Research Article

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Traveling Wave Solutions to the Burgers-$\alpha\beta$ Equations

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Abstract: The Burgers-$\alpha\beta$ equation, which was first introduced by Holm and Staley [4], is considered in the special case where $\nu = 0$ and $b = 3$. Traveling wave solutions are classified to the Burgers-$\alpha\beta$ equation containing four parameters b, α, ν, and β, which is a nonintegrable nonlinear partial differential equation that coincides with the usual Burgers equation and viscous b-family of peakon equation, respectively, for two specific choices of the parameter $\beta = 0$ and $\beta = 1$. Under the decay condition, it is shown that there are smooth, peaked and cusped traveling wave solutions of the Burgers-$\alpha\beta$ equation with $\nu = 0$ and $b = 3$ depending on the parameter β. Moreover, all traveling wave solutions without the decay condition are parametrized by the integration constant $k_1 \in \mathbb{R}$. In an appropriate limit $\beta = 1$, the previously known traveling wave solutions of the Degasperis–Procesi equation are recovered.

Keywords: Traveling Wave Solutions, Burgers-$\alpha\beta$ Equation, Degasperis–Procesi Equation

MSC 2010: 35Q35, 35Q53, 35B65, 37K45

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1 Introduction

Consider the Burgers-$\alpha\beta$ equation [4] with $\nu = 0$ and $b = 3$:

$$u_t - \alpha^2 u_{txx} + (1 + 3\beta)uu_x = \alpha^2(3u_x u_{xx} + uu_{xxx}), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where the subscripts denote the partial derivatives with respect to the spatial coordinate x and temporal coordinate t, and α, β are real parameters. Indeed, using the transformations $t \mapsto \frac{1}{\alpha}t$ and $x \mapsto \frac{1}{\alpha}x$, we can rewrite (1.1) as

$$u_t - u_{txx} + (1 + 3\beta)uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

It is easy to see that (1.2) is the more general case compared to the Degasperis–Procesi equation. The Degasperis–Procesi equation is a special case of (1.2) with $\beta = 1$. The formal integrability of the Degasperis–Procesi equation was obtained in [2] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to the Camassa–Holm peakons [2]. The Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm shallow water equation [1, 3]. An inverse scattering approach for computing N-peakon solutions of the Degasperis–Procesi equation was presented in [8]. Its traveling wave solutions were investigated in [6, 9].

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Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $u = (1 - \partial_x^2)^{-1}m = p * m$, where $m := u - u_{xx}$ and $*$ denotes the convolution product on $\mathbb{R}$, given by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y)dy.$$

This formulation allows us to define a weak form of (1.2) as follows:

$$u_t + \partial_x\Big(\frac{1}{2}u^2 + p * \Big(\frac{3\beta}{2}u^2\Big)\Big) = 0, \quad x \in \mathbb{R},\ t > 0. \tag{1.3}$$

We also note that the peaked solitons are not classical solutions of (1.2) with $\beta = 1$. They satisfy the Degasperis–Procesi equation in the weak form (1.3) with $\beta = 1$.

Recently, Holm and Staley in [4] studied traveling wave solutions of the Burgers-$\alpha\beta$ equation with $(3 - b)\beta = 1$ and $\nu = 0$. The aim of the present paper is to classify all weak traveling wave solutions of the Burgers-$\alpha\beta$ equation with $b = 3$ and $\nu = 0$ by the idea used in [5, 7].

Our main results of this paper are Theorem 2.3 (traveling wave solution with decay; Figures 1 and 2) and Theorems 3.1–3.4 (traveling wave solution without decay; Figures 3, 4 and 5)

This paper is organized as follows. In Section 2, we classify the traveling wave solutions of (1.2) under the decay condition. In particular, we show the existence of peaked traveling wave solutions for $\beta = 1$. In Section 3, we categorize the traveling wave solutions of (1.2) without the decay condition, by using an analogous analysis in [6]. Finally, we give our concluding remarks in Section 4.

2 Traveling Wave Solutions With the Decay Condition

In this section, we study all weak traveling wave solutions of (1.2), i.e. solutions of the form

$$u(t, x) = \varphi(x - ct), \quad c \in \mathbb{R} \tag{2.1}$$

for some $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \to 0$ as $|x| \to \infty$. Note that if $\varphi(x - ct)$ is a traveling wave solution of (1.2), then $-\varphi(x + ct)$ is also a traveling wave solution of (1.2). Substituting (2.1) into (1.2) and integrating it, we have

$$(c - \varphi)\varphi_{xx} - \varphi_x^2 = c\varphi - \frac{1 + 3\beta}{2}\varphi^2, \tag{2.2}$$

We rewrite (2.2) as

$$\frac{1}{2}[(\varphi - c)^2]_{xx} = -c\varphi + \frac{1 + 3\beta}{2}\varphi^2.$$

Now we give the definition of a traveling wave solution of (1.2).

Definition 2.1. A function $\varphi(x - ct) \in H^1(\mathbb{R})$ is a nontrivial traveling wave solution of (1.2) with $c \in \mathbb{R}$ and $\varphi \to 0$ as $|x| \to \infty$.

The following lemma deals with the regularity of the traveling wave solutions. The idea is inspired by the study of the traveling waves of the Camassa–Holm equation [5].

