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Nonlinear Parametric Robin Problems with Combined Nonlinearities

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Abstract

We consider a nonlinear parametric Robin problem driven by the p-Laplacian. We assume that the reaction exhibits a concave term near the origin. First we prove a multiplicity theorem producing three solutions with sign information (positive, negative and nodal) without imposing any growth condition near $\pm\infty$ on the reaction. Then, for problems with subcritical reaction, we produce two more solutions of constant sign, for a total of five solutions. For the semilinear problem (that is, for $p = 2$), we generate a sixth solution but without any sign information. Our approach is variational, coupled with truncation, perturbation and comparison techniques and with Morse theory.

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1 Introduction

Let $\Omega \subseteq \mathcal{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$\begin{cases} -\Delta_p u(z) = f(z, u(z), \lambda) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p}(z) + \beta(z)|u(z)|^{p-2}u(z) = 0 \text{ on } \partial\Omega\,. \end{cases} \tag{P_λ}$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega),\ 1 < p < \infty.$$

Also $\frac{\partial u}{\partial n_p}$ denotes the nonlinear boundary derivative defined by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathcal{R}^N} \text{ for all } u \in W^{1,p}(\Omega),$$

with $n(z)$ being the outward unit normal at $z \in \partial\Omega$.

The reaction $f(z, x, \lambda)$ is a Carathéodory function of $(z, x) \in \Omega \times \mathcal{R}$ (that is, for all $x \in \mathcal{R}$ and all $\lambda > 0$, the mapping $z \longmapsto f(z, x, \lambda)$ is measurable, while for almost all $z \in \Omega$ and all $\lambda > 0$, the application $x \longmapsto f(z, x, \lambda)$ is continuous) and $\lambda > 0$ is a parameter, which may enter in the reaction in a nonlinear fashion. Our hypotheses on $f(z, x, \lambda)$ imply the presence of a concave term near the origin (that is, a term exhibiting $(p-1)$-superlinear growth near zero). In the second multiplicity theorem (see Theorem 4.1), we also assume that $x \longmapsto f(z, x, \lambda)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti-Robinowitz condition (AR-condition for short). So, in the second multiplicity theorem of this work, we have the combined effects of concave and convex nonlinearities. In fact, a special case of our reaction is the function

$$f(z, x, \lambda) = f(x, \lambda) = \lambda|x|^{q-2}x + |x|^{r-2}x$$

with $\lambda > 0$ and

$$1 < q < p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p. \end{cases}$$

Such reactions were first considered by Ambrosetti, Brezis and Cerami [4] in equations driven by the Dirichlet Laplacian and by Garcia Azorero, Manfredi and Peral Alonso [12] in equations driven by the Dirichlet p-Laplacian. Both works focus on positive solutions and prove bifurcation-type results for them. Multiplicity results for Dirichlet equations driven by the p-Laplacian and with concave terms, were also proved by Gasinski and Papageorgiou [15], Guo and Zhang [17] and Motreanu, Motreanu and Papageorgiou [23]. All the aforementioned works consider forms of the reaction in which the parameter enters linearly.

Recently Papageorgiou and Rădulescu [26] studied a different class of coercive parametric Robin problems without concave terms in the reaction and proved multiplicity theorems providing sign information for all the solutions. Bifurcation phenomena for the positive solutions of nonlinear Robin problems like (P_λ), were proved in the very recent work of Papageorgiou and Rădulescu [27]. Yet another class of parametric p-Laplacian Robin problems, were studied by Duchateau [9], who obtained two nontrivial solutions, but with no sign information. We also refer to Ahmad [1] and Ahmad, Lazer and Paul [2] for pioneering contributions to the qualitative theory of nonlinear partial differential equations of elliptic type.

Our approach is variational based on the critical point theory, combined with suitable truncation, perturbation and comparison techniques and Morse theory (critical groups). In the next section, for the convenience of the reader, we briefly review the main mathematical tools that we will use in the sequel.

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Cerami condition" (the C-condition for short), if the following holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathcal{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ, which compensates for the fact that the ambient space X need not be locally compact (in general, X is infinite dimensional). It allows us to prove a deformation theorem and from it to derive the minimax theory for the critical values of φ. Prominent in that theory, is the so-called "mountain pass theorem", due to Ambrosetti and Rabinowitz [5]. Here we state the result in a slightly more general form (see Gasinski and Papageorgiou [13]).

Theorem 2.1 *Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho >$ 0,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = \eta_\rho,$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geq \eta_\rho$ and c is a critical values of φ.

In this paper, we will be dealing with the Sobolev space $W^{1,p}(\Omega)$ and with the Banach space $C^1(\overline{\Omega})$. By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ given by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space, with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider $(N-1)$-dimensional Hausdorff (surface) measure denoted by $\sigma(\cdot)$. Using this measure, we can define the Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq \infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the trace map, such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Moreover, γ_0 is compact and $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, $\ker\gamma_0 = W_0^{1,p}(\Omega)$. In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. It is understood that all restrictions of the Sobolev functions $u \in W^{1,p}(\Omega)$ on $\partial\Omega$, are defined in the sense of traces.

On the boundary weight function $\beta(\cdot)$, we impose the following conditions:

$H(\beta) : \beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega, \beta \neq 0$.

Let $f_0 : \Omega\times\mathcal{R} \to \mathcal{R}$ be a Carathéodory function satisfying a subcritical growth condition in the $x \in \mathcal{R}$ variable, that is,

$$|f_0(z,x)| \leq a_0(z)(1 + |x|^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathcal{R},$$

with $a_0 \in L^\infty(\Omega)$, $1 < r < p^* = \begin{cases} \frac{N_p}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p. \end{cases}$

Let $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-conditional $\varphi_0 : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p d\sigma - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

The next result can be found in Papageorgiou and Rădulescu [26].

Proposition 2.1 *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\| \leq \rho_1.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega |Du|^{p-2}(Du, Dy)_{\mathcal{R}^N} dz \text{ for all } u, y \in W^{1,p}(\Omega).$$

From Papageorgiou and Kyritsi [25, p. 314], we have:

Proposition 2.2 *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined above is bounded (that is, maps bounded sets to bounded sets), demicontinuous, maximal monotone and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$).*

Let X be a Banach space and $\varphi \in C^1(X),\ c \in \mathcal{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},\ K_\varphi = \{u \in X : \varphi'(u) = 0\},\ K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

For every topological pair (Y_1, Y_2) with $Y_1 \subseteq Y_2 \subseteq X$ and every integer $k \geq 0$, by $H_k(Y_2, Y_1)$ we denote the kth singular homology group with integer coefficients. Then given an isolated $u \in K_\varphi^c$, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \text{ for all integers } k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition is independent of the particular choice of the neighborhood U.

Let $\varphi \in C^1(X)$ and assume that φ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all integers } k \geq 0,$$

where $c < \inf \varphi(K_\varphi)$. The second deformation theorem (see, for example, Gasinski and Papageorgiou [13, p. 628]), implies that the above definition is independent of the choice of the level c.

Suppose that K_φ is infinite and define

$$M(t, u) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, u) t^k \text{ for all } t \in \mathcal{R},\ \text{all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \text{ for all } t \in \mathcal{R}.$$

The Morse relation establishes that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \text{ for all } t \in \mathcal{R}, \tag{2.1}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathcal{R}$, with nonnegative integer coefficients β_k.

Finally, let us fix our notation. Given $x \in \mathcal{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$ and we have

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathcal{R}^N$, by $|\cdot|$ the norm of $\mathcal{R}^N$ and by $(\cdot, \cdot)_{\mathcal{R}^N}$ the inner product of $\mathcal{R}^N$. If $u, v \in W^{1,p}(\Omega)$ and $v \leq u$, by $[v, u]$ we denote the order interval defined by

$$[v, u] = \{y \in W^{1,p}(\Omega) : v(z) \leq y(z) \leq u(z) \text{ a.e in } \Omega\}.$$

Given a measurable function $h : \Omega \times \mathcal{R} \to \mathcal{R}$ (for example, a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega),$$

the Nemytskii operator corresponding to h. Evidently, the application $z \longmapsto N_h(u)(z)$ is measurable.

3 Multiple solutions for reactions of arbitrary growth

In this section, we prove a multiplicity theorem (a three solutions theorem) for problem (P_λ), providing sign information for all the solutions, without imposing a subcritical growth restriction on $f(z, \cdot, \lambda)$. In fact, the behavior of $x \longmapsto f(z, x, \lambda)$ near $\pm\infty$ is irrelevant in our analysis. More precisely, our hypotheses on the reaction $f(z, x, \lambda)$ are the following:

H_1 : $f : \Omega\times\mathcal{R}\times(0, \infty) \to \mathcal{R}$ is a function such that for a.a. $z \in \Omega$, all $\lambda > 0$, $f(z, 0, \lambda) = 0$ and

(i) for all $\lambda > 0$, $(z, x) \longmapsto f(z, x, \lambda)$ is a Carathéodory function;

(ii) $|f(z, x, \lambda)| \leq a(z, \lambda) + \vartheta(|x|)$ for a.a. $z \in \Omega$, all $x \in \mathcal{R}$, with $a(\cdot, \lambda) \in L^\infty(\Omega)_+$,

$$||a(\cdot, \lambda)||_\infty \to 0 \text{ as } \lambda \to 0^+,$$

$\vartheta(r) > 0$ for all $r > 0$, $r \longmapsto \vartheta(r)$ is bounded on bounded sets of $(0, \infty)$ and $\lim_{r\to 0^+} \frac{\vartheta(r)}{r^{p-1}} = 0$;

(iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda)ds$, then there exist $q = q(\lambda) \in (1, p)$ and $\delta_0 = \delta_0(\lambda)$, $c_0 = c_0(\lambda) > 0$ such that

$$qF(z, x, \lambda) \geq f(z, x, \lambda)x \geq c_0|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.$$

Remark. Evidently hypothesis $H_1(iii)$ implies the presence of a concave nonlinearity near zero. We stress that the growth of $x \longmapsto f(z, x, \lambda)$ near $+\infty$ can be arbitrary (see hypothesis $H_1(ii)$).

Examples. The following functions satisfy hypotheses H_1. For the sake of simplicity we drop the z-dependence:

$$f_1(x, \lambda) = \lambda|x|^{q-2}x + |x|^{r-2}x \text{ with } 1 < q < p < r < \infty$$

$$f_2(x, \lambda) = \begin{cases} \lambda|x|^{q-2}x & \text{if } |x| \leq 1 \\ \frac{\lambda}{2}\left[|x|^{p-2}x + |x|^{\tau-2}x\right] & \text{if } |x| > 1 \end{cases} \quad \text{with } 1 < q, \tau < p$$

$$f_3(x, \lambda) = \begin{cases} |x|^{q-2}x & \text{if } |x| \leq \rho(\lambda) \\ |x|^{r-2}x \pm \xi(\lambda) & \text{if } |x| > \rho(\lambda) \end{cases}$$

with $1 < q < p, r, \rho(\lambda) \in (0, 1)$ for all $\lambda > 0, \rho(\lambda) \to 0^+$ as $\lambda \to 0^+$ and $\xi(\lambda) = \rho(\lambda)^{q-1} - \rho(\lambda)^{r-1}$.

First we produce two nontrivial solutions of constant sign.

