U\|_{\mathrm{BMO}}\|g\|_{\mathcal{H}^1} \le C([\Phi^{\frac{2}{p+2}}(u)]_{p/(p+2)+1})\|U\|_{\mathrm{BMO}}\Big[\bar{I}_1^{\frac12}(I_1^{\frac12} + \hat{I}_1^{\frac12}) + I_1^{\frac12} I_2^{\frac12}\Big].$$

A simple use of Young's inequality to the right-hand side then gives (2.3).

Therefore, in the rest of the proof we need only establish (2.7). We then write $g = g_1 + g_2$ with $g_i = \operatorname{div} V_i$, setting

$$V_1 = \Phi(u)|DU|^{p+1}\Big(\Phi(u)|DU|^{p-1}DU - \fint_{B_\varepsilon} \Phi(u)|DU|^{p-1}DU\,dx\Big),$$

$$V_2 = \Phi(u)|DU|^{p+1}\fint_{B_\varepsilon} \Phi(u)|DU|^{p-1}DU\,dx.$$

Let us consider $g_1$ first and define $h = \Phi(u)|DU|^{p-1}DU$. For any $y \in \Omega$ and $B_\varepsilon = B_\varepsilon(y) \subset \Omega$, we use integration by parts, the property of $\phi_\varepsilon$ and then Hölder's inequality for any $s > 1$ to get

$$\begin{aligned}
|g_1 * \phi_\varepsilon(y)| &= \Big|\int_{B_\varepsilon(y)} D\phi\Big(\frac{x-y}{\varepsilon}\Big)(h - h_{B_\varepsilon(y)})\Phi(u)|DU|^{p+1}\,dx\Big| \\
&\le \frac{C_1}{\varepsilon}\Big|\fint_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}|\Phi(u)|DU|^{p+1}\,dx\Big| \\
&\le \frac{C_1}{\varepsilon}\Big(\fint_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}|^s\,dx\Big)^{\frac1s}\Big(\fint_{B_\varepsilon(y)} \Phi^{s'}(u)|DU|^{(p+1)s'}\,dx\Big)^{\frac{1}{s'}}.
\end{aligned}$$

There is a constant $C$ such that $|Dh| \le |\Phi_u(u)||Du||DU|^p + p\Phi|DU|^{p-1}|D^2U|$. Poincaré–Sobolev's inequality, with $s_* = ns/(n+s)$, then gives

$$\begin{aligned}
\frac{C_1}{\varepsilon}\Big(\fint_{B_\varepsilon} |h - h_{B_\varepsilon}|^s\,dx\Big)^{\frac1s} &\le C\Big(\fint_{B_\varepsilon} |Dh|^{s_*}\,dx\Big)^{\frac{1}{s_*}} \\
&\le C\Big[\fint_{B_\varepsilon} |\Phi_u(u)|^{s_*}|Du|^{s_*}|DU|^{ps_*}\,dx + \fint_{B_\varepsilon} \Phi^{s_*}|DU|^{(p-1)s_*}|D^2U|^{s_*}\,dx\Big]^{\frac{1}{s_*}}.
\end{aligned} \tag{2.8}$$

Using the above estimates in (2.8), we get

$$\frac{C_1}{\varepsilon}\Big(\fint_{B_\varepsilon} |h - h_{B_\varepsilon}|^s\,dx\Big)^{\frac{1}{s}} \le C[\Psi_1 + \Psi_2],$$

where

$$\Psi_1(y) = \big(M(|\Phi_u(u)|^{s_*}|DU|^{ps_*}|Du|^{s_*})(y)\big)^{\frac{1}{s_*}},$$
$$\Psi_2(y) = \big(M(\Phi^{s_*}|DU|^{(p-1)s_*}|D^2U|^{s_*})(y)\big)^{\frac{1}{s_*}}.$$

Setting

$$\Psi_3(y) = \big(M(\Phi^{s_*}(u)|DU|^{(p+1)s_*})(y)\big)^{\frac{1}{s_*}}$$

and putting these estimates together, we thus have

$$\sup_{\varepsilon>0,}|g_1 * \phi_\varepsilon| \le C[\Psi_1 + \Psi_2]\Psi_3. \tag{2.9}$$

In the sequel, we will denote

$$F = \Phi|DU|^{p+1}, \quad \bar F = |\Phi_u||DU|^{p+1}, \quad f = \Phi|Du|^{p+1}, \quad \bar f = |\Phi_u||Du|^{p+1}. \tag{2.10}$$

Take $s = 2n/(n-1)$, then $s_* = s' = 2n/(n+1)$. With these notations and the definition of $\bar I_1$, we can use Young's inequality and then (2.4), because $2 > 2n/(n+1) = s_*$, to get

$$\Big(\int_\Omega \Psi_1^2\,dx\Big)^{\frac{1}{2}} \le C\Big(\|M(\bar F^{s_*})\|_{\frac{2}{s_*}}^{\frac{1}{s_*}} + \|M(\bar f^{s_*})\|_{\frac{2}{s_*}}^{\frac{1}{s_*}}\Big) \le C(\|\bar F^2\|_2 + \|\bar f^2\|_2) \le C\bar I_1^{\frac{1}{2}}.$$

Similarly,

$$\Big(\int_\Omega \Psi_3^2\,dx\Big)^{\frac{1}{2}} \le C\|\Phi^2|DU|^{2(p+1)}\|_2 \le CI_1^{\frac{1}{2}}.$$

Furthermore, (2.4) also gives

$$\Big(\int_\Omega \Psi_2^2\,dx\Big)^{\frac{1}{2}} = \|M(\Phi^{s_*}|DU|^{(p-1)s_*}|D^2U|^{s_*})\|_{\frac{2}{s_*}}^{\frac{1}{s_*}} \le C\|\Phi^2|DU|^{p-1}|D^2U|\|_2.$$

Therefore, by Holder's inequality, the above estimates and the notations (2.1) and (2.2), we get

$$\int_\Omega \sup_\varepsilon|g_1 * \phi_\varepsilon|\,dx \le C\Big[I_1^{\frac{1}{2}}\bar I_1^{\frac{1}{2}} + I_1^{\frac{1}{2}}I_2^{\frac{1}{2}}\Big]. \tag{2.11}$$

We now turn to $g_2$ and note that $|\operatorname{div} V_2| \le C(J_1 + J_2)$ for some constant $C$ and

$$J_1 := |\Phi_u(u)||DU|^{p+1}|Du|J_3, \quad J_2 := \Phi|DU|^p|D^2U|J_3,$$

with

$$J_3 := \Big|\fint_{B_\varepsilon(y)} \Phi|DU|^p\,dx\Big|.$$

We will estimate $\|\phi_\varepsilon * J_1\|_{L^1(\Omega)}$ and $\|\phi_\varepsilon * J_2\|_{L^1(\Omega)}$. The calculations for these estimates are similar, we consider $J_1$ first and denote

$$K = |\Phi_u(u)||DU|^{p+1}, \quad L = |Du|.$$

We first observe that

$$J_3(y) \le M^*(\Phi|DU|^p)(x) \quad \text{for } x \in B_\varepsilon(y).$$

Here, $M^*$ is the *uncentered* Hardy–Littlewood maximal operator. We then have

$$\phi_\varepsilon * J_1(x) \le \phi_\varepsilon * K(x)L(x)M^*(\Phi|DU|^p)(x).$$

Therefore,

$$\int_\Omega |\phi_\varepsilon * J_1(x)|\,dx \le \int_\Omega K(x)L(x)M^*(\Phi|DU|^p)(x)\,dx = \int_\Omega K(L\Phi^{\frac{1}{p+1}})\big(M^*(\Phi|DU|^p)\Phi^{\frac{-1}{p+1}}\big)\,dx.$$

We then apply Hölder's inequality to the last integral to get

$$\int_\Omega |\phi_\varepsilon * J_1|\,dx \le \|K\|_{L^2(\Omega)}\|L\Phi^{\frac{1}{p+1}}\|_{L^{2(p+1)}(\Omega)}\big\|M^*(\Phi|DU|^p)\Phi^{\frac{-1}{p+1}}\big\|_{L^{\frac{2(p+1)}{p}}(\Omega)}. \tag{2.12}$$

Concerning the last term on the right-hand side of (2.12), we note that $w := \Phi^{\frac{-2}{p}}$ is an $A_q$ weight for $q = 2(p+1)/p$. Indeed, because $1 - q' = -p/(p+2)$ and $q - 1 = (p+2)/p$, we see easily that

$$\fint_B w\,dx\Big(\fint_B w^{1-q'}\,dx\Big)^{q-1} = \fint_B \Phi^{\frac{-2}{p}}\,dx\Big(\fint_B \Phi^{\frac{2}{p+2}}\,dx\Big)^{\frac{p+2}{p}} \le \Big([\Phi^{\frac{2}{p+2}}]_{p/(p+2)+1}\Big)^{\frac{p+2}{p}}.$$

Therefore, $[w]_q$ is bounded by a constant depending on $[\Phi^{\frac{2}{p+2}}]_{p/(p+2)+1}$.

By Muckenhoupt's theorem (2.5), we can find a constant $C([w]_q) \sim [w]_q$ such that

$$\big\|M^*(\Phi|DU|^p)\Phi^{\frac{-1}{p+1}}\big\|^{\frac{p}{2(p+1)}}_{L^{\frac{2(p+1)}{p}}(\Omega)} = \int_\Omega (M^*(\Phi|DU|^p))^q w\,dx \le C([w]_q)\int_\Omega \Phi^q|DU|^{pq}w\,dx.$$

Note that from the definition of $q = 2(p+1)/p$ and $w := \Phi^{\frac{-2}{p}}$, we have $\Phi^q|DU|^{pq}w = |\Phi^2 DU|^{2p+2}$.

Hence, using the above estimate for the last integral in (2.12), we derive

$$\int_\Omega |\phi_\varepsilon * J_1|\,dx \le C_\Phi\|K\|_{L^2(\Omega)}\|L\Phi^{\frac{1}{p+1}}\|_{L^{2(p+1)}(\Omega)}\|F\|_2^{\frac{p}{p+1}}, \tag{2.13}$$

where $C_\Phi$ is a constant depending on $[w]_q$ or $[\Phi^{\frac{2}{p+2}}]_{p/(p+2)+1}$.

