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Symmetry of Entire Solutions to the Allen-Cahn Equation

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Abstract

In this paper, we shall prove even symmetry of monotone entire solutions to the balanced Allen-Cahn equation with one spatial variable. Related results for the unbalanced Allen-Cahn equation are also discussed.

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1 Introduction

In this paper, we consider entire solutions of the Allen-Cahn equation with a double well potential. By an entire solution we mean a solution not only defined in the entire Euclidean space but also entirely in time. There are three types of well-known entire solutions: homogeneous solutions $v(x, y, t) = u(t)$, which only depends on time, stationary solutions which depends only on (x, y), and traveling wave solutions, which are of the form $v(x, y, t) = u(x, y - ct)$, where $(x, y) \in \mathbb{R}^{n+1}$, $t \in \mathbb{R}$. In particular, traveling wave solutions to the Allen-Cahn equation have been intensively studied recently. (See, e.g., [23], [24], [12], [37], [38], [40], [41], [36], [30], [22] and references therein.) There exist also other types of entire solutions to the Allen-Cahn equation. For example, the existence of entire solutions with two traveling fronts approaching in opposite directions from negative infinity in time and annihilating in a finite time is shown for the Allen-Cahn equation with a balanced or an unbalanced double well potential in [14] and [13] respectively. In this paper, we shall classify all entire solutions which are bistable, i.e., connecting the two stable states, of the Allen-Cahn equation in one spatial variable. To be more precise, we consider

$$v_t = \Delta_{x,y} v - F'(v), \quad |v| \le 1, \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where F is a $C^{3,\beta}$ double-well potential, i.e., F satisfies

$$\begin{cases} F'(-1) = F'(1) = 0, & F''(-1) > 0, F''(1) > 0 \\ F'(s) > 0,\ s \in (-1, \theta); & F'(s) < 0,\ s \in (\theta, 1) \end{cases} \tag{1.2}$$

for some $\theta \in (0, 1)$. Without loss of generality, we may assume that $F(-1) = 0$ and $\theta = 0$. If $F(1) = F(-1) = 0$, F is called a balanced double well potential. Otherwise, it is called an unbalanced double well potential, and in this case we may assume that $F(1) > F(-1) = 0$ without loss of generality.

A typical example of balanced double well potential is $F(u) = \frac{1}{4}(1 - u^2)^2, \quad u \in \mathbb{R}$, while a typical unbalanced double well potential is $F(u) = \frac{1}{4}(1 - u^2)^2 - a(u^3/3 - u)$ with $a \in (-1, 0)$. Note that $F'(u) = (u - a)(u^2 - 1)$ in the latter case.

If we consider a traveling wave solution $v(x, y, t) = u(x, y - ct)$ with a constant speed $c > 0$ and $(x, y) \in \mathbb{R}^{n+1}$, then u satisfies an elliptic equation

$$\Delta_x u + u_{yy} + cu_y - F'(u) = 0, \quad |u| \le 1, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.3}$$

On the other hand, in (1.1), if we drop one spatial diemension, say, the one represented by variable y, we may reverse the direction of time and denote the new variable by $y = -t$ and let $u(x, y) = v(x, -t)$. We may also restrict the range of u to $[-1, 1]$. Then, It is obvious that u satisfies an reverse parabolic equation

$$\Delta_x u + u_y - F'(u) = 0, \quad |u| \le 1, \quad (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{1.4}$$

It is clear that (1.4) and (1.3) are related but of different types (elliptic and parabolic respectively). Naturally, there are some differences regarding solutions of these equations. For example, there does not exist an entire solution connecting -1 and 1, which is homogeneous in x, while there exists such a traveling wave solution. On the other hand, we shall see that there are many similarities of solutions to (1.4) and (1.3) with the same spatial dimensions for x. After all, solutions to (1.3) are special entire solutions to (1.1), while (1.4) is just a rewritting of (1.1) with a spatial dimension lowered by one.

We may assume that the entire solution u under consideration is monotone in time t and hence in y. Without loss of generality, we assume

$$u_y(x, y) > 0, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.5}$$

We may also assume that the solution u connects two stable states, i.e.,

$$\lim_{y \to \pm\infty} u(x, y) = \pm 1, \quad x \in \mathbb{R}^n. \tag{1.6}$$

We note that the limit condition above does not need to be uniform in x. Indeed, we shall see that the limits are not uniform.

When $n = 1$, it is well-known that there exists a unique speed $c_0 \geq 0$ such that (1.3) has a unique solution $g(y)$ (up to translation) satisfying the monotone condition (1.5), i.e.,

$$\begin{cases} g''(s) + c_0 g'(s) - F'(g(s)) = 0, \quad s \in \mathbb{R}, \\ \lim_{s\to\infty} g(s) = 1, \quad \lim_{s\to-\infty} g(s) = -1, \end{cases} \tag{1.7}$$

where $c_0 = 0$ in the balanced case and $c_0 > 0$ in the unbalanced case. We may assume that $g(0) = 0$. The solution g is non-degenerate in the sense that the linearized operator has a kernel spanned only by g'.

