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Solitary Waves for a Class of Quasilinear Schrödinger Equations Involving Vanishing Potentials

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Abstract

In this paper we study the existence of weak positive solutions for the following class of quasilinear Schrödinger equations

$$-\Delta u + V(x)u - [\Delta(u^2)]u = h(u) \quad \text{in} \quad \mathbb{R}^N,$$

where h satisfies some "mountain–pass" type assumptions and V is a nonnegative continuous function. We are interested specially in the case where the potential V is neither bounded away from zero, nor bounded from above. We give a special attention to the case when V may eventually vanish at infinity. Our arguments are based on penalization techniques, variational methods and Moser iteration scheme.

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1 Introduction

The problem

$$-\Delta u + V(x)u - [\Delta(u^2)]u = h(u) \quad \text{in} \quad \mathbb{R}^N \tag{1.1}$$

where $N \geq 3$, arises in various branches of mathematical physics and it has been the subject of extensive study in recent years (cf. [12, 13, 14, 22, 23]). Part of the interest is due to the fact that solutions of (1.1) are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(y)\psi - h(|\psi|^2)\psi - \kappa[\Delta\rho(|\psi|^2)]\rho'(|\psi|^2)\psi \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where V is a given potential, κ is a real constant and h, ρ are real functions.

Our work was motivated by some papers that have appeared in the recent years concerning the study of nonlinear Schrödinger equations. These works used only variational approach and followed the seminal work of P. Rabinowitz [31]. The semilinear case corresponding to $\kappa = 0$ has also been studied extensively in recent years, see for example [1, 9, 26, 31] and references therein. For quasilinear equations of the form (1.1) we refer the reader to the recent papers [15, 19, 20, 21, 27] and their references for a discussion on the subject. We also mention the very recent works [25, 32] where the authors studied multiplicity and high energy solutions for a class of quasilinear Shrödinger equations more general than (1.2).

All the works mentioned above are built on the assumption that the potential V is bounded away from zero. The main purpose of the present paper is to extend and complement the results in [2, 4, 5, 6] to quasilinear Schrödinger equations with unbounded or vanishing potentials of form (1.1). More precisely, we will assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function satisfying

(V_1) There exist $\mu > 0$ and $R > 1$ such that

$$\frac{1}{R^{N+2}} \inf_{|x|\geq R} |x|^{N+2} V(x) \geq \mu.$$

Let $2^* = 2N/(N-2)$ the critical Sobolev exponent. It was first proved in an earlier work of J. Liu et al. [27] that $2(2^*) = 4N/(N-2)$ behaves as a critical exponent for modified Schrödinger equations of the form

$$\Delta u + V(x)u - u\Delta(u^2) = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N$$

in the sense that this equation has no positive solution with $\int u^2|\nabla u|^2 \, dx < \infty$ in $H^1(\mathbb{R}^N)$ provided that $x \cdot \nabla V(x) \geq 0$ and $p \geq 2(2^*)$ (see also [17, 18] for related problems involving critical growth). In view of this result, it is natural to assume that $h : [0, +\infty) \to [0, +\infty)$ is a continuous function satisfying the following conditions:

(h_1) $\displaystyle\lim_{s\to 0^+} \frac{sh(s)}{s^{4N/(N-2)}} = 0.$

(h_2) There exists $p \in (4, 4N/(N-2))$ such that

$$\limsup_{s\to+\infty} \frac{sh(s)}{s^{p-1}} < \infty.$$

(h_3) There exists $\theta > 2$ such that

$$0 < 2\theta H(s) := 2\theta \int_0^s h(t)dt \le sh(s), \quad \forall\, s > 0.$$

The main result of this paper is stated as follows:

Theorem 1.1 *Suppose that (V_1) and $(h_1)-(h_3)$ are satisfied. Then, there exists $\mu^* > 0$ such that Problem (1.1) possesses a positive solution for all $\mu \ge \mu^*$.*

Remark 1.1 As we already mentioned, quasilinear equations like (1.2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ρ. They are one of the main objects of the quantum physics, because they appear in problems involving nonlinear optics, plasma physics and condensed matter physics (cf. [10], [28]). We mention that our work was also motivated by the recent papers of A. Ambrosetti, V. Felli and A. Malchiodi [4] and V. Benci, C. Grisanti and A. M. Micheletti [8], where the authors have investigated existence of solutions for a class of nonlinear Schrödinger equations when V tends to zero at infinity, which corresponds to the case $\kappa = 0$ in (1.2).

Remark 1.2 Here we present some examples of functions satisfying our hypotheses. For $\alpha > 2(2^*)$ and p as in (h_2), the function

$$h(s) = \begin{cases} s^{\alpha-1}, & \text{if} \quad 0 \le s \le 1 \\ s^{p-2}, & \text{if} \quad s \ge 1 \end{cases}$$

is an example of a nonlinearity satisfying hypotheses $(h_1) - (h_3)$. On the other hand, if $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that

$$\liminf_{|x|\to\infty} V(x) > 0,$$

then hypothesis (V_1) holds. In particular this occurs when V is coercive. Another example is $V(x) = \mu\eta(x)\min\{1, (R/|x|)^{N+2}\}$ for some $\mu > 0$, $R > 1$ and $\eta \in C^\infty(\mathbb{R}^N, [0,1])$ such that $\eta \equiv 1$ in $\mathbb{R}^N \backslash B(0,R)$.

Remark 1.3 1. From the hypotheses (h_1) and (h_2), there is a constant $c_0 > 0$ such that

$$|sh(s)| \le c_0|s|^{2(2^*)} \quad \text{and} \quad |sh(s)| \le c_0|s|^{p-1} \quad \text{for all} \quad s \in \mathbb{R}.$$

2. Condition (V_1) guarantees that $\mathcal{Z} = \left\{x \in \mathbb{R}^N : V(x) = 0\right\} \subset B_R(0)$. Therefore $\mathcal{Z}$ is a compact subset of $\mathbb{R}^N$.

We use variational methods to prove the existence of solutions for problem (1.1). Our argument can be divided in several steps: In the first step, our goal is to overcome the loss of compactness typical of this class of problems that are defined on the unbounded domain $\mathbb{R}^N$. For that, in Subsection 2.1 we apply the penalization method in order to obtain a modified functional $\widehat{I}$ which satisfies the Palais-Smale compactness condition. This functional $\widehat{I}$ is not well defined in the Sobolev space $H^1(\mathbb{R}^N)$ because of the term $|u|^2|\nabla u|^2$. In order to get rid of this term, in the second step we perform a change of variables to get a suitable functional J well defined in an useful Orlicz space (see Subsection 2.2). In the third step we verify the mountain–pass geometry and the Palais-Smale condition is proved on the fourth step. In fifth step we prove the existence of a mountain–pass critical point for the functional J. To conclude the proof of our main result, we will show that this critical point of J will eventually be a solution of the original equation with the help of an uniform L^∞–estimate which will be obtained via Moser iteration scheme.