Recalling the definition of $K = \bar{F}$ (see (2.10)) and $L = |Du|$, we see that

$$\|L\Phi^{\frac{1}{p+1}}\|_{L^{2(p+1)}(\Omega)} = \|f\|^{\frac{1}{p+1}}_{L^2(\Omega)}.$$

Therefore, Young's inequality yields

$$\int_\Omega \sup_\varepsilon|\phi_\varepsilon * J_1|\,dx \le C_\Phi\|\bar{F}\|_2\|f\|_2^{\frac{1}{p+1}}\|F\|_2^{\frac{p}{p+1}} \le C_\Phi\bar{I}_1^{\frac{1}{2}}(I_1^{\frac{1}{2}} + \hat{I}_1^{\frac{1}{2}}). \tag{2.14}$$

Next, for $J_2 = \Phi|DU|^{p-1}|D^2U||DU|J_3$ we repeat the calculation for $J_1$, using

$$K = |\Phi(u)||DU|^{p-1}|D^2U|, \quad L = |DU|.$$

We then obtain an estimate similar to (2.13) for $\|\phi_\varepsilon * J_2\|_{L^1(\Omega)}$. Now, with the new definitions of $K, L$, we have

$$\|K\|_{L^2(\Omega)} = I_1^{\frac{1}{2}}, \quad \|L\Phi^{\frac{1}{p+1}}\|_{L^{2(p+1)}(\Omega)} = \|F\|^{\frac{1}{p+1}}_{L^2(\Omega)}.$$

We then obtain the inequality

$$\int_\Omega \sup_\varepsilon|\phi_\varepsilon * J_2|\,dx \le C_\Phi I_1^{\frac{1}{2}}I_2^{\frac{1}{2}}. \tag{2.15}$$

Combining the estimates (2.14), (2.15), we derive

$$\int_\Omega \sup_\varepsilon|g_2 * \phi_\varepsilon|\,dx \le C_\Phi\Big[\bar{I}_1^{\frac{1}{2}}(I_1^{\frac{1}{2}} + \hat{I}_1^{\frac{1}{2}}) + I_1^{\frac{1}{2}}I_2^{\frac{1}{2}}\Big]. \tag{2.16}$$

The above and (2.11) yield

$$\int_\Omega \sup_\varepsilon |g * \phi_\varepsilon|\,dx \le C_\Phi\Big[\bar I_1^{\frac12}(I_1^{\frac12} + \hat I_1^{\frac12}) + I_1^{\frac12} I_2^{\frac12}\Big].$$

We thus proved (2.7) and the proof of Lemma 2.1 is complete. □

**Remark 2.2.** By approximation (see [16]), Lemma 2.1 also holds for $u \in W^{1,2}(\Omega)$ and $U \in W^{2,2}(\Omega)$ provided that the quantities $I_1$, $I_2$ and $\hat I_1$ defined in (2.1) and (2.2) are finite. Furthermore, if $\Phi$ is a constant then $\bar I_1 = 0$ on the right-hand side of (2.3). Thus if $u = U$ and $\Phi$ is a constant then Lemma 2.1, with small $\varepsilon$, clearly gives

$$\int_\Omega |DU|^{2p+2}\,dx \le C\|U\|^2_{\mathrm{BMO}(\Omega)} \int_\Omega |DU|^{2p-2}|D^2U|^2\,dx.$$

This is the Gagliardo–Nirenberg inequality (1.1) established in [16].

**Remark 2.3.** In other applications, we may need a similar version of the lemma with the usual gradient operator $D$ being replaced by a general differential operator $\mathbf{D}$, e.g. a weighted linear combination of $D_{x_i}$. One can easily see that the proof is virtually unchanged if a certain Poincaré–Sobolev inequality used in (2.8) holds. Namely,

$$\frac{1}{\varepsilon}\Big(\fint_{B_\varepsilon} |h - h_{B_\varepsilon}|^s\,dx\Big)^{\frac1s} \le C\Big(\fint_{B_\varepsilon} |\mathbf{D}h|^{s_*}\,dx\Big)^{\frac{1}{s_*}}$$

holds for some $s_*$, depending on $s$, such that $s', s_* < 2$. Of course, $DU$, $Du$ will be accordingly replaced by $\mathbf{D}U$, $\mathbf{D}u$.

To study the regularity of solutions, assuming that their BMO norms in small balls are small, we have the following local version of Lemma 2.1.

**Lemma 2.4.** *Let $u, U : \Omega \to \mathbb{R}^m$ be vector-valued functions with $u \in C^1(\Omega)$, $U \in C^2(\Omega)$, and let $\Phi : \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ function such that the condition* (GN) *in Lemma 2.1 holds. For any ball $B_t$ in $\Omega$ we set*

$$I_1(t) := \int_{B_t} \Phi^2(u)|DU|^{2p+2}\,dx, \quad \hat I_1(t) := \int_{B_t} \Phi^2(u)|Du|^{2p+2}\,dx,$$

$$\bar I_1(t) := \int_{B_t} |\Phi_u(u)|^2(|DU|^{2p+2} + |Du|^{2p+2})\,dx,$$

$$I_2(t) := \int_{B_t} \Phi^2(u)|DU|^{2p-2}|D^2U|^2\,dx.$$

*Consider any ball $B_s$ concentric with $B_t$, $0 < s < t$, and any nonnegative $C^1$ function $\psi$ such that $\psi = 1$ in $B_s$ and $\psi = 0$ outside $B_t$. Then, for any $\varepsilon > 0$ there are positive constants $C_{\varepsilon,\Phi}$, $C_\varepsilon$ such that*

$$\begin{aligned} I_1(s) \le \varepsilon[I_1(t) + \hat I_1(t)] &+ C_{\varepsilon,\Phi}\|U\|^2_{\mathrm{BMO}(B_t)}[\bar I_1(t) + I_2(t)] \\ &+ C_\varepsilon \|U\|_{\mathrm{BMO}(B_t)} \sup_{x\in B_t}|D\psi(x)|^2 \int_{B_t} |\Phi|^2(u)|DU|^{2p}\,dx. \end{aligned} \tag{2.17}$$

*Proof.* We revisit the proof of Lemma 2.1. Integrating by parts, noting that $\psi = 0$ on $\partial\Omega$, we have

$$\int_\Omega \Phi^2(u)\psi^2|DU|^{2p+2}\,dx = -\int_\Omega U\,\mathrm{div}(\Phi^2(u)\psi^2|DU|^{2p}DU)\,dx.$$

Again, we will show that $g = \mathrm{div}(\Phi^2\psi^2|DU|^{2p}DU)$ belongs to the Hardy space $\mathcal{H}^1$. We write $g = g_1 + g_2$ with $g_i = \mathrm{div}\,V_i$, setting

$$V_1 = \Phi(u)\psi|DU|^{p+1}\Big(\Phi(u)\psi|DU|^{p-1}DU - \fint_{B_\varepsilon} \Phi(u)\psi|DU|^{p-1}DU\,dx\Big),$$

$$V_2 = \Phi(u)\psi|DU|^{p+1}\fint_{B_\varepsilon} \Phi(u)\psi|DU|^{p-1}DU\,dx.$$

In estimating $V_1$ we follow the proof of Lemma 2.1 and replace $\Phi(u)$ by $\Phi(u)\psi(x)$. There will be some extra terms in the proof in computing $D(\Phi(u)\psi)$. In particular, in estimating $Dh$ in the right-hand side of (2.8) we have the following term and it can be estimated as follows:

$$\Big(\fint_{B_\varepsilon} \Phi^{s_*}(u)|D\psi|^{s_*}|DU|^{ps_*}\,dx\Big)^{\frac{1}{s_*}} \le \sup_{x\in B_t}|D\psi|\Big(\fint_{B_\varepsilon} \Phi^{s_*}(u)|DU|^{ps_*}\,dx\Big)^{\frac{1}{s_*}}.$$

We then use the following inequality, via Young's inequality, in the right-hand side of (2.9) (with $\Omega = B_t$):

$$\sup|D\psi|\int_{B_t} \Psi_1 M(\Phi^{s_*}(u)|DU|^{ps_*})^{\frac{1}{s_*}}\,dx \le C\Big[\int_{B_t} \Psi_1^2\,dx + \sup_{x\in B_t}|D\psi|^2\int_{B_t} M(\Phi^{s_*}(u)|DU|^{ps_*})^{\frac{2}{s_*}}\,dx\Big].$$

The last integral can be bounded via (2.4) by

$$\sup_{x\in B_t}|D\psi|^2\int_{B_t} \Phi^2(u)|DU|^{2p}\,dx.$$

Using the fact that $|\psi| \le 1$ and taking $\Omega$ to be $B_t$ and omitting the obvious parameter $t$ in the sequel, the previous proof can go on and (2.11) now becomes

$$\int_{B_t} \sup_\varepsilon|g_1 * \phi_\varepsilon|\,dx \le C\big[I_1^{\frac12}\bar{I}_1^{\frac12} + I_1^{\frac12}I_2^{\frac12}\big] + C(\varepsilon)\sup_{B_t}|D\psi|^2\int_{B_t} \Phi^2(u)|DU|^{2p}\,dx. \tag{2.18}$$

Similarly, in considering $g_2 = \operatorname{div} V_2$, we will have an extra term $\Phi(u)|D\psi||DU|^{p+1}J_3$ in $J_1$. We then use the estimate

$$\sup_\varepsilon|\phi_\varepsilon * \Phi(u)||D\psi||DU|^{p+1}J_3 \le \sup_{B_t}|D\psi|M(\Phi(u)|DU|^{p+1})M(\Phi(u)|DU|^p),$$

and, via Young's inequality and (2.4),

$$\int_{B_t} \sup_\varepsilon|\phi_\varepsilon * \Phi(u)||D\psi||DU|^{p+1}J_3\,dx \le \varepsilon I_1(t) + C(\varepsilon)\sup_{B_t}|D\psi|^2\int_{B_t} \Phi^2(u)|DU|^{2p}\,dx.$$

Therefore estimate (2.16) is now (2.18) with $g_1$ being replaced by $g_2$. Combining the estimates for $g_1, g_2$ and using Young's inequality, we get

$$\int_{B_t} \sup_\varepsilon|g * \phi_\varepsilon|\,dx \le \varepsilon I_1(t) + C(\varepsilon)\sup_{B_t}|D\psi|^2\int_{B_t} |\Phi|^2|DU|^{2p}\,dx + C_\Phi\big[\bar{I}_1^{\frac12}(I_1^{\frac12} + \hat{I}_1^{\frac12}) + I_1^{\frac12}I_2^{\frac12}\big].$$

The above gives an estimate for the $\mathcal{H}^1$ norm of $g$. By the Fefferman–Stein theorem, we obtain

$$\int_{B_t} \Phi^2(u)\psi^2|DU|^{2p+2}\,dx \le \varepsilon I_1(t) + C_\Phi\|U\|_{\mathrm{BMO}(B_t)}\big[\bar{I}_1^{\frac12}(I_1^{\frac12} + \hat{I}_1^{\frac12}) + I_1^{\frac12}I_2^{\frac12}\big]$$
$$+ C(\varepsilon)\|U\|_{\mathrm{BMO}(B_t)}\sup_{B_t}|D\psi|^2\int_{B_t} \Phi^2(u)|DU|^{2p}\,dx.$$

As before, we can use Young's inequality and then the fact that $\psi = 1$ in $B_s$ to obtain (2.17) and complete the proof. □

# 3 A-Priori Estimates in $W^{1,p}(\Omega)$ for $p > n$

In this section we will establish the key estimate for the proof of our main theorem. As we mentioned in the Introduction, for simplicity we will assume that $A, \hat{f}$ are independent of $x, t$. The general case can be treated similarly. Throughout this section, for some fixed $T_0 > 0$ we consider two vector-valued functions $U$, $W$ from $\Omega \times (0, T_0)$ into $\mathbb{R}^m$ and solve the system

$$U_t = \operatorname{div}(A(W, DU)) + \hat{f}(W, DU). \tag{3.1}$$

We will consider the following assumptions on $U$, $W$ and (3.1):

(U.0) $A, \hat{f}$ satisfy 1, (F) and (SG) with $u = W$ and $\zeta = DU$.