Lemma 2.2. *Let $\varphi(x - ct)$ be a traveling wave solutions of* (1.2). *Then*

$$(\varphi - c)^k \in C^j(\mathbb{R} \setminus \varphi^{-1}(c)), \quad k \geq 2^j. \tag{2.3}$$

Therefore

$$\varphi \in C^\infty(\mathbb{R} \setminus \varphi^{-1}(c)).$$

Proof. Let $v = \varphi - c$ and denote $P(v) = -2c(v + c) + (1 + 3\beta)(v + c)^2$. So $P(v)$ is a polynomial in v. Then v satisfies

$$(v^2)_{xx} = P(v).$$

Since $v \in H^1(\mathbb{R})$, we know that $(v^2)_{xx} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Hence $(v^2)_x$ is absolutely continuous and $v^2 \in C^1(\mathbb{R})$. Then $v \in C^1(\mathbb{R} \setminus v^{-1}(0))$. Moreover, we have

$$\begin{aligned}(v^k)_{xx} = (kv^{k-1}v_x)_x = \frac{k}{2}\big(v^{k-2}(v^2)_x\big)_x &= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}(v^2)_{xx}\\ &= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}P(v).\end{aligned} \tag{2.4}$$

For $k = 3$, the right-hand side of (2.4) is in $L^1_{\mathrm{loc}}(\mathbb{R})$. Thus we conclude that $v^3 \in C^1(\mathbb{R})$. For $k \geq 2^2 = 4$ we know that (2.4) implies

$$(v^k)_{xx} = \frac{k}{4}(k-2)v^{k-4}\big[(v^2)_x\big]^2 + \frac{k}{2}v^{k-2}P(v) \in C^1(\mathbb{R}).$$

Therefore $v^k \in C^1(\mathbb{R})$ for $k \geq 2^2 = 4$.

For $k \geq 2^3 = 8$ we see from the above that $v^4, v^{k-4}, v^{k-2}, v^{k-2}P(v) \in C^2(\mathbb{R})$. Moreover, we have

$$v^{k-2}v_x^2 = \frac{1}{4}(v^4)_x\frac{1}{k-4}(v^{k-4})_x \in C^1(\mathbb{R}).$$

Hence from (2.4) we deduce that

$$v^k \in C^3(\mathbb{R} \setminus v^{-1}(0)), \quad k \geq 2^3 = 8.$$

Applying the same argument to higher values of k we prove that $v^k \in C^j(\mathbb{R} \setminus v^{-1}(0))$ for $k \geq 2^j$, and hence (2.3). This completes the proof of Lemma 2.2. □

Let $\varphi_x = G$. Then (2.2) becomes

$$(G^2)_\varphi - \frac{2}{c-\varphi}G^2 = \frac{2(c\varphi - \frac{1+3\beta}{2}\varphi^2)}{c-\varphi}. \tag{2.5}$$

Solving the first-order ordinary differential equation (2.5), we have

$$\varphi_x^2 = \frac{\varphi^2\big[\frac{1+3\beta}{4}\varphi^2 - (1+\beta)c\varphi + c^2\big]}{(\varphi-c)^2} := F(\varphi). \tag{2.6}$$

Consider the polynomial

$$P(\varphi) = \varphi^2\Big[\frac{1+3\beta}{4}\varphi^2 - (1+\beta)c\varphi + c^2\Big] \tag{2.7}$$

with a double root at $\varphi = 0$. Then we can classify all traveling wave solutions of (1.2) depending on the different behaviors of this polynomial.

If $\beta = -\frac{1}{3}$, then we know that $P(\varphi)$ is the third-degree polynomial with a double zero at $\varphi = 0$ and a simple zero at $\varphi = \frac{3}{2}c$ such that $P(\varphi) = -\frac{2}{3}c\varphi^2(\varphi - \frac{3}{2}c)$.

If $\beta \neq -\frac{1}{3}$, then $P(\varphi)$ is the fourth-degree polynomial with a double zero at $\varphi = 0$ and there are the three cases

(i) $\Delta < 0$,
(ii) $\Delta = 0$,
(iii) $\Delta > 0$,

where $\Delta := c^2\beta(\beta-1)$ is the determinant of $\frac{1+3\beta}{4}\varphi^2 - (1+\beta)c\varphi + c^2$.

(i) $\Delta < 0$: For $0 < \beta < 1$, we have that $P(\varphi)$ is the fourth-degree polynomial with a double zero at $\varphi = 0$.

(ii) $\Delta = 0$:

- For $\beta = 0$, $P(\varphi) = \varphi^2(\frac{1}{2}\varphi - c)^2$ is the fourth-degree polynomial with a double zero at $\varphi = 0$ and $\varphi = 2c$.
- For $\beta = 1$, $P(\varphi) = \varphi^2(\varphi - c)^2$ is the fourth-degree polynomial with a double zero at $\varphi = 0$ and $\varphi = c$.

(iii) $\Delta > 0$: For $\beta < 0$ or $\beta > 1$, $P(\varphi) = \frac{1+3\beta}{4}\varphi^2(c - l_1 - \varphi)(c - l_2 - \varphi)$ is the fourth-degree polynomial with a double zero at $\varphi = 0$ and a simple zero at $\varphi = c - l_1$ or $\varphi = c - l_2$, where

$$l_1 = c\left[\frac{\beta-1}{1+3\beta} + \sqrt{\Big(\frac{\beta-1}{1+3\beta}\Big)^2 + \frac{\beta-1}{1+3\beta}}\right], \quad l_2 = c\left[\frac{\beta-1}{1+3\beta} - \sqrt{\Big(\frac{\beta-1}{1+3\beta}\Big)^2 + \frac{\beta-1}{1+3\beta}}\right]$$

for $c > 0$ and

$$l_1 = c\left[\frac{\beta-1}{1+3\beta} - \sqrt{\left(\frac{\beta-1}{1+3\beta}\right)^2 + \frac{\beta-1}{1+3\beta}}\right], \quad l_2 = c\left[\frac{\beta-1}{1+3\beta} + \sqrt{\left(\frac{\beta-1}{1+3\beta}\right)^2 + \frac{\beta-1}{1+3\beta}}\right]$$

for $c < 0$ are two roots of the equation

$$y^2 - \frac{2(\beta-1)}{1+3\beta}cy - \frac{\beta-1}{1+3\beta}c^2 = 0.$$

If $\beta < -\frac{1}{3}$ then

$$l_2 < 0 < c < l_1 \quad \text{or} \quad l_2 < c < 0 < l_1. \tag{2.8}$$

If $-\frac{1}{3} < \beta < 0$ then

$$l_2 < l_1 < 0 < c \quad \text{or} \quad c < 0 < l_2 < l_1. \tag{2.9}$$

If $\beta > 1$ then

$$l_2 < 0 < l_1 < c \quad \text{or} \quad c < l_2 < 0 < l_1.$$

Using the idea as introduced in [5] gives us the following conclusions:

(a) Assume $F(\varphi)$ has a simple zero at $\varphi = m$ so that $F(m) = 0$ and $F'(m) \neq 0$. The solution φ of (2.6) satisfies

$$\varphi_x^2 = (\varphi - m)F'(m) + O((\varphi - m)^2) \quad \text{as } x \to m,$$

where $f = O(g)$ as $x \to a$ means that $\left|\frac{f(x)}{g(x)}\right|$ is bounded in some interval $[a - \epsilon, a + \epsilon]$ with $\epsilon > 0$. Therefore

$$\varphi(x) = m + \frac{1}{4}(x - \xi)^2 F'(m) + O((x - \xi)^4) \quad \text{as } x \to \xi, \tag{2.10}$$

where $\varphi(\xi) = m$.

(b) If $F(\varphi)$ has a double zero at m, so that $F'(m) = 0$, $F''(m) > 0$, then

$$\varphi_x^2 = (\varphi - m)^2 F''(m) + O((\varphi - m)^3) \quad \text{as } \varphi \to m.$$

We obtain

$$\varphi(x) - m \sim \alpha \exp\left(-x\sqrt{F''(m)}\right) \quad \text{as } x \to \infty \tag{2.11}$$

for some constant α. Thus $\varphi \to m$ exponentially as $x \to \infty$.

(c) If φ approaches a double pole $\varphi(x_0) = c$ of $F(\varphi)$. Then

$$\varphi(x) - c = \alpha|x - x_0|^{1/2} + O((x - x_0)^{3/2}) \quad \text{as } x \to x_0, \tag{2.12}$$

$$\varphi_x = \begin{cases} \frac{1}{2}\alpha|x - x_0|^{-1/2} + O((x - x_0)^{1/2}) & \text{as } x \downarrow x_0, \\ -\frac{1}{2}\alpha|x - x_0|^{-1/2} + O((x - x_0)^{1/2}) & \text{as } x \uparrow x_0 \end{cases} \tag{2.13}$$

for some constant α. In particular, when F has a double pole, the solution φ has a cusp.

(d) If the evolution of φ according to (2.6) suddenly changes direction $\varphi_x \mapsto -\varphi_x$, then peaked solitary waves occur.

In view of (a)–(d), we give the following theorem on all bounded traveling wave solutions of (1.2) with decay.

Theorem 2.3. *Any bounded traveling wave of* (1.2) *with decay belongs to one of the following categories.*

(1) *For* $\beta < -\frac{1}{3}$*:*
- *If* $c > 0$*, then there is a smooth traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c - l_1$ *and a cusped traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is a smooth traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c - l_2$ *and an anticusped traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(2) *For* $\beta = -\frac{1}{3}$*:*
- *If* $c > 0$*, then there is a cusped traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is an anticusped traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(3) *For* $-\frac{1}{3} < \beta < 0$*:*
- *If* $c > 0$*, then there is a cusped traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is an anticusped traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(4) *For* $\beta = 0$*:*
- *If* $c > 0$*, then there is a cusped traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is an anticusped traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(5) *For* $0 < \beta < 1$*:*
- *If* $c > 0$*, then there is a cusped traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is an anticusped traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(6) *For* $\beta = 1$*:*
- *If* $c > 0$*, then there is a peaked traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c$.
- *If* $c < 0$*, then there is an antipeaked traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c$.

(7) *For* $\beta > 1$*:*
- *If* $c > 0$*, then there is a smooth traveling wave with decay* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = c - l_1$.
- *If* $c < 0$*, then there is a smooth traveling wave with decay* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = c - l_2$.

Proof. First, we consider $c > 0$. The other case $c < 0$ can be handled in a very similar way.

If $\beta = -\frac{1}{3}$, then (2.6) becomes

$$\varphi_x^2 = \frac{-\frac{2}{3}c\varphi^2(\varphi - \frac{3}{2}c)}{(\varphi - c)^2} := F_1(\varphi).$$

When $c > 0$, $\varphi = c$ is a double pole of $F_1(\varphi)$. Hence from (2.12) and (2.13) we see that we obtain a traveling wave with cusp at $\varphi = c$, which decays exponentially.

If $0 < \beta < 1$ and $\beta = 0$, then $\varphi = c$ is a double pole of $F(\varphi)$ and $\varphi = 0$ is a double zero of $F(\varphi)$ in (2.6). Therefore, in the same manner as above, we obtain a cusped traveling wave with $\max_{x\in\mathbb{R}} \varphi(x) = c$ for $c > 0$. If $\beta = 1$, then (2.6) becomes

$$\varphi_x^2 = \frac{\varphi^2(\varphi - c)^2}{(\varphi - c)^2} := F_2(\varphi).$$

When $c > 0$, the smooth solution can be constructed until $\varphi = c$. But it can make a sudden turn at $\varphi = c$ and so give rise to a peak. Since $\varphi = 0$ is still a double zero of $F_2(\varphi)$, we still have the exponential decay.

