Advanced Nonlinear Studies **15** (2015), 667–690

Infinitely Many Spiky Solutions for a Hénon Type Biharmonic Equation with Critical Growth

Jianghao Hao[*], **Yajing Zhang**[†]

School of Mathematical Sciences

Shanxi University, Taiyuan, Shanxi 030006, China

e-mail: hjhao@sxu.edu.cn

e-mail: zhangyj@sxu.edu.cn

Xinfu Chen[‡]

Department of Mathematics

University of Pittsburgh, Pittsburgh, PA 15260, USA

e-mail: xinfu@pitt.edu

Received in revised form 26 June 2014

Communicated by Norman Dancer

Abstract

Following the constructive method of Wei and Yan [29], with new ingredients to take care of high-space dimensions, we prove the existence of infinitely many solutions of the nonlinear biharmonic equation $\Delta^2 v = |x|^\alpha v^{\frac{n+4}{n-4}}$ in the unit ball of $\mathbb{R}^n$ $(n \geqslant 6, \alpha > 0)$ with the Navier conditions $v = \Delta v = 0$ on the boundary.

2000 Mathematics Subject Classification. 35J35, 35J40.

Key words. Bi-harmonic operator, Hénon problem, Critical Sobolev exponents, Spike

1 Introduction

Consider the following Hénon type biharmonic problem: for $u = u(x)$,

$$\Delta^2 u = |x|^\alpha u^p, \;\; u > 0 \quad \text{in } B_1, \qquad u = \Delta u = 0 \quad \text{on } \partial B_1, \tag{1.1}$$

[*]Supported by NNSFC (No. 61374089), Shanxi Scholarship Council of China(2013-013) and International Cooperation Projects of Shanxi Province (No. 2014081026).

[†]Supported by NSF of Shanxi Province(No. 2014011005-2). Corresponding author.

[‡]Supported by NSF DMS-1008905.

where $\alpha > 0$ is a constant, $p = (n+4)/(n-4)$ is the critical Sobolev exponent, and B_1 is the unit ball in $\mathbb{R}^n$ centered at the origin. This paper is focused on the proof of the following:

Theorem 1.1 *If $\alpha > 0, n \geqslant 6$, and $p = \frac{n+4}{n-4}$, problem* (1.1) *admits infinitely many solutions.*

In recent years biharmonic equations with nonlinear source terms have attracted quite a bit of attention; see, for example, [1, 4, 5, 12, 14, 15, 16, 17, 19, 27, 30, 31, 32] and the references therein. In [27], Wang proved that problem (1.1) possesses at least one non-radial solution when $n \geqslant 6$, $p = \frac{n+4}{n-4}$, and, additionally, α is large enough. With $p = \frac{n+4}{n-4}$, replacing $|x|^\alpha$ by a positive smooth function $K(x)$ and B_1 by a general bounded domain in $\mathbb{R}^6$, Ayed and Hammami [1] proved, among many beautiful estimates, the existence of a solution under certain conditions on K. Figueiredo, Santos, and Miyagaki [12] proved that (1.1) admits a classical solution if and only if $1 < p < \frac{n+4+2\alpha}{n-4}$ (see also [27]), and established the existence of solutions of the form $v = v(|y|, |z|)$ with $y \in \mathbb{R}^\ell, z \in \mathbb{R}^{n-\ell}$ for (ℓ, p) in certain ranges.

The original Hénon problem, modeling mass distribution in spherical symmetric clusters of stars [18], was formulated in 1973 in terms of the second order elliptic problem

$$-\Delta u = |x|^\alpha u^q,\ u > 0 \quad \text{in } B_1, \qquad u = 0 \text{ on } \partial B_1. \tag{1.2}$$

In 1982, Ni [21], for the first time proved rigorously the existence of radial solutions of (1.2) for each $\alpha > 0$ and $q \in (1, \frac{n+2+2\alpha}{n-2})$. There are many works concerning the energy and the profile of the ground state (constrained energy minimizer) solutions of (1.2) for either $q \approx \frac{n+2}{n-2}$ or $\alpha \gg 1$; see, for example, [2, 7, 8, 9, 10, 25, 26], and the references therein. Smets, Su, and Willem [25, 26], based on numerical discovery of Chen, Ni, and Zhou [11], proved that for every fixed $q \in (1, \frac{n+2}{n-2})$ and α large enough, or for any fixed $\alpha > 0$ and q sufficiently close to $\frac{n+2}{n-2}$ from below, any ground state solution of (1.2) is not radial. Serra [24] studied the case $q = \frac{n+2}{n-2}$ and proved the existence of non-radial positive solutions of (1.2) for α sufficiently large. Cao, Peng, and Yan [9, 10], as well as Byeon and Wang [7, 8], proved that the points of maximum of ground states approach the boundary ∂B_1 as $p \nearrow \frac{n+2}{n-2}$ or $\alpha \to \infty$; they also obtained limiting profile of the ground state. Multiple peak solutions for slightly subcritical growth was established by Pistoia and Serra [23] and Peng [22], and for slightly supercritical ($\alpha = 0$) by del Pino, Felmer and Musso [13]. Quite recently, Wei and Yan [29], using a finite dimensional reduction argument (c.f. [13, 20, 22, 23, 28]) equipped with a carefully chosen weighted L^∞ norm, produced the following elegant result:

> *When $n \geqslant 4$, $q = \frac{n+2}{n-2}$, and $\alpha > 0$,* (1.2) *admits infinitely many non-radial solutions.*

In this paper, we follow the line of the argument of Wei and Yan [29] for (1.2), carrying out the analogous analysis for (1.1). It seems to us that the analysis goes through when $4 \leqslant n \leqslant 9$ for (1.2) and $6 \leqslant n \leqslant 19$ for (1.1). For this reason, here we introduce new techniques and more estimates to complement the classical method of Wei and Yan [29] and to overcome technical difficulties arising from the case when n is large. Meanwhile, we shall simplify some of their calculations.

The solution constructed in [29] has k peaks, with k an arbitrarily large integer; see Figure 1. The possibility of the existence of multi-peak solutions is due to the local stability (modulo dilation and translation) of radially symmetric ground state in $\mathbb{R}^n$; see Bianchi and Egnell [6] for the Laplace operator and Bartsch, Weth, and Willem [3] for polyharmonic operators.

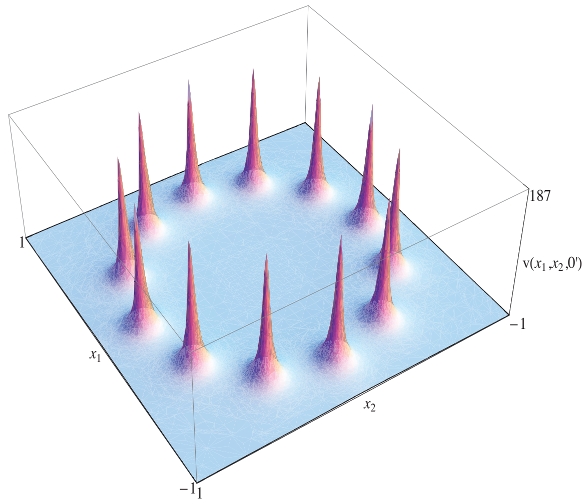

Figure 1: A Spike Solution in Space (x_1, x_2, u) with $u = v(x_1, x_2, 0, \cdots, 0)$.

Here we briefly describe the multi-spike solution. The equation $\Delta^2 v = v^p$ admits a ground state $v(x) = \Phi(x) = c_n(1+|x|^2)^{-m}$ where $m = (n-4)/2$ and c_n is a positive constant. By scaling and translation, for each parameter $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$,

$$v(x) = \frac{1}{\varepsilon^m}\Phi\Big(\frac{x-\xi}{\varepsilon}\Big) = \frac{c_n\varepsilon^m}{[\varepsilon^2 + |x-\xi|^2]^m}$$

is also a solution of $\Delta^2 v = v^p$. When $0 < \varepsilon \ll 1$, we call $\varepsilon^{-m}\Phi([x-\xi]\varepsilon^{-1})$ a spike centered at ξ. For each large enough integer k, we search for k-spike solutions (c.f. Figure 1) of the form

$$u(x) \approx \max_i \frac{1}{\varepsilon^m}\Phi\Big(\frac{x - r\mathbf{e}_i}{\varepsilon}\Big), \quad \varepsilon = \frac{1}{\lambda k^{1+\frac{1}{n-4}}}, \ r = 1 - \frac{\sigma}{k}, \tag{1.3}$$

where $\{\mathbf{e}_i\}_{i=1}^k$ are evenly distributed on the circle $x_1^2 + x_2^2 = 1, x_3 = 0, \cdots, x_n = 0$ and $(\lambda, \sigma) \approx (\lambda^*, \sigma^*)$, with λ^* and σ^* being positive constants depending only on n; see the following

Table 1: The parameter (λ^*, σ^*) with respect to space dimension n

n	5	6	7	8	9	10	11	12
σ^*	3.53056	3.01448	2.89353	2.86396	2.86511	2.87773	2.89431	2.91166
$\alpha^{\frac{1}{n-4}}\lambda^*$	0.62793	0.40943	0.35983	0.33680	0.32276	0.31300	0.30569	0.29995

n	13	14	15	16	17	18	19	20
σ^*	2.92837	2.94387	2.95796	2.97064	2.98200	2.99217	3.00127	3.00943
$\alpha^{\frac{1}{n-4}}\lambda^*$	0.29532	0.29149	0.28828	0.28550	0.28316	0.28110	0.27928	0.27766

To prove Theorem 1.1, one needs (i) an approximate solution W, (ii) invertibility of $\mathcal{L} = \Delta^2 - p|x|^\alpha W^{p-1}$, and (iii) smallness of the nonlinear term $N(\phi) := |x|^\alpha\{(W+\phi)^p - W^p - pW^{p-1}\phi\}$. When $n \leqslant 12$, $N(\phi) = O([W + |\phi|]^{p-2})\phi^2$; however, when $n \geqslant 13$, we only have $N(\phi) = O(W + |\phi|)|\phi|^{p-1})$. Our new techniques and delicate estimates are introduced mainly for the purpose of taking care of the case when n is large.

The rest of the paper is organized as follows. In Section 2, we explain the idea of the proof. In Section 3, we provide a few basic estimates. We study the operator $\mathcal{L}$ in Section 4 and a nonlinear problem involving Lagrange multipliers in Section 5. In Section 6 we calculate, as a function of (λ, σ), the gradient of an energy associated with W. Finally in Section 7, we show that the energy admits a critical point near (λ^*, σ^*), from which we obtain, for each integer $k \gg 1$, a solution of (1.1) having the form (1.3).

Remark 1.1 As in [29], Theorem 1.1 is valid when the function $|x|^\alpha$ is replaced by $K(|x|)$ where $K(\cdot)$ satisfies, for some $\delta > 0$, $K \in C([0, 1]) \cap C^2([1 - \delta, 1]), K \geqslant 0, \alpha := K'(1) > 0$.

Remark 1.2 With respect to [29], here we have made two corrections: (i) the calculation of the Green's function, e.g. the expansions of $\mathbf{E}(\mathbf{x}_0)$ and $\nabla\mathbf{E}(\mathbf{x}_0)$ in Section 6; this leads to our new conclusion that there is no explicit formula for (λ^*, σ^*); (ii) the calculation of a variation of an energy; our new estimation (7.4) indicates that it is necessary to introduce new techniques for (1.2) when $n \geqslant 10$ and for (1.1) with $n \geqslant 20$.

Remark 1.3 The problem for the case $n = 5$ for (1.1) and $n = 3$ for (1.2) is still open.

2 Idea of the proof

2.1 The approximate solution

In the sequel we use the following notation:

$$p = \frac{n+4}{n-4}, \qquad m = \frac{n-4}{2}, \qquad \tau = \frac{n-4}{n-3}, \qquad \Phi(x) = \frac{[n(n^2-4)(n-4)]^{m/4}}{(1+|x|^2)^m}.$$

Note that $\Delta^2\Phi = \Phi^p$ in $\mathbb{R}^n$. Fix a positive integer k. For $\mathbf{e} \in \mathbb{S}^{n-1}$, $\lambda > 0$ and $\sigma \in (0, k)$, we define

$$\varepsilon := \frac{k^{-1/\tau}}{\lambda}, \qquad r := 1 - \frac{\sigma}{k}, \qquad U(x; \lambda, \sigma, \mathbf{e}, k) := \frac{1}{\varepsilon^m}\Phi\Big(\frac{x - r\mathbf{e}}{\varepsilon}\Big).$$

Then $\Delta^2 U = U^p$ in $\mathbb{R}^n$. We denote by $V = \mathbb{P}U$ the projection of U defined by

$$\Delta^2(V - U) = 0 \ \text{ in } \ B_1, \qquad V = \Delta V = 0 \ \text{ on } \ \partial B_1. \tag{2.1}$$

We define

$$\begin{aligned}
\mathbf{e}_i(k) &:= \Big(\cos\tfrac{2\pi i}{k}, \sin\tfrac{2\pi i}{k},\ 0, \cdots, 0\Big), \qquad i = 0, 1, \cdots, k,\\
U_i(x; \lambda, \sigma, k) &:= U(x; \lambda, \sigma, \mathbf{e}_i(k), k), \qquad V_i(x; \lambda, \sigma, k) := \mathbb{P}U_i(\cdot\,; \lambda, \sigma, k)(x),\\
W(x; \lambda, \sigma, k) &:= \sum_{i=0}^{k-1} V_i(x; \lambda, \sigma, k) = \sum_{i=1}^{k} V_i(x; \lambda, \sigma, k),
\end{aligned} \tag{2.2}$$

$$U_{i\lambda} = \frac{\partial U_i}{\partial \lambda}, \quad U_{i\sigma} = \frac{\partial U_i}{\partial \sigma}, \qquad V_{i\lambda} = \frac{\partial V_i}{\partial \lambda}, \quad V_{i\sigma} = \frac{\partial V_i}{\partial \sigma},$$

$$Z_1 = \sum_{i=1}^{k} U_i^{p-1} U_{i\lambda} = \frac{1}{p}\Delta^2 W_\lambda, \quad Z_2 = \sum_{i=1}^{k} U_i^{p-1} U_{i\sigma} = \frac{1}{p}\Delta^2 W_\sigma. \tag{2.3}$$

We always assume that (λ, σ) is in the range

$$\lambda \in \Big[\frac{1}{2}\lambda^*,\ 2\lambda^*\Big], \qquad \sigma \in \Big[\frac{1}{2}\sigma^*,\ 2\sigma^*\Big], \tag{2.4}$$

where (λ^*, σ^*) is the unique positive root, for (λ, σ), of the algebraic system

$$L_1(\sigma) = 0, \qquad \alpha\lambda^{n-4} = AL_2(\sigma), \tag{2.5}$$

where

$$L_1(\sigma) = \sum_{i\neq 0}\Big(\frac{1}{|i\pi|^{n-4}} - \frac{1}{[(i\pi)^2 + \sigma^2]^{\frac{n-4}{2}}}\Big) - \frac{1}{\sigma^{n-4}}, \tag{2.6}$$

$$L_2(\sigma) = \sum_{i=-\infty}^{\infty} \frac{\sigma}{[\sigma^2 + (i\pi)^2]^{\frac{n-2}{2}}}, \tag{2.7}$$

$$A = \frac{A_2}{A_1}, \quad A_1 = \frac{m}{n}\int_{\mathbb{R}^n} \Phi^{p+1}(y)dy, \quad A_2 = \frac{m}{2^{n-4}}\int_{\mathbb{R}^n} \Phi(0)\Phi^p(y)dy. \tag{2.8}$$

Note that $L_1'(\sigma) = (n-4)L_2(\sigma) > 0$ for $\sigma \in (0, \infty)$, $L_1(0+) = -\infty$ and $L_1(\infty) > 0$. Hence, (2.5) admits a unique root; see Table 1 for its numerical values when $5 \leqslant n \leqslant 20$.