(U.1) $U \in L^1((0, T_0), W^{2,2}(\Omega))$ and $W(\cdot, t) \in W^{1,2}(\Omega)$ for a.e. $t \in (0, T_0)$. On the lateral boundary $\partial\Omega \times (0, T_0)$, $U$ satisfies Neumann or Dirichlet boundary conditions.

(U.2) There is a constant $C$ such that $|DW| \le C|DU|$ and $|W_t| \le C|U_t|$.

The following assumption seems to be technical but we will see in many applications that it is easy to be verified when $W$ is a BMO function, a condition will be assumed in the main result of this section.

(U.3) There is a positive $C^1$ function $\beta : \mathbb{R}^m \to \mathbb{R}$ such that the following number is finite:

$$\mathbf{\Lambda} = \sup_W \left\{ \frac{|\lambda_W(W)|}{\lambda(W)}, \frac{|\beta_W(W)|}{\beta(W)} \right\}.$$

Moreover, $\beta^{-1}(W)$ and $\lambda(W)\beta(W)$ belong to $L^r(\Omega)$ for sufficiently large $r > 1$; $(\lambda(W)\beta(W))^{\frac{1}{2}}$ is an $A_{\frac{4}{3}}$ weight. Namely, there is a continuous function $C$ on $(0, T_0)$ such that, for a.e. $t \in (0, T_0)$,

$$\|\beta^{-1}(W)\|_{L^r(\Omega)},\ \|\lambda(W)\beta(W)\|_{L^r(\Omega)},\ [(\lambda(W)\beta(W))^{\frac{1}{2}}]_{\frac{4}{3}} \le C(t).$$

(U.4) There is a constant $C$ such that

$$\iint_{\Omega\times[0,T_0)} \lambda(W)\beta(W)|DU|^2\,dz \le C. \tag{3.2}$$

To continue, we introduce the quantities

$$\Gamma(W) = \lambda(W)\frac{|\beta_W(W)|^2}{\beta(W)} + \beta(W)\frac{|\lambda_W(W)|^2}{\lambda(W)}.$$

For any fixed $t_0 > 0$ we consider $T \in (2t_0, T_0)$ and $x_0 \in \bar{\Omega}$. For $t > 0$ we will denote

$$Q_t(x_0, T, t_0) = B_t(x_0) \cap \Omega \times [T - 2t_0, T_0).$$

For $q \ge 1$ we introduce the following quantities:

$$\mathcal{A}_q(t, x_0, T, t_0) = \sup_{\tau\in[T-t_0,T_0)} \int_{B_t(x_0)\cap\Omega} \beta(W)|DU|^{2q}\,dx, \tag{3.3}$$

$$\mathcal{B}_q(t, x_0, T, t_0) = \iint_{Q_t(x_0,T,t_0)} \Gamma(W)|DU|^{2q+2}\,dz, \tag{3.4}$$

$$\mathcal{C}_q(t, x_0, T, t_0) = \iint_{Q_t(x_0,T,t_0)} \lambda(W)\beta(W)|DU|^{2q+2}\,dz, \tag{3.5}$$

$$\mathcal{H}_q(t, x_0, T, t_0) = \iint_{Q_t(x_0,T,t_0)} \lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\,dz, \tag{3.6}$$

$$\mathcal{G}_q(t, x_0, T, t_0) = \iint_{Q_t(x_0,T,t_0)} \lambda(W)\beta(W)|DU|^{2q}\,dz, \tag{3.7}$$

$$\mathcal{I}_q(x_0, T, t_0) = \iint_{Q_t(x_0,T,t_0)} \beta(W)|DU|^{2q}\,dz. \tag{3.8}$$

We also denote, for $R, t > 0$,

$$\mathcal{D}(R, t, x_0) := \|U(\cdot, t)\|_{\mathrm{BMO}(B_R(x_0)\cap\Omega)}. \tag{3.9}$$

By (SG), there is $q_0 > n/2$ such that

$$\frac{2q_0 - 2}{2q_0} = \delta_{q_0} C_*^{-1} \quad \text{for some } \delta_{q_0} \in (0, 1). \tag{3.10}$$

The main result of this section shows that if $\mathcal{D}(R, t, x_0)$ is uniformly small for sufficiently small $R$, then $\|DU\|_{L^p(\Omega)}$ can be controlled for some $p > n$.

**Proposition 3.1.** *Suppose that* (U.0)–(U.4) *hold. Assume that there exist* $t_0 > 0$ *and* $\mu_0 \in (0, 1)$, *which is sufficiently small, in terms of the constants in 1,* (F) *such that the following holds:*
(D) *There is a positive* $R_{\mu_0}$, *which may also depend on* $t_0$, $T_0$, *for which*

$$\Lambda^2 \sup_{x_0\in\bar{\Omega}, t\in[T-2t_0,T_0)} \mathcal{D}^2(R_{\mu_0}, t, x_0) \le \mu_0 \quad \text{for all } T \in [2t_0, T_0).$$

*Suppose also that for* $x_0 \in \bar{\Omega}$ *and* $T \ge t_0 > 0$ *the quantities* (3.3)–(3.8) *are finite for* $q \in [1, q_0]$, $q_0$ *is fixed in* (3.10). *Then there are* $q > n/2$ *and a constant* $C$ *depending on the constants in* (U.0)–(U.4), $q$, $R_{\mu_0}$, $t_0$, $T_0$ *and the geometry of* $\Omega$ *such that*

$$\sup_{t\in[t_0,T_0)} \int_\Omega |DU|^{2q}\, dx \le C. \tag{3.11}$$

The dependence of $C$ in (3.11) on the geometry of $\Omega$ means: $C$ depends on a number $N_{\mu_0}$ of balls $B_{R_{\mu_0}}(x_i)$, $x_i \in \bar{\Omega}$, such that

$$\bar{\Omega} \subset \bigcup_{i=1}^{N_{\mu_0}} B_{R_{\mu_0}}(x_i). \tag{3.12}$$

The proof of Proposition 3.1 relies on local estimates for the integral of $|DU|$ in finitely many balls $B_R(x_i)$ with sufficiently small radius $R$ to be determined by the geometry of $\Omega$, namely the number $N_{\mu_0}$ and the continuity of the function $\mathcal{D}$ defined in (3.9). We will establish local estimates for $DU$ in these balls and then add up the results to obtain its global estimate (3.11). In the proof, we will only consider the case when $B_R(x_i) \subset \Omega$. The boundary case ($x_i \in \partial\Omega$) is similar, invoking a reflection argument and using the fact that $\partial\Omega$ is smooth to extend the function $U$ outside $\Omega$, see Remarks 3.7 and 3.8.

In the rest of this section, let us fix a point $x_0$ in $\Omega$ and $T \ge 2t_0$. We will drop $x_0$, $T$, $t_0$ in the notations (3.3)–(3.8) and (3.9).

For any $s$, $t$ such that $0 < s < t \le R$ let $\psi$ be a cutoff function for two balls $B_s$, $B_t$ centered at $x_0$. That is, $\psi$ is nonnegative, $\psi \equiv 1$ in $B_s$ and $\psi \equiv 0$ outside $B_t$ with $|D\psi| \le 1/(t - s)$. We also fix a cutoff function $\eta$ for $t$ for $[T - 2t_0, T_0]$ and $[T - t_0, T_0]$. That is $\eta(t) = 0$ for $t \le T - 2t_0$, $\eta(t) = 1$ for $t \in (T - t_0, T_0]$ and $|\eta'(t)| \le 1/t_0$ for all $t$.

We first have the following local energy estimate result.

**Lemma 3.2.** *Assume* (U.0)–(U.2). *Assume that* $q \ge 1$ *satisfies condition* (3.10) *and that the quantities* (3.3)–(3.8) *are finite. There is a constant* $C_1(q)$ *depending also on the constants in 1 and* (F) *such that*

$$\mathcal{A}_q(s) + \mathcal{H}_q(s) \le C_1(q)\Big[\mathcal{B}_q(t) + \frac{1}{(t-s)^2}\mathcal{G}_q(t) + \frac{1}{t_0}\mathcal{I}_q(t)\Big], \quad 0 < s < t \le R. \tag{3.13}$$

*Proof.* By the assumption (U.1), we can formally differentiate (3.1) with respect to $x$, more precisely we can use difference quotients (see Remark 3.3), to get the weak form of

$$(DU)_t = \operatorname{div}(A_\zeta(W, DU)D^2U + A_W(W, DU)DWDU) + D\hat{f}(W, DU). \tag{3.14}$$

For simplicity, we will assume in the proof that $\hat{f} \equiv 0$. The presence of $\hat{f}$ will be discussed later in Remark 3.4. Testing (3.14) with $\phi = \beta(W)|DU|^{2q-2}DU\psi^2\eta$, which is legitimate since $\mathcal{H}_q$ is finite, integrating by parts in $x$ and rearranging, we have, for $Q = \Omega \times [T - 2t_0, \tau]$ with $\tau \ge T$,

$$\iint_Q \langle \phi, (DU)_t\rangle \psi^2\eta\, dz = -\iint_Q \langle A_\zeta(W, DU)D^2U + A_W(W, DU)DWDU, D\phi\rangle\, dz. \tag{3.15}$$

Firstly, we observe that

$$\begin{aligned}
2q\langle \phi, (DU)_t\rangle\eta &= \frac{d}{dt}(\beta(W)|DU|^{2q}\eta) - |DU|^{2q}\beta_W(W)W_t\eta - \beta(W)|DU|^{2q}\eta_t,\\
D\phi &= \beta(W)D(|DU|^{2q-2}DU)\psi^2 + |DU|^{2q-2}DU\beta_W DW\psi^2 + 2\beta(W)|DU|^{2q-2}DU\psi D\psi.
\end{aligned}$$

Hence, we can rewrite (3.15) as

$$\frac{1}{2q}\int_{\Omega^\tau} \beta(W)|DU|^{2q}\psi^2\,dx + \iint_Q \beta(W)\langle A_\zeta(W,DU)D^2U, D(|DU|^{2q-2}DU)\psi^2\eta\rangle\,dz$$
$$= -\iint_Q [\langle A_\zeta(W,DU)D^2U, I_1\rangle + \langle A_W(W,DU)DW, I_2\rangle]\psi^2\eta\,dz + \iint_Q I_3\,dz, \tag{3.16}$$

where $\Omega^\tau = \Omega\times\{\tau\}$.