2 The variational framework

2.1 Penalized problem

Since our goal is to prove the existence of positive solutions for (1.1), we define $h(t) = 0$ for all $t \leq 0$. Note that by assumption (h_1) the function h is continuous in whole $\mathbb{R}$. Setting $H(s) = \int_0^s h(t)\,\mathrm{d}t$, we observe that formally (1.1) is the Euler-Lagrange equation associated to the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(1 + 2|u|^2\right)|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x,$$

but the functional I is not well defined in $H^1(\mathbb{R}^N)$ because of the term $|u|^2|\nabla u|^2$. Moreover, we have the following difficulties: lack of compactness because our equation (1.1) is defined in whole $\mathbb{R}^N$ and the fact that our potential $V(x)$ may vanish near the origin or at infinity. So we will make some modifications which are appropriated to obtain a new class of problems where we can apply the mountain–pass argument (cf. [3, 30]). For that, we first will consider a reformulation of the problem following an argument introduced by del Pino and Felmer [16]. Let us denote by χ_Λ the characteristic function of the set $\Lambda \subset \mathbb{R}^N$. From now on we consider $\Lambda = B_R(0)$, where $R > 1$ is given by condition (V_1). For $k > 2\theta/(\theta - 2) > 2$ we consider the following nonnegative Carathéodory functions $\widehat{h} : \left(\mathbb{R}^N \setminus \Lambda\right) \times \mathbb{R} \to \mathbb{R}$,

$$\widehat{h}(x,t) := \min\{h(t), (V(x)/k)|t|\},$$

and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$,

$$g(x,t) := \chi_\Lambda(x)h(t) + (1 - \chi_\Lambda(x))\,\widehat{h}(x,t).$$

Next, we consider the following auxiliary problem

$$\left\{\begin{array}{l} -\Delta u + V(x)u - [\Delta(u^2)]u \;=\; g(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \;\in\; D^{1,2}(\mathbb{R}^N). \end{array}\right. \tag{2.1}$$

Setting $\widehat{H}(x,s) = \int_0^s \widehat{h}(x,t)\,\mathrm{d}t$ and $G(x,t) = \int_0^s g(x,t)\,\mathrm{d}t$, we obtain

$$G(x,t) = \chi_\Lambda(x)H(t) + (1-\chi_\Lambda(x))\,\widehat{H}(x,t).$$

We observe that formally (2.1) is the Euler-Lagrange equation associated to the functional

$$\widehat{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(1+2|u|^2\right)|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x.$$

Since the functional $\widehat{I}$ is not well defined in the space $H^1(\mathbb{R}^N)$ we have to perform a suitable change of variable to get a new problem so that we will be rid of the term $|u|^2|\nabla u|^2$ and the associated functional is well defined in a new class of function space in the next section.

Remark 2.1 Basic properties of the auxiliary function g are listed below:

1. $0 \le g(x,t) \le h(t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.
2. $g(x,t) = h(t)$ and $G(x,t) = H(t)$ for all $x \in \Lambda$.
3. $g(x,t) \le (V(x)/k)|t|$ for all $x \in \mathbb{R}^N \setminus \Lambda$ and $t \in \mathbb{R}$.
4. $G(x,t) \le (V(x)/2k)t^2$ for all $x \in \mathbb{R}^N \setminus \Lambda$ and $t \in \mathbb{R}$.

2.2 Changing the variable

From the variational point of view, the second difficulty that we have to deal with is to find an appropriate variational setting in order to apply minimax methods to study the existence of nontrivial solutions of (2.1). However, it should be pointed out that we may not apply directly such methods since the natural associated functional $\widehat{I}$ is not well defined in the usual Sobolev spaces. To overcome this difficulty, we follow the idea developed by Liu, Wang and Wang in [27], that is, we make the change of variables $v = f^{-1}(u)$ where f is defined by

$$f'(t) = \frac{1}{(1+2f^2(t))^{1/2}} \quad \text{on} \quad [0,\infty), \qquad f(t) = -f(-t) \quad \text{on} \quad (-\infty, 0].$$

Thus, after this change we obtain the following functional

$$J(v) := \widehat{I}(f(v)) \;=\; \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)f^2(v)\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(x,f(v))\,\mathrm{d}x,$$

which is well defined on the Orlicz space

$$E = \left\{ v \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x < \infty \right\}.$$

E is a Banach space (cf. Proposition 2.3) when endowed with the norm (cf. Proposition 2.2)

$$\| v \|_E = \| \nabla v \|_2 + \inf_{\varepsilon>0} \frac{1}{\varepsilon}\left[1 + \int_{\mathbb{R}^N} V(x)f^2(\varepsilon v)\,\mathrm{d}x\right]. \tag{2.2}$$

Moreover, nontrivial critical points of J correspond precisely to the positive weak solutions of the equation

$$-\Delta v = f'(v)\,[g(x, f(v)) - V(x)f(v)] \quad \text{in} \quad \mathbb{R}^N. \tag{2.3}$$

As in [29] (see also [21]) we see that if v is a weak solution for (2.3) then $u = f(v)$ is a weak solution for (2.1). Our goal here is to prove the existence of a critical point v for J satisfying $g(x, f(v)) = h(v)$.

Proposition 2.1 *Basic properties of the change of variable $f(t)$ are listed below:*

(1) f is a uniquely defined C^∞ function and invertible.

(2) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$.

(3) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$.

(4) $f(t)/t \to 1$ as $t \to 0$.

(5) $f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$.

(6) $f(t)/2 \leq tf'(t) \leq f(t)$ for all $t \geq 0$.

(7) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$.

(8) the function $f^2(t)$ is strictly convex.

(9) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1 \\ C|t|^{1/2}, & |t| \geq 1. \end{cases}$$

(10) there exist positive constants C_1 and C_2 satisfying

$$|t| \leq C_1|f(t)| + C_2|f(t)|^2 \quad \textit{for all} \quad t \in \mathbb{R}.$$

(11) $|f(t)f'(t)| \leq 1/\sqrt{2}$ for all $t \in \mathbb{R}$.

(12) $f^2(\lambda s) \leq \lambda^2 f^2(s)$, for all $s \in \mathbb{R}$ and $\lambda \geq 1$.

(13) The function $f(t)f'(t)t^{-1}$ is decreasing for $t > 0$.

(14) The function $f^3(t)f'(t)t^{-1}$ is increasing for $t > 0$.

Proof. The proof of items (1)–(11) and (13)–(14) can be seen in [19, Proposition 2.2, Corollary 2.3] (see also [15, 27]). For item (12) one can see [21, Lemma 2.1].

2.3 Properties of the Orlicz space E

In this section we collected some facts on the Orlicz space E which are crucial in our argument to prove the existence of critical points for the functional J. The next results are essential for the structure of our work.

Proposition 2.2 *(1) E endowed with the function defined in (2.1) is a normed space;*

(2) There exists a positive constant C such that for all $v \in E$,

$$\frac{\displaystyle\int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x}{1 + \left[\displaystyle\int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x\right]^{1/2}} \le C\|v\|_E; \tag{2.4}$$

(3) If $v_n \to v$ in E, then

$$\int_{\mathbb{R}^N} V(x)|f^2(v_n) - f^2(v)|\,\mathrm{d}x \to 0$$

and

$$\int_{\mathbb{R}^N} V(x)|f(v_n) - f(v)|^2\,\mathrm{d}x \to 0;$$

(4) If $v_n(x) \to v(x)$ almost everywhere and

$$\int_{\mathbb{R}^N} V(x) f^2(v_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x,$$

then

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) f^2(\xi(v_n - v))\,\mathrm{d}x\right] \to 0.$$

Proof. Items (1)–(3) can be proved using the same arguments as in [19, Proposition 3.2]. We need to prove the item (4). First of all, we observe that it is enough to prove that

$$\int_{\mathbb{R}^N} V(x) f^2(v_n - v)\,\mathrm{d}x \to 0. \tag{2.5}$$

In fact, if this convergence in (2.5) occurs, since $f^2(\lambda s) \le \lambda^2 f^2(s)$ for all $s \in \mathbb{R}$ and $\lambda \ge 1$ (cf. Proposition 2.1), it follows that

$$\frac{1}{\lambda}\left\{1 + \int_{\mathbb{R}^N} V(x) f^2\left(\lambda(v_n - v)\right)\,\mathrm{d}x\right\} \le \frac{1}{\lambda} + \frac{\lambda^2}{\lambda}\int_{\mathbb{R}^N} V(x) f^2\left(v_n - v\right)\,\mathrm{d}x \le \frac{2}{\lambda}$$

for $n \ge n_0(\lambda)$ large enough. So

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) f^2(\xi(v_n - v))\,\mathrm{d}x\right] \le \frac{2}{\lambda} \quad \text{for all} \quad n \ge n_0.$$