If $\beta < -\frac{1}{3}$, $-\frac{1}{3} < \beta < 0$, and $\beta > 1$, then (2.6) becomes

$$\varphi_x^2 = \frac{(1 + 3\beta)\varphi^2(c - \varphi - l_1)(c - \varphi - l_2)}{4(\varphi - c)^2} := F_3(\varphi).$$

When $\beta < -\frac{1}{3}$, we know that $l_2 < 0 < c < l_1$ from (2.8). $F_3(\varphi)$ has a simple zero at $\varphi = c - l_1 < 0$ and a double zero at $\varphi = 0$. Therefore from (2.10) and (2.11) we see that in this case we can obtain a smooth traveling wave with $\min_{x\in\mathbb{R}} \varphi(x) = c - l_1$ and an exponential decay to zero at infinity. Moreover, since $F_3(\varphi)$ has a double pole at $\varphi = c$, we can also obtain a cusped traveling wave with $\max_{x\in\mathbb{R}} \varphi(x) = c$, which decays exponentially.

If $-\frac{1}{3} < \beta < 0$, we know that $l_2 < l_1 < 0 < c$ from (2.9). In this case $F_3(\varphi)$ has a double pole at $\varphi = c$ and a double zero at $\varphi = 0$. Hence we obtain a cusped traveling wave with $\max_{x\in\mathbb{R}} \varphi(x) = c$, which decays exponentially.

If $\beta > 1$, we see that $l_2 < 0 < l_1 < c$ from (2.9). $F_3(\varphi)$ has a simple zero at $\varphi = c - l_1 > 0$ and a double zero at $\varphi = 0$. Therefore from (2.10) and (2.11) we see that in this case we can obtain a smooth traveling wave with $\max_{x\in\mathbb{R}} \varphi(x) = c - l_1$ and an exponential decay to zero at infinity. This completes the proof of Theorem 2.3. □

3 Traveling Wave Solutions Without the Decay Condition

In this section, we consider all weak traveling wave solutions of (1.2), i.e. solutions of the form

$$u(t, x) = \varphi(x - ct), \quad c \in \mathbb{R} \tag{3.1}$$

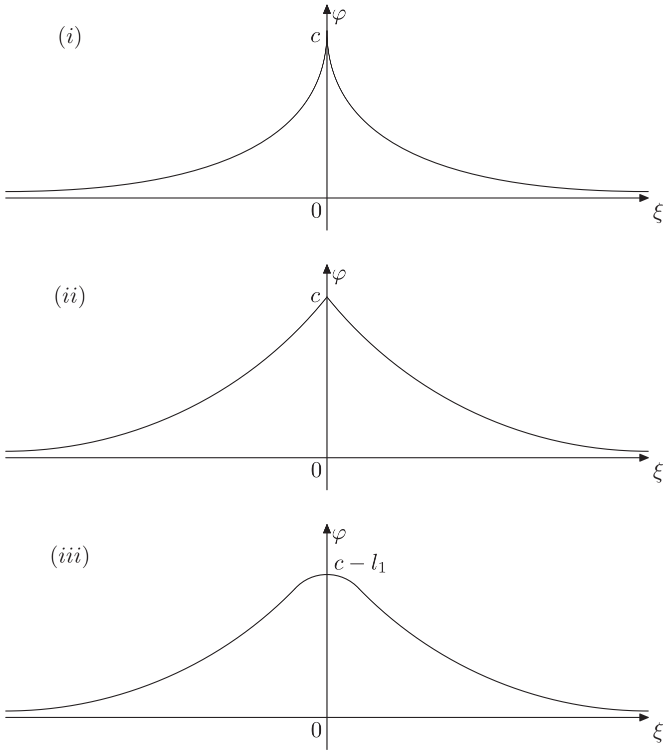

Figure 1. There are three different kinds of traveling wave solutions with decay for $c > 0$ in Theorem 2.3.
(i) Cusped traveling waves with $\max_{x\in\mathbb{R}} \varphi(x) = c$ for $\beta < 1$.
(ii) Peaked traveling waves with $\max_{x\in\mathbb{R}} \varphi(x) = c$ for $\beta = 1$.
(iii) Smooth traveling waves with $\max_{x\in\mathbb{R}} \varphi(x) = c - l_1$ for $\beta > 1$.