We will discuss the terms $I_1, I_2, I_3$ later. Let us consider the second integral on the left-hand side. By (U.0) and the uniform ellipticity of $A_\zeta(W,DU)$, we can find a constant $C_*$ such that $|A_\zeta(W,DU)\zeta| \le C_*\lambda(W)|\zeta|$. By (3.10), $\alpha = 2q-2$ satisfies

$$\frac{\alpha}{2+\alpha} = \frac{2q-2}{2q} = \delta_q C_*^{-1} = \delta_q\frac{\lambda(W)}{C_*\lambda(W)}.$$

By [1, Lemma 2.1] or [11, Lemma 6.2], for such $\alpha, q$ there is a positive constant $C(q)$ such that

$$\langle A_\zeta(W,DU)D^2U, D(|DU|^{2q-2}DU)\rangle \ge C(q)\lambda(W)|DU|^{2q-2}|D^2U|^2. \tag{3.17}$$

We then obtain from (3.16)

$$\int_\Omega \beta(W)|DU|^{2q}\psi^2\,dx + C_0(q)\iint_Q \beta(W)\lambda(W)|DU|^{2q-2}|D^2U|^2\psi^2\eta\,dz$$
$$\le -\iint_Q [\langle A_\zeta(W,DU)D^2U, I_1\rangle + \langle A_W(W,DU)DW, I_2\rangle]\eta\,dz + \iint_Q I_3\,dz. \tag{3.18}$$

The terms $I_1, I_2$ in the integrands on the right-hand side of (3.18) result from the calculation of $D\phi$ and they will be handled by Young's inequality as follows, noting that the assumption 1 gives $|A_\zeta(W,DU)| \le C|\lambda(W)|$ and $|A_W(W,DU)| \le C|\lambda_W(W)||DU|$.

Concerning $I_1$, for any $\varepsilon > 0$ we can find a constant $C(\varepsilon)$ such that

$$\big|\langle A_\zeta(W,DU)D^2U, |DU|^{2q-2}DU\beta_W DW\psi^2\rangle\big|$$
$$\le \varepsilon\lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\psi^2 + C(\varepsilon)\lambda(W)\frac{|\beta_W(W)|^2}{\beta(W)}|DW|^2|DU|^{2q}\psi^2,$$
$$\big|\langle A_\zeta(W,DU)D^2U, \beta(W)|DU|^{2q-2}DU\psi D\psi\rangle\big|$$
$$\le \varepsilon\lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\psi^2 + \lambda(W)\beta(W)|DU|^{2q}|D\psi|^2.$$

Similarly, for $I_2$ we have

$$\big|\langle A_W(W,DU)DW, \beta(W)|DU|^{2q-2}D^2U\psi^2\rangle\big|$$
$$\le \varepsilon\lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\psi^2 + \beta(W)\frac{|\lambda_W(W)|^2}{\lambda(W)}|DW|^2|DU|^{2q}\psi^2,$$
$$\big|\langle A_W(W,DU)DW, |DU|^{2q-2}DU\beta_W DW\psi^2\rangle\big| \le C|\lambda_W(W)||\beta_W(W)||DW|^2|DU|^{2q}\psi^2,$$
$$\big|\langle A_W(W,DU)DW, \beta(W)|DU|^{2q-2}DU\psi D\psi\rangle\big|$$
$$\le C\beta(W)\frac{|\lambda_W(W)|^2}{\lambda(W)}|DW|^2|DU|^{2q}\psi^2 + C\lambda(W)\beta(W)|DU|^{2q}|D\psi|^2.$$

Finally, for $I_3$, which results from the calculation of $\langle\phi, (DU)_t\rangle\eta$, we have

$$\beta(W)|DU|^{2q}|\eta_t| \le \frac{1}{t_0}\beta(W)|DU|^{2q}.$$

As we assume that $|W_t| \le C|U_t|$, we have, from the equation of $U$,

$$|W_t| \le C|A_\zeta(W,DU)||D^2U| + C|A_W(W,DU)||DW|.$$

Hence,

$$|DU|^{2q}|\beta_W(W)||W_t|\eta \le \varepsilon\lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2 + C(\varepsilon)\lambda(W)\frac{|\beta_W(W)|^2}{\beta(W)}|DU|^{2q+2}$$
$$+ C\lambda(W)\frac{|\beta_W(W)|^2}{\beta(W)}|DU|^{2q}|DW|^2 + C\beta(W)\frac{|\lambda_W(W)|^2}{\lambda(W)}|DU|^{2q}.$$

We then use the fact that $|DW| \le C|DU|$, choose $\varepsilon$ sufficiently small and put the above estimates for the terms in $I_1, I_2, I_3$ in (3.18) to obtain a number $C_1$ depending on $q$, $C(q)$ (see (3.17)) such that, for $B_s^\tau = B_s \times \{\tau\}$ and $\tau \in [T - t_0, T_0)$,

$$\int_{B_s^\tau} \beta(W)|DU|^{2q}\eta\,dx + \iint_{Q_s} \lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\eta\,dz$$
$$\le C_1\iint_{Q_t} \Gamma(W)|DU|^{2q+2}\psi^2\eta\,dz + C_1\Lambda\iint_{Q_t} \beta(W)\Big[\frac{1}{(t-s)^2}\lambda(W)|DU|^{2q} + \frac{1}{t_0}|DU|^{2q}\Big]\,dz.$$

Here, we used the definition of $\psi$ and $\Gamma(W)$. As the above holds for all $\tau \in [T - t_0, T_0)$, from the notations (3.3) and (3.4), the above gives the lemma. □

**Remark 3.3.** For $i = 1, \dots, n$ and $h \ne 0$ we denote by $\delta_{i,h}$ the difference quotient operator

$$\delta_{i,h}u = h^{-1}(u(x + he_i) - u(x)),$$

with $e_i$ being the unit vector of the $i$-th axis in $\mathbb{R}^n$. We then apply $\delta_{i,h}$ to the system for $U$ and then test the result with $|\delta_{i,h}U|^{2q-2}\delta_{i,h}U\psi^2$. The proof then continues to give the desired energy estimate by letting $h$ tend to 0.

**Remark 3.4.** If $\hat{f} \ne 0$ then there is an extra term $|D\hat{f}(W, DU)||DU|^{2q-1}\psi^2$ in (3.18). This term will give rise to similar terms in the proof. Indeed, by (1.4) in (F) with $u = W$ and $p = DU$,

$$|D\hat{f}(W, DU)| \le C\lambda^{\frac{1}{2}}(W)|D^2U| + C\frac{|\lambda_W(W)|}{\lambda^{\frac{1}{2}}(W)}|DW||DU| + |f_W(W)||DW|.$$

Therefore, by Young's inequality and (1.5), $|f_W(W)| \le C\lambda(W)$, we get

$$|D\hat{f}(W, DU)|\beta(W)|DU|^{2q-1} \le C\Big[\lambda^{\frac{1}{2}}(W)|D^2U| + C\frac{|\lambda_W(W)|}{\lambda^{\frac{1}{2}}(W)}|DW||DU| + |f_W(W)||DW|\Big]\beta(W)|DU|^{2q-1}$$
$$\le \varepsilon\lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2 + C(\varepsilon)\lambda(W)\beta(W)|DU|^{2q}$$
$$+ C\Gamma(W)|DU|^{2q+2} + C\beta(W)|DU|^{2q} + C\lambda(W)\beta(W)|DU|^{2q}.$$

Choosing $\varepsilon > 0$ sufficiently small, we then see that the proof can continue to obtain the energy estimate (3.13).

**Remark 3.5.** The energy estimate of the lemma can be established by the same argument if $A$ and $\hat{f}$ depend on $x$ and $t$. We can assume that $|A_x(t, x, u, Du)|$ and $|\hat{f}_x(t, x, u, Du)|$ satisfy the same growth as $|A|$ and $|\hat{f}|$.

**Remark 3.6.** Inspecting our proof here and the proof of [11, Lemma 6.2], we can see that the constant $C(q)$ in (3.17) is decreasing in $q$ and hence $C_1(q)$ is increasing in $q$. Note also that this is the only place we need (3.10).

**Remark 3.7.** We discuss the case when the centers of $B_\rho, B_R$ are on the boundary $\partial\Omega$. We assume that $U$ satisfies the Neumann boundary condition on $\partial\Omega$. By flattening the boundary we can assume that $B_R \cap \Omega$ is the set

$$B^+ = \{x : x = (x_1, \dots, x_n) \text{ with } x_n \ge 0 \text{ and } |x| < R\}.$$

For any point $x = (x_1, \dots, x_n)$ we denote by $\bar{x}$ its reflection across the plane $x_n = 0$, i.e., $\bar{x} = (x_1, \dots, -x_n)$. Accordingly, we denote by $B^-$ the reflection of $B^+$. For a function $u$ given on $B_+ \times (0, T)$ we denote its even reflection by $\bar{u}(x,t) = u(\bar{x},t)$ for $x \in B^-$. We then consider the even extension of $\hat{u}$ in $B = B^+ \cup B^-$:

$$\hat{u}(x,t) = \begin{cases} u(x,t) & \text{if } x \in B^+, \\ \bar{u}(x,t) & \text{if } x \in B^-. \end{cases}$$

With these notations, for $x \in B^+$ we observe that

$$U_t = \bar{U}_t, \quad \mathrm{div}_x(D_x U) = \mathrm{div}_{\bar{x}}(D_{\bar{x}} \bar{U}), \quad D_x W D_x U = D_{\bar{x}} \bar{W} D_{\bar{x}} \bar{U}.$$

Therefore, it is easy to see that $\hat{U}$ satisfies in $B$ a system similar to the one for $U$ in $B^+$. Thus, the proof can apply to $\hat{U}$ to obtain the same energy estimate near the boundary.

**Remark 3.8.** For the Dirichlet boundary condition we make use of the odd reflection $\bar{u}(x,t) = -u(\bar{x},t)$ and then define $\hat{u}$ as in Remark 3.7. Since $D_{x_i} U = 0$ on $\partial\Omega$ if $i \neq n$, we can test the system (3.14), obtained by differentiating the system of $U$ with respect to $x_i$, with $|D_{x_i} U|^{2q-2} D_{x_i} U \psi^2$ and the proof goes as before because no boundary integral terms appear in the calculation. We need only consider the case $i = n$. We observe that $D_{x_n} \hat{U}$ is the even extension of $D_{x_n} U$ in $B$ therefore $\hat{U}$ satisfies a system similar to (3.14). The proof then continues.

We now apply the local Gagliardo–Nirenberg inequality in the previous section to the functions $W$, $U$.