It is well-known that when F is balanced, g is a minimizer of the following energy functional

$$\mathbf{E}(v) := \int_{-\infty}^{\infty} [\frac{1}{2}|v'|^2 + F(v)]dx$$

in $\mathcal{H} := \{v \in H^1_{loc}(\mathbb{R}) : -1 \leq v \leq 1,\ \lim_{s\to\pm\infty} v(s) = \pm 1\}$ and

$$\mathbf{e} := \mathbf{E}(g) = \int_{-1}^{1} \sqrt{2F(u)}du < \infty.$$

There is a significant difference between the balanced and the unbalanced Allen-Cahn equation. The difference of zero speed and positive speed of one dimensional traveling wave solution g for the balanced and unbalanced potential leads to a fundamental difference of the structure of traveling fronts in higher dimensional spaces, see [22] and references therein for detailed discussion.

Entire solutions for the unbalanced Allen-Cahn equation are studied in [13], motivated by earlier papers on entire solutions of KPP equations ([28], [29]). The results may be stated as follows.

Theorem A [Chen, Guo, 2005] *Assume that F is an unbalanced double well potential satisfying* (1.2) *and* $F(1) > F(-1) = 0$. *When $n = 1$, there exists a solution $U(x, y) = U(|x|, y)$ to* (1.4), (1.5), (1.6) *such that $U_x(x, y) < 0$ for $x > 0$ and $U(0, 0) = 0$. Furthermore, if the 0-level set of U is denoted by Γ, then Γ is asymptotically linear,*

$$\lim_{y\to\infty, U(x,y)=0} y/|x| = \frac{1}{c_0}, \tag{1.8}$$

where $c_0 > 0$ is the speed of the traveling wave solution in (1.7).

Entire solutions for the balanced Allen-Cahn equation are studied in [14]. The result may be restated as follows.

Theorem B [Chen, Guo, Ninomiya, 2006] *Assume that F is a balanced double well potential satisfying* (1.2) *and* $F(-1) = F(1) = 0$. *When $n = 1$, there exists a solution $U(x, y) = U(|x|, y)$ to* (1.4), (1.5), (1.6) *such that $U_x(x, y) < 0$ for $x > 0$ and $U(0, 0) = 0$. Furthermore, if the 0-level set of U is denoted by Γ, then Γ is asymptotically a hyperbolic cosine curve, i.e., for some $A > 0$*

$$\lim_{y\to\infty, U(x,y)=0} \frac{cosh(2\mu x)}{\mu y} = A, \quad where\ \mu = \sqrt{F''(1)}. \tag{1.9}$$

Under the condition that suitable asymptotical behavior of Γ at infinity is prescribed, which may be regarded as "initial" condition at negative infinity of time, the uniqueness of entire solution has also been proven for both the balanced and unbalanced cases in [13] and [14] respectively. In this paper, we shall follow the framework of [22] and show symmetry results and hence a classification of entire solutions of the Allen-Cahn equation without prescribing the asymptotical behavior of Γ (i.e, without assuming the initial conditions at negative infinity in time). To be more precise, we shall show the following main theorems.

Theorem 1.1 *Assume that F is a balanced double well potential satisfying* (1.2) *and* $F(-1) = F(1) = 0$. *Suppose $n = 1$ and u satisfies* (1.4), (1.5) *and* (1.6). *Then, $u(x, y) = U(x, y)$ after a proper translation in (x, y), where U is the solution in Theorem B. In particular, u is evenly symmetric with respect to x and $u_x(x, y) < 0$ for $x > 0$.*

Theorem 1.2 *Assume that F is an unbalanced double well potential satisfying* (1.2) *and* $F(1) > F(-1) = 0$. *Suppose $n = 1$ and u satisfies* (1.4), (1.5) *and* (1.6). *Then, either u is the unique conical solution as in Theorem A, i.e., $u(x, y) = U(x, y)$ after a proper translation in (x, y), or u is the unique traveling wave solutions in one dimensional space, i.e., $u(x, y) = g(\pm x + c_0 y)$ after a proper translation, where g is defined in* (1.7).

The paper is organized as follows. In Section 2, the main result Theorem 1.1 shall be proved. Theorem 1.2., the classification result for entire solutions of the unbalanced Allen-Cahn equation in $\mathbb{R}^2$, will be proved in Section 3.

2 Symmetry of entire solutions of the balanced Allen-Cahn equation in $\mathbb{R}$

Throughout this section, we assume that $n = 1$ and the double well potential F is balanced, i.e., $F(-1) = F(1) = 0$. We shall prove Theorem 1.1 in three main steps, following the structure of the proof of Theorem 1.1 in [22]. First we carry out a preliminary asymptotical analysis of the level sets of the solution u and show that the slope of the 0-level curve $y = \gamma(x)$ must tend to $\pm\infty$ as x tends to $\pm\infty$. Second, we show that $y = \gamma(x)$ is asymptotically hyperbolic cosine and obtain a very detailed asymptotical formula. Lastly, we complete the proof by using the asymptotical formula of the level curve and the moving plane method.

2.1 Preliminary analysis of the level set

We first show an important lemma which asserts the integrability of of u_y.

Lemma 2.1 *Suppose that u is a solution to* (1.4), (1.5) *and* (1.6). *Then*

$$\int_{\mathbb{R}^2} u_y^2 dxdy < \infty. \tag{2.10}$$

Proof. Define

$$h(x) = \int_{\mathbb{R}} u_x u_y dy, \quad x \in \mathbb{R}.$$

Since u is bounded in $C^3(\mathbb{R}^2)$ by the standard elliptic estimates and u_y is positive, it is easy to see that $h(x)$ is well-defined and

$$|h(x)| < C, \quad x \in \mathbb{R}$$

for some constant $C > 0$.