Proposition 3.1 *If hypotheses $H(\beta)$ and H_1 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) admits two nontrivial solutions of constant sign*

$$u_0 \in int\, C_+ \text{ and } v_0 \in -int\, C_+.$$

Proof. We consider the following auxiliary Robin problem

$$-\Delta_p e(z) = 1 \text{ in } \Omega,\ \frac{\partial e}{\partial n_p} + \beta(z)e^{p-1} = 0 \text{ on } \partial\Omega,\ e > 0. \tag{3.2}$$

Let $\eta : W^{1,p}(\Omega) \to \mathcal{R}$ be the locally Lipschitz functional defined by

$$\eta(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega (u^+)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using Young's inequality with $\epsilon > 0$ (see, for example, Gasinski and Papageorgiou [13, p. 913]), we have

$$u^+(z) \le \frac{\epsilon}{p}u^+(z)^p + \frac{1}{\epsilon p'} \text{ with } \epsilon > 0 \text{ and with } \frac{1}{p} + \frac{1}{p'} = 1.$$

Therefore we have

$$\eta(u) \ge \frac{1}{p}\|Du^+\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \frac{\epsilon}{p}\|u^+\|_p^p + \frac{1}{p}\|Du^-\|_p^p + \frac{1}{p}\|u^-\|_p^p - \frac{1}{\epsilon p'}|\Omega|_N.$$

Let $\hat{\lambda}_1 > 0$ be the principal eigenvalue of the negative Robin p-Laplacian (see Papageorgiou and Rădulescu [26]). Choosing $\epsilon \in (0, \hat{\lambda}_1)$, we have

$$\begin{aligned} &\eta(u) \ge \xi_0\|u^+\|^p + \frac{1}{p}\|u^-\|^p - \frac{1}{\epsilon p'}|\Omega|_N \text{ for some } \xi_0 > 0 \\ \Rightarrow\quad &\eta(\cdot) \text{ is coercive.} \end{aligned}$$

Also, using the Sobolev embedding theorem and the compactness of the trace map we see that η is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem we can find $e \in W^{1,p}(\Omega)$ such that

$$\eta(e) = \inf[\eta(v) : v \in W^{1,p}(\Omega)]\,. \tag{3.3}$$

Let $\hat{u}_1$ be the positive L^p-normalized (that is, $\|\hat{u}_1\|_p = 1$) principal eigenfunction of the negative Robin p-Laplacian. We know that $\hat{u}_1 \in \operatorname{int} C_+$ (see Papageorgiou and Rădulescu [26]). We have

$$\eta(t\hat{u}_1) = \frac{t^p}{p}\hat{\lambda}_1 - t\|\hat{u}_1\|_1 \text{ (recall } \|\hat{u}_1\|_p = 1).$$

Choosing $t \in (0,1)$ small, we see that

$$\begin{aligned} &\eta(t\hat{u}_1) < 0 \text{ (recall } p > 1) \\ \Rightarrow\quad &\eta(e) < 0 = \eta(0) \text{ (see (3.3)), hence } e \ne 0. \end{aligned}$$

From (3.3) we have

$$0 \in \partial\eta(e)$$

with ∂ denoting the subdifferential in the sense of Clarke [7] of the locally Lipschitz functional $\eta(\cdot)$. From Clarke [7, p. 39] we have

$$\langle A(e), h\rangle - \int_\Omega (e^-)^{p-1} h dz + \int_{\partial\Omega} \beta(z)(e^+)^{p-1} h d\sigma \leq \int_\Omega \chi_{\{e\geq 0\}}(z) h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.4}$$

In (3.4) we choose $h = -e^- \in W^{1,p}(\Omega)$. We obtain

$$\|e^-\|^p = 0, \text{ hence } e \geq 0, \ e \neq 0.$$

Then (3.4) becomes

$$\begin{aligned} &\langle A(e), h\rangle + \int_{\partial\Omega} \beta(z) e^{p-1} h d\sigma \leq \int_\Omega h dz \text{ for all } h \in W^{1,p}(\Omega),\\ \Rightarrow \quad &\langle A(e), h\rangle + \int_{\partial\Omega} \beta(z) e^{p-1} h d\sigma = \int_\Omega h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.5}$$

Let $\langle\cdot,\cdot\rangle_0$ denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$ (recall $W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'} = 1$). From the representation theorem for the elements of $W^{-1,p'}(\Omega)$ (see, for example, Gasinski and Papageorgiou [13, p. 212]), we have

$$\Delta_p e \in W^{-1,p'}(\Omega).$$

Using integration by parts, we have

$$\langle A(e), h\rangle = \left\langle -\Delta_p e, h\right\rangle_0 \text{ for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

Using this equality in (3.5) and recalling that $h|_{\partial\Omega} = 0$ for all $h \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} &\left\langle -\Delta_p e, h\right\rangle_0 = \int_\Omega h dz \text{ for all } h \in W_0^{1,p}(\Omega),\\ \Rightarrow \quad &-\Delta_p e(z) = 1 \text{ for a.a. } z \in \Omega. \end{aligned} \tag{3.6}$$

We can apply the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [13, p. 210]) and obtain

$$\langle A(e), h\rangle + \int_\Omega (\Delta_p e) h dz = \left\langle \frac{\partial e}{\partial n_p}, h\right\rangle_{\partial\Omega} \text{ for all } h \in W^{1,p}(\Omega), \tag{3.7}$$

where by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ we denote the duality brackets for the pair $\left(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega)\right)$. Returning to (3.5) and using (3.7), we have

$$\begin{aligned} &-\int_\Omega (\Delta_p e) h dz + \left\langle \frac{\partial e}{\partial n_p}, h\right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) e^{p-1} h d\sigma = \int_\Omega h dz \\ &\qquad\qquad \text{for all } h \in W^{1,p}(\Omega),\\ \Rightarrow \quad &\left\langle \frac{\partial e}{\partial n_p}, h\right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) e^{p-1} h d\sigma = 0 \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (3.6))}. \end{aligned} \tag{3.8}$$

Since $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$, from (3.8) it follows that

$$\frac{\partial e}{\partial n_p} + \beta(z)e^{p-1} = 0 \text{ on } \partial\Omega. \tag{3.9}$$

From (3.6) and (3.9) it follows that $e \in W^{1,p}(\Omega)$ is a nontrivial positive solution of the auxiliary problem (3.2). From Winkert [29], we have $e \in L^\infty(\Omega)$ and then Theorem 2 of Lieberman [19] implies that $e \in C_+\backslash\{0\}$. From (3.6), we have

$$\begin{aligned} &\Delta_p e(z) \leq 0 \text{ a.e. in } \Omega, \\ \Rightarrow\quad & e \in \operatorname{int} C_+ \text{ (see Vazquez [28]).} \end{aligned}$$

Claim 3.1 *There exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$, we can find $\hat{\xi} = \hat{\xi}(\lambda) > 0$ for which*

$$||a(\cdot, \lambda)||_\infty + \vartheta(\hat{\xi}||e||_\infty) < \hat{\xi}^{p-1}.$$

Suppose that the Claim is not true. Then we can find $\lambda_n \to 0^+$ such that

$$||a(\cdot, \lambda_n)||_\infty + \vartheta(\xi||e||_\infty) \geq \xi^{p-1} \text{ for all } n \geq 1, \text{ all } \xi > 0.$$

We let $n \to \infty$ and use hypothesis $H_1(ii)$ to obtain

$$\begin{aligned} &\vartheta(\xi||e||_\infty) \geq \xi^{p-1} \text{ for all } \xi > 0, \\ \Rightarrow\quad & \frac{\vartheta(\xi||e||_\infty)}{\xi^{p-1}} \geq 1 \text{ for all } \xi > 0, \end{aligned}$$

which contradicts hypothesis $H_1(ii)$. This proves the Claim.

Let $\lambda \in (0, \lambda^*)$ and set $\bar{u} = \hat{\xi}e \in \operatorname{int} C_+$. We have

$$\begin{aligned} \Delta_p \bar{u}(z) &= \hat{\xi}^{p-1} > ||a(\cdot, \lambda)||_\infty + \vartheta(\hat{\xi}||e||_\infty) \text{ (see the Claim)} \\ &\geq f(z, \bar{u}(z), \lambda) \text{ for a.a. } z \in \Omega \text{ (see hypothesis } H_1(ii)). \end{aligned} \tag{3.10}$$

We consider the following truncation-perturbation of $f(z, \cdot, \lambda)$:

$$\hat{f}_+(z, x, \lambda) = \begin{cases} 0 & \text{if } x < 0 \\ f(z, x, \lambda) + x^{p-1} & \text{if } 0 \leq x \leq \bar{u}(z) \\ f(z, \bar{u}(z), \lambda) + \bar{u}(z)^{p-1} & \text{if } \bar{u}(z) < x. \end{cases} \tag{3.11}$$

This is a Carathéodory function. We set $\hat{F}_+(z, x, \lambda) = \int_0^x \hat{f}_+(z, s, \lambda)ds$ and consider the C^1-functional $\hat{\varphi}_+^\lambda : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\hat{\varphi}_+^\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}_+(z, u, \lambda)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.11) and hypothesis $H(\beta)$, we see that $\hat{\varphi}_+^\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+^\lambda(u_0) = \inf\left[\hat{\varphi}_+^\lambda(u) : u \in W^{1,p}(\Omega)\right] = \hat{m}_+^\lambda. \tag{3.12}$$

Let $\delta_0 > 0$ be as postulated by hypothesis $H_1(iii)$ and let $\bar{m} = \min_{\overline{\Omega}} \bar{u} > 0$ (recall $\bar{u} \in \operatorname{int} C_+$). Let $\hat{\delta} = \min\{\delta_0, \bar{m}\}$ and choose $t \in (0,1)$ small such that $t\hat{u}_1(z) \in (0, \hat{\delta}]$ for all $z \in \overline{\Omega}$ (recall $\hat{u}_1 \in \operatorname{int} C_+$ is the L^p-normalized principal eigenfunction of the negative Robin p-Laplacian). We have

$$\begin{aligned} \hat{\varphi}_+^\lambda(t\hat{u}_1) &= \frac{t^p}{p}\|D\hat{u}_1\|_p^p + \frac{t^p}{p}\int_{\partial\Omega}\beta(z)\hat{u}_1^p d\sigma - \int_\Omega F(z, t\hat{u}_1, \lambda)dz \text{ (see (3.11))} \\ &\leq \frac{\hat{\lambda}_1}{p}t^p - \frac{c_0 t^q}{q}\|\hat{u}_1\|_q^q \\ &\qquad \text{(see [26], hypothesis } H_1(iii) \text{ and recall } \|\hat{u}_1\|_p = 1). \end{aligned}$$

Since $p > q$, by choosing $t \in (0,1)$ even smaller if necessary, we obtain

$$\begin{aligned} & \hat{\varphi}_+^\lambda(t\hat{u}_1) < 0, \\ \Rightarrow \quad & \hat{\varphi}_+^\lambda(u_0) < 0 = \hat{\varphi}_+^\lambda(0) \text{ (see (3.12)), hence } u_0 \neq 0. \end{aligned}$$

From (3.12), we have

$$\begin{aligned} & (\hat{\varphi}_+^\lambda)'(u_0) = 0, \\ \Rightarrow \quad & \langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h dz + \int_{\partial\Omega}\beta(z)(u_0^+)^{p-1}h d\sigma = \\ & = \int_\Omega \hat{f}_+(z, u_0, \lambda)h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.13}$$

In (3.13) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \|Du_0^-\|_p^p + \|u_0^-\|_p^p = 0 \text{ (see (3.11))}, \\ \Rightarrow \quad & u_0 \geq 0, \ u_0 \neq 0. \end{aligned}$$

Also in (3.13) we choose $h = (u_0 - \bar{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(u_0), (u_0-\bar{u})^+\rangle + \int_\Omega u_0^{p-1}(u_0-\bar{u})^+dz + \int_{\partial\Omega}\beta(z)u_0^{p-1}(u_0-\bar{u})^+d\sigma \\ = \ & \int_\Omega [f(z,\bar{u},\lambda) + \bar{u}^{p-1}](u_0-\bar{u})^+dz \text{ (see (3.11))} \\ \leq \ & \int_\Omega [\hat{\xi}^{p-1} + \bar{u}^{p-1}](u_0-\bar{u})^+dz \text{ (see hypothesis } H_1(ii) \text{ and the Claim)} \\ = \ & \langle A(\bar{u}), (u_0-\bar{u})^+\rangle + \int_\Omega \bar{u}^{p-1}(u_0-\bar{u})^+dz + \int_{\partial\Omega}\beta(z)\bar{u}^{p-1}(u_0-\bar{u})^+d\sigma \\ & \qquad \text{(recall the definition of } \bar{u} \in \operatorname{int} C_+), \\ \Rightarrow \quad & \langle A(u_0) - A(\bar{u}), (u_0-\bar{u})^+\rangle + \int_\Omega (u_0^{p-1} - \bar{u}^{p-1})(u_0-\bar{u})^+dz + \\ & \int_{\partial\Omega}\beta(z)(u_0^{p-1} - \bar{u}^{p-1})(u_0-\bar{u})^+d\sigma \leq 0, \\ \Rightarrow \quad & |\{u_0 > \bar{u}\}|_N = 0, \text{ hence } u_0 \leq \bar{u}. \end{aligned}$$

So, we have proved that

$$u_0 \in [0, \bar{u}] = \{u \in W^{1,p}(\Omega) : 0 \le u(z) \le \bar{u}(z) \text{ a.e. in } \Omega\},\ u_0 \ne 0.$$