**Lemma 3.9.** *Let $B_s, B_t$ be two concentric balls in $\Omega$ with radii $t > s > 0$ and $\psi$ be a $C^1$ cutoff function for two balls $B_s, B_t$. Let $W \in C^1(\Omega)$ and $U \in C^2(\Omega)$ such that there is a constant $C$ such that $|DW| \le C|DU|$. Furthermore, assume that $[\lambda(W)\beta(W)]^{\frac{1}{p+2}}$ belongs to the $A_{p/(p+2)+1}$ class. There is a constant $C_{\lambda,\beta}$ depending on $[(\lambda(W)\beta(W))^{\frac{1}{p+2}}]_{p/(p+2)+1}$ such that*

$$\mathcal{B}_p(s) + \mathbf{\Lambda}^2 \mathcal{C}_p(s) \le \varepsilon \mathbf{\Lambda}^2 \mathcal{C}_p(t) + C(\varepsilon) \mathbf{\Lambda}^2 C_{\lambda,\beta} \sup_{\tau \in [T-2t_0, T_0)} \|U(\cdot,\tau)\|^2_{\mathrm{BMO}(B_t)} \Big[\mathcal{B}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2} \mathcal{G}_p(t)\Big]. \tag{3.19}$$

*Proof.* Let $u = W$ and the function $\Phi(W)$ in Lemma 2.4 be $[\lambda(W)\beta(W)]^{\frac{1}{2}}$. The assumption that

$$[\Phi(W)]^{\frac{2}{p+2}} = [\lambda(W)\beta(W)]^{\frac{1}{p+2}}$$

belongs to the $A_{p/(p+2)+1}$ class makes the lemma applicable here.

We now redefine

$$I_1(t) := \int_{B_t} \Phi^2(W)|DU|^{2p+2}\,dx, \quad \bar{I}_1(t) := \int_{B_t} |\Phi_W(W)|^2 |DU|^{2p+2}\,dx,$$

and note that, since $|DW| \le C|DU|$, the quantities $\hat{I}_1(t), \bar{I}_1(t)$ in Lemma 2.4 are majorized respectively by the above $I_1(t), \bar{I}_1(t)$. Hence, we can choose $\varepsilon$ sufficiently small in Lemma 2.4 to obtain a constant $C_\Phi \sim C_{\lambda,\beta}$ such that

$$\begin{aligned} I_1(s) \le \varepsilon I_1(t) + C(\varepsilon) C_\Phi \|U\|^2_{\mathrm{BMO}(B_t)} [\bar{I}_1(t) + I_2(t)] \\ + C(\varepsilon) C_\Phi \|U\|_{\mathrm{BMO}(B_t)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} \lambda(W)|DU|^{2p}\,dx. \end{aligned}$$

It is clear that $|\Phi_W(W)|^2 \sim \Gamma(W)$ so that

$$\bar{I}_1(t) + I_2(t) \sim \int_{B_t} (\lambda(W)\beta(W)|DU|^{2p-2}|D^2U|^2 + |\Gamma(W)||DU|^{2p+2})\,dx.$$

We then have

$$\begin{aligned} \int_{B_s} \Phi^2(W)|DU|^{2p+2}\,dx \le \varepsilon \int_{B_t} \Phi^2(W)|DU|^{2p+2}\,dx \\ + \mathbf{\Lambda}^2 C_{\lambda,\beta} \|U\|^2_{\mathrm{BMO}(B_t)} \int_{B_t} (\lambda(W)\beta(W)|DU|^{2p-2}|D^2U|^2 + |\Gamma(W)||DU|^{2p+2})\,dx \\ + \mathbf{\Lambda}^2 C_{\lambda,\beta} \|U\|_{\mathrm{BMO}(B_t)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} \lambda(W)\beta(W)|DU|^{2p}\,dx. \end{aligned}$$

Multiplying the above inequality with $\mathbf{\Lambda}^2\eta$, integrating the result over $[T-2t_0, T_0)$ and using the notations (3.3)–(3.8) (with $Q_t = B_t \times [T-2t_0, T_0)$), we see that the above implies

$$\mathbf{\Lambda}^2\mathcal{C}_p(s) \le \varepsilon\mathbf{\Lambda}^2\mathcal{C}_p(t) + C(\varepsilon)\mathbf{\Lambda}^2 C_{\lambda,\beta} \sup_{\tau\in[T-2t_0,T_0)} \|U(\cdot,\tau)\|^2_{\mathrm{BMO}(B_t)}\left[\mathcal{B}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right].$$

Because $\Gamma(W) \le \mathbf{\Lambda}^2\Phi^2(W)$, we have $\mathcal{B}_p(s) \le \mathbf{\Lambda}^2\mathcal{C}_p(s)$. We see that the above gives (3.19). □

Let us recall the following elementary iteration result (e.g., see [7, Lemma 6.1, p.192]).

**Lemma 3.10.** *Let $f, g, h$ be bounded nonnegative functions in the interval $[\rho, R]$ with $g, h$ being increasing. Assume that for $\rho \le s < t \le R$ we have*

$$f(s) \le \varepsilon_0 f(t) + [(t-s)^{-\alpha}g(t) + h(t)]$$

*with $\alpha > 0$ and $0 \le \varepsilon_0 < 1$. Then*

$$f(\rho) \le c(\alpha,\varepsilon_0)[(R-\rho)^{-\alpha}g(R) + h(R)].$$

*The constant $c(\alpha,\varepsilon_0)$ can be taken to be $(1-\nu)^{-\alpha}(1-\nu^{-\alpha}\nu_0)^{-1}$ for any $\nu$ satisfying $\nu^{-\alpha}\nu_0 < 1$.*

We then have another lemma for the main proof of this section.

**Lemma 3.11.** *Let $F, G, g, h$ be bounded nonnegative functions in the interval $[\rho, R]$ with $g, h$ being increasing. Assume that for $\rho \le s < t \le R$ we have*

$$F(s) \le \varepsilon[F(t) + G(t)] + [(t-s)^{-\alpha}g(t) + h(t)], \tag{3.20}$$
$$G(s) \le C[F(t) + (t-s)^{-\alpha}g(t) + h(t)] \tag{3.21}$$

*with $C \ge 0$, $\alpha, \varepsilon > 0$. If $2C\varepsilon < 1$ then there is constant $c(C,\alpha,\varepsilon)$ such that*

$$F(s) + G(s) \le c(C,\alpha,\varepsilon)[(t-s)^{-\alpha}g(t) + h(t)], \quad \rho \le s < t \le R. \tag{3.22}$$

*Proof.* Let $\varepsilon_0 = 2C\varepsilon$. We obtain from (3.20)

$$CF(s) \le \frac{\varepsilon_0}{2}[F(t) + G(t)] + C[(t-s)^{-\alpha}g(t) + h(t)]. \tag{3.23}$$

Let $t_1 = (s+t)/2$ and use (3.23) with $s$ being $t_1$ and (3.21) with $t$ being $t_1$ to obtain a constant $C_1$ such that

$$G(s) \le \frac{\varepsilon_0}{2}[F(t) + G(t)] + C_1[(t-s)^{-\alpha}g(t) + h(t)]. \tag{3.24}$$

Of course, we can assume that $C \ge 1$ so that (3.23) and (3.24) give

$$F(s) + G(s) \le \varepsilon_0[F(t) + G(t)] + C_1[(t-s)^{-\alpha}g(t) + h(t)].$$

Thus, if $\varepsilon_0 < 1$ or $2C\varepsilon < 1$ then Lemma 3.10 applies with $f(t) = F(t) + G(t)$ to give

$$F(\rho) + G(\rho) \le c(\alpha,\varepsilon)[(R-\rho)^{-\alpha}g(R) + h(R)].$$

Obviously, the above argument holds if we replace the interval $[\rho, R]$ by any subinterval $[s, t]$. The above inequality then gives (3.22). □

*Proof of Proposition 3.1.* For any $R > 0$ we denote ($C_{\lambda,\beta}$ is defined in Lemma 3.9)

$$\varepsilon_0(R) = \mathbf{\Lambda}^2 C_{\lambda,\beta} \sup_{\tau\in[t_0,T_0)} \|U(\cdot,\tau)\|^2_{\mathrm{BMO}(B_R)}. \tag{3.25}$$

Fix some $q_0 > n/2$ as in the proposition and let $\mu_0, R_0 := R_0 > 0$ in (D) be such that

$$C_1(q_0)\varepsilon_0(R_0) = C_1(q_0)C_{\lambda,\beta}\mathbf{\Lambda}^2 \sup_{\tau\in[t_0,T_0)} \|U(\cdot,\tau)\|^2_{\mathrm{BMO}(B_{R_0})} < \frac{1}{2}, \tag{3.26}$$

where $C_1(q_0)$ is the constant in (3.13). We recall that (see Remark 3.6) $C_1(q)$ is increasing in $q$ so that if (3.26) holds then there is $\mu_* \in (0,1)$ such that

$$C_1(q)\varepsilon_0(R) < \frac{\mu_*}{2} < \frac{1}{2}, \quad 1 \le q \le q_0,\ R \in (0, R_0].$$

By (3.19) and the notation (3.25), we have, for any $T \ge 2t_0 > 0$,

$$\mathcal{B}_p(s) + \mathbf{\Lambda}^2\mathcal{C}_p(s) \le \varepsilon_0(R_0)\Big(\mathcal{H}_q(t) + \mathcal{B}_q(t) + \frac{1}{(t-s)^2}\mathcal{G}_q(t)\Big)$$

for all $s, t$ such that $0 < s < t \le R_0$.

On the other hand, if $q$ satisfies (3.10), then (3.13) gives (from now on $C_1 = C_1(q)$)

$$\mathcal{H}_q(s) \le C_1\mathcal{B}_q(t) + \frac{C_1}{(t-s)^2}\mathcal{G}_q(t) + C_1\frac{1}{t_0}\mathcal{I}_q(t), \quad 0 < s < t \le R_0.$$

It is clear that the above two estimates imply (3.20) and (3.21) of Lemma 3.11 with $F(t) = \mathcal{B}_p(t) + \mathbf{\Lambda}^2\mathcal{C}_p(t)$, $G(t) = \mathcal{H}_q(t)$, $g(t) = \mathcal{G}_q(t)$ and $h(t) = t_0^{-1}\mathcal{I}_q(t)$. Thus, the assumption (3.26) on $\varepsilon_0$ and (3.22) of Lemma 3.11 provide a constant $C_2$ depending on $\mu_*$, $C_1$ such that

$$\mathcal{H}_q(s) + \mathcal{B}_p(s) + \mathbf{\Lambda}^2\mathcal{C}_p(s) \le \frac{C_2}{(t-s)^2}\mathcal{G}_q(t) + C_2\frac{1}{t_0}\mathcal{I}_q(t), \quad 0 < s < t \le R_0,$$

or

$$\mathcal{H}_q(s) + \mathcal{B}_p(s) \le \frac{C_2}{(t-s)^2}\mathcal{G}_q(t) + C_2\frac{1}{t_0}\mathcal{I}_q(t), \quad 0 < s < t \le R_0.$$

For $t = 2s$ the above gives (if $q$ satisfies (3.10))

$$\mathcal{H}_q(s) + \mathcal{B}_q(s) \le C_3 \iint\limits_{Q_{2s}} \Big(\frac{1}{s^2}\lambda(W)|DU|^{2q} + \frac{1}{t_0}|DU|^{2q}\Big)\,dz, \quad 0 < s \le \frac{R_0}{2}.$$