Note that due to (1.6), we have

$$\lim_{y\to\pm\infty} u_x = 0, \quad \lim_{y\to\pm\infty} u_y = 0, \quad x \in \mathbb{R}.$$

Differentiating $h(x)$ with respect to x and using the equation, we obtain

$$\begin{aligned} h'(x) &= \int_{\mathbb{R}} (u_{xx}u_y + u_x u_{xy})dy \\ &= \int_{\mathbb{R}} [\frac{\partial}{\partial y}(F(u) + \frac{1}{2}u_x^2) - u_y^2]dy \\ &= -\int_{\mathbb{R}} u_y^2 dy. \end{aligned} \tag{2.11}$$

Then

$$\int_a^b \int_{\mathbb{R}} u_y^2 dydx = h((a) - h(b). \tag{2.12}$$

The bound of $h(x)$ immediately leads to the integrability of u_y^2 in $\mathbb{R}^2$. ■

Due to (1.5) and (1.6), the 0-level set of u is a C^3 graph of a function defined in $\mathbb{R}$. We let $y = \gamma(x), x \in \mathbb{R}$ be such a function. The next lemma asserts that the slope of $y = \gamma(x)$ must tend to infinity as x goes to infinity.

Lemma 2.2 *There holds*

$$\lim_{|x|\to\infty} |\gamma'(x)| = \infty. \tag{2.13}$$

Proof. Since u is bounded in $C^3(\mathbb{R}^2)$, Lemma 2.1 implies that

$$\lim_{|x|\to\infty} u_y(x,y) = 0, \quad \text{uniformly in } y \in \mathbb{R}.$$

Now assume that (2.13) is not true. Then there exists a sequence $\{x_m\}$ such that $|x_m|$ goes to infinity and

$$\lim_{m\to\infty} \gamma'(x_m) = k_0$$

for some constant k_0.

We shall translate u along this sequence of x_m. Define

$$u_m(x,y) = u(x + x_m, y + \gamma(x_m)), \quad (x,y) \in \mathbb{R}^2.$$

By the standard theory for parabolic equations, we know that u_m is bounded in $C^{3,\beta}(\mathbb{R}^2)$. Then there is a subsequence, which we still denote by $\{x_m\}$, such that u_m converges to a function u_* in $C^3_{loc}(\mathbb{R}^2)$. It is easy to see that $u_*(0,0) = 0$, $\frac{\partial}{\partial y}u_*(x,y) = 0$, $(x,y) \in \mathbb{R}^2$. Then $u_*(x,y) = g_*(x)$ for some C^3 function g_* which is a solution to the one dimensional stationary Allen-Cahn equation

$$u_{xx} - F'(u) = 0, \quad x \in \mathbb{R}. \tag{2.14}$$

Furthermore, since $u(x,\gamma(x)) = 0$ and hence

$$u_x(x,\gamma(x)) + u_y(x,\gamma(x))\gamma'(x) = 0, \quad x \in \mathbb{R},$$

we obtain

$$g_*'(0) = \lim_{m\to\infty} u_x(x_m,\gamma(x_m)) = -\lim_{n\to\infty} u_y(x_m,\gamma(x_m))\gamma'(x_m) = 0.$$

Then we conclude that $g_* \equiv 0$. We claim that this will lead to a contradiction.

As in the proof of Lemma 2.1, we define

$$h_m(x) = \int_{-\infty}^{0} \frac{\partial u_m}{\partial x}\frac{\partial u_m}{\partial y}dy.$$

It is easy to see that $|h_m(x)| < C$ for some constant independent of both x and m. We can also derive

$$h_m'(x) = \frac{1}{2}(\frac{\partial u_m}{\partial x})^2(x,0) + F(u_m(x,0)) - \int_{-\infty}^{0}(\frac{\partial u_m}{\partial y})^2 dy, \quad x \in \mathbb{R}.$$

For any fix $R > 0$, in view of (2.10) and the boundedness of h_m, we have

$$\int_{-R}^{R}[\frac{1}{2}(\frac{\partial u_m}{\partial x})^2(x,0) + F(u_m(x,0))]]dx < C$$

for some constant C independent of m, R.

Letting m go to infinity, we obtain $2F(0)R \leq C$, which is a contradiction. The proof of the lemma is then complete. ■

Indeed, we conclude that the level curve must be of one of the following four possibilities:

(i) $\lim_{x\to\infty}\gamma'(x) = +\infty$, $\lim_{x\to-\infty}\gamma'(x) = -\infty$;

(ii) $\lim_{x\to\infty}\gamma'(x) = +\infty$, $\lim_{x\to-\infty}\gamma'(x) = +\infty$;

(iii) $\lim_{x\to\infty}\gamma'(x) = -\infty$, $\lim_{x\to-\infty}\gamma'(x) = -\infty$;

(iv) $\lim_{x\to\infty}\gamma'(x) = -\infty$, $\lim_{x\to-\infty}\gamma'(x) = +\infty$.

Moreover, it can also be concluded by the arguments above that the profile of u along the level curve must be approximately the one dimensional transition layer $g(x)$ or $g(-x)$. To be more precise, we define

$$u_s(x,y) := u(x+s, y+\gamma(s)), \quad (x,y) \in \mathbb{R}^2.$$

The following lemma holds.