Since $\lambda \geq 1$ is arbitrary we can conclude that

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) f^2(\xi(v_n - v))\,\mathrm{d}x\right] \to 0 \quad \text{as } n \to \infty.$$

Now, we are going to prove (2.5). Given $\varepsilon > 0$, take $r > 0$ and $A = B[0, r] \subset \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N \setminus A} V(x) f^2(v)\,\mathrm{d}x \leq \frac{\varepsilon}{2}.$$

We can see that

$$\int_{\mathbb{R}^N \setminus A} V(x) f^2(v_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N \setminus A} V(x) f^2(v)\,\mathrm{d}x$$

and so for n large enough,

$$\int_{\mathbb{R}^N \setminus A} V(x) f^2(v_n)\,\mathrm{d}x < \varepsilon.$$

Using the convexity of the function f^2 and Item (12) in Proposition 2.1, we get for n large enough,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus A} V(x) f^2(v_n - v)\,\mathrm{d}x &\leq \frac{1}{2}\int_{\mathbb{R}^N \setminus A} \left[V(x) f^2(2v_n) + V(x) f^2(2v)\right]\mathrm{d}x \\
&\leq \frac{4}{2}\int_{\mathbb{R}^N \setminus A} \left[V(x) f^2(v_n) + V(x) f^2(v)\right]\mathrm{d}x \leq 4\varepsilon.
\end{aligned}$$

On the other hand, as in [27, Proposition 2.1], it can be proved the integral equicontinuity of the sequence $(V(x) f^2(v_n))$. That is, for each $\sigma > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that if $U \subset \mathbb{R}^N$, $|U| \leq \delta$ and $n \geq n_0$, then

$$\int_U V(x) f^2(v_n)\,\mathrm{d}x \leq \sigma.$$

For $\varepsilon > 0$ given above, take $\delta > 0$ such that if $U \subset \mathbb{R}^N$ with $|U| \leq \delta$ then for n large enough,

$$\int_U V(x) f^2(v_n - v)\,\mathrm{d}x \leq \varepsilon.$$

For the set $\mathcal{Z} = V^{-1}(0)$, defined in Remark 1.3, observe that there is $\alpha > 0$ small enough such that

$$\mathcal{Z}_\alpha = \{x \in \mathbb{R}^N : dist(x, \mathcal{Z}) < \alpha\} \subset B(0, R) \quad \text{and} \quad |\mathcal{Z}_\alpha - \mathcal{Z}| < \delta/2.$$

Since $V(x) > 0$ in the compact set $A \setminus \mathcal{Z}_\alpha = A \cap \mathcal{Z}_\alpha^c$, we have $\zeta = \min_{A \setminus \mathcal{Z}_\alpha} V(x) > 0$. Consider the sets

$$A_{n,1} = \{x \in A : |v_n(x) - v(x)| > a\} \quad \text{and} \quad A_{n,2} = \{x \in A : |v_n(x) - v(x)| \leq a\},$$

with $a > 0$ to be chosen appropriately. We have

$$\int_{A_{n,1}} V(x) f^2(v_n - v)\,\mathrm{d}x \leq \int_{\mathcal{Z}_\alpha \cap A_{n,1}} V(x) f^2(v_n - v)\,\mathrm{d}x + \int_{\mathcal{Z}_\alpha^c \cap A_{n,1}} V(x) f^2(v_n - v)\,\mathrm{d}x$$

and, since f^2 is an even function and increasing for $t \geq 0$, we get

$$C \geq \int_{\mathbb{R}^N} V(x)f^2(v_n - v)\,\mathrm{d}x \;\geq\; \int_{\mathcal{Z}_\alpha^c \cap A_{n,1}} V(x)f^2(v_n - v)\,\mathrm{d}x \;\geq\; \zeta f^2(a)|\mathcal{Z}_\alpha^c \cap A_{n,1}|.$$

Since $f^2(t) \geq c|t|$ for all $|t| > 1$, we can choose $a > 0$ such that $C\left(\zeta f^2(a)\right)^{-1} < \delta/2$. Thus,

$$|\mathcal{Z}^c \cap A_{n,1}| = |\mathcal{Z}_\alpha^c \cap A_{n,1}| + |(\mathcal{Z}_\alpha \backslash \mathcal{Z}) \cap A_{n,1}| < \delta.$$

Therefore, we conclude that for n large enough,

$$\int_{A_{n,1}} V(x)f^2(v_n - v)\,\mathrm{d}x = \int_{\mathcal{Z}^c \cap A_{n,1}} V(x)f^2(v_n - v)\,\mathrm{d}x \leq \varepsilon.$$

On the other hand, by the Lebesgue's dominated convergence theorem, we have

$$\int_{A_{n,2}} V(x)f^2(v_n - v)\,\mathrm{d}x \to 0.$$

Therefore for n large enough,

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)f^2(v_n - v)\,\mathrm{d}x &= \int_A V(x)f^2(v_n - v)\,\mathrm{d}x + \int_{\mathbb{R}^N \backslash A} V(x)f^2(v_n - v)\,\mathrm{d}x \\
&\leq \int_{A_{n,1}} V(x)f^2(v_n - v)\,\mathrm{d}x + \int_{A_{n,2}} V(x)f^2(v_n - v)\,\mathrm{d}x + 4\varepsilon \\
&\leq 6\varepsilon.
\end{aligned}$$

This result implies (2.5) and completes the proof of this proposition.

Proposition 2.3 *E is a Banach space.*

Proof. First observe that the map: $v \mapsto f(v)$ from E into $L^{2(2^*)}(\mathbb{R}^N)$ is continuous. Indeed, since

$$|f(t)| \leq c\sqrt{|t|} \quad \forall\, t \in \mathbb{R} \quad \text{and} \quad |\nabla(f^2(v))| = 2|f(v)f'(v)||\nabla v| \leq c|\nabla v| \quad \forall v \in E,$$

we obtain $f^2(v) \in D^{1,2}(\mathbb{R}^N)$. So, by the Sobolev-Gagliardo-Nirenberg inequality we get

$$\|f(v)\|_{2(2^*)} = \|f^2(v)\|_{2^*}^{1/2} \leq C\|\nabla(f^2(v))\|_2^{1/2} \leq C\|\nabla v\|_2^{1/2} \leq C\|v\|_E^{1/2}, \tag{2.6}$$

for all $v \in E$. Let (v_n) be a Cauchy sequence in E. Using the continuous embedding

$$E \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N), \tag{2.7}$$

and the completeness of $D^{1,2}(\mathbb{R}^N)$, there exists $v \in D^{1,2}(\mathbb{R}^N)$ such that $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. By the inequality (2.4) in Proposition 2.2 we obtain

$$\int_{\mathbb{R}^N} V(x)f^2(v_n)\,\mathrm{d}x \leq C$$

which together with Fatou's Lemma implies

$$\int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)f^2(v_n)\,\mathrm{d}x \le C,$$

and consequently $v \in E$. By the inequality (2.4), given $\varepsilon \ge 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} V(x)f^2(v_n - v_m)\,\mathrm{d}x \le \varepsilon \quad \text{for all} \quad n, m \ge n_0.$$

Fixing $m > n_0$ and applying Fatou's Lemma we have

$$\int_{\mathbb{R}^N} V(x)f^2(v_m - v)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)f^2(v_m - v_n)\,\mathrm{d}x \le \varepsilon,$$

which implies that

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} V(x)f^2(v_m - v)\,\mathrm{d}x = 0.$$

Thus, as in the proof of Item (4) in Proposition 2.2, we get

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x)f^2(\xi(v_n - v))\,\mathrm{d}z\right] \to 0,$$

and consequently $v_n \to v$ in E. This is the end of the proof.