Using this estimate for $\mathcal{B}_q(t)$ in (3.13), with $s = R_0/4$ and $t = R_0/2$, respectively, we derive

$$\mathcal{A}_q\Big(\frac{R_0}{4}\Big) + \mathcal{H}_q\Big(\frac{R_0}{4}\Big) \le C_4 \iint\limits_{Q_{R_0/2}} \Big(\frac{1}{R_0^2}\lambda(W)|DU|^{2q} + \frac{1}{t_0}|DU|^{2q}\Big)\,dz. \tag{3.27}$$

Now, we will argue by induction to obtain a bound for $\mathcal{A}_q$ for some $q > n/2$. If some $q$ with $q \ge 1$ satisfies (3.10), then we can find a constant $C_q$ and $t_q \ge t_0$ such that

$$\iint\limits_{\Omega\times[T-2t_q,T_0)} (\lambda(W)\beta(W)|DU|^{2q} + \beta(W)|DU|^{2q})\,dz \le C_q \tag{3.28}$$

and that (3.26) holds. Then (3.27) implies a similar bound for $\mathcal{A}_q(R_1)$, $\mathcal{H}_q(R_1)$, $R_1 = \frac{R_0}{4}$. We now can cover $\Omega$ by $N_{R_1}$ balls $B_{R_1}$, see (3.12), and add up the estimates for $\mathcal{A}_q(R_1)$, $\mathcal{H}_q(R_1)$ to obtain ($t_q$ is $t_0$)

$$\sup_{t\in[T-t_q,T_0)} \int\limits_{\Omega} \beta(W)|DU|^{2q}\,dx + \iint\limits_{\Omega\times[T-2t_q,T_0)} \lambda(W)\beta(W)|DU|^{2q-2}|D^2U|^2\,dz \le C. \tag{3.29}$$

For some $\alpha \in (0,1)$ to be determined later let $p = \alpha q$. By Young's inequality and (3.29), we obtain a constant $C$ such that

$$\iint\limits_{\Omega\times[T-2t_q,T_0)} |DU|^{2p-2}|D^2U|^2\,dz \le C(\alpha) \iint\limits_{\Omega\times[T-2t_q,T_0)} (1 + |DU|^{2q-2})|D^2U|^2\,dz \le C. \tag{3.30}$$

Here, we have used the fact that $\lambda(W)\beta(W)$ is bounded from below so that the second integral in the left-hand side of (3.30) is bounded by the second integral on the left-hand side of (3.29), which also holds for $q = 1$ thanks to our assumption (3.2).

Using Hölder's inequality, the assumption that $\beta(W)$ belongs to $L^r(\Omega)$ for $r$ sufficiently large and the bound in (3.29), we obtain

$$\int_\Omega |DU|^{2p}\,dx \le \Big(\int_\Omega \beta(W)^{\frac{-\alpha}{1-\alpha}}\,dx\Big)^{1-\alpha}\Big(\int_\Omega \beta(W)|DU|^{2q}\,dx\Big)^{\alpha} \le C.$$

By Sobolev's inequality, setting $Q_q = \Omega \times [T - t_q, T_0)$, we get

$$\iint_{Q_q} |DU|^{2p(1+\frac{2}{n})}\,dz \le \Big(\sup_{t\in[T-t_q,T_0)} \int_{B_{R_1}} |DU|^{2p}\,dx\Big)^{\frac{2}{n}} \iint_{Q_q} |DU|^{2p-2}|D^2U|^2\,dz.$$

We derive from the above three estimates that

$$\iint_{Q_q} |DU|^{2p(1+\frac{2}{n})}\,dz \le C.$$

For $p_* \in (1, 1+\frac{2}{n})$ we write $p_* = \gamma + (1-\gamma)(1+\frac{2}{n})$ and use Hölder's inequality to get

$$\iint_{Q_q} \lambda(W)\beta(W)|DU|^{2pp_*}\,dz \le \|\lambda(W)\beta(W)\|_{L^{\frac{1}{\gamma}}(Q_q)}\Big(\iint_{Q_q} |DU|^{2p(1+\frac{2}{n})}\,dz\Big)^{1-\gamma}. \tag{3.31}$$

Since $\lambda(W)$ is bounded from below and $\lambda(W)\beta(W)$ belongs to $L^r(\Omega)$ for $r$ sufficiently large, the above estimate yields

$$\iint_{Q_q} \beta(W)|DU|^{2pp_*}\,dz \le C. \tag{3.32}$$

We now choose and fix $\alpha, \gamma$ such that $pp_* = qq_*$ for some $q_* > 1$. This is the case if $\alpha$ is close to 1 and $\gamma$ is close to 0, so that

$$q_* = \alpha\Big[\gamma + (1-\gamma)\Big(1+\frac{2}{n}\Big)\Big] > 1.$$

From (3.31) and (3.32), we see that (3.28) holds again with the exponent $q$ and the interval $[T - 2t_q, T_0)$ being $qq_*$ and $[T - t_q, T_0)$, respectively.

By our assumption (3.2), estimate (3.28) holds for $q = 1$. For integers $k = 0, 1, 2, \dots$ we define $L_k = q_*^k$ and repeat the argument finitely many times, with the same choice of $\alpha, \gamma$, as long as $L_k$ satisfies (3.10), $q_k \le q_0$ and $(\lambda(W)\beta(W))^{1/(L_k+2)}$ is an $A_p$ weight with $p = \frac{L_k}{L_k+2} + 1$. The last condition holds because $L_k \ge 1$ and because of the assumption that $(\lambda(W)\beta(W))^{\frac{1}{3}}$ is an $A_{\frac{4}{3}}$ weight (see Remark 3.12). We then find an integer $k_0$ such that

$$\sup_{t\in[T-2^{-k}t_q,T_0)} \int_\Omega \beta(W)|DU|^{2L_k}\,dx \le C(C_q, R_0, t_0, N_{R_1}), \quad k = 0, \dots, k_0.$$

Obviously, we can choose $\alpha, \gamma$ such that $L_{k_0} \in (\frac{n}{2}, q_0]$. Now, let $p_0 \in (\frac{n}{2}, L_{k_0})$ and write $p_0 = \alpha_0 L_{k_0}$ for some $\alpha_0 \in (0, 1)$. By Hölder's inequality and the above estimate, if $\tau \ge T - 2^{-k_0}t_q$, then

$$\int_\Omega |DU(x,\tau)|^{2p_0}\,dx \le \Big(\int_\Omega \beta(W)^{\frac{-\alpha_0}{1-\alpha_0}}\,dx\Big)^{1-\alpha_0}\Big(\int_\Omega \beta(W)|DU|^{2L_{k_0}}\,dx\Big)^{\alpha_0} \le C.$$

It is clear from the proof that the integer $k_0$ does not depend on $t_0$ so that we can divide the interval $[T - 2t_0, T - t_0]$ into $k_0$ equal length subintervals and repeat the argument to see that the above estimate holds for $\tau \ge T - t_0$ and $T \ge 2t_0$. This gives (3.11) and the proof is complete. □

**Remark 3.12.** By Hölder's inequality and the definition of $A_p$ weights it is easy to see that if $w$ is an $A_p$ weight for some $p > 1$ then $w^\delta$ is an $A_q$ weight for any $\delta \in (0, 1)$ and $q \in (1, p)$.

# 4 Local and Global Existence of Strong Solutions

In this section, we consider the system

$$\begin{cases} u_t = \operatorname{div}(A(x,t,u,Du)) + \hat{f}(x,t,u,Du) & \text{in } Q = \Omega \times (0,T_0), \\ u(x,0) = U_0(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

and $u$ satisfies homogeneous Dirichlet or Neumann boundary conditions on $\partial\Omega \times (0, T_0)$.

Throughout this section we will assume that $A, \hat{f}$ satisfy 1, (F) and (SG).

We first apply our estimates in the previous section to show that Amann's conditions in [2, 3] can be weakened under some mild extra assumptions which naturally occur in applications. We consider the case when $A(t,x,u,Du)$ is linear in $Du$ and (4.1) satisfies the assumptions in Amann's works (we refer the reader to [2, 3] for the precise statements) so that local existence results hold. We have the following global existence result.

**Theorem 4.1.** *Assume that $A(t,x,u,Du) = A(t,x,u)Du$ for some full $m \times m$ matrix $A(t,x,u)$ satisfying the assumptions in* [2, 3]. *Let* $(0, T_0)$ *be the maximal existence time interval for the solution $u$ of* (4.1). *Assume further that there is a positive $C^1$ function $\beta : \mathbb{R}^m \to \mathbb{R}$ such that the following number is finite:*

$$\Lambda = \sup_{(x,t)\in\Omega\times(0,T_0)} \left\{ \frac{|\lambda_W(x,t,W)|}{\lambda(x,t,W)}, \frac{|\beta_W(W)|}{\beta(W)} \right\}. \tag{4.2}$$

*Suppose that there is a sufficiently large $r > 1$ and a continuous function $C(t)$ such that for a.e. $t \in (0, T_0)$ and $u$ as a function in $x$ the following estimates hold:*

$$\|\beta^{-1}(u)\|_{L^r(\Omega)},\ \|\lambda(x,t,u)\beta(u)\|_{L^r(\Omega)},\ [(\lambda(x,t,u)\beta(u))^{\frac{1}{2}}]_{\frac{4}{3}} \le C(t). \tag{4.3}$$

*In addition, we assume that*

(M) *for any given $\mu_0 > 0$ there is a positive $R_{\mu_0}$, which may also depend on $t_0, T_0$, for which*

$$\Lambda^2 \sup_{x_0\in\bar{\Omega},t\in(0,T_0)} \|u(\cdot,t)\|^2_{\mathrm{BMO}(B_{x_0})} \le \mu_0.$$

*Then $u$ exists globally, i.e. $T_0 = \infty$.*

*Proof.* It is clear that the assumptions of the theorem imply those of Proposition 3.1. The bound (3.11) then shows that the $W^{1,q}$ norm, with some $q > n$, of $u(\cdot,t)$ does not blow up in $(0, T_0)$ so that Amann's results can apply here to give the global existence of $u$. □

On the other hand, if $A$ is nonlinear in $Du$, Amann's results can not apply here and we can alternatively establish local and global existence results for (4.1) using fixed point theories. To this end, we embed the systems (4.1) in the following family of systems with $\sigma \in [0, 1]$:

$$\begin{cases} U_t = \operatorname{div}(\hat{A}_\sigma(x,t,U,DU)) + \hat{F}_\sigma(x,t,U,DU) & \text{in } Q = \Omega \times (0,T_0), \\ U(x,0) = U_0(x) & \text{in } \Omega, \\ U \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega \times (0,T_0). \end{cases} \tag{4.4}$$

We will introduce a family of maps $\mathcal{T}(\sigma,\cdot)$, $\sigma \in [0, 1]$, acting in some suitable Banach space $\mathcal{X}$ such that strong solutions to (4.4) are their fixed points.