In the sequel, $O(1)$ denotes a generic quantity that is bounded by a constant depending only on n, α, and ρ (to be introduced later). For notational simplicity, we shall suppress the dependence on parameters (λ, σ, k), abbreviating $\mathbf{e}_i(k)$, $U_i(x; \lambda, \sigma, k)$, $V_i(x; \lambda, \sigma, k)$, $W(x; \lambda, \sigma, k)$ simply as $\mathbf{e}_i$, $U_i(x)$, $V_i(x)$, $W(x)$.

2.2 A nonlinear problem

We work on the following spaces of functions with symmetry:

$$\begin{aligned}
\mathcal{H} &= \Big\{\phi \in C(\overline{B_1}) \ :\ \phi(Re^{\mathbf{i}(\pm\theta + \frac{2\pi}{k})},\ x') = \phi(Re^{\mathbf{i}\theta},\ |x'|\,, 0, \cdots, 0)\Big\},\\
\mathcal{H}_0 &= \Big\{\phi \in \mathcal{H} \ :\ \phi \in C^2(\overline{B_1}), \quad \phi = \Delta\phi = 0 \ \text{ on } \partial B_1, \quad \langle Z_1, \phi\rangle = \langle Z_2, \phi\rangle = 0\Big\}.
\end{aligned}$$

Here we have used $(Re^{\mathbf{i}\theta}, x') \in \mathbb{C}\times\mathbb{R}^{n-2} \cong \mathbb{R}^n$ and $\langle \phi, \psi\rangle := \int_{B_1} \phi(x)\psi(x)\,dx$. We consider, for (c_1, c_2, v),

$$(c_1, c_2, v - W) \in \mathbb{R}^2 \times \mathcal{H}_0, \qquad \Delta^2 v = |x|^\alpha |v|^p + \lambda c_1 Z_1 + \varepsilon k c_2 Z_2 \ \text{ in } \ B_1. \tag{2.9}$$

Note that $u := v$ is a solution of (1.1) if $c_1 = c_2 = 0$. We shall choose appropriate λ and σ such that $c_1 = c_2 = 0$. Let $\phi = v - W$. The above equation for (c_1, c_2, v) can be written as the equation for

$$(c_1, c_2, \phi) \in \mathbb{R}^2 \times \mathcal{H}_0,\ \mathcal{L}\phi - \lambda c_1 Z_1 - \varepsilon k c_2 Z_2 = F := F_0 + F_1 + \mathcal{L}_1\phi + N(\phi), \tag{2.10}$$

where

$$\mathcal{L}\phi := \Delta^2\phi - pW^{p-1}\phi, \quad \mathcal{L}_1\phi := (|x|^\alpha - 1)pW^{p-1}\phi, \tag{2.11}$$

$$F_0 := W^p - \Delta^2 W, \;\; F_1 := (|x|^\alpha - 1)W^p, \tag{2.12}$$

$$N(\phi) := |x|^\alpha\{|W+\phi|^p - W^p - pW^{p-1}\phi\}. \tag{2.13}$$

To solve the nonlinear problem (2.10), we first investigate the linear problem: given $f \in \mathcal{H}$, find $(c_1, c_2, \phi) \in \mathbb{R}^2 \times \mathcal{H}_0$ such that $\mathcal{L}\phi - \lambda c_1 Z_1 - \varepsilon k\, c_2 Z_2 = f$ in B_1. We shall show in Section 4 that there is a unique solution which satisfies, for every $\rho \in [\tau, n-4)$, $\|\varepsilon^m\phi\|_\rho + |c_1| + |c_2| = O(1)\varepsilon^{m+4}\|f\|_{\rho+4}$, where $\|\cdot\|_\rho$ is defined by

$$\big\|f\big\|_\rho := \max_{|x|\leqslant 1}\frac{|f(x)|}{\omega_\rho(x)}, \quad \omega_\rho(x) := \sum_{i=1}^k \frac{1}{(1+|x-\mathbf{x}_i|/\varepsilon)^\rho}, \quad \mathbf{x}_i := \Big(1 - \frac{\sigma}{k}\Big)\mathbf{e}_i. \tag{2.14}$$

Remark 2.1 In the literature, e.g. [13, 29], weights are fixed; in the current situation, it corresponds to $\|\cdot\|_{**} = \|\cdot\|_{\rho^*+4}$ and $\|\cdot\|_* := \|\cdot\|_{\rho^*}$ with $\rho^* := m + \tau$. However, it seems to us that the norms used only cover the case $6 \leqslant n \leqslant 19$. For $n \geqslant 20$, we need to use extra weights and introduce new techniques.

2.3 The algebraic equation for the Lagrange multipliers

Taking the inner product of $\lambda c_1 Z_1 + \varepsilon k c_2 Z_2 = \mathcal{L}\phi - F$ with $\lambda V_{0\lambda}$ and $\varepsilon k V_{0\sigma}$ we find that

$$\begin{bmatrix} \lambda^2\ \langle V_{0\lambda}, Z_1\rangle & \lambda\varepsilon k\,\langle V_{0\lambda}, Z_2\rangle \\ \lambda\varepsilon k\,\langle V_{0\sigma}, Z_1\rangle & (\varepsilon k)^2\langle V_{0\sigma}, Z_2\rangle \end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix} = \begin{bmatrix} \lambda\ \langle V_{0\lambda}, \mathcal{L}\phi - F\rangle \\ \varepsilon k\,\langle V_{0\sigma}, \mathcal{L}\phi - F\rangle\end{bmatrix}. \tag{2.15}$$

For each fixed $k \gg 1$, we shall show that the right-hand side vanishes at some special $(\lambda, \sigma) \approx (\lambda^*, \sigma^*)$, so $c_1 = c_2 = 0$. Setting $u := v$ we have $\Delta^2 u = |x|^\alpha |u|^p$ in B_1. Upon using the maximum principle, we find that $-\Delta u > 0$ and $u > 0$ in B_1 so u is a solution of (1.1).

3 Preliminary

First we provide a few basic algebraic facts.

Lemma 3.1 *For each $\alpha \in (0, n)$ and $\beta > n - \alpha$ there exists a constant $C = C(n, \alpha, \beta)$ such that*

$$\int_{\mathbb{R}^n}\frac{|X-Y|^{-\alpha}\,dY}{(1+|Y-Z|)^\beta} \leqslant C\begin{cases} (1+|X-Z|)^{n-\alpha-\beta} & \text{if } \beta < n, \\ (1+|X-Z|)^{-\alpha}\ln[2+|X-Z|] & \text{if } \beta = n, \\ (1+|X-Z|)^{-\alpha} & \text{if } \beta > n. \end{cases}$$

This is an extension of [28, Lemma B.2] and [17, Corollary 3.9], so the proof is omitted.

Lemma 3.2 *When $s > 1$, there exists a constant $C = C(s)$ such that, for each $a > 0$ and $b \in \mathbb{R}$,*

$$\Big||a+b|^s - a^s\Big| \leqslant C\{a^{s-1}|b| + |b|^s\}, \tag{3.1}$$

$$\Big||a+b|^s - a^s - sa^{s-1}b\Big| \leqslant C\{a^{s-2}b^2 + |b|^s\}. \tag{3.2}$$

The assertion follows by considering (i) $a > 2|b|$ and (ii) $0 < a \leqslant 2|b|$.

Next we study the weight ω_ρ and one of its applications.

Lemma 3.3 *Let* (λ, σ) *be as in* (2.4) *and* $k \geqslant 2\sigma$ *be a positive integer.*

1. *Set* $r = 1 - \sigma/k$, $\mathbf{x}_i = r\mathbf{e}_i$, $\mathbf{x}_i^* = \mathbf{e}_i/r$ *and* $\Omega_0 := \{x \in B_1 : |x - \mathbf{x}_0| \leqslant |x - \mathbf{x}_i|\ \forall\, i\}$. *Then*

$$\sum_{i=1}^{k} \frac{1}{|x - \mathbf{x}_i^*|^\rho} + \sum_{i=1}^{k-1} \frac{1}{|x - \mathbf{x}_i|^\rho} = O(1) \begin{cases} k^\rho & \text{if } \rho > 1, \\ k \ln k & \text{if } \rho = 1, \\ k & \text{if } \rho < 1, \end{cases} \quad \forall\, x \in \overline{\Omega_0}. \tag{3.3}$$

2. *Let* $\|\cdot\|_\rho$ *be as in* (2.14). *If* $\rho < \tilde{\rho}$, *then* $\omega_{\tilde{\rho}} < \omega_\rho$ *and* $\|\cdot\|_\rho \leqslant \|\cdot\|_{\tilde{\rho}}$. *Also,* $\omega_\tau = O(1)$ *and*

$$\omega_\rho \leqslant \begin{cases} \omega_{\hat{\rho}}^{\rho/\hat{\rho}} & \text{if } \rho \geqslant \hat{\rho} > 0, \\ \omega_\tau^{\frac{\hat{\rho}-\rho}{\hat{\rho}-\tau}}\ \omega_{\hat{\rho}}^{\frac{\rho-\tau}{\hat{\rho}-\tau}} & \text{if } \tau \leqslant \rho < \hat{\rho}. \end{cases} \tag{3.4}$$

3. *For every* $\rho \in (0, n-4)$, *the solution* ϕ *of* $\Delta^2\phi = f$ *in* B_1 *with* $\phi = \Delta\phi = 0$ *on* ∂B_1 *satisfies*

$$\|\phi\|_\rho = O(1)\varepsilon^4 \|f\|_{\rho+4}.$$

Proof. (1) For $x \in \bar{B}_1$, it is easy to see that

$$|x - \mathbf{x}_i^*| \geqslant \max\{|x - \mathbf{x}_i|,\ \ 1 - |\mathbf{x}_i|\} = \max\{|x - \mathbf{x}_i|, \sigma k^{-1}\}. \tag{3.5}$$

When $x \in \bar{\Omega}_0$, $2|x - \mathbf{x}_i| \geqslant |x - \mathbf{x}_0| + |x - \mathbf{x}_i| \geqslant |\mathbf{x}_i - \mathbf{x}_0|$, so

$$|x - \mathbf{x}_i^*| \geqslant |x - \mathbf{x}_i| \geqslant \frac{1}{2}|\mathbf{x}_i - \mathbf{x}_0| = r \sin\frac{\pi i}{k} \geqslant \frac{1}{2}\sin\frac{\pi i}{k} \geqslant \frac{\min\{i, k-i\}}{k}.$$

It then follows that when $x \in \bar{\Omega}_0$,

$$\sum_{i=0}^{k-1} \frac{1}{|x - \mathbf{x}_i^*|^\rho} + \sum_{i=1}^{k-1} \frac{1}{|x - \mathbf{x}_i|^\rho} \leqslant \frac{k^\rho}{\sigma^\rho} + 4\sum_{i=1}^{[k/2]} \frac{k^\rho}{i^\rho}.$$

The first assertion of the lemma thus follows.

(2) The monotonicity of ω_ρ and $\|\cdot\|_\rho$ in ρ follows by the definition (2.14).

If $\rho = \tau$ and $x \in \Omega_0$, $\omega_\tau(x) \leqslant 1 + \sum_{i=1}^{k-1} \varepsilon^\tau |x - \mathbf{x}_i|^{-\tau} = 1 + O(1)\varepsilon^\tau k = O(1)$.

If $0 < \hat{\rho} \leqslant \rho$, then $\omega_\rho(x) \leqslant \omega_{\hat{\rho}} \max_i [1 + |x - \mathbf{x}_i|/\varepsilon]^{\hat{\rho}-\rho} \leqslant \omega_{\hat{\rho}}\omega_{\hat{\rho}}^{(\rho-\hat{\rho})/\hat{\rho}} = \omega_{\hat{\rho}}^{\rho/\hat{\rho}}$.

If $\hat{\rho} > \rho \geqslant \tau$, the Hölder inequality gives $\omega_\rho \leqslant \omega_\tau^{\frac{\hat{\rho}-\rho}{\hat{\rho}-\tau}} \omega_{\hat{\rho}}^{\frac{\rho-\tau}{\hat{\rho}-\tau}}$. This proves the second assertion.