Then (3.13) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma = \int_\Omega f(z, u_0, \lambda) h dz \qquad (3.14)$$
$$\text{for all } h \in W^{1,p}(\Omega) \text{ (see (3.11))}.$$

From (3.14), as before, using the nonlinear Green's identity, we infer that u_0 is a nontrivial positive solution of problem (P_λ) with $\lambda \in (0, \lambda^*)$. Again, the nonlinear regularity theory (see Lieberman [19]), implies that $u_0 \in C_+ \backslash \{0\}$. Note that hypotheses $H_1(ii), (iii)$ imply that given $\rho > 0$, we can find $\xi_\rho = \xi_\rho(\lambda) > 0$ such that

$$f(z, x, \lambda)x + \xi_\rho |x|^p \ge 0 \text{ for a.a. } z \in \Omega,\ \text{all } |x| \le \rho.$$

So, we have

$$\begin{aligned} & -\Delta_p u_0(z) + \xi_\rho u_0(z)^{p-1} = f(z, u_0(z), \lambda) + \xi_\rho u_0(z)^{p-1} \ge 0 \text{ a.e. in } \Omega, \\ \Rightarrow\quad & \Delta_p u_0(z) \le \xi_\rho u_0(z)^{p-1} \text{ a.e. in } \Omega, \\ \Rightarrow\quad & u_0 \in \operatorname{int} C_+ \text{ (see Vazquez [28])}. \end{aligned}$$

In a similar fashion, we let $\underline{u} = -\hat{\xi} e \in -\operatorname{int} C_+$ and consider the following truncation-perturbation of $f(z, \cdot, \lambda)$

$$\hat{f}_-(z, x, \lambda) = \begin{cases} f(z, \underline{u}(z), \lambda) + |\underline{u}(z)|^{p-2}\underline{u}(z) & \text{if } x < \underline{u}(z) \\ f(z, x, \lambda) + |x|^{p-2}x & \text{if } \underline{u}(z) \le x \le 0 \\ 0 & \text{if } 0 < x. \end{cases}$$

Using the Carathéodory function $(z, x) \longmapsto \hat{f}_-(z, x, \lambda)$ and reasoning as above via the direct method, we obtain a second nontrivial constant sign solution $v_0 \in \operatorname{int} C_+$, $\underline{u} \le v_0$. □

Next we will produce a third nontrivial solution for (P_λ) $(\lambda \in (0, \lambda^*))$ which is nodal (that is, sign changing). To this end, first we show that problem (P_λ) has extremal constant sign solutions, that is, a smallest nontrivial positive solution and a biggest nontrivial negative solution. To this end, we need to strengthen the hypotheses on $f(z, \cdot, \lambda)$. So, the new conditions on the reaction, are the following:

$H_2 : f : \Omega\times\mathcal{R}\times(0, \infty) \to \mathcal{R}$ is a function such that for a.a. $z \in \Omega$, all $\lambda > 0$, $f(z, 0, \lambda) = 0$, hypotheses $H_2(i), (ii), (iii)$ are the same as the corresponding hypotheses $H_1(i), (ii), (iii)$ and (iv) for every $\lambda > 0$, we can find $c_1 = c_1(\lambda) > 0$, $c_2 = c_2(\lambda) > 0$ and $r = r(\lambda) \in (p, p^*)$ such that

$$f(z, x, \lambda)x \ge c_1|x|^q - c_2|x|^r \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathcal{R}.$$

This extra unilateral growth condition on $f(z, \cdot, \lambda)$, leads to the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u(z) = c_1|u(z)|^{q-2}u(z) - c_2|u(z)|^{r-2}u(z) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega. \end{cases} \qquad (3.15)$$

Proposition 3.2 *If hypotheses $H(\beta)$ hold, then problem (3.15) has a unique nontrivial positive solution $\overline{u}_2 \in int\, C_+$ and since the equation is odd, $\bar{v}_* = -\bar{u}_* \in -int\, C_+$ is the unique nontrivial negative solution of (3.15).*

Proof. First we show the existence of a nontrivial positive solution. To this end, let $\psi_+ : W^{1,p}(\Omega) \to \mathcal{R}$ be the C^1-functional defined by

$$\psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \frac{c_1}{q}\|u^+\|_q^q + \frac{c_2}{r}\|u^+\|_r^r \quad \text{for some } u \in W^{1,p}(\Omega).$$

By virtue of hypothesis $H(\beta)$, we have

$$\begin{aligned}\psi_+(u) &\geq \frac{1}{p}\|u\|^p + \left[\frac{c_2}{r}\|u^+\|_r^r - \frac{1}{p}\|u^+\|_p^p - \frac{c_1}{q}\|u^+\|_q^q\right]\\ &\geq \frac{1}{p}\|u\|^p + \left[\frac{c_2}{r}\|u^+\|_r^r - c_3\left(\|u^+\|_r^p + \|u^+\|_r^q\right)\right]\\ &\qquad \text{for some } c_3 > 0 \text{ (recall } q < p < r)\\ &= \frac{1}{p}\|u\|^p + \left[\frac{c_2}{r}\|u^+\|_r^{r-p} - c_3 - \frac{c_3}{\|u^+\|_r^{p-q}}\right]\|u^+\|_r^p.\end{aligned} \tag{3.16}$$

Since $q < p < r$, from (3.16) it is clear that ψ_+ is coercive. Also, ψ_+ is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_* \in W^{1,p}(\Omega)$ such that

$$\psi_+(\bar{u}_*) = \inf[\psi_+(u) : u \in W^{1,p}(\Omega)]. \tag{3.17}$$

As before (see the proof of Proposition 3.1) and since $q < p < r$, for $t \in (0,1)$ small we have

$$\begin{aligned}&\psi_+(t\hat{u}_1) < 0,\\ \Rightarrow\ &\psi_+(\bar{u}_*) < 0 = \psi_+(0) \text{ (see (3.17)), hence } \bar{u}_* \neq 0.\end{aligned}$$

From (3.17), we have

$$\begin{aligned}&\psi_+'(\bar{u}_*) = 0,\\ \Rightarrow\ &\langle A(\bar{u}_*), h\rangle - \int_\Omega (\bar{u}_*^-)^{p-1}hdz + \int_{\partial\Omega}\beta(z)(\bar{u}_*^+)^{p-1}hd\sigma =\\ &c_1\int_\Omega (\bar{u}_*^+)^{q-1}hdz - c_2\int_\Omega(\bar{u}_*^+)^{r-1}hdz \text{ for all } W^{1,p}(\Omega).\end{aligned} \tag{3.18}$$

In (3.18) we choose $h = -\bar{u}_*^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\|D\bar{u}_*^-\|_p^p + \|\bar{u}_*^-\|_p^p = 0,\\ \Rightarrow\ &\bar{u}_* \geq 0,\ \bar{u}_* \neq 0.\end{aligned}$$

Then equation (3.18) becomes

$$\langle A(\bar{u}_*), h\rangle + \int_{\partial\Omega}\beta(z)\bar{u}_*^{p-1}hd\sigma = c_1\int_\Omega \bar{u}_*^{q-1}hdz - c_2\int_\Omega \bar{u}_*^{r-1}hdz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.19}$$

From (3.19) as before (see the proof of Proposition 3.1), we infer that $\bar{u}_*$ is a nontrivial positive solution of (3.15) and the nonlinear regularity theory (see Lieberman [19] and Winkert [29]) implies that $\bar{u}_* \in C_+\backslash\{0\}$. We have

$$\begin{aligned} & -\Delta_p \bar{u}_*(z) \geq -c_2 \bar{u}_*(z)^{r-1} \text{ a.e. in } \Omega, \\ \Rightarrow \quad & \Delta_p \bar{u}_*(z) \leq c_2 \|\bar{u}_*\|_\infty^{r-p} \bar{u}_*(z)^{p-1} \text{ a.e. in } \Omega, \\ \Rightarrow \quad & \bar{u}_* \in \operatorname{int} C_+ \text{ (see Vazquez [28]).} \end{aligned}$$

Next we show the uniqueness of this nontrivial positive solution. To this end, let $\sigma_+ : L^1(\Omega) \to \overline{\mathcal{R}} = \mathcal{R} \cup \{+\infty\}$ be the integral functional defined by

$$\sigma_+ = \begin{cases} \dfrac{1}{p}\|Du^{1/p}\|_p^p + \dfrac{1}{p}\displaystyle\int_{\partial\Omega} \beta(z) u d\sigma & \text{if } u \geq 0,\ u^{1/p} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 1 of Diaz and Saa [8] and hypotheses $H(\beta)$, we see that σ_+ is convex and lower semicontinuous.

Let $\bar{y}_* \in W^{1,p}(\Omega)$ be another nontrivial positive solution of problem (3.15). Again we can show that $\bar{y}_* \in \operatorname{int} C_+$. So, for any $h \in C^1(\overline{\Omega})$ and for $|t| \leq 1$ small, we have

$$\bar{u}_*^p + th,\ \bar{y}_*^p + th \in \operatorname{dom} \sigma_+ = \{u \in L^1(\Omega) : \sigma_+(u) < +\infty\}.$$

The functional σ_+ is Gâteaux differentiable at $\bar{u}_*^p$ and at $\bar{y}_*^p$ in the direction h. Moreover, via the chain rule and the nonlinear Green's identity, we have

$$\begin{aligned} \sigma_+'(\bar{u}_*^p)(h) &= \frac{1}{p}\int_\Omega \frac{-\Delta_p \bar{u}_*}{\bar{u}_*^{p-1}} h dz \\ \sigma_+'(\bar{y}_*^p)(h) &= \frac{1}{p}\int_\Omega \frac{-\Delta_p \bar{y}_*}{\bar{y}_*^{p-1}} h dz \text{ for all } h \in W^{1,p}(\Omega) \end{aligned}$$

(recall that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$). The convexity of σ_+ implies the monotonicity of σ_+' and so we have

$$\begin{aligned} 0 \quad &\leq \quad \frac{1}{p}\int_\Omega \left[\frac{-\Delta_p \bar{u}_*}{\bar{u}_*^{p-1}} - \frac{-\Delta_p \bar{y}_*}{\bar{y}_*^{p-1}}\right](\bar{u}_*^p - \bar{y}_*^p) dz \\ &= \quad \frac{1}{p}\int_\Omega \left[c_1\left(\frac{1}{\bar{u}_*^{p-q}} - \frac{1}{\bar{y}_*^{p-q}}\right) - c_2(\bar{u}_*^{r-p} - \bar{y}_*^{r-p})\right](\bar{u}_*^p - \bar{y}_*^p) dz \leq 0 \\ \Rightarrow \quad & \quad \bar{u}_* = \bar{y}_* \text{ (since } x \longmapsto \frac{c_1}{x^{p-q}} - c_2 x^{r-p} \text{ is strictly decreasing on } (0,\infty)). \end{aligned}$$

So, the nontrivial positive solution $\bar{u}_* \in \operatorname{int} C_+$ of (3.15) is unique.

Since equation (3.15) is odd, it follows that $\bar{v}_* = -\bar{u}_* \in -\operatorname{int} C_+$ is the unique nontrivial negative solution. □

In what follows, by $S_+(\lambda)$ (resp. $S_-(\lambda)$) we denote the set of nontrivial positive (resp. negative) solutions of problem (P_λ). From Proposition 3.1, we know that for all $\lambda \in (0, \lambda^*)$

$$\emptyset \neq S_+(\lambda) \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_-(\lambda) \subseteq -\operatorname{int} C_+.$$

Moreover, as in Filippakis, Kristaly and Papageorgiou [11], exploiting the monotonicity of A (see Proposition 2.2), we know that

- $S_+(\lambda)$ is downward directed (that is, if $u_1, u_2 \in S_+(\lambda)$, then we can find $u \in S_+(\lambda)$ such that $u \leq u_1,\ u \leq u_2$).
- $S_-(\lambda)$ is upward directed (that is, if $v_1, v_2 \in S_-(\lambda)$, then we can find $v \in S_-(\lambda)$ such that $v_1 \leq v,\ v_2 \leq v$).

Proposition 3.3 *If hypotheses $H(\beta)$ and H_2 hold and $\lambda \in (0, \lambda^*)$, then*

(a) $\bar{u}_* \leq u$ for all $u \in S_+(\lambda)$;

(b) $v \leq \bar{v}_*$ for all $v \in S_+(\lambda)$.