Lemma 2.3 *The translated solution $u_s(x,y)$ converges in $C^3_{loc}(\mathbb{R}^2)$ to either $g(x)$ or $g(-x)$ as $|s|$ tends to infinity.*

2.2 The exponential decay of u and a Hamiltonian identity

In this subsection, we shall show that solution u must decay exponentially to ± 1 as the distance from the the level set $y = \gamma(x)$ tends to infinity. The exponential decay of u will be used to prove a version of Hamiltonian identity for equation (1.4). This type of analysis was first carried out in [12] for the axially symmetric traveling wave solutions. Their arguments are slightly modified and presented here for the convenience of the reader.

Due to the double well potential condition of F, there exist two constants α^+, α^- such that $-1 < \alpha^- < 0 < \alpha^+ < 1$ and

$$F''(s) > \mu_0 > 0, \quad s \in [-1, \alpha^-] \cup [\alpha^+, 1].$$

for some constant $\mu_0 > 0$.

Define

$$\begin{aligned}
&\Omega^+ := \{(x,y) \in \mathbb{R}^2 : u(x,y) \geq \alpha^+\}, \quad \Omega^- := \{(x,y) \in \mathbb{R}^2 : u(x,y) \leq \alpha^-\}, \\
&\Omega^0 := \{(x,y) \in \mathbb{R}^2 : \alpha^- \leq u(x,y) \leq \alpha^+\}, \quad \Omega^0_y := \{x \in \mathbb{R} : \alpha^- \leq u(x,y) \leq \alpha^+\}, \\
&\gamma^\alpha := \{(x,y) \in \mathbb{R}^2 : y = \gamma^\alpha(x),\ u(x, \gamma^\alpha(x)) = \alpha\}, \quad \alpha \in (-1,1).
\end{aligned}$$

By Lemmas 2.2 and 2.3, it is easy to see that $meas(\Omega^0_y) < K < \infty$ for some constant K independent of y. Indeed, there exists a positive constant $Y_0 > 0$ and two C^3 functions $x = k_i(y), i = 1,2$ such that $\gamma^0 \cap \{(x,y) \in \mathbb{R}^2 : |y| > Y_0\}$ can be expressed as the graph of $k_i(y)$, i.e.,

$$\gamma^0 \cap \{(x,y) \in \mathbb{R}^2 : |y| > Y_0\} = \{(x,y) : x = k_i(y),\ |y| > Y_0,\ i = 1,2\}.$$

In Case (i), both k_1 and k_2 are defined for $Y > Y_0$, while in Case (iv), k_1 and k_2 are defined for $Y < -Y_0$. We may assume that $k_1(y) < k_2(y)$ in these two cases.

In Case (ii) and (iii), k_1 is defined for $y > Y_0$ and k_2 is defined for $y < -Y_0$.

In all cases, we have

$$|x - k_1(y)| < K, \text{ or } |x - k_2(y)| < K, \quad \forall x \in \Omega^0_y,\ |y| > Y_0. \tag{2.15}$$

Now we can state the exponential decay of u as the following lemma.

Lemma 2.4 *There exist constants C and $\nu > 0$ such that*

$$\begin{cases} |u^2-1| + |\nabla u| + |\nabla^2 u| \le Ce^{-\nu d(x,y)}, & |y| > Y_0 \\ |u^2-1| + |\nabla u| + |\nabla^2 u| \le Ce^{-\nu|x|}, & |y| \le Y_0. \end{cases} \tag{2.16}$$

where $d(x,y) := \min\{|x-k_1(y)|,\ |x-k_2(y)|\}$ for $|y| > Y_0$.

Proof. Let

$$w(x,y) = 1 \mp u(x,y) > 0,\ (x,y) \in \Omega^{\pm}.$$

Then, by the definition of μ_0 and $\Omega^{\pm}$, it is easy to see that

$$w_{xx} + w_y - \mu_0 w = (\frac{F'(\pm 1) - F'(u)}{\pm 1 - u} - \mu_0) \cdot w \ge 0, \quad (x,y) \in \Omega^{\pm}.$$

For any rectangular domain $D_R := \{(x,y) : |x| < \sqrt{\frac{\mu_0}{2}}R,\ |y| < R\}$, we consider the function

$$B(x,y) = e^{-\mu_0 R/2} \cosh(\frac{\mu_0}{2} y) \cosh(\sqrt{\frac{\mu_0}{2}} x), \quad (x,y) \in D_R.$$

Straight forward computations reveal that

$$B_{xx} + B_y - \mu_0 B < 0 \text{ in } D_R, \quad B \ge 1 \text{ on } \partial D_R.$$

Now for any $(x_0,y_0) \in \Omega^{\pm}$, let $R = cR(x_0,y_0)$, where $R(x_0,y_0)$ is the distance from (x_0,y_0) to Ω^0 and $c = c(\mu_0)$ is a constant so that D_R remains in $\Omega^{\pm}$. Now compare $w(x,y)$ with $B(x-x_0, y-y_0)$ in $D_R(x_0,y_0) := \{(x,y) : (x-x_0, y-y_0) \in D_R\}$. Then the maximum principle implies that

$$w(x,y) \le B(x-x_0, y-y_0), \quad (x,y) \in D_R(x_0,y_0).$$

In particular, we have $w(x_0,y_0) \le B(0,0) = e^{-\mu_0 R/2}$. In view of (2.13), (2.15) and the definition of k_1, k_2, we know that, for $R(x,y) \ge K$, there exists some constant $\mu_3 \in (0,1)$ such that $R(x,y) \ge \mu_3 d(x,y)$ when $|y| > Y_0$ and $R(x,y) \ge \mu_3|x|$ when $|y| \le Y_0$.