(3) Let $\Gamma(z) = C_n|z|^{4-n}$ be the fundamental solution of $(-\Delta)^2$ and $G(x, y)$ be the Green's function of $(-\Delta)^{-2}$ associated with the Navier boundary condition. By comparison, $0 \leqslant G(x, y) \leqslant \Gamma(x - y)$. Hence,

$$\phi(x) = \int_{B_1} G(x, y) f(y) dy = O(1)\|f\|_{\rho+4} \int_{\mathbb{R}^n} \frac{\omega_{\rho+4}(y)}{|x - y|^{n-4}} dy.$$

By the change of variables $Y = y/\varepsilon$ and Lemma 3.1 with $\alpha = n-4$ and $\beta = \rho+4$, we have

$$\int_{\mathbb{R}^n} \frac{\omega_{\rho+4}(y)}{|x-y|^{n-4}} dy = \sum_{i=1}^{k} \int_{\mathbb{R}^n} \frac{|Y - x/\varepsilon|^{4-n} \varepsilon^4 dY}{(1+|Y - \mathbf{x}_i/\varepsilon|)^{\rho+4}} \leqslant C\varepsilon^4 \omega_\rho(x). \tag{3.6}$$

Hence, $|\phi| = O(1)\varepsilon^4 \|f\|_{\rho+4}\omega_\rho$. This implies that $\|\phi\|_\rho = O(1)\varepsilon^4\|f\|_{\rho+4}$ and completes the proof. □

Now we study $V = \mathbb{P}U$. Note that for $U(x) = \varepsilon^{-m}\Phi(y)$ with $y = (x-\mathbf{x})/\varepsilon$ and $\mathbf{x} = (1-\sigma/k)\mathbf{e}$,

$$\lambda U_\lambda = mU + (x-\mathbf{x})\cdot\nabla U = \frac{1-|y|^2}{1+|y|^2}\, mU, \quad \varepsilon k U_\sigma = \varepsilon \mathbf{e}\cdot\nabla U = \frac{-2\mathbf{e}\cdot y}{1+|y|^2} mU, \tag{3.7}$$

$$\varepsilon\nabla U = \frac{-2myU}{1+|y|^2}, \quad -\varepsilon^2\Delta U = \frac{4|y|^2+2n}{(1+|y|^2)^2} mU, \quad |\varepsilon\nabla U| + |U_\lambda| + |\varepsilon k U_\sigma| = O(1)U. \tag{3.8}$$

Lemma 3.4 *Let $k \geqslant 2\sigma$ be a positive integer, (λ,σ) be constants satisfying* (2.4)*, and $U(x) = \varepsilon^{-m}\Phi(y)$ with $y = (x - r\mathbf{e})/\varepsilon$ where $|\mathbf{e}| = 1$, $\varepsilon = k^{-1/\tau}/\lambda$, $r = 1-\sigma/k$. Let $V = \mathbb{P}U$ be the solution of* (2.1)*. Set*

$$\mathbf{x} = r\mathbf{e}, \qquad \mathbf{x}^* = r^{-1}\mathbf{e}, \quad \Gamma^* = \varepsilon^m\Phi(0)|x-\mathbf{x}^*|^{4-n}, \quad \zeta = U - V.$$

Then in B_1, $0 < \zeta \leqslant U$ and

$$\frac{\Gamma^*}{U} = O(1), \qquad \frac{\zeta}{\Gamma^*} = 1 + O(1)(\varepsilon k)^2, \qquad \frac{\zeta_\lambda}{\Gamma^*} = O(1), \qquad \frac{\zeta_\sigma}{\Gamma^*} = O(1). \tag{3.9}$$

Proof. Note that $\Delta^2\zeta = 0 < U^p = \Delta^2 U$ in B_1, and $\Delta\zeta = \Delta U$ and $\zeta = U$ on ∂B_1. It follows by comparison that $0 < -\Delta\zeta < -\Delta U$ and $0 < \zeta < U$ in B_1.

Next, for $x \in B_1$, $|x-\mathbf{x}^*| \geqslant \max\{\sigma/k, |x-\mathbf{x}|\}$. The first equation in (3.9) then follows by

$$\frac{\Gamma^*(x)}{U(x)} = \Big(\frac{\varepsilon^2 + |x-\mathbf{x}|^2}{|x-\mathbf{x}^*|^2}\Big)^m = O(1) \quad \forall\, x \in \bar{B}_1.$$

The point $\mathbf{x}^*$ has the special property $r|x-\mathbf{x}^*| = |x-\mathbf{x}|$ for $x \in \partial B_1$. Set $z = |x-\mathbf{x}|/\varepsilon$. Then $z \geqslant \sigma/(\varepsilon k)$. Therefore, on ∂B_1, using $\zeta = U$ and $\Delta\zeta = \Delta U$ we obtain

$$\begin{aligned} \frac{\zeta}{\Gamma^*} &= \frac{r^{-2m}|z|^{2m}}{[1+|z|^2]^m} = \Big(1-\frac{\sigma}{k}\Big)^{-2m}\Big(1-\frac{1}{1+|z|^2}\Big)^m = 1 + O(1)(\varepsilon k)^2, \\ \frac{r^{n-2}\Delta\zeta}{\Delta\Gamma^*} &= \Big(\frac{n/2}{1+|z|^2}+1\Big)\Big(1-\frac{1}{1+|z|^2}\Big)^{n/2} = 1 + O(1)(\varepsilon k)^4. \end{aligned}$$

Here we use $k^{-1} = (\lambda\varepsilon k)^{n-4}$ and $n \geqslant 6$. Since $\Delta^2\zeta = 0 = \Delta^2\Gamma^*$ in B_1, for suitably large M, using $-(1 \pm M(\varepsilon k)^4)\Delta\Gamma^*$ and $(1 \pm M(\varepsilon k)^2)\Gamma^*$ as sub/supersolutions for $-r^{n-2}\Delta\zeta$ and ζ respectively, we obtain

$$[1 - M(\varepsilon k)^4]\Delta\Gamma^* \geqslant r^{n-2}\Delta\zeta \geqslant [1 + M(\varepsilon k)^4]\Delta\Gamma^*,$$

and

$$[1 - M(\varepsilon k)^2]\Gamma^* \leqslant \zeta \leqslant [1 + M(\varepsilon k)^2]\Gamma^*.$$

These give the second equation in (3.9). Next, on ∂B_1,

$$\begin{aligned}\frac{\zeta_\lambda}{\zeta} &= \frac{U_\lambda}{U} = \frac{\partial \ln U}{\partial \lambda} = \frac{m}{\lambda}\,\frac{\varepsilon^2 - |x-\mathbf{x}|^2}{\varepsilon^2 + |x-\mathbf{x}|^2} = -\frac{m}{\lambda} + O(1)(\varepsilon k)^2,\\ \frac{\Delta\zeta_\lambda}{\Delta\zeta} &= \frac{(\Delta U)_\lambda}{\Delta U} = \frac{\partial \ln(-\Delta U)}{\partial\lambda} = -\frac{m}{\lambda} + O(1)(\varepsilon k)^2.\end{aligned}$$

It then follows by comparison that $\Delta\zeta_\lambda = [-m/\lambda + O(1)(\varepsilon k)^2]\Delta\zeta$ and $\zeta_\lambda = [-m/\lambda + O(1)(\varepsilon k)^2]\zeta$ on $\bar{B}_1$. Similarly, using $\mathbf{x} = r\mathbf{e}$ and $r = 1 - \sigma/k$, we find that when $x \in \partial B_1$,

$$\frac{\zeta_\sigma}{\zeta} = -\frac{2m}{k}\,\frac{(x-\mathbf{x})\cdot\mathbf{e}}{\varepsilon^2 + |x-\mathbf{x}|^2} = \frac{O(1)}{k[1-r]} = O(1), \quad \frac{\Delta\zeta_\sigma}{\Delta\zeta} = O(1).$$

Hence, by comparison, $\Delta\zeta_\sigma = O(1)\Delta\zeta$ and $\zeta_\sigma = O(1)\zeta$ on $\bar{B}$. This completes the proof. □

Lemma 3.5 *Let W be as in* (2.2). *For $x \in \Omega_0 := \{x \in B_1 : |x - \mathbf{x}_0| \leqslant |x - \mathbf{x}_i|\ \forall\, i\}$ and $y = (x - \mathbf{x}_0)/\varepsilon$,*

$$\varepsilon^m W(x) - \Phi(y) = \sum_{i=1}^{k-1} \frac{\Phi(0)\varepsilon^{n-4}}{|x-\mathbf{x}_i|^{n-4}} - \sum_{i=1}^{k} \frac{\Phi(0)\varepsilon^{n-4}}{|x-\mathbf{x}_i^*|^{n-4}} + \frac{O(1)(\varepsilon k)^2}{k} = \frac{O(1)}{k}, \tag{3.10}$$

$$\lambda\varepsilon^m W_\lambda = m\Phi(y) + y\cdot\nabla\Phi(y) + O(1)k^{-1}, \quad \varepsilon k\varepsilon^m W_\sigma = \mathbf{e}_0\cdot\nabla\Phi(y) + O(1)\varepsilon. \tag{3.11}$$

Proof. Let $\zeta_i = U_i - V_i$, $\Gamma_i = \Phi(0)\varepsilon^m|x-\mathbf{x}_i|^{4-n}$ and $\Gamma_i^* = \Phi(0)\varepsilon^m|x-\mathbf{x}_i^*|^{4-n}$. Then,

$$W = \sum_{i=1}^{k} V_i = U_0 + \sum_{i=1}^{k-1}\Gamma_i - \sum_{i=0}^{k-1}\Gamma_i^* + \sum_{i=1}^{k-1}\big(U_i - \Gamma_i\big) + \sum_{i=0}^{k-1}\big(\Gamma_i^* - \zeta_i\big).$$

When $x \in \bar{\Omega}_0$ and $i = 1, \cdots, k-1$, $U_i - \Gamma_i = O(1)(\varepsilon/|x-\mathbf{x}_i|)^2\Gamma_i = O(1)(\varepsilon k)^2\Gamma_i$. Hence,

$$\sum_{i=1}^{k-1}\Big|U_i - \Gamma_i\Big| + \sum_{i=0}^{k-1}\Big|\Gamma_i^* - \zeta_i\Big| = O(1)(\varepsilon k)^2\Big\{\sum_{i=1}^{k-1}\Gamma_i + \sum_{i=0}^{k-1}\Gamma_i^*\Big\} = \frac{O(1)(\varepsilon k)^{n-2}}{\varepsilon^m},$$

by (3.9) and (3.3). Since $\varepsilon^m U_0 = \Phi(y)$ and $k^{-1} = (\lambda\varepsilon k)^{n-4}$, this gives (3.10). Using $W = \sum_{i=1}^k [U_i - \zeta_i]$, (3.7), $\varepsilon^m U_{i\lambda} = O(1)|y_i|^{4-n}$ and $\varepsilon k\varepsilon^m U_{i\sigma} = O(1)|y_i|^{3-n}$ for $y_i = (x-\mathbf{x}_i)/\varepsilon$, (3.9), and (3.3), we obtain the estimates (3.11). This completes the proof of the Lemma. □

Lemma 3.6 *Let W be as in* (2.2) *and Z_i be as in* (2.3). *Then*

(i) *for each $\rho \geqslant \tau$, there exists $\theta > 0$ such that $\|\phi W^{p-1}\|_{\rho+4+\theta} = O(1)\varepsilon^{-4}\|\phi\|_\rho$;*

(ii) $\|Z_1\|_{n+4} + \varepsilon k\|Z_2\|_{n+5} = O(1)\varepsilon^{-mp}$.

Proof. (i) Since $m(p-1) = 4$ and $0 \leqslant W < \sum_{i=1}^k U_i = O(1)\varepsilon^{-m}\omega_{n-4}$, $|W^{p-1}\phi| = O(1)\varepsilon^{-4}\|\phi\|_\rho\omega_{n-4}^{p-1}\omega_\rho$.

(1) If $\rho \geqslant n-4-8\tau$, set $\hat\rho = \rho + 8\tau \geqslant n-4$, then by (3.4),

$$\omega_{n-4}^{p-1}\omega_\rho = O(1)\omega_{\hat\rho}^{[(n-4-\tau)(p-1)+(\rho-\tau)]/(\hat\rho-\tau)} = O(1)\omega_{\hat\rho}.$$

(2) If $\tau \leqslant \rho < n-4-8\tau$, set $\hat\rho = (\rho+8)/2+\sqrt{(\rho+8)^2/4-8\tau}$. Then $8/\hat\rho+(\rho-\tau)/(\hat\rho-\tau) = 1$ and $\tau \leqslant \hat\rho < n-4$. Hence,

$$\omega_{n-4}^{p-1}\omega_\rho = O(1)\omega_{\hat\rho}^{(n-4)(p-1)/\hat\rho+(\rho-\tau)/(\hat\rho-\tau)} = O(1)\omega_{\hat\rho}.$$

Thus, $|W^{p-1}\phi| = O(1)\varepsilon^{-4}\|\phi\|_\rho\omega_{\hat\rho}$, so $\|W^{p-1}\phi\|_{\hat\rho} = O(1)\varepsilon^{-4}\|\phi\|_\rho$ where $\theta = \hat\rho - \rho - 4 > 0$.