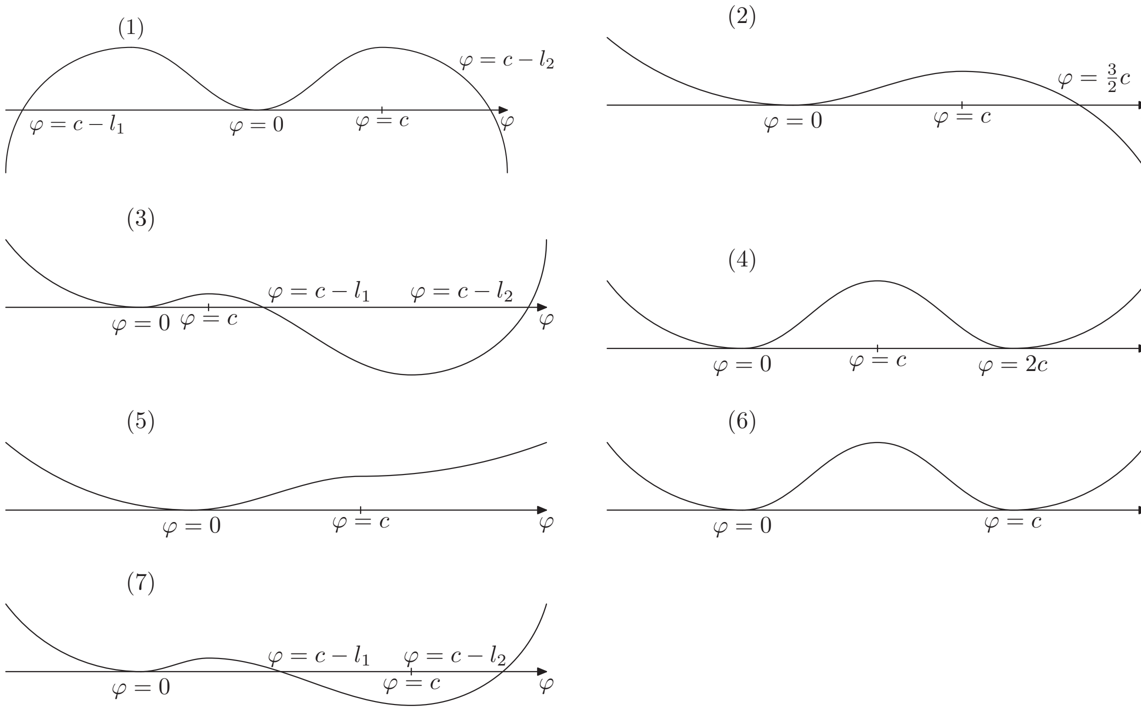

Figure 2. The graph of the polynomial (2.7) displayed for different values of β. The seven cases give rise to the categories (1)–(7) in Theorem 2.3.

without the decay condition at infinity. Note that if $\varphi(x-ct)$ is a traveling wave solution of (1.2), then $-\varphi(x+ct)$ is also a traveling wave solution of (1.2). Thus we only consider traveling wave with a positive speed $c>0$. Substituting (3.1) into (1.2) and integrating it, we have

$$(c-\varphi)\varphi_{xx}-\varphi_x^2=c\varphi-\frac{1+3\beta}{2}\varphi^2+k_1, \tag{3.2}$$

where k_1 is an integration constant. Let $\varphi_x=G$. Then (3.2) becomes

$$(G^2)_\varphi-\frac{2}{c-\varphi}G^2=\frac{2(c\varphi-\frac{1+3\beta}{2}\varphi^2+k_1)}{c-\varphi}. \tag{3.3}$$

Solving the first-order ordinary differential equation (3.3), we have

$$\varphi_x^2=\frac{(\varphi-c)^2[\frac{1+3\beta}{4}\varphi^2+\frac{\beta-1}{2}c\varphi+\frac{\beta-1}{4}c^2-k_1]+k_2}{(\varphi-c)^2}:=H(\varphi),$$

where k_2 is an integration constant.

Consider the polynomial

$$P_1(\varphi)=(\varphi-c)^2\Big[\frac{1+3\beta}{4}\varphi^2+\frac{\beta-1}{2}c\varphi+\frac{\beta-1}{4}c^2-k_1\Big]$$

with a double root at $\varphi=c$. Then we can classify all traveling wave solutions of (1.2) depending on the different behaviors of this polynomial. Once k_1 is fixed, a change in k_2 will shift the graph vertically up or down. Hence we can easily determine which k_2 yield bounded traveling waves. There are qualitatively different cases.

Case 1. We consider $\beta=-\frac{1}{3}$. Then $P_1(\varphi)=(\varphi-c)^2(-\frac{2}{3}c\varphi-\frac{1}{3}c^2-k_1)$ becomes a third-degree polynomial with a double zero at $\varphi=c$.

Arguments similar to the ones in [5] and (2.10)–(2.13) give us the following theorem for $\beta=-\frac{1}{3}$.

Theorem 3.1. *Let $\beta=-\frac{1}{3}$ and $c>0$. Any bounded traveling wave of (1.2) belongs to one of the following categories. The waves are parametrized by $k_1\in\mathbb{R}$ as follows:*

(1) *If $k_1\le -c^2$, then there are no bounded solutions of (1.2).*

(2) *If $k_1>-c^2$, then there exist a one-parameter group of cusped periodic traveling waves and one cusped traveling wave with decay.*

Proof. If $\beta=-\frac{1}{3}$, we have

$$P_1(\varphi)=-\frac{2}{3}c(\varphi-c)^2\Big(\varphi+\frac{1}{2}c+\frac{3k_1}{2c}\Big)=-\frac{2}{3}c(\varphi-c)^3-(c^2+k_1)(\varphi-c)^2. \tag{3.4}$$

Since $c>0$ and $k_1\le -c^2$, we see that

$$P_1(\varphi)'=-2c(\varphi-c)^2-2(\varphi-c)(c^2+k_1)<0 \quad \text{for } \varphi<c.$$

Therefore, $P_1(\varphi)$ is decreasing for $\varphi<c$. There are no bounded solutions for any k_2.

If $-c^2<k_1<-\frac{c^2}{3}$, $k_1=-\frac{c^2}{3}$, and $k_1>-\frac{c^2}{3}$, then $P_1(\varphi)$ has a double zero at $\varphi=c$ and a simple zero at $\varphi=-\frac{c^2+3k_1}{2c}>0$, $\varphi=-\frac{c^2+3k_1}{2c}=0$, and $\varphi=-\frac{c^2+3k_1}{2c}<0$, respectively. Hence there are cusped traveling waves for some $k_2>0$. This completes the proof of Theorem 3.1. □

Case 2. Consider $\beta\neq-\frac{1}{3}$. We know that

$$P_1(\varphi)=(\varphi-c)^2\Big[\frac{1+3\beta}{4}\varphi^2+\frac{\beta-1}{2}c\varphi+\frac{\beta-1}{4}c^2-k_1\Big] \tag{3.5}$$

is a fourth-degree polynomial. We distinguish two cases:

(i) $\beta>-\frac{1}{3}$,

(ii) $\beta<-\frac{1}{3}$.