2.4 Mountain–pass structure for functional J

2.4.1 Regularity results

Proposition 2.4 *The Euler-Lagrange functional J associated with (2.3) satisfies the following conditions:*

1. *J is well defined and continuous in E.*
2. *J is Gateaux-differentiable in E and its derivative is given by*

$$J'(v)(w) = \int_{\mathbb{R}^N} \Big(\nabla v \nabla w + V(x)f(v)f'(v)w\Big)\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x, f(v))f'(v)w\,\mathrm{d}x$$

for all $v, w \in E$.

3. *For $v \in E$, $J'(v) \in E'$ and if $v_n \to v$ in E then $J'(v_n) \to J'(v)$ in the weak-$*$topology of E', that is, for each $w \in E$ we have*

$$\langle J'(v_n), w\rangle \to \langle J'(v), w\rangle.$$

Proof. The proof is essentially the same as in [19, Proposition 2.5].

2.4.2 Mountain–pass geometry

It is standard to prove that J has a mountain–pass geometry. We include the proof here for completeness. For this next result we consider the functional $Q : E \to \mathbb{R}$ defined by

$$Q(v) = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)f^2(v)\right) \mathrm{d}x$$

and the set

$$S(\rho) := \{v \in E : \ Q(v) = \rho^2\}.$$

Lemma 2.1 *The functional J has the mountain–pass geometry, that is, J satisfies:*

1. *There are $\rho,\ \alpha > 0$, such that $J(v) \geq \alpha$ if $v \in S(\rho)$.*
2. *There is $\varphi \in E$ such that $J(t\varphi) \to -\infty$ as $t \to +\infty$.*

Proof. Since $G(x, t) \leq H(t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, for $v \in S(\rho)$ we have

$$J(v) = \frac{1}{2}Q(v) - \int_{\mathbb{R}^N} G(x, f(v))\,\mathrm{d}x = \frac{1}{2}\rho^2 - \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x.$$

By Remark 1.3 and Item (7) in Proposition 2.1 we have

$$H(f(t)) \leq c_0|f(t)|^{2(2^*)} \leq C|t|^{2^*} \quad \text{for all} \quad t \in \mathbb{R}.$$

Then, using the Sobolev Inequality we obtain

$$J(v) \geq \frac{1}{2}\rho^2 - C\int_{\mathbb{R}^N} |v|^{2^*} dx \geq \frac{1}{2}\rho^2 - CQ(v)^{2^*/2} = \rho^2\left(\frac{1}{2} - C\rho^{2^*-2}\right).$$

Choosing $\rho > 0$ such that $(1/2) - C\rho^{2^*-2} > 0$, we conclude the first part of the proof.

Now, we will prove Item 2. From hypothesis (h_3), we see that there are positive constants C_1, C_2 such that

$$H(s) \geq C_1 s^{2\theta} - C_2 \quad \text{for all} \quad s \geq 0. \tag{2.8}$$

Choosing a nontrivial function $\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\mathrm{supp}(\varphi) \subset \Lambda$, (2.8) implies that

$$J(t\varphi) \leq \frac{t^2}{2}\int_\Lambda \left(|\nabla\varphi|^2 + V(x)|\varphi|^2\right) \mathrm{d}x - C_1\int_\Lambda |f(t\varphi)|^{2\theta}\,\mathrm{d}x + C_2|\Lambda|,$$

where $|\Lambda|$ denotes the Lebesgue measure of Λ. Using the property (6) in Proposition 2.1, it follows that $f(s)/s$ is decreasing for $s > 0$. Since $0 \leq t\varphi(x) \leq t$ for $x \in \Lambda$ and $t \geq 0$, we obtain $f(t\varphi(x)) \geq f(t)\varphi(x)$, and so

$$J(t\varphi) \leq \frac{t^2}{2}\left[\int_\Lambda \left(|\nabla\varphi|^2 + V(x)|\varphi|^2\right) \mathrm{d}x - C_1\frac{f(t)^{2\theta}}{t^2}\int_\Lambda \varphi^{2\theta}\,\mathrm{d}x + \frac{C_2}{t^2}|\Lambda|\right].$$

Given that $\theta > 2$ and $f(t) \geq C\sqrt{t}$ for some positive constant and all $t > 1$, we see that

$$\lim_{t\to+\infty} \frac{f(t)^{2\theta}}{t^2} \geq C \lim_{t\to+\infty} t^{\theta-2} = +\infty.$$

So, we get $\lim_{t\to\infty} J(t\varphi) = -\infty$, as desired. This is the end of the proof.

2.5 On the Palais-Smale condition for functional J

A sequence (v_n) in E is a Palais-Smale sequence for J if $|J(v_n)| \leq C$, uniformly in n, while $\| J'(v_n) \| \to 0$ as $n \to \infty$.

Lemma 2.2 *Suppose that $(h_1) - (h_3)$ hold. Then any Palais-Smale sequence for J is bounded in E.*

Proof. Let $(v_n) \subset E$ be a Palais-Smale sequence for J. Thus, given $\delta > 0$ for n large we have

$$J(v_n) - \frac{1}{\theta}\langle J'(v_n), v_n\rangle \leq \delta \| v_n \|_E + C. \tag{2.9}$$

Using Item (6) in Proposition 2.1 we get $0 \leq f(t)f'(t)t \leq f^2(t)$, for all $t \in \mathbb{R}$. It follows from Remark 2.1 that

$$\begin{aligned} J(v_n) - \frac{1}{\theta}\langle J'(v_n), v_n\rangle \quad \geq \quad & \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x \\ & -\frac{1}{2\theta}\int_{\Lambda}\left[2\theta H(f(v_n)) - 2h(f(v_n))f'(v_n)v_n\right]\,\mathrm{d}x \\ & -\frac{1}{2k}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x. \end{aligned}$$

Since $k \geq 2\theta/(\theta - 2)$, using the condition (h_3) we see that

$$J(v_n) - \frac{1}{\theta}\langle J'(v_n), v_n\rangle \geq \frac{1}{2k}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x. \tag{2.10}$$

Given the elementary inequality $s^{1/2} < 1 + s$ for $s \geq 0$, we get the estimate

$$\begin{aligned} \| v_n \|_E \quad & \leq \quad \left(\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x\right)^{1/2} + 1 + \int_{\mathbb{R}^N}V(x)f^2(v_n)\,\mathrm{d}x \\ & \leq \quad 2 + \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x. \end{aligned} \tag{2.11}$$

Choosing $\delta > 0$ such that $C(\theta, \delta) := 1/(2k) - \delta > 0$ and using (2.9)-(2.11), we obtain

$$2\delta + C \geq C(\theta, \delta)\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x,$$

which implies that

$$\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x \leq \tilde{C} = C(\theta, C).$$

From estimate (2.11) we get that (v_n) is bounded in E.