Proof. (a) Let $u \in S_+(\lambda)$ and consider the following Carathéodory function

$$k_+(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ c_1 x^{q-1} - c_2 x^{r-1} + x^{p-1} & \text{if } 0 \leq x \leq u(z) \\ c_1 u(z)^{q-1} - c_2 u(z)^{r-1} + u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.20}$$

We set $K_+(z, x) = \int_0^x k_+(z, s)ds$ and consider the C^1-functional $\tau_+ : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\tau_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega k_+(z, u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.20) and hypotheses $H(\beta)$, it is clear that τ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_* \in W^{1,p}(\Omega)$ such that

$$\tau_+(\hat{u}_*) = \inf[\tau_+(u) : u \in W^{1,p}(\Omega)]. \tag{3.21}$$

Since $q < p < r$, for $t \in (0, 1)$ small we have

$$\begin{aligned} & \tau_+(t\hat{u}_1) < 0, \\ \Rightarrow\ & \tau_+(\hat{u}_*) < 0 = \tau_+(0) \text{ (see (3.21)), hence } \hat{u}_* \neq 0. \end{aligned}$$

Also, from (3.21) we have

$$\begin{aligned} & \tau_+'(\hat{u}_*) = 0 \\ \Rightarrow\ & \langle A(\hat{u}_*), h\rangle + \int_\Omega |\hat{u}_*|^{p-2}\hat{u}_* h dz + \int_{\partial\Omega} \beta(z)(\hat{u}_*^+)^{p-1} h d\sigma = \\ & = \int_\Omega k_+(z, \hat{u}_*) h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.22}$$

As in the proof of Proposition 3.1, in (3.22) first we choose $h = -\hat{u}_*^- \in W^{1,p}(\Omega)$ and then $h = (\hat{u}_* - u)^+ \in W^{1,p}(\Omega)$, to show that

$$\begin{aligned} & \hat{u}_* \in [0, u],\ \hat{u}_* \neq 0, \\ \Rightarrow\quad & \hat{u}_* \text{ is a nontrivial positive solution of (3.15) (see (3.20))}, \\ \Rightarrow\quad & \hat{u}_* = \bar{u}_* \text{ (see Proposition 3.2)}, \\ \Rightarrow\quad & \bar{u}_* \leq u \text{ for all } u \in S_+(\lambda). \end{aligned}$$

Similarly we show that $v \leq \bar{v}_*$ for all $v \in S_-(\lambda)$. □

We will use this proposition to establish the existence of extremal constant sign solutions for problem (P_λ) $(\lambda \in (0, \lambda^*))$.

Proposition 3.4 *If hypotheses $H(\beta)$ and H_2 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits extremal constant sign solutions*

$$u_*^\lambda \in int\, C_+ \text{ and } v_*^\lambda \in -int\, C_+.$$

Proof. Since $S_+(\lambda)$ is downward directed, without any loss of generality we may assume that there exists $c_4 > 0$ such that $||u||_\infty \leq c_4$ for all $u \in S_+(\lambda)$. Then from Dunford and Schwartz [10, p. 336] we know that we can find $\{u_n\}_{n\geq 1} \subseteq S_+(\lambda)$ such that

$$\inf S_+(\lambda) = \inf_{n\geq 1} u_n.$$

We have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega f(z, u_n, \lambda) h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{3.23}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (3.24) and using hypotheses $H(\beta)$ and $H_2(ii)$ (recall that $\{u_n\}_{n\geq 1} \subseteq L^\infty(\Omega)$ is bounded), we see that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded and so, we may assume that

$$u_n \xrightarrow{w} u_*^\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_*^\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.24}$$

In (3.23) we choose $h = u_n - u_*^\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.24). Then

$$\begin{aligned} & \lim_{n\to\infty} \left\langle A(u_n), u_n - u_*^\lambda \right\rangle = 0, \\ \Rightarrow\quad & u_n \to u_*^\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.2)}. \end{aligned} \tag{3.25}$$

So, if in (3.23) we pass to the limit as $n \to \infty$ and use (3.25), then

$$\left\langle A(u_*^\lambda), h\right\rangle + \int_{\partial\Omega} \beta(z)(u_*^\lambda)^{p-1} h d\sigma = \int_\Omega f(z, u_*^\lambda, \lambda) h dz \tag{3.26}$$
$$\text{for all } h \in W^{1,p}(\Omega) \text{ (see hypothesis } H_2(i)).$$

Also, from Proposition 3.3 we have

$$\begin{aligned} &\bar{u}_* \leq u_n \text{ for all } n \geq 1, \\ \Rightarrow\quad &\bar{u}_* \leq u_*^\lambda, \text{ hence } u_*^\lambda \neq 0. \end{aligned} \tag{3.27}$$

From (3.26) and (3.27), as before we infer that

$$u_*^\lambda \in S_+(\lambda) \text{ and } u_*^\lambda = \inf S_+(\lambda).$$

Similarly we obtain $v_\lambda^* \in S_-(\lambda)$, $v_*^\lambda = \sup S_-(\lambda)$. □

Now that we have the extremal constant sign solutions, we can produce a nodal solution. The idea is to use variational methods to locate a nontrivial solution of (P_λ) in the order interval $[v_*^\lambda, u_*^\lambda]$, which is distinct from v_*^λ and u_*^λ. Then the extremality of v_*^λ and u_*^λ, implies that this third nontrivial solution is necessarily nodal (that is, sign changing).

Proposition 3.5 *If hypotheses $H(\beta)$ and H_2 hold and $\lambda \in (0, \lambda^*)$ then problem (P_λ) admits a nodal solution $y_0 \in C^1(\overline{\Omega})$.*

Proof. We consider the following truncation-perturbation of the reaction $f(z, \cdot, \lambda)$:

$$w_\lambda(z,x) = \begin{cases} f(z, v_*^\lambda(z), \lambda) + v_*^\lambda(z)^{p-1} & \text{if } x < v_*^\lambda(z) \\ f(z,x,\lambda) + |x|^{p-2}x & \text{if } v_*^\lambda(z) \leq x \leq u_*^\lambda(z) \\ f(z, u_*^\lambda(z).\lambda) + u_*^\lambda(z)^{p-1} & \text{if } u_*^\lambda(z) < x. \end{cases} \tag{3.28}$$

This is a Carathéodory function. We set $W_\lambda(z,x) = \int_0^x w_\lambda(z,s)ds$.

Also, we consider a corresponding truncation of the boundary term, namely the Carathéodory function

$$\gamma_\lambda(z,x) = \begin{cases} |v_*^\lambda(z)|^{p-2}v_*^\lambda(z) & \text{if } x < v_*^\lambda(z) \\ |x|^{p-2}x & \text{if } v_*^\lambda(z) \leq x \leq u_*^\lambda(z) \\ u_*^\lambda(z)^{p-1} & \text{if } u_*^\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathcal{R}. \tag{3.29}$$

We set $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)ds$.

We consider the C^1-functional $\xi_\lambda : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\xi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \int_{\partial\Omega} \beta(z)\Gamma_\lambda(z,u)d\sigma - \int_\Omega W_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

In addition, we consider also the positive and negative truncations of $w_\lambda(z,\cdot)$ and of $\gamma_\lambda(z,\cdot)$. So, we define

$$w_\lambda^\pm(z,x) = w_\lambda(z, \pm x^\pm) \text{ and } \gamma_\lambda^\pm(z,x) = \gamma_\lambda(z, \pm x^\pm).$$

These are Carathéodory functions. We set $W_\lambda^\pm(z,x) = \int_0^x w_\lambda^\pm(z,s)ds$ and $\Gamma_\lambda^\pm(z,x) = \int\limits_0^x \gamma_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $\xi_\lambda^\pm : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\xi_\lambda^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \int\limits_{\partial\Omega} \beta(\lambda)\Gamma_\lambda^\pm(z,u)d\sigma - \int_\Omega W_\lambda^\pm(z,u)dz$$

$$\text{for all } u \in W^{1,p}(\Omega).$$

Claim 1. $K_{\xi_\lambda} \subseteq [v_*^\lambda, u_*^\lambda],\ K_{\xi_\lambda^+} = \{0, u_*^\lambda\},\ K_{\xi_\lambda^-} = \{0, v_*^\lambda\}$.

Let $u \in K_{\xi_\lambda}$. Then

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uhdz + \int_{\partial\Omega} \beta(z)\gamma_\lambda(z,u)hd\sigma = \int_\Omega w_\lambda(z,u)hdz \tag{3.30}$$

for all $h \in W^{1,p}(\Omega)$.

In (3.30) we choose $h = (u - u_*^\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(u), (u-u_*^\lambda)^+\rangle + \int_\Omega u^{p-1}(u-u_*^\lambda)^+dz + \int_{\partial\Omega} \beta(z)(u_*^\lambda)^{p-1}(u-u_*^\lambda)^+d\sigma \\
= \ & \int_\Omega \left[f(z,u_*^\lambda,\lambda) + (u_*^\lambda)^{p-1}\right](u-u_*^\lambda)^+dz \text{ (see (3.28), (3.29))} \\
= \ & \langle A(u_*^\lambda), (u-u_*^\lambda)^+\rangle + \int_\Omega (u_*^\lambda)^{p-1}(u-u_*^\lambda)^+dz + \int_{\partial\Omega} \beta(z)(u_*^\lambda)^{p-1}(u-u_*^\lambda)^+d\sigma \\
& \text{(since } u_*^\lambda \in S_+(\lambda)), \\
\Rightarrow \ & \langle A(u) - A(u_*^\lambda), (u-u_*^\lambda)^+\rangle + \int_\Omega (u^{p-1} - (u_*^\lambda)^{p-1})(u-u_*^\lambda)^+dz = 0, \\
\Rightarrow \ & |\{u > u_*^\lambda\}|_N = 0, \text{ hence } u \leq u_*^\lambda.
\end{aligned}$$

Also, in (3.30) we choose $h = (v_*^\lambda - u)^+ \in W^{1,p}(\Omega)$ and obtain $u \geq v_*^\lambda$. Therefore,

$$\begin{aligned}
& u \in [v_*^\lambda, u_*^\lambda], \\
\Rightarrow \ & K_{\xi_\lambda} \subseteq [v_*^\lambda, u_*^\lambda].
\end{aligned}$$

In a similar fashion, we show that

$$K_{\xi_\lambda^+} \subseteq [0, u_*^\lambda] \text{ and } K_{\xi_\lambda^-} \subseteq [v_*^\lambda, 0].$$

The extremality of v_*^λ and u_*^λ implies that

$$K_{\xi_\lambda^+} = \{0, u_*^\lambda\} \text{ and } K_{\xi_\lambda^-} = \{0, v_*^\lambda\}.$$

This proves Claim 1.

Claim 2. $u_*^\lambda \in \operatorname{int} C_+$ and $v_*^\lambda \in -\operatorname{int} C_+$ are local minimizers of the functional ξ_λ.

Evidently the functional ξ_λ^+ is coercive (see (3.28) and (3.29)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_*^\lambda \in W^{1,p}(\Omega)$ such that

$$\xi_\lambda^+(\hat{u}_*^\lambda) = \inf\left[\xi_\lambda^+(u) :\ u \in W^{1,p}(\Omega)\right]. \tag{3.31}$$

Since $q < p$, as in the proof of Proposition 3.1, by choosing $t \in (0,1)$ small (at least such that $t\hat{u}_1(z) \leq \min_{\overline{\Omega}} u_*^\lambda$; recall $\hat{u}_1,\ u_*^\lambda \in \operatorname{int} C_+$), we have

$$\begin{aligned}
& \xi_\lambda^+(t\hat{u}_1) < 0, \\
\Rightarrow \ & \xi_\lambda^+(\hat{u}_*^\lambda) < 0 = \xi_\lambda^+(0) \text{ (see (3.31)), hence } \hat{u}_*^\lambda \neq 0.
\end{aligned}$$

From (3.31) we have

$$\begin{aligned} &\hat{u}_*^\lambda \in K_{\xi_\lambda^+} \setminus \{0\}, \\ \Rightarrow \quad &\hat{u}_*^\lambda = u_*^\lambda \in \operatorname{int} C_+ \text{ (see Claim 1).} \end{aligned}$$

Note that $\xi_\lambda^+\Big|_{C_+} = \xi_\lambda\Big|_{C_+}$. Therefore u_*^λ is a local $C^1(\overline{\Omega})$-minimizer of ξ_λ. Invoking Proposition 2.1, we conclude that u_*^λ is a local $W^{1,p}(\Omega)$-minimizer of ξ_λ. Similarly we show that $v_*^\lambda \in -\operatorname{int} C_+$ is a local minimizer of ξ_λ, using this time the functional ξ_λ^-. This proves Claim 2.