Hence we derive

$$|u^2-1| \le C_0 e^{-\nu d(x,y)}, \quad |y| > Y_0; \quad |u^2-1| \le C_0 e^{-\nu|x|}, \quad |y| \le Y_0.$$

for $\nu = c\mu_0/2$ and some constant $C_0 > 0$. Then (2.16) follows from the standard estimates for parabolic equations. ■

With the exponential decay of u, we can define

$$\rho(y) = \rho(y;u) := \int_{\mathbb{R}} [\frac{1}{2}u_x^2 + F(u)]dx, \quad y \in \mathbb{R}.$$

The following Hamiltonian identity holds.

Lemma 2.5 *For any $y_0, y \in \mathbb{R}$, there holds the following Hamiltonian identity*

$$\rho(y) - \rho(y_0) = \int_{y_0}^{y} \int_{\mathbb{R}} u_y^2 dxdy. \tag{2.17}$$

2.3 Only Case (i) is valid

Using the exponential decay (2.16), the Hamiltonian identity (2.17) and Lemma 2.3, we can exclude the Cases (ii)-(iv) in subsection 2.1.

Lemma 2.6 *Assume that u is a solution to* (1.4), (1.5) *and* (1.6), *and the graph of $y = \gamma(x)$ is the 0-level set of u. Then*

$$\lim_{x\to\infty} \gamma'(x) = \infty, \quad \lim_{x\to-\infty} \gamma'(x) = -\infty. \tag{2.18}$$

Proof. In Case (ii), using the exponential decay (2.16) and Lemma 2.3, we can compute straight forwardly

$$\lim_{y\to\infty} \rho(y) = \lim_{s\to\infty} \rho(0; u_s) = \int_{-\infty}^{\infty} [\frac{1}{2}(|g'|^2(x) + F(g(x))]dx = \mathbf{e}$$

and

$$\lim_{y_0\to-\infty} \rho(y_0) = \lim_{s\to-\infty} \rho(0; u_s) = \int_{-\infty}^{\infty} [\frac{1}{2}(|g'|^2(x) + F(g(x))]dx = \mathbf{e}.$$

Then the Hamiltonian identity (2.17) leads to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u_y^2 dxdy = 0.$$

This is a contradiction. Case (iii) can be excluded similarly.

In Case (iv), we have

$$\lim_{y\to\infty} \rho(y) = 0$$

and

$$\begin{aligned}\lim_{y_0\to-\infty} \rho(y_0) &= \lim_{y_0\to-\infty} \int_{-\infty}^{0} [\frac{1}{2}u_x^2 + F(u)]dx \\ &+ \lim_{y_0\to-\infty} \int_{0}^{\infty} [\frac{1}{2}u_x^2 + F(u)]dx \\ &= 2\int_{-\infty}^{\infty} [\frac{1}{2}(|g'|^2(x) + F(g(x))]dx = 2\mathbf{e}.\end{aligned}$$

This leads to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u_y|^2 dxdy = -2\mathbf{e} < 0.$$

This is a contradiction, and the lemma is proven. ■

2.4 The level set curve is asymptotically hyperbolic cosine

In this subsection, we shall show that the 0-level set $y = \gamma(x)$ of u is asymptotically hyperbolic cosine. It is more convenient to write the level set as the graph of functions $x = k_1(y)$, $x = k_2(y)$ for $y > Y_0$ and show that they are logarithmic. In the previous subsection, we have already derived properties for $y = \gamma(x)$ which can be rewritten for $x = k_i(y)$ as follows

$$\begin{cases} k_1'(y) < 0, \quad k_2'(y) > 0, \quad \text{for } y > y_0 \\ \lim_{y\to\infty} k_1(y) = -\infty, \quad \lim_{y\to\infty} k_2(y) = \infty \\ \lim_{y\to\infty} k_1'(y) = \lim_{y\to\infty} k_2'(y) = 0. \end{cases} \tag{2.19}$$

We shall prove the following asymptotical formulas for $k_i(y), i = 1, 2$.

Lemma 2.7 *There holds*

$$\begin{cases} k_1(y) = -\dfrac{1}{2\mu} ln(y) + C_1 + o(1), \quad as\ y \to \infty \\ k_2(y) = \dfrac{1}{2\mu} ln(y) + C_2 + o(1), \quad as\ y \to \infty \end{cases} \tag{2.20}$$

for some constants C_1, C_2, where $\mu = \sqrt{F''(1)} > 0$.