In order to define the maps $\mathcal{T}(\sigma,\cdot)$, we will use the notations $\partial_1 g(x,t,u,\zeta)$, $\partial_2 g(x,t,u,\zeta)$ to denote the partial derivatives of a function $g(x,t,u,\zeta)$ with respect to its variables $u, \zeta$.

To begin, let $Q = \Omega \times (0, T_0)$ and $u_0$ be the strong solution to the *linear* parabolic system

$$\begin{cases} (u_0)_t = \operatorname{div}(\partial_2 A(x,t,0,0)Du_0) + \partial_2\hat{f}(x,t,0,0)Du_0 + \partial_1\hat{f}(x,t,0,0)u_0 & \text{in } Q, \\ u_0(x,0) = U_0(x) & \text{in } \Omega, \\ u_0 \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega \times (0,T_0). \end{cases}$$

It is well known that $u_0$ is in $C((0,T_0), C^2(\Omega))$ and $(u_0)_t \in C(Q)$. Furthermore, for any $\tau_0 > 0$ and $\alpha \in (0,1)$, $u_0$ is Hölder continuous in $t$ and $Du_0$ is Hölder continuous in $x$ with any exponent $\alpha \in (0,1)$ in $\Omega \times (\tau_0, T_0)$ for any $\tau_0 > 0$.

Fixing some $\alpha_0 \in (0,1)$ and $\tau_0 > 0$ as in (M), we consider the Banach spaces

$$\mathcal{X}_1 = L^2((0,T_0), W^{1,2}(\Omega)), \quad \mathcal{X}_2 = C(\Omega \times [0,\tau_0]), \quad \mathcal{X}_3 = C^{\alpha_0, \frac{\alpha_0}{2}}(\Omega \times (\tau_0, T_0)),$$

and

$$\mathcal{X} = \{v : v \in \mathcal{X}_1 \cap \mathcal{X}_2,\ Dv \in \mathcal{X}_3\}$$

with norm

$$\|v\|_{\mathcal{X}} = \|v\|_{\mathcal{X}_1} + \|v\|_{\mathcal{X}_2} + \|Dv\|_{\mathcal{X}_3}.$$

Since the dependence of $A, \hat{f}$ on $x, t$ is not important in what follows, we will omit them in the notations and calculation below for the simplicity of our presentation.

For each $v \in \mathcal{X}$ and $\sigma \in [0,1]$, we denote $w = v + u_0$ and define

$$A_\sigma(w) = \int_0^1 \partial_2 A(\sigma w, t\sigma Dw)\,dt, \quad F_{1,\sigma}(w) = \int_0^1 \partial_1 \hat{f}(t\sigma w, 0)\,dt, \quad F_{2,\sigma}(w) = \int_0^1 \partial_2 \hat{f}(\sigma w, t\sigma Dw)\,dt.$$

For any given $v \in \mathcal{X}$ and $w = v + u_0$ let $u = \mathcal{T}(\sigma, v)$ be the weak solution $u$ to the *linear* parabolic system

$$u_t = \operatorname{div}(A_\sigma(w)D(u+u_0)) + F_{2,\sigma}(w)D(u+u_0) + F_{1,\sigma}(w)(u+u_0) - (u_0)_t \tag{4.5}$$

in $Q = \Omega \times (0, T_0)$ and $u$ satisfies the initial and boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times [0, T_0).$$

Clearly, if $u^{(\sigma)}$ is a fixed point of $\mathcal{T}(\sigma, \cdot)$ for some $\sigma \in (0,1]$, i.e. $u^{(\sigma)} = \mathcal{T}(\sigma, u^{(\sigma)})$, then $U = u^{(\sigma)} + u_0$ solves

$$U_t = \operatorname{div}(A_\sigma(U)DU) + F_{2,\sigma}(U)DU + F_{1,\sigma}(U)U. \tag{4.6}$$

We will assume that $A, \hat{f}$ satisfy 1 and (F) so that $A(\sigma U, 0) = 0$ and $\hat{f}(0,0) = 0$. Hence,

$$A_\sigma(U)DU = \int_0^1 \partial_2 A(\sigma U, t\sigma DU)\,dt\,DU = \sigma^{-1}A(\sigma U, \sigma DU) \tag{4.7}$$

and

$$F_{2,\sigma(U)}DU + F_{1,\sigma}(U)U = \int_0^1 \partial_2 \hat{f}(\sigma U, t\sigma DU)\,dt\,DU + \int_0^1 \partial_1 \hat{f}(t\sigma U, 0)\,dt\,U = \sigma^{-1}\hat{f}(\sigma U, \sigma DU). \tag{4.8}$$

Therefore, for $\sigma \in (0,1]$ we will define

$$\begin{aligned} \hat{A}_\sigma(U,\zeta) &= \sigma^{-1}A(\sigma U, \sigma\zeta), & \hat{F}_\sigma(U,\zeta) &= \sigma^{-1}\hat{f}(\sigma U, \sigma\zeta),\\ \hat{A}_0(U,\zeta) &= \partial_2 A(0,0)\zeta, & \hat{F}_0(U,\zeta) &= \partial_2\hat{f}(0,0)\zeta + \partial_1\hat{f}(0,0)U. \end{aligned} \tag{4.9}$$

We then consider the following family of systems for $\sigma \in [0,1]$:

$$\begin{cases} U_t = \operatorname{div}(\hat{A}_\sigma(U, DU)) + \hat{F}_\sigma(U, DU) & \text{in } Q = \Omega \times (0, T_0),\\ U(x,0) = U_0(x) & \text{in } \Omega,\\ U(x,t) = 0 & \text{in } \partial\Omega \times (0, T_0). \end{cases} \tag{4.10}$$

By (4.9), we can see that $u_0$ solves (4.10) for $\sigma = 0$.

**Theorem 4.2.** *We assume that* $A, \hat{f}$ *satisfy 1,* (F) *and* (SG). *For some* $T_0 > 0$ *we assume that there is* $t_0 \in (0, T_0)$ *such that* $\lambda(t, u)$ *is bounded for* $t \in (0, t_0)$. *As in Theorem 4.1, we suppose that the conditions* (4.2), (4.3) *and* (M) *uniformly hold for* $\sigma \in [0, 1]$ *with* $u$ *being a solution* $U$ *of* (4.10). *Namely, there is a sufficiently large* $r > 1$ *and a continuous function* $C(t)$ *such that for a.e.* $t \in (0, T_0)$ *and* $U$ *as a function in* $x$ *the following hold:*

$$\|\beta^{-1}(U)\|_{L^r(\Omega)}, \quad \|\lambda(x,t,U)\beta(U)\|_{L^r(\Omega)}, \quad [(\lambda(x,t,U)\beta(U))^{\frac{1}{2}}]_{\frac{4}{3}} \le C(t),$$

*and*

(M') *for any given* $\mu_0 > 0$ *there is a positive* $R_{\mu_0}$, *which may also depend on* $T_0$, *for which*

$$\mathbf{\Lambda}^2 \sup_{x_0 \in \bar{\Omega}, t \in (0, T_0)} \|U(\cdot, t)\|^2_{\mathrm{BMO}(B_{x_0})} \le \mu_0. \tag{4.11}$$

*Proof.* We will use Leray–Schauder's fixed point index theory to establish the existence of a fixed point of $\mathcal{T}(1,\cdot)$, which is a strong solution to (4.1) and the above theorem then follows. The main ingredient of the proof is to establish a uniform estimate for the fixed points of $\mathcal{T}(\sigma,\cdot)$ in $\mathcal{X}$. To this end, we need a crucial Hölder regularity for these fixed points in $\Omega \times (0, T_0)$. We will make use of Proposition 3.1 which provides such regularity for these fixed points. However, this estimate holds for $t \ge t_0$ if we have some information on their spatial derivatives in early time, namely $[t_0/2, t_0]$, so that the quantities in the energy estimate of Lemma 3.2 are finite. This is the main reason for the assumption that $\lambda(x, t, u)$ is bounded when $t$ is near 0 which together with (M') and the results in [6] will give the needed boundedness of the spatial derivatives near $t = 0$.

We will establish the following facts:
(i) $\mathcal{T}(\sigma,\cdot) : \mathcal{X} \to \mathcal{X}$ is compact for $\sigma \in (0, 1]$.
(ii) $\mathcal{T}(0,\cdot) : \mathcal{X} \to \mathcal{X}$ is a constant map.
(iii) A fixed point $u = \mathcal{T}(\sigma, u)$ is a solution to (4.10). For $\sigma = 1$, such fixed points are solutions to (4.1).
(iv) There is $M > 0$ such that any fixed point $u^{(\sigma)} \in \mathcal{X}$ of $\mathcal{T}(\sigma,\cdot)$, $\sigma \in [0, 1]$, satisfies $\|u^{(\sigma)}\|_{\mathcal{X}} < M$.

Once (i)–(iv) are established, the theorem follows from the Leray–Schauder index theory. Indeed, we let $\mathbf{B}$ be the ball centered at 0 with radius $M$ of $\mathcal{X}$ and consider the Leray–Schauder indices

$$i(\mathcal{T}(\sigma,\cdot), \mathbf{B}, \mathcal{X}) := \deg(\mathrm{Id} - \mathcal{T}(\sigma,\cdot), \mathbf{B}, 0),$$

where the right-hand side denotes the Leray–Schauder degree with respect to zero of the vector field $\mathrm{Id} - \mathcal{T}(\sigma,\cdot)$. This degree is well defined on the closure of the open set $\mathbf{B} \subset \mathcal{X}$ because $\mathcal{T}(\sigma,\cdot)$ is compact (see (i)) and $\mathrm{Id} - \mathcal{T}(\sigma,\cdot)$ does not have zero on $\partial\mathbf{B}$ (see (iv)).

By the homotopy invariance of the indices and (ii), we have

$$i(\mathcal{T}(\sigma,\cdot), \mathbf{B}, \mathcal{X}) = i(\mathcal{T}(0,\cdot), \mathbf{B}, \mathcal{X}) = 1.$$

Thus, $\mathcal{T}(\sigma,\cdot)$ has a fixed point in $\mathbf{B}$ for all $\sigma \in [0, 1]$. Our theorem then follows from (iii).

Using regularity properties of solutions to linear parabolic systems with continuous coefficients (see, e.g., [5]), we see that (i) holds. Checking (ii) and (iii) is fairly standard and straightforward.

To check (iv), let $u^{(\sigma)} \in \mathcal{X}$ be a fixed point of $\mathcal{T}(\sigma,\cdot)$, $\sigma \in [0, 1]$. We need only consider the case $\sigma > 0$. We now denote $W = \sigma(u^{(\sigma)} + u_0)$ and $U = u^{(\sigma)} + u_0$ and need to show that $\|U\|_{\mathcal{X}}$ is uniformly bounded for $\sigma \in [0, 1]$. First of all, the uniform boundedness for $\|U\|_{\mathcal{X}_1}$, or equivalently $\|DU\|_{L^2(Q)}$, is fairly standard. We multiply the systems (4.6) with $U$ and integrate over $Q$. A simple use of integration by parts and Young's inequality shows that $\|DU\|_{L^2(Q)}$ can be estimated by the integrals over $Q$ of $f(W)|U|$. By (F), $|f_u(u)| \le C\lambda(u)$ so that $f(W)|U| \le C\lambda(W)|U|^2$. By our assumptions, $\lambda(W)\beta(W)$ and $\beta^{-1}(W)$ are in $L^r(\Omega)$ for some large $r \ge 1$ with its norms being uniformly bounded, a simple use of Hölder's inequality then shows that $\lambda(W)$ satisfies the same properties. Similarly, $U$ is BMO so that it is in $L^q(\Omega)$ for all $q \ge 1$. Hölder's inequality then gives a uniform bound for the integral of $\lambda(W)|U|^2$ and then of $\|DU\|_{L^2(Q)}$.