(ii) Note by (2.3) and (3.7) that $Z_1 = O(1)\varepsilon^{-mp}\omega_{n+4}$ and $\varepsilon k Z_2 = O(1)\varepsilon^{-mp}\omega_{n+5}$. Hence, $\|Z_1\|_{n+4} = O(1)\varepsilon^{-mp}$ and $\varepsilon k\|Z_2\|_{n+5} = O(1)\varepsilon^{-mp}$. The assertion of the Lemma thus follows. □

Lemma 3.7 *There are positive constants $a_1(n)$ and $a_2(n)$ that depend only on n such that*

$$M{:=}\begin{bmatrix}\lambda^2\,\langle Z_1, V_{0\lambda}\rangle & \lambda\varepsilon k\,\langle Z_2, V_{0\lambda}\rangle\\ \lambda\varepsilon k\,\langle Z_1, V_{0\sigma}\rangle & (\varepsilon k)^2\,\langle Z_2, V_{0\sigma}\rangle\end{bmatrix}=\begin{bmatrix}a_1(n) & 0\\ 0 & a_2(n)\end{bmatrix}+\begin{bmatrix}O(k^{-1}) & O(k^{-1})\\ O(\varepsilon) & O(\varepsilon)\end{bmatrix}. \tag{3.12}$$

Proof. Denote Z_1 by Z_λ and Z_2 by Z_σ, so $Z_\ell = \sum_{i=1}^k U_i^{p-1}U_{i\ell}$. By symmetry, for $t = \lambda$ or σ,

$$\langle Z_\ell, V_{0t}\rangle = \sum_{i=0}^{k-1}\int_{B_1} U_i^{p-1}U_{i\ell}V_{0t}\,dx = \sum_{i=0}^{k-1}\int_{B_1} U_0^{p-1}U_{0\ell}V_{it}\,dx = \int_{B_1} U_0^{p-1}U_{0\ell}W_t\,dx. \tag{3.13}$$

Since $mp = m - \frac{1}{2} + \frac{9}{2}$ and $\varepsilon^m U_0(x) = \Phi(y) = O(1)|y|^{4-n}$ with $y = (x-\mathbf{x}_0)/\varepsilon$, by Lemma 3.1 and (3.3)

$$\int_{|x-\mathbf{x}_0|>\frac{\sigma_0}{k}} U_0^p\sum_{i=0}^{k-1} U_i dx = O(1)\int_{|y|>\frac{\sigma_0}{\varepsilon k}}\sum_{i=0}^{k-1}\frac{|y-(\mathbf{x}_i-\mathbf{x}_0)/\varepsilon|^{4-n}dy}{(\varepsilon k)^{-\frac{9}{2}}|y|^{n-\frac{1}{2}}} = O(1)(\varepsilon k)^n \tag{3.14}$$

where $\sigma_0 = \min\{1,\sigma\}$. Note that $B_{\sigma_0/k}(\mathbf{x}_0) = \{x : |x-\mathbf{x}| < \sigma_0/k\} \subset \Omega_0$. The assertion of the lemma thus follows from (3.13), (3.11), (3.7), (3.8), (3.14), $\int_{\mathbb{R}^n}\varepsilon^{-m}U_0^p dx = O(1)$, and identities $\int_{\mathbb{R}^n} U_0^{p-1}U_{0\lambda}U_{0\sigma}\,dx = 0$,

$$\lambda^2\int_{\mathbb{R}^n} U_0^{p-1}U_{0\lambda}^2\,dx = a_1(n) := \int_{\mathbb{R}^n}\Phi^{p-1}(y)|m\Phi(y) + y\cdot\nabla\Phi(y)|^2 dy,$$

$$(\varepsilon k)^2\int_{\mathbb{R}^n} U_0^{p-1}U_{0\sigma}^2\,dx = a_2(n) := \frac{1}{n}\int_{\mathbb{R}^n}\Phi^{p-1}(y)|\nabla\Phi(y)|^2 dy. \qquad □$$

4 A linear problem

Let $\mathcal{L}$ be the linear operator defined in (2.11) with W be given by (2.2). Here we solve the linear problem: Given $f \in \mathcal{H}$, find (c_1, c_2, ϕ) such that

$$(c_1, c_2, \phi) \in \mathbb{R}^2 \times \mathcal{H}_0, \qquad \mathcal{L}\phi - \lambda c_1 Z_1 - \varepsilon k c_2 Z_2 = f \text{ in } B_1. \tag{4.1}$$

Theorem 4.1 *Assume that $n \geqslant 6$ and $\rho \in [\tau, n-4)$. Then there exist positive constants k_0 and C that depend only on n and ρ such that when $k \geqslant k_0$ and (λ, σ) satisfy* (2.4)*, problem* (4.1) *with $f \in \mathcal{H}$ admits a unique solution. Moreover, the solution satisfies*

$$\varepsilon^m \|\phi\|_\rho + |c_1| + |c_2| \leqslant C\, \varepsilon^{m+4} \|f\|_{\rho+4}. \tag{4.2}$$

Proof. By Fredholm alternative, we need only establish the a priori estimate (4.2). To this end, we assume that $(c_1, c_2, \phi, f) \in \mathbb{R}^2 \times \mathcal{H}_0 \times \mathcal{H}$ is a quadruple satisfying (4.1).

1. We first estimate c_i. Multiplying (4.1) by V_{0t} ($t = \lambda$ or σ) and integrating over B_1, we obtain

$$\lambda \langle V_{0t}, Z_1 \rangle c_1 + \varepsilon k \langle V_{0t}, Z_2 \rangle c_2 = \langle \mathcal{L}\phi - f, V_{0t} \rangle = \langle \phi, \mathcal{L} V_{0t} \rangle - \langle f, V_{0t} \rangle. \tag{4.3}$$

Using $|V_{0\lambda}| + \varepsilon k |V_{0\sigma}| = O(1) U_0 = O(1) \varepsilon^m |x - \mathbf{x}_0|^{4-n}$, $|f| \leqslant \|f\|_{\rho+4} \omega_{\rho+4}$, and (3.6), we obtain

$$\Big|\langle f, V_{0\lambda} \rangle\Big| + \varepsilon k \Big|\langle f, V_{0\sigma} \rangle\Big| = O(1) \varepsilon^m \|f\|_{\rho+4} \int_{B_1} \frac{\omega_{\rho+4}(x)}{|x - \mathbf{x}_0|^{n-4}} dx = O(1) \|f\|_{\rho+4} \varepsilon^{m+4}.$$

As shown in (7.2) and (7.3) in Section 7, we have $|\langle \phi, \mathcal{L} V_{0\lambda} \rangle| + (\varepsilon k) |\langle \phi, \mathcal{L} V_{0\sigma} \rangle| = O(\varepsilon k) \varepsilon^m \|\phi\|_\rho$. As the matrix M in (3.12) has an $O(1)$ inverse, we obtain from (4.3) that

$$|c_1| + |c_2| = O(\varepsilon k) \varepsilon^m \|\phi\|_\rho + O(1) \|f\|_{\rho+4} \varepsilon^{m+4}. \tag{4.4}$$

2. Next, using $\Delta^2 \phi = [p W^{p-1} \phi + \lambda c_1 Z_1 + \varepsilon k c_2 Z_2] + f$, linearity, Lemma 3.3 (3), and Lemma 3.6 (with some $\theta \in (0, 4)$) we obtain

$$\begin{aligned} |\phi| &= O(1) \varepsilon^4 (\|p W^{p-1} \phi\|_{\rho+4+\theta} + |c_1| \|Z_1\|_{n+4} + |c_2| (\varepsilon k) \|Z_2\|_{n+4}) \omega_{\rho+\theta} \\ &\quad + O(1) \varepsilon^4 \|f\|_{\rho+4} \omega_\rho \\ &= O(1) \{\|\phi\|_\rho + \varepsilon^{-m} [|c_1| + |c_2|]\} \omega_{\rho+\theta} + O(1) \varepsilon^4 \|f\|_{\rho+4} \omega_\rho. \end{aligned}$$

Using (4.4) and $\omega_{\rho+\theta} < \omega_\rho$, we obtain

$$\frac{|\phi(x)|}{\omega_\rho(x)} \leqslant C \Big\{ \|\phi\|_\rho \frac{\omega_{\rho+\theta}(x)}{\omega_\rho(x)} + \varepsilon^4 \|f\|_{\rho+4} \Big\} \qquad \forall\, x \in \Omega. \tag{4.5}$$

3. Suppose the a priori estimate (4.2) is not true. Then along a sequence of integer $k \to \infty$, there are (λ_k, σ_k) satisfying (2.4) and $(c_{1k}, c_{2k}, \phi_k, f_k) \in \mathbb{R}^2 \times \mathcal{H}_0 \times \mathcal{H}$ such that $\max\{\|\phi_k\|_\rho, \varepsilon_k^{-m} [|c_{1k}| + \varepsilon_k k |c_{2k}|]\} = 1$ and $\varepsilon_k^4 \|f_k\|_{\rho+4} \to 0$ along the sequence $k \to \infty$. By (4.4), $(|c_{1k}| + |c_{2k}|) \varepsilon_k^{-m} \to 0$ as $k \to \infty$. Hence, $\|\phi_k\|_\rho = 1$. Consequently, $\|\phi_k\|_{L^\infty} \leqslant \|\omega_\rho\|_{L^\infty} \leqslant \|\omega_\tau\|_{L^\infty} = O(1)$.

Let $z_k \in \bar{\Omega}_0$ be a point such that $1 = \|\phi_k\|_\rho = |\phi_k(z_k)| / \omega_\rho(z_k)$. By (4.5), $\omega_\rho(z_k) \leqslant C \omega_{\rho+\theta}(z_k)$. Since $\omega_{\rho+\theta} \leqslant (\min_i [1 + |x - \mathbf{x}_i| / \varepsilon])^{-\theta} \omega_\rho$, we have $|z_k - \mathbf{x}_0| / \varepsilon_k \leqslant L := C^{1/\theta} - 1$.

Now set $u_k(z) = \phi_k(\mathbf{x}_0 + \varepsilon_k z)$. Then using $m(p-1) = 4$ and (3.10) we obtain, for $z \in B_{\sigma/(\varepsilon_k k)}$,

$$\Delta_z^2 u_k - p(\Phi(z) + o(1))^{p-1} u_k = [c_{1k} \varepsilon_k^{-m}][\varepsilon_k^{mp} Z_1] + [c_{2k} \varepsilon^{-m}][\varepsilon_k^{mp} \varepsilon_k k Z_2] + \varepsilon_k^4 f_k.$$

Along a subsequence of $k \to \infty$, u_k converges uniformly in any compact subset of $\mathbb{R}^n$ to a solution u of

$$\Delta^2 u(z) - p\Phi^{p-1}(z)u(z) = 0 \ \text{ in } \mathbb{R}^N, \quad [1+L]^{-\rho} \leqslant \max_{x\in\mathbb{R}^n} |u(x)| = O(1). \tag{4.6}$$

4. We estimate the decay rate of u. Since $n \geqslant 6$, there exists $\delta \in (\tau, 1)$ such that $\tau < 4 - \frac{8\delta}{n-4} < n-4$. When $x \in \Omega_i := \{x \in B_1 : |x - \mathbf{x}_i| \leqslant |x - \mathbf{x}_j| \ \forall\, j\}$, using (3.3) we obtain, with $d_i(x) = 1 + |x - \mathbf{x}_i|/\varepsilon$,

$$\varepsilon^4 W^{p-1}(x) = \Big(\frac{O(1)}{d_i^{n-4}} + \frac{O(1)}{d_i^{n-4-\delta}} \sum_{j\neq i} \frac{\varepsilon^\delta}{|x - \mathbf{x}_j|^\delta}\Big)^{\frac{8}{n-4}} = \frac{O(1)}{d_i^8} + \frac{O(1)[\varepsilon^\delta k]^{\frac{8}{n-4}}}{d_i^{8[1-\delta/(n-4)]}}.$$

Hence, for $x \in \Omega_0$, $\varepsilon = \varepsilon_k$, and $z = (x - \mathbf{x}_0)/\varepsilon$, by Lemma 3.1 and Lemma 3.3 (3),

$$\begin{aligned} u_k(z) = \phi_k(x) &= O(1)(-\Delta)^{-2}[pW^{p-1}\phi_k] + O(1)[\varepsilon^4\|f\|_{\rho+4} + \varepsilon^{-m}[|c_1| + |c_2|]\omega_\rho \\ &= \frac{1}{\varepsilon^4}\sum_{i=0}^{k-1}\int_{\Omega_i} \frac{|\phi_k(y)|}{|x-y|^{n-4}}\Big(\frac{O(1)}{d_i(y)^8} + \frac{O(1)[\varepsilon^\delta k]^{\frac{8}{n-4}}}{d_i(y)^{8[1-\delta/(n-4)]}}\Big)dy + o(1) \\ &= \int_{\{\mathbf{x}_0+\varepsilon Y\in\Omega_0\}} \frac{O(1)|u_k(Y)| \quad dY}{|Y-z|^{n-4}(1+|Y|)^8} + \sum_{i=1}^{k-1} \frac{O(1)\|\phi_k\|_{L^\infty}\ln k}{d_i(x)^{\min\{4,n-4\}}} \\ &\qquad + O(1)\|\phi_k\|_{L^\infty}[\varepsilon^\delta k]^{\frac{8}{n-4}}\omega_{4-\frac{8\delta}{n-4}} + o(1), \end{aligned}$$

since $4 < 8(1 - \frac{\delta}{n-4}) < n$. As $\varepsilon^\delta k = o(1)$, $\sum_{i=1}^{k-1} d_i^{-\rho} = O(1)(\varepsilon k)^\rho$ for $\rho > 1$ and $x \in \Omega_0$, $\omega_\rho = O(1)$ for $\rho \geqslant \tau$, and $4 - \frac{8\delta}{n-4} > \tau$, sending $k \to \infty$ we derive that

$$|u(z)| = \int_{\mathbb{R}^n} \frac{O(1)|u(Y)|\,dY}{|z-Y|^{n-4}[1+|Y|]^8} \qquad \forall\, z \in \mathbb{R}^n.$$

Starting from $u = O(1)$, we use Lemma 3.1 finding that $|u(x)| = O(1)(1+|x|)^{-n_1}$ where $n_1 = \min\{4, n-4\}$ if $n \neq 8$ and $n_1 = 3$ if $n = 8$. Suppose $|u(x)| = O(1)[1+|x|]^{-n_i}$ $(0 < n_i < n-4)$. Then by Lemma 3.1, we derive that $|u(x)| = O(1)[1+|x|]^{-n_{i+1}}$ with $n_{i+1} = \min\{n_i + 4, n-4\}$. After finite steps, we derive that

$$|u(x)| = O(1)[1+|x|]^{4-n} \qquad \forall\, x \in \mathbb{R}^n. \tag{4.7}$$

5. Note that $u(z)$ is even with respect to $z_2, \cdots, z_n$. Also $\phi \in \mathcal{H}_0$ and symmetry implies that

$$\int_{\Omega_0} \phi\Delta^2 W_\lambda dx = 0, \qquad \int_{\Omega_0} \phi\Delta^2 W_\sigma dx = 0,$$

since $\Delta^2 W_\lambda = (\Delta^2 W)_\lambda = (\sum_{i=1}^k U_i^p)_\lambda = p\sum_{i=1}^k U_i^{p-1}U_{i\lambda} = pZ_1$ and $\Delta^2 W_\sigma = pZ_2$. Using (3.11) one can show that u is perpendicular, under the inner product $(\phi,\psi) = \int_{\mathbb{R}^n} \Delta\phi(z)\Delta\psi(z)dz$, to $\nabla\Phi(z)$ and $m\Phi(z) + z\cdot\nabla\Phi(z)$. However, we know by [3, Theorem 2.1] that there is no such solution of (4.6) and (4.7). This contradiction completes the proof. □

Remark 4.1 Step **4** for (4.7) is new in the literature. It extends the range of ρ in (4.2) to $[\tau, n-4)$.