Without any loss of generality, we may assume $\xi_\lambda(v_*^\lambda) \le \xi_\lambda(u_*^\lambda)$ (the analysis is similar if the opposite inequality holds). Because of Claim 2, we can find $\rho \in (0,1)$ small such that

$$\xi_\lambda(v_*^\lambda) \le \xi_\lambda(u_*^\lambda) < \inf\left[\xi_\lambda(u) : \ \|u - u_*^\lambda\| = \rho\right] = \eta_\rho^\lambda, \ \|v_*^\lambda - u_*^\lambda\| > \rho \tag{3.32}$$

(see Gasinski and Papageorgiou [14], proof of Theorem 2.12). From (3.28) and (3.29) it is clear that ξ_λ is coercive, hence it satisfies the C-conditions. This fact and (3.32), permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &y_0 \in K_{\xi_\lambda} \text{ and } \eta_\rho^\lambda \le \xi_\lambda(y_0), \\ \Rightarrow \quad &y_0 \in [v_*^\lambda, u_*^\lambda], \ y_0 \notin \{v_*^\lambda, u_*^\lambda\} \text{ (see Claim 1 and (3.32)),} \\ \Rightarrow \quad &y_0 \text{ is a solution of } (P_\lambda) \text{ (see (3.28), (3.29))} \\ &\qquad\qquad \text{and} \\ &y_0 \in C^1(\overline{\Omega}) \text{ (by the nonlinear regularity theory).} \end{aligned}$$

It remains to show that $y_0 \neq 0$. Since y_0 is a critical point of mountain pass type for the functional ξ_λ, we have

$$C_1(\xi_\lambda, y_0) \neq 0. \tag{3.33}$$

Next we compute the critical groups of ξ_λ at the origin. We mention that Moroz [22] was the first to compute the critical groups of functionals defined on $H_0^1(\Omega)$ and concave near the origin. Jiu and Su [18] extended the work of Moroz to functionals defined on $W_0^{1,p}(\Omega)$.

Claim 3. $C_k(\xi_\lambda, 0) = 0$ for all $k \ge 0$.

From (3.28) and hypothesis $H_2(iii)$, we see that

$$W_\lambda(z,x) \ge \frac{c_0}{q}|x|^q - c_5|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathcal{R}, \text{ some } c_5 > 0. \tag{3.34}$$

For all $u \in W^{1,p}(\Omega)$ and $t > 0$, we have

$$\xi_\lambda(tu) \le \frac{t^p}{p}\|Du\|_p^p + \frac{t^p}{p}\|u\|_p^p + \frac{t^p}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma + c_5 t^r \|u\|_r^r - \frac{c_0 t^q}{q}\|u\|_q^q \tag{3.35}$$

(see (3.29) and (3.34)).

Since $q < p < r$, from (3.35) it is clear that we can find $t^* = t^*(u) \in (0,1)$ such that

$$\xi_\lambda(tu) < 0 \text{ for all } t \in (0, t^*). \tag{3.36}$$

Suppose $u \in W^{1,p}(\Omega)$, $0 < ||u|| \leq 1$ and $\xi_\lambda(u) = 0$. Then

$$\begin{aligned}
\frac{d}{dt}\xi_\lambda(tu)\Big|_{t=1} &= \langle \xi'_\lambda(u), u\rangle \\
&= ||Du||_p^p + ||u||_p^p + \int_{\partial\Omega} \beta(z)\gamma_\lambda(z,u)ud\sigma - \int_\Omega w_\lambda(z,u)udz \\
&= \left(1-\frac{q}{p}\right)||Du||_p^p + \left(1-\frac{q}{p}\right)||u||_p^p + \int_{\partial\Omega}\beta(z)\left[\gamma_\lambda(z,u)u - q\Gamma_\lambda(z,u)\right]d\sigma \\
&\qquad - \int_\Omega \left[w_\lambda(z,u)u - qW_\lambda(z,u)\right]dz \text{ (since } \xi_\lambda(u)=0) \\
&\geq \left(1-\frac{q}{p}\right)||u||^p + \int_\Omega \left[qW_\lambda(z,u) - w_\lambda(z,u)u\right]dz \text{ (see (3.29)} \\
&\qquad \text{and hypotheses } H(\beta)) \\
&\geq c_6||u||^p - c_7||u||^r \text{ for some } c_6, c_7 > 0 \text{ with } r > p \\
&\qquad \text{(see hypothesis } H_2(iii) \text{ and (3.28)).}
\end{aligned}$$

Since $r > p$, we see that for $\rho \in (0,1)$ small we have

$$\frac{d}{dt}\xi_\lambda(tu)\Big|_{t=1} > 0 \text{ for all } u \in W^{1,p}(\Omega) \text{ with } 0 < ||u|| \leq \rho,\ \xi_\lambda(u) = 0. \tag{3.37}$$

Let $u \in W^{1,p}(\Omega)$ with $0 < ||u|| \leq \rho$ and $\xi_\lambda(u) = 0$. We will show that

$$\xi_\lambda(tu) \leq 0 \text{ for all } t \in [0,1]. \tag{3.38}$$

Suppose that (3.38) does not hold. Then we can find $t_0 \in (0,1)$ such that $\xi_\lambda(t_0u) > 0$. Recall that $\xi_\lambda(u) = 0$. So, we can find $t_1 \in (t_0, 1]$ such that $\xi_\lambda(t_1u) = 0$. Define

$$t_* = \min\{t \in (t_0, 1] : \xi_\lambda(tu) = 0\} > t_0 > 0.$$

Then we have

$$\xi_\lambda(tu) \geq 0 \text{ for all } t \in [t_0, t_*]. \tag{3.39}$$

Let $v = t_*u$. We have $0 < ||v|| \leq ||u|| \leq \rho$ and $\xi_\lambda(v) = 0$. So, from (3.37) it follows that

$$\frac{d}{dt}\xi_\lambda(tv)\Big|_{t=1} > 0. \tag{3.40}$$

From (3.39) we have

$$\begin{aligned}
&\xi_\lambda(v) = \xi_\lambda(t_*u) = 0 < \xi_\lambda(tu) \text{ for all } t \in [t_0, t_*), \\
\Rightarrow\ &\frac{d}{dt}\xi_\lambda(tv)\Big|_{t=1} = t_*\frac{d}{dt}\xi_\lambda(tu)\Big|_{t=t_*} = t_*\lim_{t\to t_*^-}\frac{\xi_\lambda(tu)}{t-t_*} \leq 0.
\end{aligned} \tag{3.41}$$

Comparing (3.40) and (3.41), we reach a contradiction. This proves (3.38).

By choosing $\rho \in (0,1)$ even smaller if necessary, we can have $K_{\xi_\lambda} \cap \overline{B}_\rho = \{0\}$ (here $\overline{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\| \le \rho\}$). Let $h : [0,1] \times (\xi_\lambda^0 \cap \overline{B}_\rho) \to \xi_\lambda^0 \cap \overline{B}_\rho$ be the continuous function defined by

$$h(t,u) = (1-t)u \text{ for all } (t,u) \in [0,1] \times (\xi_\lambda^0 \cap \overline{B}_\rho).$$

From (3.38) we see that this deformation is well-defined and shows that the set $\xi_\lambda^0 \cap \overline{B}_\rho$ is contractible in itself.

Consider $u \in \overline{B}_\rho$ with $\xi_\lambda(u) > 0$. We show that there exists unique $t(u) \in (0,1)$ such that

$$\xi_\lambda(t(u)u) = 0. \tag{3.42}$$

Since $\xi_\lambda(u) > 0$ and $t \longmapsto \xi_\lambda(tu)$ is continuous, from (3.36) and Bolzano's theorem, we see that we can find $t(u) \in (0,1)$ such that (3.42) holds. We need the uniqueness of the $t(u)$. Suppose $0 < \hat{t}_1 = t(u)_1 < \hat{t}_2 = t(u)_2 < 1$ both satisfy (3.42). Then from (3.38), we have

$$\mu(t) = \xi_\lambda(t\hat{t}_2 u) \le 0 \text{ for all } t \in [0,1].$$

Then $\hat{t}_1/\hat{t}_2 \in (0,1)$ is a maximizer of the function μ and so

$$\begin{aligned} & \frac{d}{dt}\mu(t)\Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} = 0, \\ \Rightarrow \quad & \frac{\hat{t}_1}{\hat{t}_2}\frac{d}{dt}\xi_\lambda(t\hat{t}_2 u)\Big|_{t=\frac{\hat{t}_1}{\hat{t}_2}} = \frac{d}{dt}\xi_\lambda(t\hat{t}_1 u)\Big|_{t=1} = 0, \end{aligned}$$

which contradicts (3.37). This proves the uniqueness of $t(u) \in (0,1)$ satisfying (3.42). The uniqueness of $t(u) \in (0,1)$ implies that

$$\begin{aligned} &\xi_\lambda(tu) < 0 \text{ for } t \in (0, t(u)) \text{ (see (3.36))} \\ &\xi_\lambda(tu) > 0 \text{ for } t \in (t(u), 1] \text{ (see (3.42) and recall } \xi_\lambda(u) > 0). \end{aligned} \tag{3.43}$$

We introduce the function $\vartheta : \overline{B}_\rho \setminus \{0\} \to (0,1]$ defined by

$$\vartheta(u) = \begin{cases} 1 & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \xi_\lambda(u) \le 0 \\ t(u) & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \xi_\lambda(u) > 0. \end{cases} \tag{3.44}$$

It is easily seen that $\vartheta(\cdot)$ is continuous. Then using $\vartheta(\cdot)$, we can define the map $\tau : \overline{B}_\rho \setminus \{0\} \to (\xi_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\}$ by setting

$$\tau(u) = \begin{cases} u & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \xi_\lambda(u) \le 0 \\ \vartheta(u)u & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \xi_\lambda(u) > 0. \end{cases} \tag{3.45}$$

The continuity of $\vartheta(\cdot)$ implies that $\tau(\cdot)$ is continuous too. Also, we have

$$\begin{aligned} & \tau\Big|_{(\xi_\lambda^0 \cap \overline{B}_\rho)\setminus\{0\}} = id\Big|_{(\xi_\lambda^0 \cap \overline{B}_\rho)\setminus\{0\}} \text{ (see (3.45))}, \\ \Rightarrow \quad & (\xi_\lambda^0 \cap \overline{B}_\rho) \setminus \{0\} \text{ is a retract of } \overline{B}_\rho \setminus \{0\}, \text{ with retraction } \tau. \end{aligned}$$

But $\overline{B}_\rho \setminus \{0\}$ is contractible in itself. Hence so is $(\xi^0_\lambda \cap \overline{B}_\rho) \setminus \{0\}$. Recalling that $\xi^0_\lambda \cap \overline{B}_\rho$ is contractible in itself (it was established earlier), we have

$$H_k(\xi^0_\lambda \cap \overline{B}_\rho, (\xi^0_\lambda \cap \overline{B}_\rho) \setminus \{0\}) = 0 \text{ for all } k \geq 0$$

(see Granas and Dugundji [16, p. 389]),

$$\Rightarrow \quad C_k(\xi_\lambda, 0) = 0 \text{ for all } k \geq 0.$$

This proves Claim 3.

From Claim 3 and (3.33) we infer that $y_0 \neq 0$. Since $y_0 \in [v^\lambda_*, u^\lambda_*]$, $y_0 \notin \{v^\lambda_*, u^\lambda_*\}$ it follows that $y_0 \in C^1(\overline{\Omega})$ is a nodal solution of problem (P_λ) with $\lambda \in (0, \lambda^*)$. □

So, we can state the following multiplicity theorem for problem (P_λ). Note that our result provides sign information for all the solutions produced and the reaction satisfies a very general growth condition and no asymptotic conditions at $\pm\infty$ are imposed.