Lemma 2.3 *Suppose that $(h_1) - (h_3)$ hold. Then J satisfies the Palais-Smale condition, that is, if $(v_n) \subset E$ is an arbitrary Palais-Smale sequence for J than there exists a subsequence of (v_n) that converges in E.*

Proof. Let $(v_n) \subset E$ be a Palais-Smale sequence for J. From Lemma 2.2 we know that (v_n) is bounded in E, and so in $D^{1,2}(\mathbb{R}^N)$. We can assume that there exists $v \in D^{1,2}(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \text{ in } D^{1,2}(\mathbb{R}^N), \quad v_n \rightharpoonup v \text{ in } L^{2^*}(\mathbb{R}^N), \quad \text{and} \quad v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N.$$

Thus, by (2.4) and Fatou's Lemma we obtain

$$\int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)f^2(v_n)\,\mathrm{d}x \le C,$$

which implies that $v \in E$. Now, since f^2 is convex we see that the functional $Q(u)$ is convex and so, by [24, Lemma 15.3], we have

$$\begin{aligned} \frac{1}{2}Q(v) - \frac{1}{2}Q(v_n) &\ge \frac{1}{2}\langle Q'(v_n), v - v_n\rangle \\ &= \int_{\mathbb{R}^N} \nabla v_n(\nabla v - \nabla v_n)\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)(v - v_n)\,\mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)f^2(v)\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)f^2(v_n)\right)\mathrm{d}x \\ &\qquad \ge \int_{\mathbb{R}^N} g(x, f(v_n))f'(v_n)(v - v_n)\,\mathrm{d}x + \langle J'(v_n), v - v_n\rangle. \end{aligned} \tag{2.12}$$

We need to estimate the sequence

$$\int_{\mathbb{R}^N} g(x, f(v_n))f'(v_n)(v - v_n)\,\mathrm{d}x.$$

At first we observe that $g(x, f(v_n))f'(v_n)$ is bounded in $L^{2N/(N+2)}(\mathbb{R}^N)$. In fact, using Remarks 1.3 and 2.1 and Proposition 2.1 we see that for all $t \in \mathbb{R}$,

$$0 \le g(x, f(t))f'(t) \le h(f(t))f'(t) \le c_0|f(t)|^{2(2^*)-1}f'(t) \le c|f(t)|^{2(2^*-1)} \le C|t|^{2^*-1}.$$

Since $(2^* - 1)2N/(N + 2) = 2^*$, using the Sobolev inequality we obtain

$$\int_{\mathbb{R}^N} |g(x, f(v_n))f'(v_n)|^{2N/(N+2)}\,\mathrm{d}x \le C\int_{\mathbb{R}^N} |v_n|^{2^*}\mathrm{d}x \le C\|\nabla v_n\|_2^{2^*} \le C\|v_n\|_E^{2^*} \le C'.$$

For each $\varepsilon > 0$, let $r > R$ be such that

$$\max\left\{[2k/(k-1)]\omega_N^{1/N}C,\, C'\right\}\left(\int_{|x|\ge r} |v(x)|^{2^*}\,\mathrm{d}x\right)^{1/2^*} < \frac{\varepsilon}{4} \tag{2.13}$$

where C is a positive constant such that $\| v_n \|_E \le C$ for all n and ω_N is the volume of the unitary ball in $\mathbb{R}^N$. For this r we get

$$\int_{|x|\ge 2r} |g(x, f(v_n))f'(v_n)v| dx \le \|g(\cdot\,, f(v_n))f'(v_n)\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\|v\|_{L^{2^*}(\mathbb{R}^N\setminus B_r)} < \frac{\varepsilon}{4}.$$

Let $\eta = \eta_r \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a function verifying $\text{supp}(\eta) \subseteq B_r^c(0)$, $\eta \equiv 1$ in $B_{2r}^c(0)$ and $|\nabla\eta(x)| \leq 1/r$ for all $x \in \mathbb{R}^N$. Since (v_n) is bounded in E, the sequence (ηv_n) is also bounded in E and then $\langle J'(v_n), (\eta v_n)\rangle = o_n(1)$, that is,

$$\int_{\mathbb{R}^N} [\nabla v_n \nabla(\eta v_n) + V(x) f(v_n) f'(v_n)(\eta v_n)] \mathrm{d}x = \int_{\mathbb{R}^N} (\eta v_n) g(x, f(v_n)) f'(v_n) \mathrm{d}x + o_n(1).$$

Once $\eta \equiv 0$ in $B_r(0)$ and $\Lambda = B_R(0) \subset B_r(0)$, the last equality combined with the property given in item (3) Remark 2.1 yields

$$\begin{aligned}\int_{|x|\geq r} & \eta\left[|\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right] \mathrm{d}x \\ & \leq \frac{1}{k}\int_{|x|\geq r} \eta V(x) f(v_n) f'(v_n) v_n \,\mathrm{d}x - \int_{|x|\geq r} v_n \nabla v_n \nabla \eta \,\mathrm{d}x + o_n(1)\end{aligned}$$

and so,

$$\begin{aligned}\left(1 - \frac{1}{k}\right) & \int_{|x|\geq r} \eta[|\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n] \,\mathrm{d}x \\ & \leq \frac{1}{r}\int_{r\leq|x|\leq 2r} |v_n||\nabla v_n| \,\mathrm{d}x + o_n(1).\end{aligned} \tag{2.14}$$

Since we have $H^1(B_{2r}(0)) = D^{1,2}(B_{2r}(0))$, we get $v_n \in L^2(B_{2r}(0))$ and by Holder's inequality we obtain

$$\int_{r\leq|x|\leq 2r} |v_n||\nabla v_n| \,\mathrm{d}x \leq \| \nabla v_n \|_2 \left(\int_{r\leq|x|\leq 2r} |v_n|^2 \,\mathrm{d}x\right)^{1/2}.$$

Now, given that $v_n \rightharpoonup v$ in $H^1(B_{2r} \backslash B_r)$ and due to the Rellich-Kondrachov Compactness Theorem, we have $v_n \to v$ in $L^s(B_{2r} \setminus B_r)$, up to a subsequence, for all $1 \leq s < 2^*$. Using that $\|\nabla v_n\|_2 \leq C$, it follows that

$$\limsup_n \int_{r\leq|x|\leq 2r} |v_n||\nabla v_n| \,\mathrm{d}x \leq C\left(\int_{r\leq|x|\leq 2r} |v|^2 \,\mathrm{d}x\right)^{1/2}. \tag{2.15}$$

On the other hand, invoking again Holder's inequality, we get

$$\left(\int_{r\leq|x|\leq 2r} |v|^2 \,\mathrm{d}x\right)^{1/2} \leq \left(\int_{r\leq|x|\leq 2r} |v|^{2^*} \,\mathrm{d}x\right)^{1/2^*} |B_{2r} \setminus B_r|^{1/N}. \tag{2.16}$$

Recalling that $|B_{2r} \setminus B_r| \leq |B_{2r}| = \omega_N 2^N r^N$, from (2.15) and (2.16) we obtain

$$\limsup_n \int_{r\leq|x|\leq 2r} |v_n||\nabla v_n| \,\mathrm{d}x \leq 2Cr\omega_N^{1/N}\left(\int_{r\leq|x|\leq 2r} |v|^{2^*} \,\mathrm{d}x\right)^{1/2^*}. \tag{2.17}$$

By (2.13), (2.14) and (2.17), it follows that

$$\limsup_n \int_{|x|\geq r} \eta\left(|\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n\right) \mathrm{d}x < \frac{\varepsilon}{4}.$$

Since $g(x,t) \le (V(x)|t|)/k$ in $\mathbb{R}^N \backslash B_r$, $\eta \equiv 1$ in $\mathbb{R}^N \backslash B_{2r}$ and $k > 1$ we get

$$0 \le \limsup_n \int_{|x|\ge 2r} g(x, f(v_n)) f'(v_n) v_n \, \mathrm{d}x \; < \; \frac{\varepsilon}{4}. \tag{2.18}$$

Now, using $(h_1) - (h_2)$, the compactness of the embedding $H^1(B_{2r}) \hookrightarrow L^{\frac{p-2}{2}\frac{2N}{N+2}}(B_{2r})$ and the Dominated Convergence Theorem, it leads us to

$$g(x, f(v_n)) f'(v_n) \to g(x, f(v)) f'(v) \quad \text{in} \quad L^{2N/(N+2)}(B_{2r}).$$

By the Holder inequality and $\|v - v_n\|_{2^*} \le C_1$, it follows that

$$\lim_{n\to\infty} \int_{|x|\le 2r} \left[g(x, f(v_n)) f'(v_n) - g(x, f(v)) f'(v) \right] (v - v_n) \, \mathrm{d}x = 0. \tag{2.19}$$