5 The nonlinear problem

Here we solve the nonlinear problem (2.9). Set $\phi = v - W$. Then (2.9) is equivalent to (2.10). We shall use Theorem 4.1 and a contraction mapping theorem to solve (2.10).

Lemma 5.1 *Let N be defined in* (2.13). *If $\rho \geqslant \tau$, then for every $\phi, \psi \in \mathcal{H}$,*

$$\begin{aligned} &\|N(\phi)-N(\psi)\|_{\rho+4} \\ =\ & O(1)\max\{\varepsilon^m(\|\phi\|_{\rho_1}+\|\psi\|_{\rho_1}), [\varepsilon^m(\|\phi\|_{\rho_1}+\|\psi\|_{\rho_1})]^{p-1}\}\varepsilon^{-4}\|\phi-\psi\|_\rho \end{aligned} \tag{5.1}$$

where

$$\rho_1 = m + \tau\ \max\Big\{1,\ \frac{m}{\rho+4-\tau}\Big\}. \tag{5.2}$$

Proof. Note that $\rho_1 < 2m = n-4$ so $W = O(1)\varepsilon^{-m}\omega_{n-4} = O(1)\varepsilon^{-m}\omega_{\rho_1}$. There exists $t \in [0,1]$ such that

$$N(\phi) - N(\psi) \quad = \quad p\,|x|^\alpha\,\{\, s|W + t\phi + (1-t)\psi)|^{p-1} - W^{p-1}\,\}\,(\phi - \psi),$$

where $s = 1$ if $W + t\phi + (1-t)\psi > 0$ and $s = -1$ if $W + t\phi + (1-t)\psi \leqslant 0$. By (3.1) we obtain

$$\begin{aligned} & N(\phi) - N(\psi) \\ =\ & O(1)\max\{W^{p-2}[|\phi| + |\psi|], [|\phi| + |\psi|]^{p-1}\}|\phi-\psi| \\ =\ & O(1)\max\{\varepsilon^{m(2-p)}(\|\phi\|_{\rho_1}+\|\psi\|_{\rho_1}), (\|\phi\|_{\rho_1}+\|\psi\|_{\rho_1})^{p-1}\}\|\phi-\psi\|_\rho\omega_{\rho_1}^{p-1}\omega_\rho. \end{aligned}$$

Finally, we use (3.4):

(i) If $\rho + 4 - \tau \geqslant m$, then $\rho_1 \leqslant \rho + 4$, so

$$\omega_{\rho_1}^{p-1}\omega_\rho = O(1)\omega_{\rho+4}^{[(\rho_1-\tau)(p-1)+(\rho-\tau)]/(\rho+4-\tau)} = O(1)\omega_{\rho+4}.$$

(ii) If $\rho + 4 - \tau \leqslant m$, then $\rho_1 \geqslant \rho + 4$, so

$$\omega_{\rho_1}^{p-1}\omega_\rho = O(1)\omega_{\rho+4}^{\rho_1(p-1)/(\rho+4)+(\rho-\tau)/(\rho+4-\tau)} = O(1)\omega_{\rho+4}.$$

Upon using $m(p-1) = 4$, we then obtain the assertion of the lemma. □

Lemma 5.2 *Let F_1 be as in* (2.12). *Then* $\|F_1\|_{n+(3n-16)/(n-3)} = O(1)k^{-1}\varepsilon^{-mp}$.

Proof. Note that $1 - t^\alpha = O(1)(1-t)$ for $t \in [0,1]$. For $x \in B_1$,

$$1-|x|^\alpha = O(1)(1-|x|) = O(1)\big(1-r+\min_i |x-\mathbf{x}_i|\big) = \frac{O(1)}{k}\ \min_i\Big\{1+\frac{|x-\mathbf{x}_i|}{\varepsilon}\Big\}^\tau. \tag{5.3}$$

Hence, using $W = O(1)\varepsilon^{-m}\omega_{n-4}$ we obtain

$$\begin{aligned} F_1 \quad &= \quad \big(|x|^\alpha - 1\big)W^p = \frac{O(1)}{k}\Big(\min_i\Big\{1 + \frac{|x-\mathbf{x}_i|}{\varepsilon}\Big\}^{\tau/p}\ \varepsilon^{-m}\omega_{n-4}\Big)^p \\ &= \quad O(1)k^{-1}\varepsilon^{-mp}\omega_{n-4-\frac{\tau}{p}}^p = O(1)k^{-1}\varepsilon^{-mp}\omega_{n+\frac{3n-16}{n-3}}, \end{aligned}$$

by the interpolation (3.4) with $\rho = n-4-\tau/p$ and $\hat{\rho} = n + (3n-16)/(n-3)$. This completes the proof. □

Lemma 5.3 *Let $\mathcal{L}_1$ be as in* (2.11)*. Then for each $\rho \geqslant \tau$,*

$$\|\mathcal{L}_1\phi\|_{\rho+4} = O(1)k^{-1}\varepsilon^{-4}\|\phi\|_\rho.$$

Proof. By (5.3) and the interpolation (3.4) for cases $n-4-\frac{\tau}{p-1} < \rho+4$ and $n-4-\frac{\tau}{p-1} \geqslant \rho+4$,

$$\begin{aligned}\mathcal{L}_1\phi(x) &:= (|x|^\alpha - 1)pW^{p-1}\phi\\ &= O(1)k^{-1}\varepsilon^{-m(p-1)}\omega_{n-4-\frac{\tau}{p-1}}^{p-1}\|\phi\|_\rho\omega_\rho\\ &= O(1)k^{-1}\varepsilon^{-4}\|\phi\|_\rho\omega_{\rho+4}^{1+\nu},\end{aligned}$$

for some $\nu > 0$. The assertion of the Lemma thus follows. □

Lemma 5.4 *For each $\rho \in [0, n-4)$, $\|F_0\|_{\rho+4} = O(1)\varepsilon^{-mp}k^{-\gamma(\rho)}$, where $\gamma(\rho) := \min\{1, \frac{n-\rho}{n-4}\}$.*

Proof. As $W = \sum_{i=1}^k V_i$ and $\Delta^2 W = \sum_{i=1}^k U_i^p$, we have

$$F_0 := W^p - \Delta^2 W = \Big\{\Big(\sum_{i=1}^k V_i\Big)^p - \sum_{i=1}^k V_i^p\Big\} - \sum_{i=1}^k [U_i^p - V_i^p] =: J_1 - J_2.$$

By (3.9), $0 \leqslant U_i - V_i = \zeta_i = O(1)\Gamma_i^* = O(1)\varepsilon^m k^{n-4}$. Hence, for $\gamma = \min\{1, \frac{n-\rho}{n-4}\}$,

$$\begin{aligned}0 < J_2 &:= \sum_{i=1}^k (U_i^p - V_i^p) \leqslant p\sum_{i=1}^k U_i^{p-\gamma}(U_i - V_i)^\gamma\\ &= O(1)\varepsilon^{-m(p-\gamma)}\omega_{(n-4)(p-\gamma)}(\varepsilon^m k^{n-4})^\gamma = O(1)\varepsilon^{-mp}k^{-\gamma}\omega_{\rho+4}.\end{aligned}$$

For J_1, we need only consider $x \in \Omega_0$. Set $V_c = \sum_{i=1}^{k-1} V_i$. Then, by (3.1),

$$0 \leqslant J_1 := \Big(\sum_{i=0}^{k-1} V_i\Big)^p - \sum_{i=0}^{k-1} V_i^p \leqslant (V_0 + V_c)^p - V_0^p = O(1)[V_0^{p-1}V_c + V_c^p].$$

Note that for $\theta \in [0, n-5)$, with $d_i = 1 + |x - \mathbf{x}_i|/\varepsilon$,

$$V_0^{p-1}V_c = \frac{O(1)}{\varepsilon^{mp}d_0^8}\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}} = \frac{O(1)}{\varepsilon^{mp}d_0^{8+\theta}}\sum_{i=1}^{k-1}\Big(\frac{\varepsilon}{|x - \mathbf{x}_i|}\Big)^{n-4-\theta} = \frac{O(1)}{\varepsilon^{mp}}(\varepsilon k)^{n-4-\theta}\omega_{8+\theta},$$

by (3.3) with $\rho = n-4-\theta > 1$. Taking $\theta = 0$ when $0 \leqslant \rho \leqslant 4$ and $\theta = \rho - 4$ when $4 \leqslant \rho < n-4$, we obtain $V_0^{p-1}V_c = O(1)\varepsilon^{-mp}k^{-\gamma(\rho)}\omega_{\rho+4}$, since $\lambda\varepsilon k = k^{-1/(n-4)}$.

Finally using Hölder inequality, (3.3), and $(n-\rho)/(p-1) = (n-\rho)(n-4)/8 > 1$, we obtain

$$\begin{aligned}V_c^p &= \frac{O(1)}{\varepsilon^{mp}}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^p = \frac{O(1)}{\varepsilon^{mp}}\sum_{i=1}^{k-1}\frac{1}{d_i^{\rho+4}}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{(n-\rho)/(p-1)}}\Big)^{p-1}\\ &= O(1)\varepsilon^{-mp}\omega_{\rho+4}(\varepsilon k)^{n-\rho}.\end{aligned}$$

Combining these estimates we obtain the assertion of the lemma. □

Theorem 5.1 *There exists a large integer k_1 such that for every integer $k \geqslant k_1$ and (λ, σ) satisfying* (2.4)*,* (2.9) *admits a unique solution (c_1, c_2, v) satisfying $v = W + \phi$ and*

$$|c_1|+|c_2|+\varepsilon^m\|\phi\|_\rho = O(1)k^{-\gamma(\rho)} \ \forall\,\rho \in [\tau, n-4) \ \Big(\gamma(\rho) := \min\Big\{1, \frac{n-\rho}{n-4}\Big\}\Big). \tag{5.4}$$

Proof. **1.** First we establish the existence of a unique solution and the $\|\phi\|_\rho$ estimate for $\rho \in [m+\tau, n-4)$. Fix $\nu \in (1, \min\{p, 2\})$ and define

$$\mathbf{X}_\rho := \{\phi \in \mathcal{H} \ : \ \varepsilon^m\|\phi\|_\rho \leqslant k^{-\gamma(\rho)/\nu}\}.$$

For each $\phi \in \mathbf{X}_\rho$, set $f = F_0 + F_1 + \mathcal{L}_1\phi + N(\phi)$. Note by (5.2) that in the current case $\rho_1 = m + \tau \leqslant \rho$ so $\|\phi\|_{\rho_1} \leqslant \|\phi\|_\rho$. Hence, by Lemma 5.1 with $\psi \equiv 0$ and Lemmas 5.2—5.4,

$$\|f\|_{\rho+4} = O(1)\varepsilon^{-mp}\{k^{-\gamma(\rho)} + k^{-1} + k^{-1-\gamma(\rho)/\nu} + k^{-\frac{\gamma(\rho)}{\nu}\min\{p,2\}}\} = O(1)\varepsilon^{-mp}k^{-\gamma(\rho)}.$$

Define (c_1, c_2, ψ) as the unique solution of $\mathcal{L}\psi - \lambda c_1 Z_1 - \varepsilon k c_2 Z_2 = f$ in $\mathbb{R}^2 \times \mathcal{H}_0$. By Theorem 4.1 and equation $m + 4 = mp$ we have $\varepsilon^m\|\psi\|_\rho + |c_1| + |c_2| = O(1)\varepsilon^{4+m}\|f\|_{\rho+4} = O(1)k^{-\gamma(\rho)}$ when $k \geqslant k_0$. We now define $\mathbf{T}\phi := \psi$. Then $\mathbf{T}$ maps $\mathbf{X}_\rho$ to itself if $k \gg 1$. In addition, by Lemma 3.3 (3), Lemma 5.3, and Lemma 5.1 with $\|\phi\|_{\rho_1} \leqslant \|\phi\|_\rho \leqslant k^{-\gamma(\rho)/\nu}$, we find that for any $\phi_1, \phi_2 \in \mathbf{X}_\rho$,

$$\begin{aligned} \|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_\rho &= O(1)\varepsilon^4[\|\mathcal{L}_1(\phi_1 - \phi_2)\|_{\rho+4} + \|N(\phi_1) - N(\phi_2)\|_{\rho+4}] \\ &= O(1)[k^{-1} + k^{-\frac{\gamma(\rho)}{\nu}\min\{1,p-1\}}]\|\phi_1 - \phi_2\|_\rho. \end{aligned}$$

Hence, $\mathbf{T}$ is a contraction if $k \gg 1$. Consequently, when $k \gg 1$, by the contraction mapping theorem, there exists a unique fixed point of $\mathbf{T}$ in $\mathbf{X}_\rho$, which gives a unique solution of (2.10). The solution satisfies

$$\varepsilon^m\|\phi\|_\rho + |c_1| + |c_2| = O(1)k^{-\gamma(\rho)} \quad \forall\,\rho \in [m+\tau, n-4). \tag{5.5}$$

2. Next for the unique solution given in the previous step, we estimate the $\|\phi\|_\rho$ norm for $\rho \in [\tau, m+\tau]$. Using Theorem 4.1, Lemma 5.1 with $\psi \equiv 0$ and Lemmas 5.2—5.4, we obtain

$$\begin{aligned} &\varepsilon^m\|\phi\|_\rho \\ =\ & O(1)\varepsilon^{mp}\{\|F_0\|_{\rho+4} + \|F_1\|_{\rho+4} + \|\mathcal{L}_1\phi\|_{\rho+4} + \|N(\phi)\|_{\rho+4}\} \\ =\ & O(1)k^{-\gamma(\rho)} + O(1)k^{-1}\varepsilon^m\|\phi\|_\rho + O(1)\max\{\varepsilon^m\|\phi\|_{\rho_1}, (\varepsilon^m\|\phi\|_{\rho_1})^{p-1}\}\varepsilon^m\|\phi\|_\rho. \end{aligned}$$

Note from (5.2) that $\rho_1 \geqslant m+\tau$, so by (5.5), $\max\{\varepsilon^m\|\phi\|_{\rho_1}, (\varepsilon^m\|\phi\|_{\rho_1})^{p-1}\} = O(1)k^{-\gamma(\rho_1)\min\{p-1,1\}} = o(1)$. Hence, (5.4) holds. This completes the proof. □

Remark 5.1 Step **2** is new in the literature. It extends (5.4) from $\rho \in [m+\tau, n-4)$ to $\rho \in [\tau, n-4)$.