Theorem 3.1 *If hypotheses $H(\beta)$ and H_2 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) admits at least three distinct nontrivial solutions*

$$u_0 \in int\, C_+,\ v_0 \in -int\, C_+ \text{ and } y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

4 Five and six nontrivial solutions

In this section, we assume that the reaction $f(z, \cdot, \lambda)$ exhibits subcritical growth and satisfies certain asymptotic conditions at $\pm\infty$ which imply that $x \longmapsto f(z, x, \lambda)$ is $(p-1)$-superlinear. However, we do not employ the usual in such cases AR-condition (see [5]). Instead, we use an alternative condition (see hypothesis $H_3(iv)$), which incorporates in our framework superlinear reaction with "slow" growth near $\pm\infty$.

The new hypotheses on $f(z, x, \lambda)$ are the following:

H_3: $f : \Omega \times \mathcal{R} \times (0, \infty) \to \mathcal{R}$ is a function such that for a.a. $z \in \Omega$, all $\lambda > 0$, $f(z, 0, \lambda) = 0$ and

(i) for all $\lambda > 0$, $(z, x) \longmapsto f(z, x, \lambda)$ is a Carathéodory function;

(ii) $|f(z, x, \lambda)| \leq a(z, \lambda) + c|x|^{r-1}$ for a.a. $z \in \Omega$, all $x \in \mathcal{R}$, with $a(\cdot, \lambda) \in L^\infty(\Omega)_+$,

$$\|a(\cdot, \lambda)\|_\infty \to 0 \text{ as } \lambda \to 0^+,$$

$c > 0$ and $p < r < p^*$;

(iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds$, then

$$\lim_{x \to \pm\infty} \frac{F(z, x, \lambda)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) if $k_\lambda(z, x) = f(z, x, \lambda)x - pF(z, x, \lambda)$, then there exists $\beta^*_\lambda \in L^1(\Omega)_+$ such that

$$k_\lambda(z, x') \leq k_\lambda(z, x) + \beta^*_\lambda(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x' \leq x \text{ or } x' \leq x \leq 0;$$

(v) there exist $q = q(\lambda) \in (1, p)$ and $\delta_0 = \delta_0(\lambda)$, $c_0 = c_0(\lambda) > 0$ such that

$$qF(z, x, \lambda) \geq f(z, x, \lambda)x \geq c_0|x|^q \text{ for a.a. } z \in \Omega, \text{ all } a \leq |x| \leq \delta_0;$$

(vi) for every $\rho > 0$ and $\lambda > 0$, there exists $\xi_\rho^\lambda > 0$ such that for a.a. $z \in \Omega$,

$$x \longmapsto f(z, x, \lambda) + \xi_\rho^\lambda |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark. Hypothesis $H_3(iii)$ implies that for a.a. $z \in \Omega$, all $\lambda > 0$, the primitive $F(z, \cdot, \lambda)$ is p-superlinear. Hypotheses $H_3(iii), (iv)$ imply that the reaction $x \longmapsto f(z, x, \lambda)$ is $(p-1)$-superlinear (see Li and Yang [20, Lemma 2.4]). A slightly more restrictive version of hypothesis $H_3(iv)$ was used earlier by Miyagaki and Souto [21] and Li and Yang [20].

Examples. The following functions satisfy hypotheses H_3. As before, for the sake of simplicity we drop the z-dependence:

$$f_1(x) = \lambda |x|^{q-2}x + |x|^{r-2}x \text{ with } 1 < q < p < r < p^*,$$
$$f_2(x) = \lambda |x|^{q-2}x + |x|^{p-2}x \left[\ln|x| + \frac{1}{p}\right] \text{ with } 1 < q < p.$$

Note that f_2 does not satisfy the AR-condition (see [5]).

Under the above conditions, we can prove a multiplicity theorem producing five nontrivial solutions, all with sign information.

Theorem 4.1 *If hypotheses $H(\beta)$ and H_3 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least five nontrivial solutions*

$$u_0, \hat{u} \in int\, C_+,\ u_0 \leq \hat{u},\ u_0 \neq \hat{u}$$
$$v_0, \hat{v} \in -int\, C_+,\ \hat{v} \leq v_0,\ v_0 \neq \hat{v}$$
$$y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

Proof. From Theorem 3.1, we know that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least three nontrivial solutions

$$u_0 \in \text{int}\, C_+,\ v_0 \in -\text{int}\, C_+ \text{ and } y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

By virtue of Proposition 3.4, without any loss of generality, we may assume that u_0 and v_0 are extremal constant sign solutions.

We will use $u_0 \in \text{int}\, C_+$ and $v_0 \in -\text{int}\, C_+$ to produce two more nontrivial constant sign solutions.

First we produce a second positive solution. To this end, using $u_0 \in \text{int}\, C_+$ we introduce the following truncation-perturbation of the reaction $f(z, \cdot, \lambda)$:

$$g_\lambda^+(z, x) = \begin{cases} f(z, u_0(z), \lambda) + u_0(z)^{p-1} & \text{if } x < u_0(z) \\ f(z, x, \lambda) + x^{p-1} & \text{if } u_0(z) \leq x. \end{cases} \tag{4.46}$$

We also introduce a corresponding truncation of the boundary term:

$$\eta_+(z,x) = \begin{cases} \beta(z)u_0(z)^{p-1} & \text{if } x < u_0(z) \\ \beta(z)x^{p-1} & \text{if } u_0(z) \le x, \end{cases} \tag{4.47}$$

for all $(z,x) \in \partial\Omega \times \mathcal{R}$. Both are Carathéodory functions. We set

$$G_\lambda^+(z,x) = \int_0^x g_\lambda^+(z,s)ds \text{ and } H_+(z,x) = \int_0^x \eta_+(z,s)ds$$

and consider the C^1-functional $\tau_\lambda^+ : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\tau_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \int_{\partial\Omega} H_+(z,u)d\sigma - \int_\Omega G_\lambda^+(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Claim 1. We may assume that $u_0 \in \operatorname{int} C_+$ is a local minimizer of the functional τ_λ^+.
Let $\overline{u} \in \operatorname{int} C_+$ be as in the proof of Proposition 3.1. We know that

$$u_0 \in [0, \overline{u}].$$

We introduce the following truncations of $g_\lambda^+(z,\cdot)$ and $\eta_+(z,\cdot)$:

$$\hat{g}_\lambda^+(z,x) = \begin{cases} g_\lambda^+(z,x) & \text{if } x < \overline{u}(z) \\ g_\lambda^+(z,\overline{u}(z)) & \text{if } \overline{u}(z) \le x, \end{cases} \tag{4.48}$$

$$\hat{\eta}_+(z,x) = \begin{cases} \eta_+(z,x) & \text{if } x < \overline{u}(z) \\ \eta_+(z,\overline{u}(z)) & \text{if } \overline{u}(z) \le x, \end{cases} \tag{4.49}$$

for all $(z,x) \in \partial\Omega \times \mathcal{R}$.

Both are Carathéodory functions. We set

$$\hat{G}_\lambda^+(z,x) = \int_0^x \hat{g}_\lambda^+(z,s)ds \text{ and } \hat{H}_+(z,x) = \int_0^x \hat{\eta}_+(z,s)ds$$

and consider the C^1-functional $\hat{\tau}_\lambda^+ : W^{1,p}(\Omega) \to \mathcal{R}$ defined by

$$\hat{\tau}_\lambda^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \int_{\partial\Omega} \hat{H}_+(z,u)d\sigma - \int_\Omega \hat{G}_\lambda^+(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (4.48) and (4.49) it is clear that $\hat{\tau}_\lambda^+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\tau}_\lambda^+(\hat{u}_0) = \inf\left[\hat{\tau}_\lambda^+(u) : u \in W^{1,p}(\Omega)\right], \\ \Rightarrow\ & (\hat{\tau}_\lambda^+)'(\hat{u}_0) = 0, \\ \Rightarrow\ & \langle A(\hat{u}_0), h\rangle + \int_\Omega |\hat{u}_0|^{p-2}\hat{u}_0 h dz + \int_{\partial\Omega} \hat{\eta}_+(z,\hat{u}_0)h d\sigma = \int_\Omega \hat{g}_\lambda^+(z,\hat{u}_0)h dz \qquad (4.50) \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (4.50), first we choose $h = (u_0 - \hat{u}_0)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
& \langle A(\hat{u}_0), (u_0 - \hat{u}_0)^+\rangle + \int_\Omega |\hat{u}_0|^{p-2}\hat{u}_0(u_0 - \hat{u}_0)^+ dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}(u_0 - \hat{u}_0)^+ d\sigma \\
= & \int_\Omega \left[f(z, u_0, \lambda) + u_0^{p-1}\right](u_0 - \hat{u}_0)^+ dz \text{ (see (4.46), (4.47), (4.48), (4.49))} \\
= & \langle A(u_0), (u_0 - \hat{u}_0)^+\rangle + \int_\Omega u_0^{p-1}(u_0 - \hat{u}_0)^+ dz + \int_{\partial\Omega} \beta(z)u_0^{p-1}(u_0 - \hat{u}_0)^+ d\sigma \\
& \qquad \text{(since } u_0 \in S_+(\lambda)) \\
\Rightarrow & \langle A(u_0) - A(\hat{u}_0), (u_0 - \hat{u}_0)^+\rangle + \int_\Omega (u_0^{p-1} - |\hat{u}_0|^{p-2}\hat{u}_0)(u_0 - \hat{u}_0)^+ dz = 0, \\
\Rightarrow & |\{u_0 > \hat{u}_0\}|_N = 0, \text{ hence } u_0 \le \hat{u}_0.
\end{aligned}$$

Next in (4.50) we choose $h = (\hat{u}_0 - \overline{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
& \langle A(\hat{u}_0), (\hat{u}_0 - \overline{u})^+\rangle + \int_\Omega \hat{u}_0^{p-1}(\hat{u}_0 - \overline{u})^+ dz + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(u_0 - \overline{u})^+ d\sigma \\
= & \int_\Omega \left[f(z, \overline{u}, \lambda) + \overline{u}^{p-1}\right](\hat{u}_0 - \overline{u})^+ dz \text{ (see (4.46), (4.47), (4.48), (4.49))} \\
\le & \int_\Omega \left[\xi^{p-1} + \overline{u}^{p-1}\right](\hat{u}_0 - \overline{u})^+ dz \text{ (see the Claim in the proof of Proposition 3.1)} \\
= & \langle A(\overline{u}), (\hat{u}_0 - \overline{u})^+\rangle + \int_\Omega \overline{u}^{p-1}(\hat{u}_0 - \overline{u})^+ dz + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(\hat{u}_0 - \overline{u})^+ d\sigma, \\
\Rightarrow & \langle A(\hat{u}_0) - A(\overline{u}), (\hat{u}_0 - \overline{u})^+\rangle + \int_\Omega (\hat{u}_0^{p-1} - \overline{u}^{p-1})(\hat{u}_0 - \overline{u})^+ dz \le 0, \\
\Rightarrow & |\{\hat{u}_0 > \overline{u}\}|_N = 0, \text{ hence } \hat{u}_0 \le \overline{u}.
\end{aligned}$$

So, we have proved that

$$\hat{u}_0 \in [u_0, \overline{u}].$$

Then by virtue of (4.46) – –(4.49), equation (4.50) becomes

$$\begin{aligned}
& \langle A(\hat{u}_0), h\rangle + \int_{\partial\Omega} \beta(z)\hat{u}_0^{p-1} h d\sigma = \int_\Omega f(z, \hat{u}_0, \lambda) h dz \text{ for all } h \in W^{1,p}(\Omega), \\
\Rightarrow & \hat{u}_0 \in S_+(\lambda).
\end{aligned}$$

If $\hat{u}_0 \neq u_0$, then this is the desired second nontrivial positive solution of (P_λ) and $u_0 \le \hat{u}_0$. Therefore, we may assume that $\hat{u}_0 = u_0$. For $\delta > 0$, let $u_0^\delta = u_0 + \delta \in \operatorname{int} C_+$. Let $\rho = \|\overline{u}\|_\infty$

and let $\xi_\rho^\lambda > 0$ be as postulated by hypothesis $H_3(v)$. Then

$$\begin{array}{rl} & -\Delta_p u_0^\delta + \xi_\rho^\lambda (u_0^\delta)^{p-1} \\ \leq & -\Delta_p u_0 + \xi_\rho^\lambda u_0^{p-1} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\ = & f(z, u_0, \lambda) + \xi_\rho^\lambda u_0^{p-1} + \chi(\delta) \\ < & \hat{\xi}^{p-1} + \xi_\rho^\lambda \overline{u}^{p-1} \text{ for } \delta > 0 \text{ small} \\ & \qquad \text{(see the Claim in the proof of Proposition 3.1)} \\ = & -\Delta_p \overline{u} + \xi_\rho^\lambda \overline{u}^{p-1} \text{ a.e. in } \Omega, \\ \Rightarrow & u_0^\delta \leq \overline{u} \text{ for } \delta > 0 \text{ small, hence } \overline{u} - u_0 \in \operatorname{int} C_+. \end{array}$$

So, we have proved that

$$u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, \overline{u}].$$

Note that $\tau_\lambda^+\Big|_{[0,\overline{u}]} = \hat{\tau}_\lambda^+\Big|_{[0,\overline{u}]}$ (see (4.46), (4.47), (4.48), (4.49)). So, it follows that

$$\begin{array}{rl} & u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \tau_\lambda^+, \\ \Rightarrow & u_0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \tau_\lambda^+ \text{ (see Proposition 2.1).} \end{array}$$

This proves Claim 1.