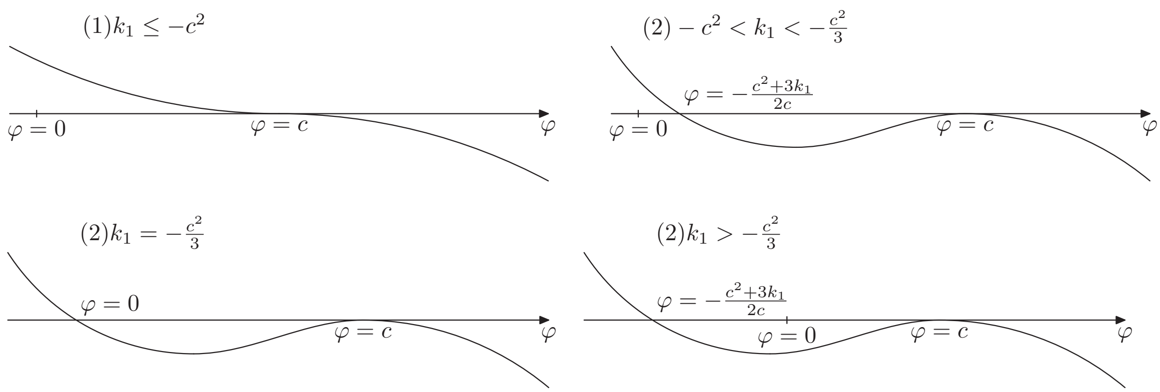

Figure 3. The graph of the polynomial (3.4) displayed for different values of k_1. The four cases give rise to the categories (1)–(2) in Theorem 3.1.

We need the determinant $\Delta := \frac{1-\beta^2}{2}c^2 + (1+3\beta)k_1$ of the second-degree polynomial

$$\frac{1+3\beta}{4}\varphi^2 + \frac{\beta-1}{2}c\varphi + \frac{\beta-1}{4}c^2 - k_1.$$

(i) $\beta > -\frac{1}{3}$:

$$\Delta > 0\colon k_1 > \frac{\beta^2-1}{2(1+3\beta)}c^2, \quad \Delta = 0\colon k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2, \quad \Delta < 0\colon k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2.$$

(ii) $\beta < -\frac{1}{3}$:

$$\Delta > 0\colon k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2, \quad \Delta = 0\colon k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2, \quad \Delta < 0\colon k_1 > \frac{\beta^2-1}{2(1+3\beta)}c^2.$$

The idea as introduced in [5] and (2.10)–(2.13) give us the following theorems.

Theorem 3.2. *Let $\beta > -\frac{1}{3}$ and $c > 0$. Any bounded traveling wave of (1.2) belongs to one of the following categories. The waves are parametrized by $k_1 \in \mathbb{R}$ as follows:*

(1) *If $k_1 \le -\frac{c^2}{2(1+3\beta)}$, then there are no bounded solutions of (1.2).*
(2) *If $-\frac{c^2}{2(1+3\beta)} < k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2$, then there exist a one-parameter group of smooth periodic traveling waves and one smooth traveling wave with decay.*
(3) *Let $k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2$.*
   - *If $\beta > 0$, then there exist a one-parameter group of smooth periodic traveling waves and one peaked solitary wave.*
   - *If $-\frac{1}{3} < \beta \le 0$, then there are no bounded solutions of (1.2).*
(4) *If $\frac{\beta^2-1}{2(1+3\beta)}c^2 < k_1 < \frac{3\beta-1}{2}c^2$, then there exist a one-parameter group of smooth periodic traveling waves, one peaked periodic traveling wave, a one-parameter group of cusped periodic traveling waves, and one cusped traveling wave with decay.*
(5) *Let $k_1 = \frac{3\beta-1}{2}c^2$.*
   - *If $\beta > 0$, then there exist a one-parameter group of cusped periodic traveling waves and one cusped traveling wave with decay.*
   - *If $-\frac{1}{3} < \beta \le 0$, then there are no bounded solutions of (1.2).*
(6) *If $k_1 > \frac{3\beta-1}{2}c^2$, then there exist a one-parameter group of cusped periodic traveling waves, one cusped traveling with decay, a one-parameter group of anticusped periodic traveling waves, and one anticusped traveling wave with decay.*

(C) *(Composite waves) Any countable number of cuspons, anticuspons, and peakons from the categories* (1)–(6) *corresponding to the same value of* k_1 *may be joined at points where* $\varphi = c$ *to form a composite wave* φ. *If the Lebesgue measure* $\mu(\varphi^{-1}(c))$ *equals* 0, *then* φ *is a traveling wave of* (1.2).

(S) *(Stumpons) For* $k_1 = \frac{3\beta-1}{2}c^2$ *the composite waves are traveling waves of* (1.2) *even if the Lebesgue measure* $\mu(\varphi^{-1}(c))$ *exceeds* 0. *Consequently, these waves may contain intervals where* $\varphi \equiv c$.

Proof. Let $\beta > -\frac{1}{3}$ and $c > 0$.

(1) If $k_1 \le -\frac{c^2}{2(1+3\beta)}$, a direct computation gives us

$$P_1(\varphi)' = (1+3\beta)(\varphi - c)\Big(\varphi - \frac{c}{1+3\beta}\Big)^2 - (\varphi - c)\Big(\frac{c^2}{1+3\beta} + 2k_1\Big) < 0 \quad \text{for } \varphi < c.$$

There are no bounded traveling wave solutions of (1.2) for any k_2 since $P_1(\varphi)$ is decreasing for $\varphi < c$.

(2) If $-\frac{c^2}{2(1+3\beta)} < k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2$, there are smooth traveling waves for some negative k_2.

(3) Let $k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2$. If $\beta > 0$, then there are a peaked solitary wave for $k_2 = 0$ and smooth traveling waves for some negative k_2. If $-\frac{1}{3} < \beta \le 0$, there are no bounded solutions for any k_2.