Finally, (2.18) and (2.19) result in

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[g(x, f(v_n)) f'(v_n) - g(x, f(v)) f'(v) \right] (v - v_n) \, \mathrm{d}x = 0. \tag{2.20}$$

Thus, taking the limit in (2.12),

$$\begin{aligned} \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x \quad &+ \quad \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, \mathrm{d}x \\ &\le \quad \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) f^2(v) \, \mathrm{d}x. \end{aligned}$$

On the other hand, by the weak lower semicontinuity of the norm and Fatou's Lemma we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x \quad &\le \quad \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x \\ \int_{\mathbb{R}^N} V(x) f^2(v) \, \mathrm{d}x \quad &\le \quad \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, \mathrm{d}x. \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x \quad &= \quad \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x \\ \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, \mathrm{d}x \quad &= \quad \int_{\mathbb{R}^N} V(x) f^2(v) \, \mathrm{d}x. \end{aligned}$$

Using (4) of Proposition 2.2 we get, up to a subsequence,

$$\inf_{\xi>0} \frac{1}{\xi} \left[1 + \int_{\mathbb{R}^N} V(x) f^2(\xi(v_n - v)) \, \mathrm{d}x \right] \to 0$$

which implies that $v_n \to v$ in E and completes the proof.

Hereafter, we denote by B the unitary ball in $\mathbb{R}^N$, that is, $B = B_1(0)$, and by $I_0 : H_0^1(B) \to \mathbb{R}$ the functional

$$I_0(v) = \frac{1}{2}\int_B \left(|\nabla v|^2 + \max_B (V(x), 1) f^2(v)\right) \mathrm{d}x - \int_B H(f(v)) \,\mathrm{d}x.$$

Using arguments analogous to those above, we see that I_0 has the mountain–pass geometry and so, the mountain–pass level associated with I_0 is well defined, that is,

$$d = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_0(\gamma(t)),$$

where

$$\Gamma_0 = \{\gamma \in C\left([0,1], H_0^1(B)\right) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\},$$

with $e \in H_0^1(B) \setminus \{0\}$ verifying $I_0(e) < 0$.

Remark 2.2 We observe that $J(u) \leq I_0(u)$ for all $u \in H_0^1(B)$. In particular we have $J(e) \leq I_0(e) < 0$. We denote by m_J the mountain–pass level associated with J, that is,

$$m_J = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E)) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\}.$$

It is easily seen that $0 < m_J \leq d$.

In order to prove existence of a critical point for J we will use the following version of the mountain–pass theorem, which is a consequence of the Ekeland Variational Principle as developed in [7] (see also [24]).

Proposition 2.5 *Le E be a Banach space and $\Phi \in C(E, \mathbb{R})$, Gateaux-differentiable for all $v \in E$ with G-derivative Φ' continuous from the norm topology of E to the weak $*$ topology of E'. Suppose also that Φ satisfies (P.–S.) condition and $\Phi(0) = 0$. Let S be a closed subset of E which disconnects (archwise) E. Let $v_0 = 0$ and $v_1 \in E$ be points belonging to distinct connected components of $E\backslash S$. Suppose that*

$$\inf_S \Phi \geq \alpha > 0 \quad \textit{and} \quad \Phi(v_1) \leq 0.$$

Then, Φ possesses a critical value c which can be characterized as

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \alpha,$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \quad \textit{and} \quad \gamma(1) = v_1\}.$$

Proposition 2.6 *There is $\phi \in E\backslash\{0\}$ a critical point for J such that $J(\phi) = m_J$.*

Proof. It follows from Lemma 2.1 and 2.3 and Proposition 2.5.

2.6 A priori estimate

Lemma 2.4 *Let $v \in E$ be a critical point of the functional J at the minimax level m_J. Then v satisfies the estimate*

$$\|v\|_E \quad \leq \quad 2 + 2kd.$$

Proof. It is enough to combine (2.10) and (2.11) and the fact that $m_J \leq d$.

The next proposition is crucial to our arguments because it establishes an important estimate involving the $L^\infty-$norm of the solution v of the problem (2.1). The following proposition is a version of a result due to H. Brézis and T. Kato [11] which is suitable to apply to our class of problems.

Proposition 2.7 *Let $a \in L^q(\mathbb{R}^N)$, $2q > N$, and $v \in E$ be a weak solution of the problem*

$$-\Delta v + V(x)f(v)f'(v) = g(x, f(v))f'(v) \quad in \quad \mathbb{R}^N. \tag{2.21}$$

Suppose that

$$|g(x, f(v))| \quad \leq \quad a(x)|v|, \quad a.e. \quad x \in \mathbb{R}^N.$$

Then there exists $M = M(N, q, \|a\|_q) > 0$ such that

$$\|v\|_\infty \leq M \, \|v\|_{2^*} \,. \tag{2.22}$$

Proof. For each $m \in \mathbb{N}$ and $\beta > 1$, let

$$A_m \quad = \quad \{x \in \mathbb{R}^N : |v(x)|^{\beta-1} \leq m\},$$

and

$$v_m = \begin{cases} v|v|^{2(\beta-1)} & \text{in} \quad A_m \\ m^2 v & \text{in} \quad B_m = \mathbb{R}^N \setminus A_m. \end{cases}$$

From the definition of v_m we have $v_m|_{\partial B_m} = v_m|_{\partial A_m}$, and so $v_m \in D^{1,2}(\mathbb{R}^N)$. Moreover, since $|v_m| \leq m^2|v|$, for each $m \in \mathbb{N}$ we get

$$\int_{\mathbb{R}^N} V(x) f^2(v_m)\, \mathrm{d}x \;\leq\; m^4 \int_{\mathbb{R}^N} V(x) f^2(v)\, \mathrm{d}x \;<\; \infty$$

which implies that $v_m \in E$. Calculating the ∇v_m, we have

$$\nabla v_m = \begin{cases} (2\beta - 1)|v|^{2(\beta-1)}\nabla v & \text{in} \quad A_m \\ m^2 \nabla v & \text{in} \quad B_m. \end{cases}$$

and so

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m\, \mathrm{d}x \;=\; (2\beta - 1) \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2\, \mathrm{d}x + m^2 \int_{B_m} |\nabla v|^2\, \mathrm{d}x. \tag{2.23}$$

Applying this test function v_m in (2.21), we obtain

$$\int_{\mathbb{R}^N} \left(\nabla v \nabla v_m + V(x) f(v) f'(v) v_m\right) \mathrm{d}x \quad = \quad \int_{\mathbb{R}^N} g(x, f(v)) f'(v) v_m \,\mathrm{d}x.$$

Furthermore, defining

$$\omega_m = \begin{cases} v|v|^{\beta-1} & \text{in} \quad A_m \\ mv & \text{in} \quad B_m = \mathbb{R}^N \setminus A_m, \end{cases}$$

it follows that $\omega_m \in E$ and

$$\nabla\omega_m = \begin{cases} \beta|v|^{\beta-1}\nabla v & \text{in} \quad A_m \\ m\nabla v & \text{in} \quad B_m. \end{cases}$$

Observe that

$$\int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x = \beta^2 \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}x + m^2 \int_{B_m} |\nabla v|^2 \,\mathrm{d}x. \tag{2.24}$$

From (2.23) and (2.24) we get

$$\int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla v \nabla v_m \,\mathrm{d}x = \int_{A_m} \left(\beta^2 - 2\beta + 1\right) |v|^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}x.$$