By virtue of Claim 1, we can find $\rho \in (0,1)$ small such that

$$\tau_\lambda^+(u_0) < \inf \left[\tau_\lambda^+(u) : \|u - u_0\| = \rho\right] = \eta_\lambda^+. \tag{4.51}$$

Claim 2. The functional τ_λ^+ satisfies the C-condition.

Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\tau_\lambda^+(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1; \tag{4.52}$$

$$(1 + \|u_n\|)(\tau_\lambda^+)'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{4.53}$$

From (4.53) we have

$$\left| \langle A(u_n), h \rangle + \int_\Omega |u_n|^{p-2} u_n h dz + \int_{\partial\Omega} \eta_+(z, u_n) h d\sigma - \int_\Omega g_\lambda^+(z, u_n) h dz \right| \leq$$

$$\leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{4.54}$$

In (4.54) first we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Using (4.46) and (4.47), we obtain

$$\begin{array}{rl} & \|u_n^-\|^p \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1, \\ \Rightarrow & \{u_n^-\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \end{array} \tag{4.55}$$

Next in (4.54) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\|Du_n^+\|_p^p - \|u_n^+\|_p^p - \int_{\partial\Omega} \eta_+(z, u_n^+)u_n^+ d\sigma + \int_{\Omega} g_\lambda^+(z, u_n^+)u_n^+ dz \leq \varepsilon_n \quad \text{for all } n \geq 1. \tag{4.56}$$

On the other hand from (4.52) and (4.55), we have

$$\|Du_n^+\|_p^p + \|u_n^+\|_p^p + \int_{\partial\Omega} pH_+(z, u_n^+)d\sigma - \int_{\Omega} pG_\lambda^+(z, u_n^+)dz \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \geq 1. \tag{4.57}$$

Adding (4.56) and (4.57), we obtain

$$\int_{\partial\Omega} [pH_+(z, u_n^+) - \eta_+(z, u_n^+)u_n^+]d\sigma + \int_{\Omega} [g_\lambda^+(z, u_n^+)u_n^+ - pG_\lambda^+(z, u_n^+)]dz \leq$$
$$\leq M_4 \text{ for some } M_4 > 0, \text{ all } n \geq 1. \tag{4.58}$$

From (4.47) we see that

$$pH_+(z, x) - \eta_+(z, x)x = \begin{cases} (p-1)\beta(z)u_0(z)^{p-1}x & \text{if } x \in [0, u_0(z)] \\ (p-1)\beta(z)u_0(z)^p & \text{if } u_0(z) < x. \end{cases} \tag{4.59}$$

Using (4.59) in (4.58), we obtain

$$\int_{\Omega} [g_\lambda^+(z, u_n^+)u_n^+ - pG_\lambda^+(z, u_n^+)]dz \leq M_4 \text{ for all } n \geq 1,$$
$$\Rightarrow \int_{\Omega} [f(z, u_n^+, \lambda)u_n^+ - pF(z, u_n^+, \lambda)]dz \leq M_5 \text{ for some } M_5 > 0, \quad \text{all } n \geq 1 \text{ (see (4.46))}. \tag{4.60}$$

Using (4.60), we will show that $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Arguing by contradiction, suppose that this is not true. By passing to a subsequence if necessary, we may assume that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.61}$$

First we assume that $y \neq 0$. Let $Z(y) = \{z \in \Omega : y(z) = 0\}$. Then

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \Omega \setminus Z(y).$$

Then hypothesis $H_3(iii)$ implies that

$$\frac{F(z, u_n^+(z), \lambda)}{\|u_n^+\|^p} = \frac{F(z, u_n^+(z), \lambda)}{u_n^+(z)^p} y_n(z)^p \to +\infty \text{ for a.a. } z \in \Omega \setminus Z(y), \text{ as } n \to \infty.$$

Using this convergence and Fatou's lemma (see hypothesis $H_3(iii)$), we have

$$\lim_{n\to\infty}\int_\Omega \frac{F(z,u_n^+,\lambda)}{||u_n^+||^p}dz = +\infty. \tag{4.62}$$

But from (4.52) and (4.55) we have

$$-\frac{1}{p}||u_n^+||^p - \int_{\partial\Omega} H_+(z,u_n^+)d\sigma + \int_\Omega G_\lambda^+(z,u_n^+)dz \le M_6 \text{ for some } M_6>0, \text{ all } n\ge 1.$$

$$\Rightarrow \int_\Omega F(z,u_n^+,\lambda)dz \le \frac{1}{p}||u_n^+||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u_n^+)^p d\sigma + M_7$$

$$\text{for some } M_7>0, \text{ all } n\ge 1 \text{ (see (4.46) and (4.47))}$$

$$\Rightarrow \int_\Omega \frac{F(z,u_n^+,\lambda)}{||u_n^+||^p}dz \le M_8 \text{ for some } M_8>0, \text{ all } n\ge 1. \tag{4.63}$$

Comparing (4.62) and (4.63) we reach a contradiction.

So, we may assume that $y=0$. Let $k>0$ and let $w_n=(2kp)^{1/p}y_n$ for all $n\ge 1$. Evidently $w_n\to 0$ in $L^r(\Omega)$ as $n\to\infty$ (see (4.61)). Hence

$$\int_\Omega G_\lambda^+(z,w_n)dz \to 0 \text{ as } n\to\infty. \tag{4.64}$$

Let $n_0\in\mathbb{N}$ be such that

$$(2kp)^{1/p}\frac{1}{||u_n^+||^p} < 1 \text{ for all } n\ge n_0. \tag{4.65}$$

Also, let $t_n\in[0,1]$ be such that

$$\tau_\lambda^+(t_nu_n^+) = \max_{0\le t\le 1}\tau_\lambda^+(tu_n^+) \text{ for all } n\ge 1. \tag{4.66}$$

Then from (4.65) and (4.66), we have

$$\begin{aligned}\tau_\lambda^+(t_nu_n^+) &\ge \tau_\lambda^+(w_n)\\ &\ge 2k - \int_\Omega G_\lambda^+(z,w_n)dz \text{ (see hypotheses } H(\beta)),\end{aligned}$$

$$\Rightarrow \tau_\lambda^+(t_nu_n^+) \ge k \text{ for all } n\ge n_1\ge n_0 \text{ (see (4.64)).}$$

But $k>0$ is arbitrary. So, we infer that

$$\tau_\lambda^+(t_nu_n^+)\to+\infty \text{ as } n\to\infty. \tag{4.67}$$

Observe that $\{\tau_\lambda^+(u_n^+)\}_{n\geq 1} \subseteq \mathcal{R}$ is bounded (see (4.52) and (4.55)). Also, $\tau_\lambda^+(0) = 0$. Hence from (4.67) it follows that $t_n \in (0,1)$ for all $n \geq 1$. So, we have

$$\begin{aligned} & \frac{d}{dt}\tau_\lambda^+(tu_n^+)\Big|_{t=t_n} = 0 \\ \Rightarrow \quad & t_n^p\|u_n^+\|^p + \int_{\partial\Omega} \eta_+(z, t_nu_n^+)(t_nu_n^+)d\sigma = \int_\Omega g_\lambda^+(z, t_nu_n^+)(t_nu_n^+)dz \quad \text{for all } n \geq n_1. \end{aligned} \tag{4.68}$$

By hypothesis $H_3(iv)$, we have

$$\begin{aligned} & \int_\Omega k_\lambda(z, t_nu_n^+)dz \leq \int_\Omega k_\lambda(z, u_n^+)dz + \|\beta_\lambda^*\|_1 \text{ for all } n \geq 1, \\ \Rightarrow \quad & \int_\Omega k_\lambda(z, t_nu_n^+)dz \leq M_9 \text{ for some } M_9 > 0, \text{ all } n \geq 1 \text{ (see (4.60))}, \\ \Rightarrow \quad & \int_\Omega [g_\lambda^+(z, t_nu_n^+)(t_nu_n^+) - pG_\lambda^+(z, t_nu_n^+)]dz \leq M_{10} \text{ for some } M_{10} > 0, \text{ all } n \geq 1 \\ & \qquad \text{(see (4.46))} \\ \Rightarrow \quad & \|t_nu_n^+\|^p + \int_{\partial\Omega} pH_+(z, t_nu_n^+)d\sigma - \int_\Omega pG_\lambda^+(z, t_nu_n^+)]dz \leq M_{11} \\ & \qquad \text{for some } M_{11} > 0, \text{ all } n \geq n_1 \text{ (see (4.59) and (4.68))}, \\ \Rightarrow \quad & p\tau_\lambda^+(t_nu_n^+) \leq M_{11} \text{ for all } n \geq n_1. \end{aligned} \tag{4.69}$$

Comparing (4.67) and (4.69), we reach a contradiction. This proves Claim 2.

Hypothesis $H_3(iii)$ implies that

$$\tau_\lambda^+(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{4.70}$$

Then (4.51), (4.70) and Claim 2 permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$(\tau_\lambda^+)'(\hat{u}) = 0 \text{ and } \eta_\lambda^+ \leq \tau_\lambda^+(\hat{u}). \tag{4.71}$$

From (4.51) and (4.71) it follows that $\hat{u} \neq u_0$. Also, from the equality in (4.71), we have

$$\langle A(\hat{u}), h\rangle + \int_\Omega |\hat{u}|^{p-2}\hat{u}hdz + \int_{\partial\Omega} \eta_+(z, \hat{u})hd\sigma = \int_\Omega g_\lambda^+(z, \hat{u})hdz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{4.72}$$

In (4.72) we choose $h = (u_0 - \hat{u})^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
& \langle A(\hat{u}), (u_0 - \hat{u})^+ \rangle + \int_\Omega |\hat{u}|^{p-2}\hat{u}(u_0 - \hat{u})^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - \hat{u})^+ d\sigma \\
= \quad & \int_\Omega [f(z, u_0, \lambda) + u_0^{p-1}](u_0 - \hat{u})^+ dz \text{ (see (4.46), (4.47))} \\
= \quad & \langle A(\hat{u}), (u_0 - \hat{u})^+ \rangle + \int_\Omega u_0^{p-1}(u_0 - \hat{u})^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - \hat{u})^+ d\sigma \\
\Rightarrow \quad & \langle A(u_0) - A(\hat{u}), (u_0 - \hat{u})^+ \rangle + \int_\Omega [u_0^{p-1} - |\hat{u}|^{p-2}\hat{u}](u_0 - \hat{u})^+ dz = 0 \\
\Rightarrow \quad & |\{u_0 > \hat{u}\}|_N, \text{ hence } u_0 \le \hat{u}, \ \hat{u} \neq u_0.
\end{aligned}
$$

Then (4.72) becomes

$$
\langle A(\hat{u}), h \rangle + \int_{\partial\Omega} \beta(z) \hat{u}^{p-1} h d\sigma = \int_\Omega f(z, \hat{u}, \lambda) h dz \text{ for all } h \in W^{1,p}(\Omega)
$$

(see (4.46) and (4.47))

$$
\Rightarrow \quad \hat{u} \in S_+(\lambda) \subseteq \operatorname{int} C_+.
$$

Similarly, using $v_0 \in -\operatorname{int} C_+$, introducing

$$
g_\lambda^-(z, x) = \begin{cases} f(z, x, \lambda) + |x|^{p-2}x & \text{if } x < v_0(z) \\ f(z, v_0(z), \lambda) + |v_0(z)|^{p-2}v_0(z) & \text{if } v_0(z) \le x \end{cases}
$$

$$
\text{and} \quad \eta_-(z, x) = \begin{cases} \beta(z)|x|^{p-2}x & \text{if } x < v_0(z) \\ \beta(z)|v_0(z)|^{p-2}v_0(z) & \text{if } v_0(z) \le x, \end{cases}
$$

for all $(z, x) \in \partial\Omega \times \mathcal{R}$ and reasoning as above, we produce a second nontrivial negative solution $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \le v_0$, $v_0 \neq \hat{v}$. □

In the semilinear case (that is, $p = 2$) and under stronger regularity conditions on the reaction $x \longmapsto f(z, x, \lambda)$, we can improve Theorem 4.1 and produce six nontrivial solutions. However, we are unable to determine the sign of the sixth solution.