Next, as we assume that $\lambda(t, W)$ is bounded and $U$ is VMO in $\Omega \times (0, \tau_0]$, the argument in [6] applies here to show that $U(x, t)$ is uniformly Hölder continuous in $\Omega \times (0, \tau_0]$. Therefore, $\|U\|_{\mathcal{X}_2}$ is uniformly bounded.

Concerning $\|U\|_{\mathcal{X}_3}$, we will show that Proposition 3.1 can be applied to the systems (4.10). As $U = u^{(\sigma)} + u_0$ and $W = \sigma U$, with $u^{(\sigma)} \in \mathcal{X}$, the conditions (U.1) and (U.2) are clearly verified.

From (4.7), (4.8) and the assumption that $A, \hat{f}$ satisfy 1 and (F) we see that (U.0) is verified. Indeed, we will show that $\hat{A}_\sigma(U, \zeta)$ and $\hat{F}_\sigma(U, \zeta)$ satisfy the structural conditions 1 and (F). Firstly,

$$\langle \hat{A}_\sigma(U, \zeta), \zeta\rangle = \langle \sigma^{-1}A(\sigma U, \sigma\zeta), \zeta\rangle = \langle \sigma^{-2}A(\sigma U, \sigma\zeta), \sigma\zeta\rangle \geq \lambda(\sigma U)|\zeta|^2,$$
$$\|\hat{A}_\sigma(U, \zeta)\| = \sigma^{-1}\|A(\sigma U, \sigma\zeta)\| \leq C_*\lambda(\sigma U)|\zeta|,$$
$$\left\|\frac{\partial}{\partial U}\hat{A}_\sigma(U, \zeta)\right\| = \|\partial_1 A(\sigma U, \zeta)\| \sim \lambda(\sigma U)|\zeta|.$$

Therefore $\hat{A}_\sigma$ satisfies 1 with $u = \sigma U$.

Secondly,

$$|\hat{F}_\sigma(U, \zeta)| \leq \sigma^{-1}(\lambda^{\frac{1}{2}}(\sigma U)|\sigma\zeta| + f(\sigma U)) \leq \lambda(\sigma U)|\zeta| + \sigma^{-1}f(\sigma U),$$

and $|\partial_U(\sigma^{-1}f(\sigma U))| = |f_u(\sigma U)| \leq \lambda(\sigma U)$. Here, $f_u(u)$ denotes the derivative of $f(u)$ with respect to its variable $u$. Also,

$$\begin{aligned}|D\hat{F}_\sigma(U, \zeta)| &\leq C\sigma^{-1}\Big(\lambda^{\frac{1}{2}}(\sigma U)|D(\sigma\zeta)| + \frac{\lambda_u(\sigma U)}{\lambda^{\frac{1}{2}}(\sigma U)}|D(\sigma U)||\sigma\zeta| + |f_u(\sigma U)||D(\sigma U)|\Big)\\ &\leq C\Big(\lambda^{\frac{1}{2}}(\sigma U)|D\zeta| + \frac{\lambda_u(\sigma U)}{\lambda^{\frac{1}{2}}(\sigma U)}|DU||\zeta| + |f_u(\sigma U)||DU|\Big).\end{aligned}$$

Hence, $\hat{F}_\sigma(U, \zeta)$ satisfies (F). We see that (U.0) is verified for $U$ and $W = \sigma U$.

In addition, since $u^{(\sigma)} \in \mathcal{X}$, $u^{(\sigma)}$ is bounded and VMO near $t = 0$, the results in [6] show that $u^{(\sigma)}$ is Hölder continuous in $Q$ and $Du^{(\sigma)} \in C^{\beta,\frac{\beta}{2}}(\Omega \times (t_1, T_0))$ for any $t_1 > 0$ and some $\beta \in (0, 1)$. Hence, $u^{(\sigma)} = \mathcal{T}(\sigma, u^{(\sigma)})$ is the solution to the linear system (4.5), whose coefficients with $v$ being $u^{(\sigma)}$ are in $C^1_{\mathrm{loc}}(\Omega \times (0, T_0))$, so that $U = u^{(\sigma)} + u_0$ belongs to $W^{2,2}_{\mathrm{loc}}(Q)$. Therefore, because $W(\cdot, t)$, $U(\cdot, t)$ belong to $C^1(\Omega)$ for $t > 0$, the quantities in the energy estimate of Lemma 3.2 are finite for all $q \geq 1$.

Finally, it is clear that (4.11) in the assumption (M') of our theorem gives the condition (D) of the proposition. More importantly, the uniform bound in (4.11) then gives some positive constants $\mu_0$, $R(\mu_0)$ such that the proposition applies to all $W$, $U$.

Therefore, Proposition 3.1 applies to $W = \sigma(u^{(\sigma)} + u_0)$, $U = u^{(\sigma)} + u_0$ and gives a uniform estimate for $\|u^{(\sigma)}(\cdot, t)\|_{W^{1,2q}(\Omega)}$ for some $q > n/2$ and all $t \in (t_0, T_0)$ and $\sigma \in [0, 1]$. By Sobolev's imbedding theorems this shows that $U$ is Hölder continuous with its norm uniformly bounded with respect to $\sigma \in [0, 1]$. Again, the results in [6] imply that $Du^{(\sigma)} \in C^{\alpha,\frac{\alpha}{2}}(\Omega \times (\tau_0, T_0))$ for any $\alpha \in (0, 1)$ and its norm is uniformly bounded. We then obtain a uniform estimate for $\|u^{(\sigma)}\|_{\mathcal{X}_3}$ and (iv) is verified. The proof of Theorem 4.2 is complete. □

**Remark 4.3.** We applied Proposition 3.1 to strong solutions in the space $\mathcal{X}$ so that $U$, $DU$ are bounded and the key quantities $\mathcal{B}$, $\mathcal{H}$ are finite. However, the bound provided by the proposition did not involve the supremum norms of $U$, $DU$ but the BMO norm of $U$ in (M') and the constants in 1 and (F).

We conclude this paper by considering the case when $\lambda(x, t, u) \sim (\lambda_0 + |u|)^M$ for some $\lambda_0, M > 0$ when $t$ is large. We will first show that the conditions, with the exception of (M'), are easily verifiable if the solutions are uniformly BMO. Condition (M') will be discussed in Remark 4.5.

We then recall the following result from [9, Theorem 6] on the connection between BMO functions and weights: *Let $\Psi$ be a positive function such that $\Psi$, $\Psi^{-1}$ are BMO. Then $\Psi$ belongs to $\bigcap_{\gamma>1} A_\gamma$ and $[\Psi]_\gamma$ is bounded by a constant depending on $[\Psi]_{\mathrm{BMO}}$ and $[\Psi^{-1}]_{\mathrm{BMO}}$.*

**Theorem 4.4.** *Assume that $\lambda(x, t, u)$ is bounded in $(0, t_0]$ for some $t_0 > 0$ and $\lambda(x, t, u) \sim (\lambda_0 + |u|)^M$ for $t \geq t_0$ and some $\lambda_0, M > 0$. Suppose that $\|U(\cdot, t)\|_{\mathrm{BMO}(\Omega)}$ is bounded on $(0, T_0)$ and for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that*

$$\lambda_0^{-1}\|U(\cdot, t)\|_{\mathrm{BMO}(B_{R_\varepsilon})} \leq \varepsilon \quad \text{for all } B_{R_\varepsilon} \subset \Omega \text{ and } t \in (0, T_0).$$

*Then there is a strong solution in $\Omega \times (0, T_0)$.*

*Proof.* As in the proof of Theorem 4.2, we need only to show that Proposition 3.1 can apply for $t \geq t_0$. For $W = \sigma U$ with $\sigma \in [0, 1]$ assumptions (U.0)–(U.2) are clearly satisfied. We will show that the condition (U.3) holds here. To this end, we choose $\beta(W) \sim (\lambda_0 + |W|)^{-M+2\varepsilon_0}$ with $\varepsilon_0 \in (0, 1)$. If $M \leq 2$, we can take $\beta(W) \equiv 1$. It is clear that the constant $\mathbf{\Lambda} \sim \lambda_0^{-1}$.

Let $w := (\lambda(W)\beta(W))^{\frac{1}{2}}$. Since $w \sim (\lambda_0 + |W|)^{\varepsilon_0}$, with $\varepsilon_0 \in (0, 1)$, the assumption that $W$ is BMO implies that $w$ is BMO. Also, $w^{-1}$ is BMO because $w$ is bounded from below. By the aforementioned result in [9], $w$ is an $A_p$ weight for all $p > 1$. Therefore, $w = (\lambda(W)\beta(W))^{\frac{1}{2}}$ is in $A_{\frac{4}{3}}$ class. On the other hand, it is well known that if $W$ belongs to the BMO space then it belongs to $L^p(\Omega)$ for any $p > 1$. Here, $\beta^{-1}(W)$ and $\lambda(W)\beta(W)$ have polynomial growth in $W$ so that they also belong to $L^p(\Omega)$ for any $p > 1$. We have shown that (U.3) is verified.

Moreover, it is easy to see that (3.28) holds for $q = 1$ by testing the system with $U$ and then using the fact that $\beta(W)$ is bounded from above. The proof of Theorem 4.4 is complete. □

**Remark 4.5.** To establish the uniform continuity condition (M') one can try to establish a uniform boundedness of $\|DU\|_{L^n(\Omega)}$ and apply Poincaré's inequality to see that $U$ is VMO. If this can be done then one can argue by contradiction to obtain (M'). We sketch the idea of the proof here. If (M') is not true then along a sequence $\sigma_n, t_n, r_n, r_n > 0$, $U_n(\cdot) = U(\cdot, t_n)$ converge weakly to some $U$ in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ but $\|U_n\|_{\mathrm{BMO}(B_{r_n})} > \varepsilon_0$ for some $r, \varepsilon_0 > 0$. We then have $\|U_n\|_{B_R} \to \|U\|_{B_R}$ for any given $R > 0$. It is not difficult to see that $DU \in L^n$ so that $U$ satisfies (M'). Furthermore, if $r_n < R$ then $\|U_n\|_{\mathrm{BMO}(B_r)} \leq \|U_n\|_{\mathrm{BMO}(B_R)}$. Choosing $R$ sufficiently small and letting $n$ tend to infinity, we obtain a contradiction.

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