(4) If $\frac{\beta^2-1}{2(1+3\beta)}c^2 < k_1 < \frac{3\beta-1}{2}c^2$, then there are smooth traveling waves for some negative k_2, a peaked periodic traveling wave for $k_2 = 0$, and cusped traveling wave solutions for some $k_2 > 0$.

(5) Let $k_1 = \frac{3\beta-1}{2}c^2$. If $\beta > 0$, there are cusped traveling waves for some positive k_2 and the constant $\varphi \equiv c$ is a solution for $k_2 = 0$. If $-\frac{1}{3} < \beta \le 0$, there are no bounded solutions for any k_2.

(6) If $k_1 > \frac{3\beta-1}{2}c^2$, then there are cusped and anticusped traveling waves for some positive k_2. □

Remark 3.3. In [6], Lenells categorized traveling wave solutions of the Degasperis–Procesi equation. His categories (1)–(8) correspond to our categories (1)–(6), (C), and (S) for $\beta = 1$ in Theorem 3.2.

Theorem 3.4. *Let* $\beta < -\frac{1}{3}$ *and* $c > 0$. *Any bounded traveling wave of* (1.2) *belongs to one of the following categories. The waves are parametrized by* $k_1 \in \mathbb{R}$ *as follows:*

(1) *If* $k_1 \ge -\frac{c^2}{2(1+3\beta)}$, *then there are no bounded solutions of* (1.2).

(2) *If* $\frac{\beta^2-1}{2(1+3\beta)}c^2 < k_1 < -\frac{c^2}{2(1+3\beta)}$, *then there exist a one-parameter group of cusped periodic traveling waves and one cusped traveling wave with decay.*

(3) *If* $k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2$, *then there exist a one-parameter group of smooth periodic traveling waves, one smooth traveling waves with decay, a one-parameter group of cusped periodic traveling waves, and one cusped traveling wave with decay.*

(4) *If* $\frac{3\beta-1}{2}c^2 < k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2$, *then there exist a one-parameter group of smooth periodic traveling waves, a one-parameter group of cusped periodic traveling waves, and one cusped traveling wave with decay.*

(5) *If* $k_1 = \frac{3\beta-1}{2}c^2$, *then there exist a one-parameter group of smooth periodic traveling waves and cusped periodic traveling waves.*

(6) *If* $k_1 > \frac{3\beta-1}{2}c^2$, *then there exist a one-parameter group of smooth periodic traveling waves, one peaked periodic traveling waves, and cusped periodic traveling waves.*

(C) *(Composite waves) Any countable number of cuspons, anticuspons, and peakons from the categories* (1)–(6) *corresponding to the same value of* k_1 *may be joined at points where* $\varphi = c$ *to form a composite wave* φ. *If the Lebesgue measure* $\mu(\varphi^{-1}(c))$ *equals* 0, *then* φ *is a traveling wave of* (1.2).

(S) *(Stumpons) For* $k_1 = \frac{3\beta-1}{2}c^2$ *the composite waves are traveling waves of* (1.2) *even if the Lebesgue measure* $\mu(\varphi^{-1}(c))$ *exceeds* 0. *Consequently, these waves may contain intervals where* $\varphi \equiv c$.

Proof. Let $\beta < -\frac{1}{3}$ and $c > 0$.

(1) If $k_1 \ge -\frac{c^2}{2(1+3\beta)}$, a direct computation gives us

$$P_1(\varphi)' = (1+3\beta)(\varphi - c)\Big(\varphi - \frac{c}{1+3\beta}\Big)^2 - (\varphi - c)\Big(\frac{c^2}{1+3\beta} + 2k_1\Big) > 0 \quad \text{for } \varphi < c.$$

There are no bounded traveling wave solutions of (1.2) for any k_2 since $P_1(\varphi)$ is increasing for $\varphi < c$.