2.4.1 A standard profile with two transition layers

The proof of Lemma 2.7 follows the main ideas of [12] in the derivation of similar formula for axially symmetric traveling wave solutions. Instead of only dealing with one unknown function in [12], here we need to consider the coupled functions $k_i(y)$, $i = 1, 2$. We shall approximate $u(x, y)$ as functions of x by a family of standard profiles of two transition layers for y sufficiently large. Namely, for $l_1 < l_2$ and $2l = l_2 - l_1$ sufficiently large, we define a continuous and piecewise smooth function $\phi = \phi(l_1, l_2, x)$ so that it is the solution of one dimensional Allen-Cahn equation in three segments of $\mathbb{R}$:

$$\begin{cases} \phi'' - F'(\phi) = 0, \quad x \in (-\infty, l_1) \cup (l_1, l_2) \cup (l_2, \infty) \\ \phi(x) > 0, \quad x \in (l_1, l_2); \quad \phi(x) < 0, \quad x \in (-\infty, l_1) \cup (l_2, \infty) \\ \phi(l_1) = \phi(l_2) = 0, \quad \lim_{x\to\pm\infty} \phi(x) = -1 \end{cases} \tag{2.21}$$

Below we collect some basic facts about $\phi = \phi(l_1, l_2, x)$ and related functions. Indeed, $\phi(l_1, l_2, x) = g(l_2 - x)$ for $x > l_2$ and $\phi(l_1, l_2, x) = g(x - l_1)$ for $x < l_1$. For $x \in (l_1, l_2)$, $\phi(l_1, l_2, x) = g(l, x - (l_1 + l_2)/2)$ where $g(l, x) = g(l, -x)$ can be solved explicitly by

$$g_x^2(l, x) = 2F(g(l, x)) - 2F(g(l, 0)), \quad x \in (-l, l)$$
$$\int_{g(l,x)}^{g(l,0)} \frac{ds}{\sqrt{2((F(s) - F(g(l, 0)))}} = x, \quad x \in (0, l)$$

where $0 < g(l, 0) < 1$.

Note that elementary computations can lead to $\lim_{l\to\infty} g(l,0) = 1$ and

$$l = \int_0^{g(l,0)} \frac{ds}{\sqrt{2(F(s) - F(g(l,0))}} = -\frac{\ln(1 - g(l,0))}{\mu} + A_1 + o(1)$$

as $l \to \infty$, where A_1 is a constant depending only on F. It is also easy to see that $g(l,x)$ is the minimizer of

$$\mathbf{E}_l(v) := \int_{-l}^{l} [\frac{1}{2}|v'|^2 + F(v)]dx$$

in $\mathcal{H}_l := \{v \in H_0^1([-l,l]) : 0 \leq v \leq 1,\ v(-l) = v(l) = 0\}$ when l is sufficiently large.

If we denote

$$E(l) := \mathbf{E}_l(g(l,\cdot)) = \mathbf{E}(\phi(-l,l,\cdot)) - \mathbf{e},$$

then $E(l) = \mathbf{e} + o(1)$ and

$$E_l := \frac{\partial E(l)}{\partial l} = |g'(0)|^2 - g_x^2(l,l) = 2F(g(l,0)) = 2\mathbf{e}Ae^{-2\mu l + o(1)} \tag{2.22}$$

where A is a positive constant depending only on F (see [12]).

For a piecewise continuous function $\psi(x)$ with possible jump discontinuities at $x = l_1, l_2$, we define

$$\hat{\psi} = \psi(l_1+) - \phi(l_1-), \quad \check{\psi} = \psi(l_2+) - \phi(l_2-),$$
$$\tilde{\psi} = \frac{1}{2}(\psi(l_1-) + \phi(l_1+)), \quad \bar{\psi} = \frac{1}{2}(\psi(l_2+) + \phi(l_2-)).$$

Note that $E_l = -\hat{\phi}^2_{l_1} = -\hat{\phi}^2_x = \check{\phi}^2_{l_2} = \check{\phi}^2_x$.

We also use the norm and inner product of $L^2(\mathbb{R})$, i.e.,

$$\langle\psi_1,\psi_2\rangle := \int_{\mathbb{R}} \psi_1\psi_2 dx, \quad ||\psi||^2 := \langle\psi,\psi\rangle.$$

Now we state the following lemma, which follows from Lemma 6.1 in [12].

Lemma 2.8 *For $l = (l_2 - l_1)/2 > 0$, $\phi(l_1,l_2,x)$ is smooth except at $x = l_1, l_2$ and*

$$\phi_{l_1} \leq 0, \quad \phi_{l_2} \geq 0, \quad ||\phi_{l_i}||^2 = E(l) + o(1) = \mathbf{e} + o(1), \quad i = 1,2.$$

Furthermore, there exists a constant $C > 0$ such that $\forall l > 1$

$$\sum_{i=1,2} ||\phi_{l_i}||_{L^1(\mathbb{R})} + \sum_{i,j=1,2} ||\phi_{l_il_j}||_{L^1(\mathbb{R})} + \sum_{i,j,k=1,2} ||\phi_{l_il_jl_k}||_{L^1(\mathbb{R})} \leq C;$$
$$\sum_{i=1,2} ||\phi_{l_i}|| + \sum_{i,j=1,2} ||\phi_{l_il_j}|| + \sum_{i,j,k=1,2} ||\phi_{l_il_jl_k}|| \leq C;$$
$$\sum_{i=1,2} (|\hat{\phi}_{l_i}| + |\check{\phi}_{l_i}|) + \sum_{i,j,k=1,2} (|\hat{\phi}_{l_il_j}| + |\check{\phi}_{l_il_j}|) + \sum_{i=1,2} (|\hat{\phi}_{xl_i}| + |\check{\phi}_{xl_i}|) \leq C \cdot E_l;$$
$$| < \phi_{l_1},\phi_{l_2} > | + |E_{ll}| + |E_{lll}|| \leq C \cdot E_l.$$

2.4.2 Derivation of ordinary differential equations for l_i, $i = 1, 2$

Now, for $y > Y_1$ sufficiently large, we can choose a unique pair $l_1(y) < l_2(y)$ so that

$$\|u(\cdot, y) - \phi(l_1(y), l_2(y), \cdot)\|_{L^2(\mathbb{R})} = \inf_{l_1<l_2}\{\|u(\cdot, y) - \phi(l_1, l_2, \cdot)\|\}. \tag{2.23}$$

As we shall show, the asymptotical behavior of $u(x, y)$ near $y = \infty$ can be accurately described by the dynamics of $l_i(y), i = 1, 2$. (See, e.g., [12] Section 6.1 for an intuitive explanation for the case $l_1 = -l_2$ by using invariant manifold and center manifold terminology.)