6 Variation of energy

The existence of spiky solutions is based on the following facts:

Theorem 6.1 *Let* $n \geqslant 6$, $W = W(\cdot\,;\lambda,\sigma,k)$ *be as in* (2.2) *where* (λ,σ) *satisfies* (2.4). *Define*

$$J(\lambda,\sigma,k) := \frac{1}{k}\int_{B_1}\Big(\frac{1}{2}\big|\Delta W\big|^2 - \frac{|x|^\alpha}{p+1}W^{p+1}\Big)\,dx.$$

Let $L_1(\cdot), L_2(\cdot), A_1$ *and* A_2 *be defined by* (2.6)–(2.8). *Then*

$$\frac{\partial J(\lambda,\sigma,k)}{\partial\lambda} = \frac{1}{\lambda^{n-3}k}\Big\{A_2L_1(\sigma) + O(1)(\varepsilon k)^2\Big\}, \tag{6.1}$$

$$\frac{\partial J(\lambda,\sigma,k)}{\partial\sigma} = \frac{1}{\lambda^{n-4}k}\Big\{\alpha A_1\lambda^{n-4} - A_2L_2(\sigma) + O(1)(\varepsilon k)\Big\}. \tag{6.2}$$

Proof. Define $\zeta_0 = U_0 - V_0$. It is easy to see that Γ_0^* and Γ_1^* are proportional. Hence, by (3.9), $|\zeta_0|+|\zeta_{0\lambda}|+|\zeta_{0\sigma}| = O(1)\Gamma_0^* = O(1)\Gamma_1^* = O(1)U_1$. Consequently, setting $U^c = W - U_0 = -\zeta_0 + \sum_{i=1}^{k-1}V_i$ and $d_i = 1 + |x - \mathbf{x}_i|/\varepsilon$ we obtain from Lemma 3.4 that

$$|U^c| + |U^c_\lambda| + \varepsilon k|U^c_\sigma| = O(1)\sum_{i=1}^{k-1}U_i = O(1)\varepsilon^{-m}\sum_{i=1}^{k-1}d_i^{4-n} \qquad \forall\, x\in B_1. \tag{6.3}$$

By symmetry and decomposition $W = \sum_{i=1}^k V_i = U_0 + U^c$, we can calculate, for $t = \lambda$ or $t = \sigma$,

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \frac{1}{k}\int_{B_1}\Big\{\Delta^2W - |x|^\alpha W^p\Big\}W_t\,dx = \int_{B_1}U_0^pW_t\,dx - \int_{\Omega_0}|x|^\alpha W^pW_t\,dx\\
&= \int_{B_1\setminus\Omega_0}U_0^pW_t\,dx + \int_{\Omega_0}\Big(U_0^p - |x|^\alpha W^p\Big)U^c_t\,dx\\
&\quad + \int_{\Omega_0}|x|^\alpha\Big(U_0^p - W^p + pU_0^{p-1}U^c\Big)U_{0t}\,dx\\
&\quad + \int_{\Omega_0}(1 - |x|^\alpha)U_0^pU_{0t}\,dx - \int_{\Omega_0}p|x|^\alpha U_0^{p-1}U^cU_{0t}\,dx\\
&=: I_{1t} + I_{2t} + I_{3t} + I_{4t} + I_{5t}.
\end{aligned}$$

1. The Term I_{1t}**.** Since $|V_{i\lambda}| = O(1)U_i$ and $(\lambda\varepsilon k)^{n-4} = k^{-1}$, by (3.14),

$$I_{1\lambda} = \int_{B_1\setminus\Omega_0}U_0^p\sum_{i=0}^{k-1}V_{i\lambda}\,dx = O(1)\int_{|x-\mathbf{x}_0|>\frac{\sigma_0}{k}}\sum_{i=0}^{k-1}U_0^pU_idx = \frac{O(1)(\varepsilon k)^4}{k}.$$

Similarly, as $\varepsilon kV_{i\sigma} = O(1)U_i$, $I_{1\sigma} = O(1)k^{-1}(\varepsilon k)^3$.

2. The Term I_{2t}**.** From (5.3), $1 - |x|^\alpha = O(1)k^{-1}d_0^\tau$. Hence, by (3.1), (6.3), and $m(p+1) = n$,

$$\begin{aligned}
&(U_0^p - |x|^\alpha W^p)U^c_\lambda\\
=\;& (1 - |x|^\alpha)U_0^pU^c_\lambda + |x|^\alpha[U_0^p - (U_0 + U^c)^p]U^c_\lambda\\
=\;& \frac{O(1)\varepsilon^{-n}}{kd_0^{n+4-\tau}}\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}} + O(1)\varepsilon^{-n}\Big[\frac{1}{d_0^8}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^2 + \Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^{p+1}\Big].
\end{aligned}$$

Fix $\delta \in (0,1)$. We obtain by Hölder inequality and $(n-1+\delta)/p > 1$, that for $x \in \Omega_0$,

$$\Big(\sum_{i=1}^{k-1} \frac{1}{d_i^{n-4}}\Big)^{p+1} = \Big(\sum_{i=1}^{k-1} \frac{1}{d_i^{(n-1+\delta)/p}}\Big)^p \sum_{i=1}^{k-1} \frac{O(1)}{d_i^{n+1-\delta}} = (\varepsilon k)^{n-1+\delta} \sum_{i=1}^{k-1} \frac{O(1)}{d_i^{n+1-\delta}},$$

since $\sum_{i=1}^{k-1} d_i^{-\rho} = O(1)(\varepsilon k)^\rho$ for $x \in \Omega_0$ and $\rho > 1$. Hence, using $\sum_{i=1}^{k} d_i^{4-n} = O(1)(\varepsilon k)^{n-4} = O(1)k^{-1}$,

$$\begin{aligned} I_{2\lambda} &:= \int_{\Omega_0} \Big(U_0^p - |x|^\alpha W^p\Big) U_\lambda^c \\ &= \frac{O(1)}{k}\Big(\frac{1}{k}\int_{\Omega_0} \frac{\varepsilon^{-n}\,dx}{d_0^{n+4-\tau}} + \sum_{i=1}^{k-1}\int_{\Omega_0} \frac{\varepsilon^{-n}\,dx}{d_0^8 d_i^{n-4}} + (\varepsilon k)^{3+\delta}\sum_{i=1}^{k-1}\int_{\Omega_0} \frac{\varepsilon^{-n}\,dx}{d_i^{n+1-\delta}}\Big). \end{aligned}$$

Note that

$$\begin{aligned} \int_{\Omega_0} \frac{\varepsilon^{-n} dx}{d_0^{n+4+\tau}} &\leqslant \int_{\mathbb{R}^n} \frac{dy}{[1+|y|]^{n+4+\tau}} = O(1), \\ \sum_{i=1}^{k-1}\int_{\Omega_0} \frac{\varepsilon^{-n} dx}{d_i^{n+1-\delta}} &= \sum_{i=1}^{k-1}\int_{\Omega_i} \frac{\varepsilon^{-n} dx}{d_0^{n+1-\delta}} \leqslant \int_{|y|>\sigma_0/(\varepsilon k)} \frac{dy}{|y|^{n+1-\delta}} = O(1)(\varepsilon k)^{1-\delta}, \\ \sum_{i=1}^{k-1}\int_{\Omega_0} \frac{\varepsilon^{-n}\,dx}{d_0^8 d_i^{n-4}} &= O(1)\Big\{(\varepsilon k)^4 \mathbf{1}_{\{n>8\}} + (\varepsilon k)^4 \ln k \mathbf{1}_{\{n=8\}} + (\varepsilon k)^{n-4}\mathbf{1}_{\{n<8\}}\Big\}. \end{aligned}$$

Thus, $I_{2\lambda} = O(1)k^{-1}\{k^{-1} + (\varepsilon k)^4[1+\mathbf{1}_{\{n=8\}}\ln k]\} = O(1)k^{-1}(\varepsilon k)^2$. Similarly, $I_{2\sigma} = O(1)k^{-1}(\varepsilon k)$.

3. The I_{3t} Term. Let $\theta \in (1,2)$ be a number such that $(n-4)(n-\theta)/8 > 1$. Using $W = U_0 + U^c$, $U_{0\lambda} = O(1)U_0$, (3.2) and Hölder inequality, Lemma 3.1 and (3.3), we obtain

$$\begin{aligned} I_{3\lambda} &:= \int_{\Omega_0} |x|^\alpha \Big[U_0^p - (U+U^c)^p + pU_0^{p-1}U^c\Big] U_{0\lambda}\,dx \\ &= O(1)\int_{\Omega_0} \{U_0^{p-2}|U^c|^2 + |U^c|^p\} U_0\,dx = O(1)\int_{\Omega_0} \{U_0^{p-1}|U^c|^2 + |U^c|^p U_0\}\,dx \\ &= O(1)\int_{\Omega_0} \left\{\frac{\varepsilon^{-n}}{d_0^8}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^2 + \frac{O(1)\varepsilon^{-n}}{d_0^{n-4}}\sum_{i=1}^{k-1}\frac{1}{d_i^{4+\theta}}\Big(\sum_{i=1}^{k-1} d_i^{\frac{-(n-4)(n-\theta)}{8}}\Big)^{\frac{8}{n-4}}\right\} dx \\ &= O(1)(\varepsilon k)^{n-4}\sum_{k=1}^{k-1}\int_{\Omega_0} \frac{\varepsilon^{-n} dx}{d_0^8 d_i^{n-4}} + O(1)(\varepsilon k)^{n-\theta}\sum_{i=1}^{k-1}\int_{\Omega_0}\frac{\varepsilon^{-n} dx}{d_0^{n-4} d_i^{4+\theta}} \\ &= O(1)\{(\varepsilon k)^n \mathbf{1}_{\{n>8\}} + (\varepsilon k)^n \ln k \mathbf{1}_{\{n=8\}} + (\varepsilon k)^{2n-8}\mathbf{1}_{\{n<8\}}\} + O(1)(\varepsilon k)^n \\ &= \frac{O(1)(\varepsilon k)^2}{k}. \end{aligned}$$

Similarly, $I_{3\sigma} = O(1)k^{-1}(\varepsilon k)$.

4. The I_{4t} Term. Using $\operatorname{div}\big((x-\mathbf{x}_0)U_0^{p+1}\big) = (p+1)\lambda U_0^p U_{0\lambda} = O(1)U_0^{p+1}$ we have

$$I_{4\lambda} := \int_{\Omega_0} (1-|x|^\alpha)U_0^p U_{0\lambda}\,dx = \int_{B_{\sigma_0/k}(\mathbf{x}_0)} \frac{1-|x|^\alpha}{(p+1)\lambda} \operatorname{div}\Big((x-\mathbf{x}_0)U_0^{p+1}\Big)\,dx + O(\varepsilon k)^n.$$

We use the Taylor expansion: when $|x-\mathbf{x}_0|\leqslant \sigma_0/k$,

$$1-|x|^\alpha = 1-r^\alpha-\alpha\mathbf{e}_0\cdot(x-\mathbf{x}_0)+O(k^{-2}).$$

Hence, by divergence Theorem and the fact that $U_0^pU_{0\lambda}$ is a function of $|x-\mathbf{x}_0|$,

$$\begin{aligned} I_{4\lambda} &= (1-r^\alpha)\int_{|x-\mathbf{x}_0|=\sigma_0/k}|x-\mathbf{x}_0|U_0^{p+1}\,dx+O(k^{-2})\int_{\mathbb{R}^n}U_0^{p+1}dx+O(\varepsilon k)^n \\ &= O(1)[(\varepsilon k)^n+k^{-2}]. \end{aligned}$$

Similarly, using $\varepsilon kU_{0\sigma}=O(1)U_0$ and $kU_{0\sigma}=\mathbf{e}_0\cdot\nabla U_0$ we can calculate, by the divergence theorem,

$$\begin{aligned} I_{4\sigma} &= \int_{B_{\sigma_0/k}(\mathbf{x}_0)}\left(\left\{1-|\mathbf{x}_0|^\alpha-\alpha\mathbf{e}_0\cdot(x-\mathbf{x}_0)\right\}\frac{\mathbf{e}_0\cdot\nabla U_0^{p+1}}{(p+1)k}+\frac{O(1)U_0^{p+1}}{\varepsilon k^3}\right)dx+\frac{O(1)(\varepsilon k)^n}{\varepsilon k} \\ &= \frac{\alpha}{(p+1)k}\int_{B_{\sigma_0/k}(\mathbf{x}_0)}U_0^{p+1}\,dx+\frac{O(1)(\varepsilon k)^n}{k}+\frac{O(1)}{\varepsilon k^3}+O(1)(\varepsilon k)^{n-1} \\ &= \frac{\alpha}{(p+1)k}\int_{\mathbb{R}^n}\Phi^{p+1}(y)dy+\frac{O(1)[(\varepsilon k)^{n-5}+(\varepsilon k)^3]}{k}=\frac{\alpha A_1+O(1)(\varepsilon k)}{k}, \end{aligned}$$

where $A_1=\frac{1}{p+1}\int_{\mathbb{R}^n}\Phi^{p+1}(y)dy$ is the same as that defined in (2.8) since $p+1=\frac{n}{m}$.