Using (2.23), we get the inequality

$$\left(2\beta - 1\right) \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^N} \left(\nabla v \nabla v_m + V(x) f(v) f'(v) v_m\right) \mathrm{d}x,$$

which leads us to

$$\int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x \leq \left[\frac{\left(\beta^2 - 2\beta + 1\right)}{(2\beta - 1)} + 1 \right] \int_{\mathbb{R}^N} \left(\nabla v \nabla v_m + V(x) f(v) f'(v) v_m\right) \mathrm{d}x.$$

Since (2.21) holds for v, it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x \quad &\leq \quad \frac{\beta^2}{(2\beta - 1)} \int_{\mathbb{R}^N} g(x, f(v)) f'(v) v_m \,\mathrm{d}x \\ &\leq \quad \beta^2 \int_{\mathbb{R}^N} g(x, f(v)) f'(v) v_m \,\mathrm{d}x. \end{aligned}$$

Let S be the best constant which verifies

$$\| v \|_{2^*}^2 \quad \leq \quad S \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x \qquad \text{for all} \quad v \in D^{1,2}(\mathbb{R}^N).$$

By definition of v_m we get $|v_m| \leq |v|^{2\beta-1}$ in $\mathbb{R}^N$ and so

$$|g(x, f(v)) f'(v) v_m| \leq a(x)|v|^{2\beta}.$$

Thus, for $q_1 > 1$ such that $(1/q_1) + (1/q) = 1$, $\sigma = 2^*/(2q_1) > 1$ and $\beta = \sigma$, from Holder's inequality we obtain

$$\begin{aligned}\left[\int_{A_m} |\omega_m|^{2^*} \mathrm{d}x\right]^{(N-2)/N} &\leq S\beta^2 \int_{\mathbb{R}^N} a(x)|v|^{2\beta}\,\mathrm{d}x \\ &\leq S\beta^2 \, \|a\|_q \left[\int_{\mathbb{R}^N} |v|^{2\beta q_1}\,\mathrm{d}x\right]^{1/q_1}.\end{aligned}$$

Since $|\omega_m| \leq |v|^\beta$ in $\mathbb{R}^N$ and $|\omega_m| = |v|^\beta$ in A_m, it follows that

$$\left[\int_{A_m} |v|^{\beta 2^*} \mathrm{d}x\right]^{(N-2)/N} \leq S\beta^2 \, \|a\|_q \left[\int_{\mathbb{R}^N} |v|^{2\beta q_1}\,\mathrm{d}x\right]^{1/q_1}.$$

Passing to the limit as $m \to \infty$, from Monotone Convergence Theorem we get

$$\|v\|_{\beta 2^*}^{2\beta} \leq S\beta^2 \, \|a\|_q \|v\|_{2\beta q_1}^{2\beta},$$

and so

$$\|v\|_{\beta 2^*} \leq \beta^{1/\beta} \left(S \, \|a\|_q\right)^{1/(2\beta)} \|v\|_{2\beta q_1}. \tag{2.25}$$

In other words, $v \in L^{\sigma 2^*}(\mathbb{R}^N)$ and

$$\|v\|_{2^*\sigma} \leq \sigma^{1/\sigma} \left(S \, \|a\|_q\right)^{1/(2\sigma)} \|v\|_{2^*}. \tag{2.26}$$

When $\beta = \sigma^2$ in (2.25) we get $2q_1\beta = 2^*\sigma$ and

$$\|v\|_{2^*\sigma^2} \leq \sigma^{2/\sigma^2} \left(S \, \|a\|_q\right)^{1/(2\sigma^2)} \|v\|_{2^*\sigma}. \tag{2.27}$$

The inequalities (2.26) and (2.27) imply that

$$\|v\|_{2^*\sigma^2} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}} \left(S \, \|a\|_q\right)^{\frac{1}{2}\left(\frac{1}{\sigma}+\frac{1}{\sigma^2}\right)} \|v\|_{2^*}. \tag{2.28}$$

An iteration argument, replacing β by σ^j in (2.25), leads us to

$$\|v\|_{2^*\sigma^j} \leq \sigma^{\sum_{l=1}^{j}(1/\sigma^l)} \left(S \, \|a\|_q\right)^{\frac{1}{2}\left(\sum_{l=1}^{j}(1/\sigma^l)\right)} \|v\|_{2^*}. \tag{2.29}$$

Given that

$$\sum_{l=1}^{\infty} \frac{1}{\sigma^l} = \frac{1}{\sigma - 1}$$

it follows from (2.29),

$$\|v\|_{2^*\sigma^j} \leq \sigma^{1/(\sigma-1)} \left(S \, \|a\|_q\right)^{1/(2(\sigma-1))} \|v\|_{2^*}, \quad \text{for all} \quad j \in \mathbb{N}.$$

Recalling that

$$\|v\|_\infty \quad = \quad \lim_{i\to\infty} \|v\|_i,$$

and $\sigma = 2^*/(2q_1) > 1$, we can conclude that Proposition 2.7 is valid with

$$M \quad = \quad \sigma^{1/(\sigma-1)} \left(S \ \|a\|_q\right)^{1/(2(\sigma-1))}$$

which means that $M = M(N, q, \|a\|_q)$. This is the end of the proof.

Lemma 2.5 *If $v \in E$ is a critical point of the functional J, then $v > 0$.*

Proof. Since $v \in E$ is a critical point for J we get

$$0 = \langle J'(v), v^- \rangle = \int_{\mathbb{R}^N} \left(|\nabla v^-|^2 + V(x) f(v^-) f'(v^-) v^-\right) \mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla v^-|^2 \mathrm{d}x,$$

which implies $\| v^- \|_{D^{1,2}} = 0$ and so $v \geq 0$. Since $v \in L^\infty(\mathbb{R}^N)$ the result follows from the Maximum Principle.

Lemma 2.6 *There exists $\tilde{M} > 0$ such that*

$$\|v\|_\infty \quad \leq \quad \tilde{M}$$

for any $R > 1$ and any solution v of (MP) such that $J(v) = m_J$.

Proof. Since $0 \leq g(x, t) \leq h(t) \leq c_0|t|^{p-2}$ for all $t \in \mathbb{R}$, we get

$$|g(x, f(v))| \leq c_0|f(v)|^{p-2} \leq 2^{1/4} c_0 |v|^{(p-2)/2} \leq a(x)|v|$$

where

$$a(x) \quad = \quad 2^{1/4} c_0 |v(x)|^{(p-4)/2}.$$

A direct computation shows that $a \in L^q(\mathbb{R}^N)$ for $q = 2(2^*)/(p-4) > N/2$. Then, using the Sobolev inequality and Lemma 2.4 we obtain

$$\|a\|_q \quad \leq \quad 2c_0 \ \|v\|_{2^*}^{(p-4)/4} \leq \ 2c_0 \left(S^{1/2}(2 + 2kd)\right)^{(p-4)/4}.$$

Now, for M given in Proposition 2.7 we obtain

$$\begin{aligned} M \ \|v\|_{2^*} \quad &= \quad \sigma^{1/(\sigma-1)} \left(S \ \|a\|_q\right)^{1/(2(\sigma-1))} \ \|v\|_{2^*} \\ &\leq \quad \sigma^{1/(\sigma-1)} \left(2S\, c_0 (S^{1/2}(2 + 2kd))^{(p-4)/4)}\right)^{1/(2(\sigma-1))} \left(S^{1/2}(2 + 2kd)\right) \\ &:= \quad \tilde{M} \end{aligned}$$

with $\sigma = 2^*/(2q_1)$. Observe that $\tilde{M}$ does not depend on $R > 1$ or v satisfying the conditions of this proposition. Thus, the result follows from Proposition 2.7. This is the end of the proof.