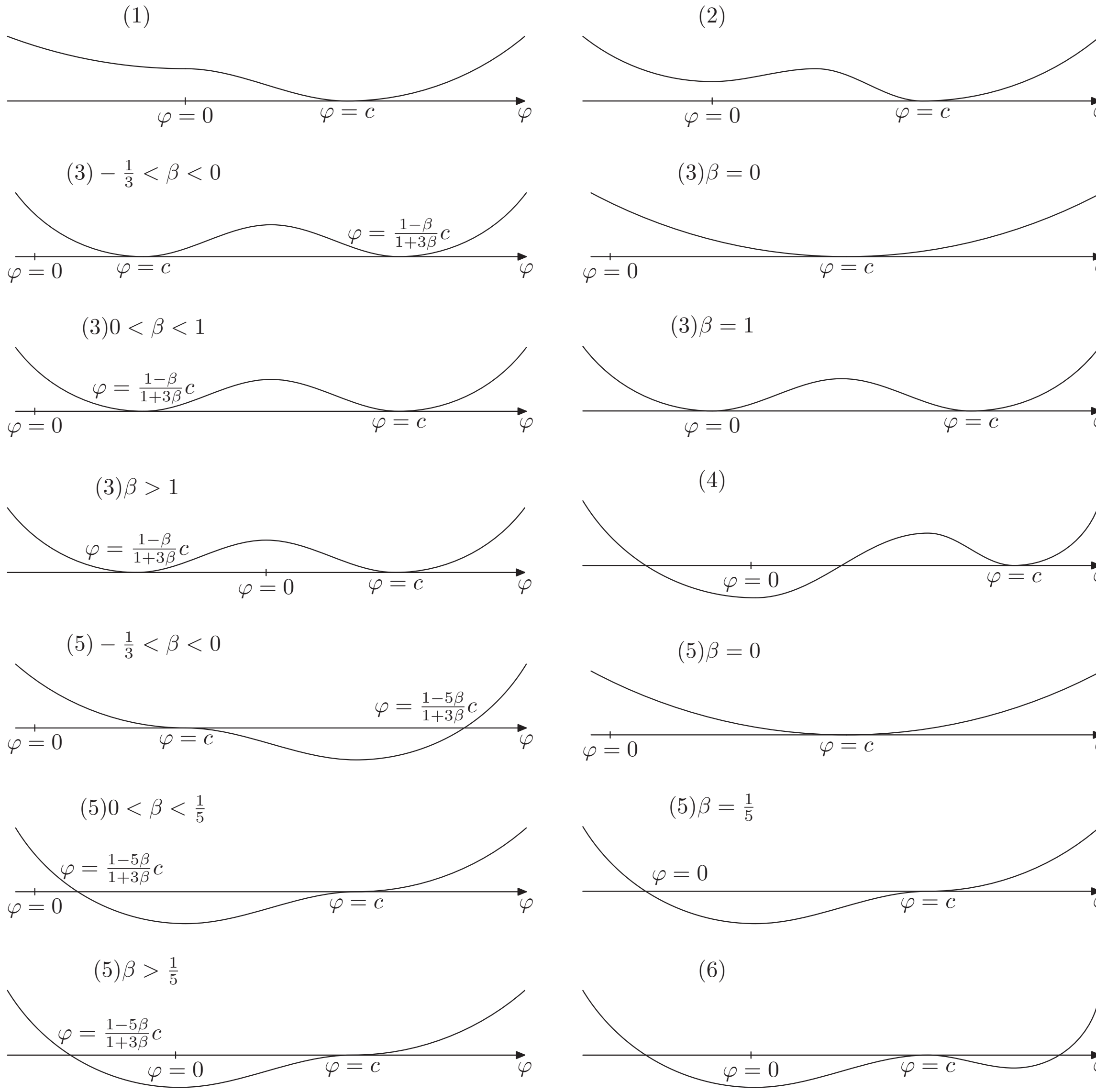

Figure 4. The polynomial (3.5) displayed for different values of k_1. All cases give rise to the categories (1)–(6) in Theorem 3.2.

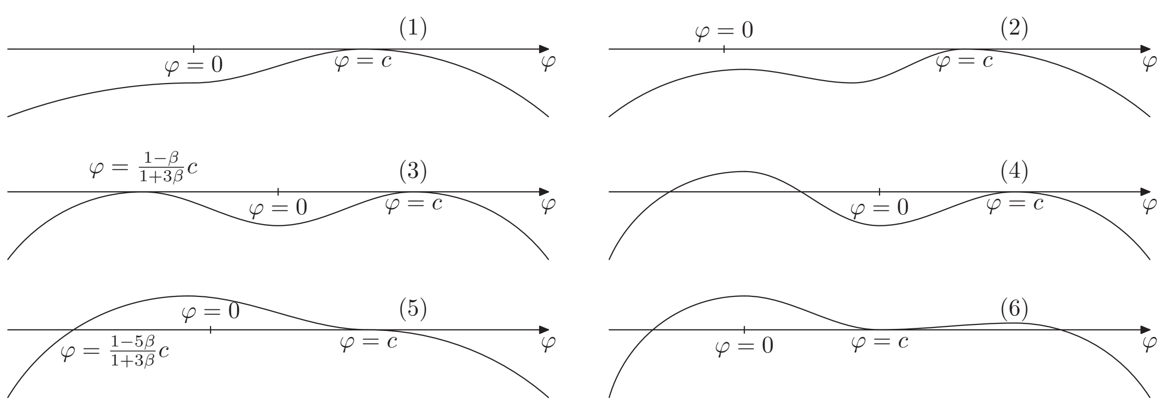

Figure 5. The polynomial (3.5) displayed for different values of k_1. The six cases give rise to the categories (1)–(6) in Theorem 3.4.

(2) If $\frac{\beta^2-1}{2(1+3\beta)}c^2 < k_1 < -\frac{c^2}{2(1+3\beta)}$, there are cusped traveling waves for some positive k_2.

(3) If $k_1 = \frac{\beta^2-1}{2(1+3\beta)}c^2$, there are smooth traveling waves and cusped traveling waves for some positive k_2.

(4) If $\frac{3\beta-1}{2}c^2 < k_1 < \frac{\beta^2-1}{2(1+3\beta)}c^2$, there are smooth periodic traveling waves for $k_2 = 0$ and cusped traveling waves for some positive k_2.

(5) If $k_1 = \frac{3\beta-1}{2}c^2$, there are smooth periodic traveling waves for some negative k_2, cusped periodic traveling waves for positive k_2, and smooth periodic traveling waves and the constant $\varphi \equiv c$ for $k_2 = 0$.

(6) If $k_1 > \frac{3\beta-1}{2}c^2$, there are smooth periodic traveling waves for some negative k_2, a peaked periodic traveling wave for $k_2 = 0$, and cusped periodic traveling waves for some positive k_2. □

4 Concluding Remarks

In this paper, we have investigated traveling wave solutions of the Burgers-$\alpha\beta$ equation with $\nu = 0$ and $b = 3$, including the well-studied integrable Degasperis–Procesi equation [2], $\beta = 1$. Hence the present paper generalizes some priori traveling wave results from [6] of the Degasperis–Procesi equation. The free parameter β and the integration constant k_1 play an important role in the type of traveling wave solutions of (1.2). Our study shows that there are three different kinds of traveling wave solutions with the decay condition to (1.2) such as cusped ($\beta < 1$), peaked ($\beta = 1$), and smooth ($\beta > 1$). Traveling wave solutions without the decay condition to (1.2) are parametrized by the integration constant k_1.

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