5. The I_{5t} Term. Suppressing the dependence on k and σ, we introduce the function

$$\mathbf{E}(x):=\sum_{i=1}^{k-1}\frac{k^{4-n}}{|x-\mathbf{x}_i|^{n-4}}-\sum_{i=0}^{k-1}\frac{k^{4-n}}{|x-\mathbf{x}_i^*|^{n-4}},\quad \mathbf{x}_i:=\left(1-\frac{\sigma}{k}\right)\mathbf{e}_i,\quad \mathbf{x}_i^*:=\left(1-\frac{\sigma}{k}\right)^{-1}\mathbf{e}_i.$$

Differentiating E with respect to x and using (3.3) we find that

$$\mathbf{E}(x)=O(1),\quad \nabla\mathbf{E}(x)=O(1)k,\quad D^2\mathbf{E}(x)=O(1)k^2\qquad \forall\, x\in\bar\Omega_0.$$

From (3.10) and $W=U_0+U^c$, we have, for $x\in\Omega_0$,

$$U^c=\Phi(0)\varepsilon^{-m}(\varepsilon k)^{n-4}\left\{\mathbf{E}(x)+O(1)(\varepsilon k)^2\right\}=\frac{\Phi(0)\varepsilon^{-m}}{\lambda^{n-4}k}\left\{\mathbf{E}(x)+O(1)(\varepsilon k)^2\right\}.$$

We now can estimate $I_{5\lambda}$ as follows: We use expansion $\mathbf{E}(x)=\mathbf{E}(x_0)+\nabla\mathbf{E}(\mathbf{x}_0)\cdot(x-\mathbf{x}_0)+O(1)k^2|x-\mathbf{x}_0|^2$. Also we note that $U_0^{p-1}U_{0\lambda}$ is a function of $|x-\mathbf{x}_0|$ and $p\lambda U_0^{p-1}U_{0\lambda}=\operatorname{div}((x-\mathbf{x}_0)U_0^p)-mU_0^p$. Hence,

$$\begin{aligned} I_{5\lambda} &:= -\int_{\Omega_0}p|x|^\alpha U_0^{p-1}U_{0\lambda}U^c dx=-\frac{p\Phi(0)}{\lambda^{n-4}k}\int_{\Omega_0}|x|^\alpha U_0^{p-1}U_{0\lambda}\varepsilon^{-m}[\mathbf{E}+O(1)(\varepsilon k)^2]dx \\ &= -\int_{B_{\sigma_0/k}(\mathbf{x}_0)}\frac{p\Phi(0)}{\lambda^{n-4}k}U_0^{p-1}U_{0\lambda}\varepsilon^{-m}\left\{\mathbf{E}(\mathbf{x}_0)+\nabla\mathbf{E}(x_0)\cdot(x-\mathbf{x}_0)\right\}dx \\ &\quad +\frac{O(1)}{k}\int_{\Omega_0}\left\{1-|x|^\alpha+(\varepsilon k)^2\left[1+\frac{|x-\mathbf{x}_0|^2}{\varepsilon^2}\right]+\mathbf{1}_{\mathbb{R}^n\setminus B_{\sigma_0/k}(\mathbf{x}_0)}\right\}U_0^p\varepsilon^{-m}dx \\ &= -\frac{\mathbf{E}(\mathbf{x}_0)\Phi(0)}{\lambda^{n-3}k}\int_{\mathbb{R}^n}\left[\operatorname{div}((x-\mathbf{x}_0)U_0^p)-mU_0^p\right]\varepsilon^{-m}dx+\frac{O(1)(\varepsilon k)^2}{k} \\ &= \frac{m\mathbf{E}(\mathbf{x}_0)}{\lambda^{n-3}k}\int_{\mathbb{R}^n}\Phi(0)\Phi^p(y)dy+\frac{O(1)(\varepsilon k)^2}{k}=\frac{2^{n-4}A_2\mathbf{E}(\mathbf{x}_0)+O(1)(\varepsilon k)^2}{\lambda^{n-3}k}, \end{aligned}$$

where $A_2 = 2^{4-n} m \int_{\mathbb{R}^n} \Phi(0)\Phi^p(y)dy$ is the same as that defined in (2.8).

Similarly, using $U_{0\sigma} = O(1)(\varepsilon k)^{-1} U_0$ and $kU_{0\sigma} = \mathbf{e} \cdot \nabla U_0$ we obtain

$$
\begin{aligned}
I_{5\sigma} &:= -\int_{\Omega_0} p|x|^\alpha U_0^{p-1} U_{0\sigma} U^c dx = -\frac{p\Phi(0)}{\lambda^{n-4}k} \int_{\Omega_0} |x|^\alpha U_0^{p-1} U_{0\sigma}[\mathbf{E} + O(1)(\varepsilon k)^2] \frac{dx}{\varepsilon^m} \\
&= -\Phi(0) \int_{B_{\sigma_0/k}(\mathbf{x}_0)} \frac{\mathbf{e}_0 \cdot \nabla U_0^p}{\lambda^{n-4}k^2} \Big\{\mathbf{E}(\mathbf{x}_0) + \nabla \mathbf{E}(x_0) \cdot (x - \mathbf{x}_0)\Big\} \frac{dx}{\varepsilon^m} \\
&\quad + \frac{O(1)}{\varepsilon k^2} \int_{\Omega_0} \Big\{ 1 - |x|^\alpha + (\varepsilon k)^2 \Big[1 + \frac{|x - \mathbf{x}_0|^2}{\varepsilon^2} \Big] + \mathbf{1}_{\mathbb{R}^n \setminus B_{\sigma_0/k}(\mathbf{x}_0)} \Big\} \frac{U_0^p \, dx}{\varepsilon^m} \\
&= \int_{B_{\sigma_0/k}(\mathbf{x}_0)} \frac{\mathbf{e}_0 \cdot \nabla [\nabla \mathbf{E}(\mathbf{x}_0) \cdot (x - \mathbf{x}_0)]}{\lambda^{n-4}k^2} \frac{\Phi(0) U_0^p dx}{\varepsilon^m} + \frac{O(1)(\varepsilon k)^2}{\varepsilon k^2} \\
&= \frac{\mathbf{e}_0 \cdot \nabla \mathbf{E}(\mathbf{x}_0)}{\lambda^{n-4}k^2} \int_{\mathbb{R}^n} \Phi(0)\Phi^p(y)dy + O(\varepsilon) = \frac{2^{n-4}A_2}{m\lambda^{n-4}k} \Big(\frac{\nabla \mathbf{E}(\mathbf{x}_0) \cdot \mathbf{e}_0}{k} + O(\varepsilon k) \Big).
\end{aligned}
$$

6. Evaluation of $\mathbf{E}(\mathbf{x}_0)$ and $\nabla \mathbf{E}(\mathbf{x}_0)$. Finally, we evaluate

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}_0) &= \sum_{i=1}^{k-1} \frac{1}{|k(\mathbf{x}_0 - \mathbf{x}_i)|^{n-4}} - \sum_{i=0}^{k-1} \frac{1}{|k(\mathbf{x}_0 - \mathbf{x}_i^*)|^{n-4}}, \\
\frac{\nabla \mathbf{E}(\mathbf{x}_0)}{k} &= (4-n) \left(\sum_{i=1}^{k-1} \frac{k(\mathbf{x}_0 - \mathbf{x}_i)}{|k(\mathbf{x}_i - \mathbf{x}_0)|^{n-2}} - \sum_{i=0}^{k-1} \frac{k(\mathbf{x}_0 - \mathbf{x}_i^*)}{|k(\mathbf{x}_0 - \mathbf{x}_i^*)|^{n-2}} \right).
\end{aligned}
$$

Note that, with $r = 1 - \sigma/k$,

$$
\begin{aligned}
|k(\mathbf{x}_i - \mathbf{x}_0)|^2 &= k^2 r^2 \{2 - 2\mathbf{e}_i \cdot \mathbf{e}_0\} = 4r^2 \Big(k \sin \frac{\pi i}{k} \Big)^2, \\
|k(\mathbf{x}_i^* - \mathbf{x}_0)|^2 &= k^2 \Big| \frac{1}{r} \mathbf{e}_i - r\mathbf{e}_0 \Big|^2 = 4 \Big\{ D^2 + \Big(k \sin \frac{i\pi}{k} \Big)^2 \Big\},
\end{aligned}
$$

where $D = \frac{k}{2}[\frac{1}{r} - r] = \sigma + O(1)k$. Using $d^2 + k^2 \sin^2 t = [d^2 + k^2 t^2][1 + O(1)\theta^2]$ for $t \in [0, \theta]$ we have

$$
\begin{aligned}
\sum_{i=1}^{[k/2]} \Big[\Big(k \sin \frac{i\pi}{k} \Big)^2 + D^2 \Big]^{-m} &= \sum_{i=1}^{[\theta k/\pi]} \frac{1 + O(1)\theta^2}{(D^2 + (\pi i)^2)^m} + O(1) \sum_{i=[\theta k/\pi]+1}^{\infty} i^{4-n} \\
&= \sum_{i=1}^{\infty} \frac{1}{[D^2 + (\pi i)^2]^m} + O(1)\theta^2 + O(1)(k\theta)^{5-n} \\
&= \sum_{i=1}^{\infty} \frac{1}{[D^2 + (\pi i)^2]^m} + O(1)k^{-\frac{2(n-5)}{n-3}}
\end{aligned}
$$

by taking $\theta = k^{-\frac{n-5}{n-3}}$. This expansion is also true if D is replaced by 0. Hence, for $L_1(\cdot)$ defined in (2.6),

$$
\mathbf{E}(\mathbf{x}_0) = 2^{4-n} \Big(L_1(D) + O(1)k^{-\frac{2(n-5)}{n-3}} \Big) = 2^{4-n} \Big(L_1(\sigma) + O(1)k^{-1} + O(1)k^{-\frac{2(n-5)}{n-3}} \Big).
$$

Similarly, we find that, for $L_2(\cdot)$ is defined in (2.7),

$$\frac{\nabla \mathbf{E}(\mathbf{x}_0)}{k} = -\frac{m\,\mathbf{e}_0}{2^{n-4}}\Big\{ L_2(\sigma) + O(1)k^{-1} + O(1)k^{-\frac{2(n-3)}{n-1}}\Big\}.$$

Collecting all estimates for I_{it}, $i = 1, \cdots, 5$, we then obtain the assertion of the theorem. □

7 Proof of Theorem 1.1

Let (c_1, c_2, v) be the solution of (2.9) given by Theorem 5.1. We now find (λ, σ) such that $(c_1, c_2) = (0, 0)$. We use the equation (2.15) which can be written as, with M given in Lemma 3.7,

$$M\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \lambda\langle\phi, \mathcal{L}V_{0\lambda}\rangle - \lambda\langle F_0 + F_1, V_{0\lambda}\rangle - \lambda\langle\mathcal{L}_1\phi, V_{0\lambda}\rangle - \lambda\langle N(\phi), V_{0\lambda}\rangle \\ \varepsilon k\langle\phi, \mathcal{L}V_{0\sigma}\rangle - \varepsilon k\langle F_0 + F_1, V_{0\sigma}\rangle - \varepsilon k\langle\mathcal{L}_1\phi, V_{0\sigma}\rangle - \varepsilon k\langle N(\phi), V_{0\sigma}\rangle \end{bmatrix}. \tag{7.1}$$

First of all, direct calculation shows that, for $t = \lambda$ or σ,

$$-\langle F_0 + F_1, V_{0t}\rangle = -\frac{1}{k}\langle F_0 + F_1, W_t\rangle = \frac{\partial J(\lambda, \sigma, k)}{\partial t}.$$

Next, since $V_{0\lambda}(x) = U_{0\lambda} - \zeta_{0\lambda} = O(1)\varepsilon^m\,|x - \mathbf{x}_0|^{4-n}$, we have, by Lemma 5.3, (3.6), and (5.4),

$$\langle\mathcal{L}_1\phi, V_{0\lambda}\rangle = O(1)\|\mathcal{L}_1\phi\|_{4+\tau}\int_{B_1}\frac{\varepsilon^m\omega_{\tau+4}(x)\,dx}{|x - \mathbf{x}_0|^{n-4}} = \frac{O(1)\|\phi\|_\tau\varepsilon^m\omega_\tau(\mathbf{x}_0)}{k} = \frac{O(1)}{k^2}.$$

Similarly, $\varepsilon k\langle\mathcal{L}_1\phi, V_{0\sigma}\rangle = O(1)k^{-2} = O(1)\varepsilon(\varepsilon k)^{n-5}$.