Lemma 2.7 *Let $v \in E$ be positive solution of (2.3) with $J(v) = m_J$. Then v satisfies*

$$v(x) \leq \frac{R^{N-2}\,\|v\|_\infty}{|x|^{N-2}} \leq \frac{R^{N-2}\tilde{M}}{|x|^{N-2}} \quad \textit{for all} \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Proof. Consider the harmonic function $\psi : \left(\mathbb{R}^N \setminus B_R(0)\right) \longrightarrow \mathbb{R}$ given by

$$\psi(x) \quad = \quad \frac{R^{N-2}\,\|v\|_\infty}{|x|^{N-2}}.$$

Since $\psi \geq 0$ we get

$$-\Delta\psi + \left(1 - \frac{1}{k}\right) V(x) f(\psi) f'(\psi) \geq 0.$$

Now, take as a test function

$$\phi = \begin{cases} (v - \psi)^+ & \text{if} \quad x \in \mathbb{R}^N \setminus B_R(0), \\ 0 & \text{if} \quad x \in B_R(0). \end{cases}$$

Observe that $v \leq \psi$ on $\partial B_R(0)$. Then we can conclude that $\phi \in D^{1,2}(\mathbb{R}^N)$. Besides that

$$J'(v)(\phi) - \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{k} V(x) f(v) f'(v)\phi \leq 0.$$

Consider $\Omega := \{x \in \mathbb{R}^N \setminus B_R(0) : v > \psi\}$. Hence, combining these estimates

$$\begin{aligned} 0 \quad &\geq \quad \int_\Omega (\nabla v - \nabla\psi)\,\nabla\phi + \left(1 - \frac{1}{k}\right) V(x)(f(v)f'(v) - f(\psi)f'(\psi))\phi \,\mathrm{d}x \\ &\geq \quad \int_\Omega \left(1 - \frac{1}{k}\right) V(x)(f(v)f'(v) - f(\psi)f'(\psi))\phi \,\mathrm{d}x. \end{aligned}$$

Using that $f^2(t)$ is strictly convex, we have that ff' is a increasing function and so

$$V(x)(f(v)f'(v) - f(\psi)f'(\psi))\phi \geq 0 \quad \text{in } \Omega.$$

Thus, the set Ω is empty and the proof is complete.

3 Proof of Theorem 1.1

From Proposition 2.6, the Problem (2.3) has a solution $v \in D^{1,2}(\mathbb{R}^N)$ at the mountain–pass level m_J and from Lemma 2.5 we know that this is a positive solution. Therefore, by definition of g, in order to show that $f(v) = u$ is a solution of problem (1.1), it is enough to verify that $f(v)$ satisfies

$$h(f(v)) \quad \leq \quad \frac{V(x)}{k} f(v) \quad \text{in} \quad \mathbb{R}^N \backslash B_R(0).$$

Indeed, from hypotheses (h_1) and (h_2), there is a constant $c_0 > 0$ such that

$$|sh(s)| \leq c_0|s|^{2(2^*)} \quad \text{for all} \quad s \in \mathbb{R}.$$

It follows from Lemma 2.7 that

$$\frac{h(f(v))}{f(v)} \leq c_0|f(v)|^{\frac{2N+4}{N-2}} \leq 2c_0|v|^{\frac{N+2}{N-2}} \leq 2c_0\tilde{M}^{\frac{N+2}{N-2}}\frac{R^{N+2}}{|x|^{N+2}} \quad \text{for all} \quad |x| \geq R.$$

Now, let $\mu^* = k2c_0\tilde{M}^{\frac{N+2}{N-2}}$ and $\mu \geq \mu^*$. Using hypothesis (V_1) we can complete the proof.

References

[1] C. O. Alves, J. M. do Ó, M. A. S. Souto, *Local mountain–pass for a class of elliptic problems involving critical growth*, Nonlinear Anal. **46** (2001), 495–510.

[2] C. O. Alves, M. A. S. Souto, *Existence of solutions for a class of elliptic equations in $\mathbb{R}^N$ with vanishing potentials*, J. Differential Equations, **252** (2012), 5555–5568.

[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349–381.

[4] A. Ambrosetti, V. Felli, A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. **7** (2005) 117–144.

[5] A. Ambrosetti, A. Malchiodi, D. Ruiz, *Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Anal. Math. **98** (2006) 317–348.

[6] A. Ambrosetti, Z. -Q. Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials*, Differ. Integral. Equations. **18** (2005) 1321–1332.

[7] J. -P. Aubin, I. Ekeland, *Applied nonlinear analysis. Pure and Applied Mathematics*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1984).

[8] V. Benci, C. R. Grisanti and A. M. Micheletti, *Existence and non existence of the ground state solution for the nonlinear Schrödinger equations with $V(\infty) = 0$*, Topol. Methods Nonlinear Anal., **26**, (2005), 203–219.

[9] H. Berestycki, P. -L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*. Arch. Rational Mech. Anal. **82** (1983), 313–345.

[10] F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, Kluwer, Dordrecht (1991).

[11] H. Brézis, T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math Pures Appl. **58** (1979) 137–151.

[12] Y. Brihaye and B. Hartmann, *Solitons on nanotubes and fullerenes as solutions of a modified nonlinear Schrödinger equation. Advances in soliton research*, 135–151, Nova Sci. Publ., Hauppauge, NY, (2006).

[13] L. Brizhik, A. Eremko, B. Piette and W. J. Zakrzewski, *Static solutions of a D-dimensional modified nonlinear Schrödinger equation*, Nonlinearity, **16** (2003), 1481–1497.

[14] L. Brüll and H. Lange, *Solitary waves for quasilinear Schrödinger equations*, Expos. Math., **4** (1986), 278–288.

[15] M. Colin, L. Jeanjean, *Solutions for a quasilinear Schrödinger equation: a dual approach*, Nonlinear Anal. **56** (2004) 212–226.

[16] M. del Pino, P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Cal. Var. **4** (1996) 121–137.

[17] J. M. do , O. H. Miyagaki, S. H. M. Soares, *Soliton solutions for quasilinear Schrödinger equations: the critical exponential case*, Nonlinear Anal. **67** (2007) 3357–3372.

[18] J. M. do , O. H. Miyagaki, S. H. M. Soares, *Soliton solutions for quasilinear Schrödinger equations with critical growth*, J. Differential Equations **248** (2010) 722–744.

[19] J. M. do Ó, U. Severo, *Quasilinear Schrödinger equations involving concave and convex nonlinearities*, Commun. Pure Appl. Anal. **8** (2009), 621–644.

[20] J. M. do Ó, U. Severo, *Solitary waves for a class of quasilinear Schrödinger equations in dimension two*, Calc. Var. **38** (2010), 275–315.

[21] E. Gloss, *Existence and concentration of positive solutions for a quasilinear equation in* $\mathbb{R}^N$, J. Math. Anal. Appl. **371** (2010), 465–484.

[22] B. Hartmann and W. J. Zakrzewski, *Electrons on hexagonal lattices and applications to nanotubes*, Phys. Rev. B, 68 (2003), 184–302.

[23] S. Kurihura, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Japan, **50** (1981), 3262–3267.

[24] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Mathématiques & Applications, **13** Springer-Verlag, Paris,(1993).

[25] Q.Li, X. Wu, *Multiple solutions for a quasilinear Schrödinger equations*, J. Math. Phys. (2014), impress.

[26] P. -L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case*. II. Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.

[27] J. Liu,Y. Wang and Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations II,*. J. Differential Equations **187** (2003), 473–493.

[28] A. F. Nikiforov and V. B. Uvarov, *Special functions of mathematical physics. A unified introduction with applications*, Birkh user Verlag, Basel, (1988).

[29] U. Severo, *Existence of weak solutions for quasilinear elliptic equations involving the p-Laplacian*, Electron. J. Differential Equations **56** (2008), 16 pp.

[30] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, (1986).

[31] P. H Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[32] X. Wu, *Multiple solutions for quasilinear Schrödinger equations with a parameter*, J. Differential Equations **256** (2014) 2619–2632.