Let

$$v(x, y) = u(x, y) - \phi(l_1(y), l_2(y), x), \quad x \in \mathbb{R}, \quad y \geq Y_1.$$

In view of Lemma 2.3, Lemma 2.4 and Lemma 2.6, we see that

$$k_1(y) - l_1(y) \to 0, \quad k_2(y) - l_2(y) \to 0, \quad \|v(\cdot, y)\| \to 0, \quad \text{as } y \to \infty. \tag{2.24}$$

Moreover, using the implicit function theorem, one can see that for $y > Y_1$ sufficiently large, the functions $l_i(y), i = 1, 2$ are smooth and satisfy

$$l_1'(y) < 0, \;\; l_2'(y) > 0, \quad \lim_{y\to\infty} l_i'(y) = 0, \;\; i = 1, 2. \tag{2.25}$$

(See, e.g., [12] Lemma 6.2 for a similar statement for the case $l_1 = -l_2$.)

It is also obvious that

$$\lim_{y\to\infty} \|(|v| + |\nabla v|)\|_{L^\infty(\mathbb{R})} = 0. \tag{2.26}$$

From (2.23) it is easy to see that

$$\langle v(\cdot, y), \phi_{l_i}(l_1(y), l_2(y), \cdot)\rangle = 0, \;\; i = 1, 2, \quad y \geq Y_1. \tag{2.27}$$

Differentiating the above identities with respect to y and dropping the variables of functions for the simplicity of notation, we obtain

$$\langle v_y, \phi_{l_i}\rangle + \sum_{j=1,2}\langle v, \phi_{l_il_j}\rangle l_j' - v\hat{\phi}_{l_i} l_1' + v\check{\phi}_{l_i} l_2' = 0, \quad i = 1, 2. \tag{2.28}$$

Differentiating (2.28) for $i = 1$ with respect to y, we have

$$\begin{aligned}
&\langle v_{yy}, \phi_{l_1}\rangle + \langle v_y, \phi_{l_1l_1}\rangle l_1' + \langle v_y, \phi_{l_1l_2}\rangle l_2' - v_y\hat{\phi}_{l_1} l_1' + v_y\check{\phi}_{l_1} l_2' \\
&+ \langle v_y, \phi_{l_1l_1}\rangle l_1' + [\langle v, \phi_{l_1l_1l_1}\rangle l_1' + \langle v, \phi_{l_1l_1l_2}\rangle l_2' - v\hat{\phi}_{l_1l_1} l_1' + v\check{\phi}_{l_1l_1} l_2']l_1' + \langle v, \phi_{l_1l_1}\rangle l_1'' \\
&+ \langle v_y, \phi_{l_1l_2}\rangle l_2' + [\langle v, \phi_{l_1l_1l_2}\rangle l_1' + \langle v, \phi_{l_1l_2l_2}\rangle l_2' - v\hat{\phi}_{l_1l_2} l_1' + v\check{\phi}_{l_1l_2} l_2']l_2' + \langle v, \phi_{l_1l_2}\rangle l_2'' \\
&- [v_y\hat{\phi}_{l_1} + v\hat{\phi}_{l_1l_1} \cdot l_1' + v\hat{\phi}_{l_1l_2} \cdot l_2'] \cdot l_1' - v\hat{\phi}_{l_1} l_1'' \\
&+ [v_y\check{\phi}_{l_1} + v\check{\phi}_{l_1l_1} \cdot l_1' + v\check{\phi}_{l_1l_2} \cdot l_2']l_2' + v\check{\phi}_{l_2l_2} l_2'' = 0.
\end{aligned}$$

By Lemma 2.8, this leads to

$$|\langle v_{yy}, \phi_{l_1}\rangle| + |\langle v_y, \phi_{l_1}\rangle| = o(1)(|l_1'| + |l_2'| + |l_1''| + |l_2''|), \quad \text{as } y \to \infty. \tag{2.29}$$

Similar computations can also be done for $i = 2$.

Now, using equation (1.4) we derive

$$v_{xx} + v_y - ((F'(v+\phi) - F'(\phi)) + \phi_y = 0, \quad (x,y) \in \mathbb{R}^2 \setminus \Gamma, y > Y_1, \tag{2.30}$$

where

$$\phi_y = \phi_{l_1} \cdot l_1' + \phi_{l_2} \cdot l_2'.$$

Multiplying (2.30) by ϕ_{l_1} and integrating over $\mathbb{R}$, we obtain

$$l_1' \|\phi_{l_1}\|^2 + \langle \phi_{l_1}, \phi_{l_2} \rangle l_2' = J_{1,1} + J_{1,2} - J_{1,3}$$

where

$$\begin{aligned} J_{1,1} &= \langle F''(\phi)v - v_{xx}, \phi_{l_1} \rangle; \\ J_{1,2} &= \langle F'(v+\phi) - F'(\phi) - F''(\phi)v, \phi_{l_1} \rangle; \\ J_{1,3} &= \langle v_y, \phi_{l_1} \rangle. \end{aligned}$$