When $\rho \in [\tau, n-4)$, using $W = \sum_{i=1}^k V_i = U_0 + U_0^c$, $|U^c| + |U_\lambda^c| = O(1)\sum_{i=1}^{k-1} U_i$, and symmetry we have

$$\begin{aligned}
\langle\phi, \mathcal{L}V_{0\lambda}\rangle &= \int_{B_1}\phi\Big\{\Delta^2 V_{0\lambda} - pW^{p-1}V_{0\lambda}\Big\}\,dx \\
&= \int_{B_1} p\phi U_0^{p-1}U_{0\lambda}\,dx - \sum_{i=1}^{k}\int_{\Omega_0} p\phi W^{p-1}V_{i\lambda}\,dx \\
&= \sum_{i=1}^{k-1}\int_{\Omega_0} p\phi U_i^{p-1}U_{i\lambda}\,dx + \int_{\Omega_0} p\phi\Big\{U_0^{p-1}U_{0\lambda} - W^{p-1}W_\lambda\Big\}dx \\
&= O(1)\|\phi\|_\rho\Big(\int_{\Omega_0}\omega_\rho U_0^{p-1}\sum_{i=1}^{k-1}U_i\,dx + \int_{\Omega_0}\omega_\rho\Big(\sum_{i=1}^{k-1}U_i\Big)^p dx\Big) \\
&= O(1)\|\phi\|_\rho\varepsilon^m\Big(\int_{\Omega_0}\frac{\omega_\rho}{d_0^8}\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\frac{dx}{\varepsilon^n} + \int_{\Omega_0}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^p w_\rho\frac{dx}{\varepsilon^n}\Big),
\end{aligned}$$

where we have used $\int_{\Omega_i} U_0^p U_{0\lambda}\omega_\rho dx = \int_{\Omega_0} U_i^p U_{i\lambda}\omega_\rho dx$. We estimate each integral on the right-hand side as follows. Using Hölder inequality, $(4+\rho-\tau)/(p-1) \geqslant 1$, (3.3), and symmetry we have

$$\begin{aligned}\int_{\Omega_0}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\Big)^p\frac{w_\rho\,dx}{\varepsilon^n} &\leqslant \int_{\Omega_0}\Big(\sum_{i=1}^{k-1}\frac{1}{d_i^{(4+\rho-\tau)/(p-1)}}\Big)^{p-1}\sum_{i=1}^{k-1}\frac{1}{d_i^{n+\tau-\rho}}\frac{\omega_\rho\,dx}{\varepsilon^n}\\ &= O(1)(\varepsilon k)^{4+\rho-\tau}\Big(1+\mathbf{1}_{\{\rho=\tau,n=6\}}\ln^4 k\Big)\sum_{i=1}^{k-1}\int_{\Omega_i}\frac{\omega_\rho}{d_0^{n+\tau-\rho}}\frac{dx}{\varepsilon^n}\\ &= O(1)(\varepsilon k)^{4+\rho-\tau}(1+\mathbf{1}_{\{\rho=\tau\}}\ln^5 k).\end{aligned}$$

Using $d_0 \leqslant d_i$ on Ω_0 and setting $\theta = \max\{0,(n+\tau)-(8+\rho)\}$ we obtain

$$\begin{aligned}\int_{\Omega_0}\frac{w_\rho(x)}{d_0^8}\sum_{i=1}^{k-1}\frac{1}{d_i^{n-4}}\frac{dx}{\varepsilon^n} &= O(1)(\varepsilon k)^{n-4-\theta}\int_{\mathbb{R}^n}\frac{w_\rho}{d_0^{8+\theta}}\frac{dx}{\varepsilon^n}\\ &= O(1)(\varepsilon k)^{n-4-\theta}\Big(1+\mathbf{1}_{\{\rho=\tau\}}\ln k\Big)\\ &= O(1)(\varepsilon k)^{\min\{n-4,4+\rho-\tau\}}(1+\mathbf{1}_{\{\rho=\tau\}}\ln k).\end{aligned}$$

In summary, using $mp = n-m$ we obtain

$$\langle\phi,\mathcal{L}V_{0\lambda}\rangle = O(1)\|\phi\|_\rho\varepsilon^m(\varepsilon k)^{\min\{n-4,4+\rho-\tau\}}(1+\mathbf{1}_{\{\rho=\tau\}}\ln^5 k)\ \forall\rho\in[\tau,n-4). \tag{7.2}$$

Similarly,

$$(\varepsilon k)\langle\phi,\mathcal{L}V_{0\sigma}\rangle = O(1)\|\phi\|_\rho\varepsilon^m(\varepsilon k)^{\min\{n-4,4+\rho-\tau\}}(1+\mathbf{1}_{\{\rho=\tau\}}\ln^5 k)\ \forall\rho\in[\tau,n-4). \tag{7.3}$$

Taking $\rho = 1$ and using (5.4), we see that

$$\Big|\langle\phi,\mathcal{L}V_{0\lambda}\rangle\Big| + (\varepsilon k)\Big|\langle\phi,\mathcal{L}V_{0\sigma}\rangle\Big| = O(1)k^{-1}(\varepsilon k)^2 = O(1)\varepsilon(\varepsilon k).$$

Finally, for $\rho\in[\tau,n-4)$, by (3.6) and (5.1) with $\psi\equiv 0$,

$$\begin{aligned}&\Big|\langle N(\phi),V_{0\lambda}\rangle\Big| + \varepsilon k\Big|\langle N(\phi),V_{0\sigma}\rangle\Big|\\ =\ & O(1)\varepsilon^m\|N(\phi)\|_{\rho+4}\int_{B_1}\omega_{\rho+4}(x)\,|x-\mathbf{x}_0|^{4-n}dx\\ =\ & O(1)\varepsilon^{m+4}\|N(\phi)\|_{\rho+4}\omega_\rho(\mathbf{x}_0)\\ =\ & O(1)\max\{\varepsilon^m\|\phi\|_{\rho_1},(\varepsilon^m\|\phi\|_{\rho_1})^{p-1}\}\varepsilon^m\|\phi\|_\rho. \end{aligned}\tag{7.4}$$

In [13, 29], the norm is fixed. It corresponds to the choice $\rho = \rho^* := m+\tau$. Then $\rho_1 = \rho^*$, so by (5.4) and (7.4), with $\nu^* = \gamma(\rho^*)\min\{p,2\} - 1/\tau$,

$$|\langle N(\phi),V_{0\lambda}\rangle| + |\varepsilon k\langle N(\phi),V_{0\sigma}\rangle| = O(1)k^{-\gamma(\rho^*)\min\{p,2\}} = O(1)\varepsilon k^{-\nu^*}.$$

Suppose $6\leqslant n\leqslant 12$. Then $p\geqslant 2$ and $\nu^* = \min\{n-5,7-2\tau\}/(n-4) \geqslant 1/(n-4)$.

Suppose $n \geqslant 12$. Then $p \leqslant 2$. We take $\rho = 4$, so by (5.4) and (7.4),

$$|\langle N(\phi), V_{0\lambda}\rangle| + |\varepsilon k\langle N(\phi), V_{0\sigma}\rangle| = O(1)k^{-1-\gamma(\rho_1)(p-1)} = O(1)\varepsilon k^{-\nu}$$

where $\nu = \gamma(\rho_1)(p-1) - 1/(n-4)$ and ρ_1 is given by (5.2) with $\rho = 4$. Using $\gamma(\rho_1) = \min\{1, \frac{n-\rho_1}{n-4}\} = \frac{n-\rho_1}{n-4}$, one finds that

$$\nu = \frac{1}{(n-4)^2}\begin{cases} 32 + (n-4)(24-7\tau)/(8-\tau) & \text{if } n \geqslant 19, \\ 3n + 20 - 8\tau & \text{if } 12 \leqslant n \leqslant 18 \end{cases} \geqslant \frac{1}{n-4}.$$

In conclusion, (7.1) can be written as, with M given in Lemma 3.7,

$$M\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda^{4-n}\begin{bmatrix} k^{-1} & 0 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} A_2L_1(\sigma) + O(1)k^{-\frac{2}{n-4}} \\ \alpha A_1\lambda^{n-4} - A_2L_2(\sigma) + O(1)k^{-\frac{1}{n-4}} \end{bmatrix}. \tag{7.5}$$

By continuity, for each large enough integer k, there exists $(\lambda, \sigma) = (\lambda^*, \sigma^*) + O(1)(k^{-\frac{1}{n-4}}, k^{-\frac{2}{n-4}})$ such that $c_1 = c_2 = 0$, from which we obtain a solution of (1.1). Now we are ready to prove the following, of which Theorem 1.1 is a direct consequence.

Theorem 7.1 *Suppose $n \geqslant 6$, $p = \frac{n+4}{n-4}$, and $\alpha > 0$. There exists a positive integer K such that for each integer $k \geqslant K$,* (1.1) *admits a solution of the form*

$$u(x) = \frac{1}{\varepsilon^m}\Big\{\max_i \Phi\Big(\frac{x - \mathbf{x}_i}{\varepsilon}\Big) + \frac{O(1)}{k}\Big\} \tag{7.6}$$

where $\varepsilon = k^{-\frac{n-3}{n-4}}/\lambda$, $\mathbf{x}_i = (1 - \sigma/k)\mathbf{e}_i$, $\mathbf{e}_i$ is defined in (2.2)*, and $(\lambda, \sigma) \approx (\lambda^*, \sigma^*)$.*

Proof. It remains to describe the profile of the solution we obtained. Indeed, taking a $\rho \in (1, \min\{n-4, 4\})$ we obtain from (5.4) that

$$\varepsilon^m\phi(x) = O(1)\varepsilon^m\|\phi\|_\rho\omega_\rho = O(1)k^{-1}\omega_\rho = O(1)k^{-1}.$$

As $\varepsilon^m u(x) = \varepsilon^m W(x) + \varepsilon^m\phi(x)$, (7.6) thus follows from (3.10). This completes the proof. □

Remark 7.1 If we follow the method in [13, 29] by taking only $\rho = \rho^* := m + \tau$, then $\varepsilon k\langle N(\phi), V_{0\sigma}\rangle = O(1)\varepsilon k^{-\nu^*}$ where $\nu^* > 0$ if and only if $6 \leqslant n \leqslant 19$; namely, the analysis breaks down when $n \geqslant 20$. Here in this paper we introduce variable weights and develop new techniques to take care of the technical difficulties.

References

[1] M. B. Ayed and M. Hammami, *On a fourth order elliptic equation with critical nonlinearity in dimension six,* Nonlinear Anal. **64** (2006), 924-957.

[2] V. Barutello, S. Secchi, and E. Serra, *A note on the radial solutions for supercritical Hénon equation*, J. Math. Anal. Appl. **341** (2008), 720-728.

[3] T. Battsch, T. Weth, and M. Willem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator*, Calc. Var. **18** (2003), 253-268.

[4] E. Berchio, F. Gazzola and T. Weth, *Critical growth biharmonic elliptic problems under Steklov-type boundary conditions,* Adv. Differ. Equ. **12** (2007), 381-406

[5] F. Bernis, J. Garcia-Azorero, I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order,* Adv. Differ. Equ. **1** (1996), 219-240.

[6] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), 18-24.

[7] J. Byeon and Z. Q. Wang, *On the Hénon equation: Asymptotic profile of ground state,* I, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), 803-828.

[8] J. Byeon and Z. Q. Wang, *On the Hénon equation: Asymptotic profile of ground state,* II, J. Differ. Equ. **216** (2005), 78-108.

[9] D. Cao and S. Peng, *The asymptotic behavior of the ground state solutions for Hénon equation,* J. Math. Anal. Appl. **278** (2003), 1-17.

[10] D. Cao, S. Peng and S. Yan , *Asymptotic behaviour of ground state solutions for the Hénon equation,* IMA J. Appl. Math. **74** (2009), 468-480.

[11] G. Chen, W. Ni, and J. Zhou, *Algorithm and visalization for solutions of nonlinear ellpitic equation*, Int. J. Bifurcat. Chaos **10** (2000), 1565-1612.

[12] D. G. de Figueiredo, E. M. dos Santos, and O. H. Miyagaki, *Sobolev spaces of symmetric functions and applications*, J. Funct. Anal. **261** (2011), 3735-3779.

[13] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem,* Calc. Var. Partial Differ. Equ. **16** (2003), 113-145.

[14] D. E. Edmuns, D. Fortunato and E. Jannelli, *Critical exponents, critical dimensions and the biharmonic operator,* Arch. Rational Mech. Anal. **112** (1990), 269-289.

[15] F. Gazzola, H.-Ch. Grunau and M. Squassina, *Existence and nonexistence results for critical growth biharmonic elliptic equations,* Calc. Var. **18** (2003), 117-143.

[16] F. Gazzola and R. Pavani, *Wide oscillations finite time blow up for solutions to nonlinear fourth order differential equations,* Arch. Rat. Mech. Anal. **207** (2013), 717-752.

[17] F. Gladiali and M. Grossi, *Supercritical elliptic problem with nonautonomous non-linearities*, J. Diff. Equ. **253** (2012), 2616-2645.

[18] M. Hénon, *Numerical experiments on the stability of spherical stellar system,* Astronom. Astrophys. **24** (1973), 229–238.

[19] C. Lin, *A classification of solutions of a conformally invariant fourth order equation in* $\mathbb{R}^n$, Comment. Math. Helv. **73** (1998), 206-231.

[20] F. H. Lin, W. M. Ni and J. Wei, *On the number of interior peak solutions for a singularly perturbed Neumann problem,* Comm. Pure Appl. Math. **60** (2007), 252-281.

[21] W. M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications,* Indiana Univ. Math. **6** (1982), 801-807.

[22] S. Peng, *Multiple boundary concentrating solutions to Dirichelet problem of Hénon equation*, Acta Math. Appl. Sin. Engl. Ser. **22** (2006), 137-162.

[23] A. Pistoia and Serra, *Multi-peak solutions for the Hénon equatiuon with slightly subcritical growth,* Math. Z. **256** (2007), 75-97.

[24] E. Serra, *Non radial positive solutions for the Hénon equation with critical growth,* Calc. Var. Partial Differ. Equ. **23** (2005), 301-326.

[25] D. Smets, J. B. Su and M. Willem, *Non-radial ground states for the Hénon equation,* Commun. Contemp. Math. **4** (2002), 467-480.

[26] D. Smets and M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems,* Calc. Var. Partial Differ. Equ. **18** (2003), 57-75.

[27] Z. Wang, *Nonradial positive solutions for a biharmonic critical growth problem,* Comm. Pure Appl. Anal. **11** (2012), 517-545.

[28] J. Wei and S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on S^N,* J. Funct. Anal. **258** (2010), 3048-3081.

[29] J. Wei and S. Yan, *Infinitely many non-radial solutions for the Hénon equation with critical growth,* Revista Mátematica Iberoamericana, **29** (2013), 997-1020.

[30] J. Wei and D. Ye, *Nonradial solutions for a conformally invariant fourth order equation in $\mathbb{R}^4$,* Calculus Var. Partial Differ. Equ. **32** (2008), 373-386.

[31] Y. Zhang, *Positive solutions of semilinear biharmonic equations with critical Sobolev exponents,* Nonlinear Anal. **75** (2012), 55-67.

[32] Y. Zhang and J. Hao, *The asymptotic behavior of the ground state solutions for biharmonic Equations,* Nonlinear Anal. **74** (2011), 2739-2749.