So, now the problem under consideration is the following:

$$
-\Delta u(z) = f(z, u(z), \lambda) \text{ in } \Omega, \ \frac{\partial u}{\partial n} + \beta(z) u = 0 \text{ on } \partial\Omega. \qquad (S_\lambda)
$$

The new hypotheses on the reaction $f(z, x, \lambda)$ are the following:

H_4: $f : \Omega \times \mathcal{R} \times (0, \infty) \to \mathcal{R}$ is a function such that for all $\lambda > 0$, $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and

(i) for all $\lambda > 0$, $(z, x) \longmapsto f(z, x, \lambda)$ is measurable and for a.a. $z \in \Omega$, $f(z, \cdot, \lambda) \in C^1(\mathcal{R})$;

(ii) $|f'_*(z,x,\lambda)| \leq a(z,\lambda)+c|x|^{r-2}$ for a.a. $z \in \Omega$, all $x \in \mathcal{R}$, all $\lambda > 0$, with $a(\cdot,\lambda) \in L^\infty(\Omega)_+$,

$$\|a(\cdot,\lambda)\|_\infty \to 0 \text{ as } \lambda \to 0^+,\ c > 0 \text{ and } 2 < r < 2^*;$$

(iii) if $F(z,x,\lambda) = \int_0^x f(z,s,\lambda)ds$, then

$$\lim_{x\to\pm\infty} \frac{F(z,x,\lambda)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iv) if $k_\lambda(z,x) = f(z,x,\lambda)x - pF(z,x,\lambda)$, then there exists $\beta^*_\lambda \in L^1(\Omega)_+$ such that

$$k_\lambda(z,x') \leq k_\lambda(z,x) + \beta^*_\lambda(z) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq x' < x \text{ or } x' < x \leq 0;$$

(v) there exist $q = q(\lambda) \in (1,p)$ and $\delta_0 = \delta_0(\lambda),\ c_0 = c_0(\lambda) > 0$ such that

$$c_0|x|^q \leq f(z,x,\lambda)x \leq qF(z,x,\lambda) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq |x| \leq \delta_0.$$

Remark. Evidently the differentiability of $f(z,\cdot,\lambda)$ and hypothesis $H_4(ii)$ imply that given $\rho > 0$, we can find $\xi^\lambda_\rho > 0$ such that for a.a. $z \in \Omega$, $x \longmapsto f(z,x,\lambda) + \xi^\lambda_\rho x$ is nondecreasing on $[-\rho,\rho]$.

Theorem 4.2 *If hypotheses $H(\beta)$ and H_4 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$ problem (S_λ) has at least six nontrivial solutions*

$$u_0, \hat{u} \in int\, C_+,\ \hat{u} - u_0 \in int\, C_+$$
$$v_0, \hat{v} \in -int\, C_+,\ v_0 - \hat{v} \in int\, C_+$$
$$y_0 \in int_{C^1(\overline{\Omega})}[v_0,u_0] \text{ nodal and } \hat{y} \in C^1(\overline{\Omega}) \setminus \{0\}.$$

Proof. From Theorem 4.1, we know that there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$ problem (S_λ) has at least five nontrivial solutions

$$u_0, \hat{u} \in \text{int}\, C_+,\ u_0 \leq \hat{u},\ u_0 \neq \hat{u},$$
$$v_0, \hat{v} \in -\text{int}\, C_+,\ \hat{v} \leq v_0,\ v_0 \neq \hat{v},$$
$$y_0 \in [v_0,u_0] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

Let $\rho = \max\{\|\hat{u}\|_\infty, \|\hat{v}\|_\infty\}$ and let $\xi^\lambda_\rho > 0$ be such that for a.a. $z \in \Omega$, the function $x \longmapsto f(z,x,\lambda) + \xi^\lambda_\rho x$ is nondecreasing on $[-\rho,\rho]$ (see the Remark after hypotheses H_4). We have

$$\begin{aligned}
& -\Delta u_\rho(z) + \xi^\lambda_\rho u_0(z) \\
= \quad & f(z,u_0(z),\lambda) + \xi^\lambda_\rho u_0(z) \\
\leq \quad & f(z,\hat{u}(z),\lambda) + \xi^\lambda_\rho \hat{u}(z) \text{ (recall } u_0 \leq \hat{u}) \\
= \quad & -\Delta\hat{u}(z) + \xi^\lambda_\rho \hat{u}(z) \text{ a.e. in } \Omega, \\
\Rightarrow \quad & \Delta(\hat{u} - u_0)(z) \leq \xi^\lambda_\rho(\hat{u} - u_0)(z) \text{ a.e. in } \Omega, \\
\Rightarrow \quad & \hat{u} - u_0 \in \text{int}\, C_+ \text{ (see Vazquez [28]).}
\end{aligned}$$

In a similar fashion, we show that

$$\begin{aligned} & v_0 - \hat{v} \in \operatorname{int} C_+,\ y_0 - v_0 \in \operatorname{int} C_+,\ u_0 - y_0 \in \operatorname{int} C_+ \\ \Rightarrow\quad & \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]. \end{aligned}$$

Let $\varphi_\lambda : H^1(\Omega) \to \mathcal{R}$ be the energy functional of problem (S_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{2}\|Du\|_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u^2 d\sigma - \int_\Omega F(z, u, \lambda)dz \text{ for all } u \in H^1(\Omega).$$

Evidently $\varphi_\lambda \in C^2(H^1(\Omega))$. Let $\overline{u} \in \operatorname{int} C_+$ and $\underline{u} \in -\operatorname{int} C_+$ be as in the proof of Proposition 3.1. Reasoning as in the first part of the proof, we can show that

$$\overline{u} - u_0 \in \operatorname{int} C_+ \text{ and } v_0 - \underline{u} \in \operatorname{int} C_+.$$

Let $\hat{\varphi}_+^\lambda$ be the C^1-functional introduced in the proof of Proposition 3.1 (now with $p = 2$). From the proof of Proposition 3.1, we know that $u_0 \in \operatorname{int} C_+$ is a minimizer of $\hat{\varphi}_+^\lambda$ and from (3.11) it follows that

$$\begin{aligned} & \varphi_\lambda\Big|_{[0,\overline{u}]} = \hat{\varphi}_+^\lambda\Big|_{[0,\overline{u}]}, \\ \Rightarrow\quad & u_0 \in \operatorname{int} C_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda, \\ \Rightarrow\quad & u_0 \in \operatorname{int} C_+ \text{ is a local } H^1(\Omega)\text{-minimizer of } \varphi_\lambda \\ & \qquad\qquad \text{(see Proposition 2.1).} \end{aligned}$$

In a similar fashion we show that $v_0 \in -\operatorname{int} C_+$ is also a local minimizer of φ_λ. Therefore, we have

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,0}\mathcal{Z} \text{ for all } k \geq 0. \tag{4.73}$$

From the proof of Theorem 4.1, we know that $\hat{u} \in \operatorname{int} C_+$ is a critical point of mountain pass type for the functional τ_λ^+. Hence

$$C_1(\tau_\lambda^+, \hat{u}) \neq 0. \tag{4.74}$$

Let $[u_0) = \{u \in H^1(\Omega) : u_0(z) \leq u(z) \text{ a.e. in } \Omega\}$. From (4.46) and (4.47) we see that

$$\varphi_\lambda\Big|_{[u_0)} = \tau_\lambda^+\Big|_{[u_0)} + \xi_+^\lambda \text{ with } \xi_+^\lambda \in \mathcal{R}. \tag{4.75}$$

Since $\hat{u} - u_0 \in \operatorname{int} C_+$, it follows from 4.75 that

$$\begin{aligned} & C_k(\varphi_\lambda\Big|_{C^1(\overline{\Omega})}, \hat{u}) = C_k(\tau_\lambda^+\Big|_{C^1(\overline{\Omega})}, \hat{u}) \text{ for all } k \geq 0, \\ \Rightarrow\quad & C_k(\varphi_\lambda, \hat{u}) = C_k(\tau_\lambda^+, \hat{u}) \text{ for all } k \geq 0 \\ & \qquad \text{(see Palais [24] and recall that } C^1(\overline{\Omega}) \text{ is dense in } H^1(\Omega)), \\ \Rightarrow\quad & C_1(\varphi_\lambda, \hat{u}) \neq 0 \text{ (see 4.74).} \end{aligned} \tag{4.76}$$

Similarly we show that

$$C_1(\varphi_\lambda, \hat{v}) \neq 0. \tag{4.77}$$

Since $\varphi_\lambda \in C^2(H^1(\Omega))$, from (4.76) and (4.77) we infer that

$$C_k(\varphi_\lambda, \hat{u}) = C_k(\varphi_\lambda, \hat{v}) = \delta_{k,1}\mathcal{Z} \text{ for all } k \geq 0 \text{ (see Bartsch [6]).} \tag{4.78}$$

Let ξ_λ be the C^1-functional introduced in the proof of Proposition 3.5. From Claim 3 of the proof of Proposition 3.5, we have

$$C_k(\xi_\lambda, 0) = 0 \text{ for all } k \geq 0. \tag{4.79}$$

We may always assume that $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are extremal constant sign solutions for problem (S_λ) (see Proposition 3.4). Then from (3.28) it follows that

$$\begin{aligned} & \xi_\lambda\Big|_{[v_0,w_0]} = \varphi_\lambda\Big|_{[v_0,u_0]}, \\ \Rightarrow\ & C_k(\xi_\lambda\Big|_{C^1(\overline{\Omega})}, 0) = C_k(\varphi_\lambda\Big|_{C^1(\overline{\Omega})}, 0) \text{ for all } k \geq 0 \\ & \qquad\qquad (\text{recall } u_0 \in \operatorname{int} C_+, v_0 \in -\operatorname{int} C_+) \\ \Rightarrow\ & C_k(\xi_\lambda, 0) = C_k(\varphi_\lambda, 0) \text{ for all } k \geq 0 \text{ (see Palais [24]),} \\ \Rightarrow\ & C_k(\varphi_\lambda, 0) = 0 \text{ for all } k \geq 0 \text{ (see (4.79)).} \end{aligned} \tag{4.80}$$

Recall that y_0 is a critical point of mountain pass type for the functional ξ_λ (see the proof of Proposition 3.5). Since $y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]$ and $\xi_\lambda\Big|_{[v_0,u_0]} = \varphi_\lambda\Big|_{[v_0,u_0]}$, as before using the results of Palais [24] and Bartsch [6], we have

$$C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathcal{Z} \text{ for all } k \geq 0. \tag{4.81}$$

Finally, using hypothesis $H_4(iv)$ and with a straightforward modification of the proof of Proposition 3.2 of Aizicovici, Papageorgiou and Staicu [3], we have

$$C_k(\varphi_\lambda, \infty) = 0 \text{ for all } k \geq 0. \tag{4.82}$$

Suppose $K_{\varphi_\lambda} = \{0, u_0, v_0, \hat{u}, \hat{v}, y_0\}$. Then from (4.73), (4.78), (4.80), (4.81), (4.82) and the Morse relation with $t = -1$ (see (2.1)), we have

$$2(-1)^0 + 2(-1)^1 + (-1) = 0, \text{ a contradiction.}$$

So, we can find $\hat{y} \in K_{\varphi_\lambda}$, $\hat{y} \notin \{0, u_0, v_0, \hat{u}, \hat{v}, y_0\}$. Then $\hat{y}$ is the sixth nontrivial solution of (S_λ) and the elliptic regularity theory implies that $y_0 \in C^1(\overline{\Omega}) \setminus \{0\}$. □

Remark. It is interesting to know if we can determine the sign of $\hat{y}$.

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