Using $E_l = -\hat{\phi}^2_{l_1} = \check{\phi}^2_{l_1}$, it can be computed that

$$J_{1,1} = v_x \hat{\phi}_{l_1} - v \hat{\phi}_{l_1 x} + v_x \check{\phi}_{l_1} - v \check{\phi}_{l_1 x} = -E_l(1 + O(|v_x| + |v|)) = E_l(1 + o(1)). \tag{2.31}$$

Here we have used (2.26), the fact

$$\hat{v_x} = -\hat{\phi_x} = \hat{\phi_{l_1}}, \quad \check{v_x} = -\check{\phi_x} = -\check{\phi_{l_1}}$$

and

$$\begin{aligned} v_x \hat{\phi}_{l_1} &= \hat{v_x} \tilde{\phi_{l_1}} + \tilde{v_x} \hat{\phi_{l_1}} = \frac{1}{2} \hat{\phi}^2_{l_1} + \tilde{v_x} \hat{\phi_{l_1}}, \\ v_x \check{\phi}_{l_1} &= \check{v_x} \bar{\phi_{l_1}} + \bar{v_x} \check{\phi_{l_1}} = -\frac{1}{2} \check{\phi}^2_{l_1} + \bar{v_x} \check{\phi_{l_1}}. \end{aligned}$$

On the other hand, we have

$$J_{1,2} = O(1)\langle v^2, \phi_{l_1} \rangle. \tag{2.32}$$

In view of (2.29), (2.31) and (2.32), we obtain

$$l_1' + o(1) l_2' = -\frac{E_l}{\mathbf{e}}(1 + o(1)) + o(1)(|l_1'| + |l_2'|) + O(1)\langle v^2, \phi_{l_1} \rangle. \tag{2.33}$$

Similarly we can obtain

$$l_2' + o(1) l_1' = -\frac{E_l}{\mathbf{e}}(1 + o(1)) + o(1)(|l_1'| + |l_2'|) + O(1)\langle v^2, \phi_{l_2} \rangle. \tag{2.34}$$

Next we shall estimate $\|v\|$.

2.4.3 Estimate of $||v||$.

Multiplying (2.30) by v and integrating on $\mathbb{R}$, we get

$$0 =< v_y, v > + < v_{xx} + \phi_{xx} - F'(v + \phi), v > .$$

It is easy to see that

$$< v_y, v >= \frac{1}{2}\frac{d\|v\|^2}{dy}, \phi_{xx} = F'(\phi),$$
$$< v_{xx}, v >= v(\hat{\phi}_x + \check{\phi}_x) - \|v_x\|^2.$$

Due to the non-degeneracy and stability property of g in $\mathbb{R}$, there holds

$$\|\psi_x\|^2 + \langle F''(\phi)\psi, \psi\rangle \geq 2\nu\|\psi\|^2 + |\tilde{\psi}|^2 + |\bar{\psi}|^2, \quad \forall\psi \perp \phi_{l_i},\ i = 1, 2 \tag{2.35}$$

for some constant $\nu > 0$ when $2l = l_2 - l_1$ is sufficiently large. (See also [12] Lemma 6.3.) By the mean value theorem, we get

$$< F''(\phi)v - F'(v + \phi) + F'(\phi), v >= O(v)\|v\|^2 = o(1)\|v\|^2.$$

Hence, we derive

$$\frac{1}{2}\frac{d\|v\|^2}{dy} - \nu\|v\|^2 \geq -ME_l^2 \tag{2.36}$$

for some positive constant M sufficiently large. Using initial condition $\|v\| \to 0$, we derive

$$\|v\|^2 \leq M_1 E_l^2$$

for some positive constant M_1.

2.4.4 Derivation of asymptotic formula for $2l = l_2 - l_1$.

Now we can write (2.33), (2.34) as

$$\begin{cases} l_1' = -Ae^{-2\mu l(y)}(1 + o(1)) + o(1)(l_2' - l_1'), \\ l_2' = Ae^{-2\mu l(y)}(1 + o(1)) + o(1)(l_2' - l_1'). \end{cases} \tag{2.37}$$

From (2.37) we can deduce

$$(1 + o(1))l' = (A + o(1))e^{-2\mu l(y)}. \tag{2.38}$$

Hence we derive

$$l(y) = \frac{1}{2\mu}\ln(y) + \frac{1}{2\mu}\ln(2\mu A) + o(1);$$
$$l'(y) = \frac{1 + o(1)}{2\mu y}, \quad E_l = \frac{1 + o(1)}{2\mu A y}, \quad l''(y) = \frac{o(1)}{2\mu y};$$
$$0 > l_1'(y) \geq -2l'(y) = -\frac{1 + o(1)}{\mu y}, \quad 0 < l_2'(y) \leq 2l'(y) = \frac{1 + o(1)}{\mu y}.$$

Thus, we obtain an explicit estimate for $\|v\|$ in term of y

$$\|v\|^2 \le \frac{O(1)}{y^2}. \tag{2.39}$$

2.4.5 Derivation of asymptotical formulas for $l_i, i = 1, 2$.

By (2.37) and asymptotic formula for l, it is easy to get

$$\begin{cases} l_1(y) = -\dfrac{1}{2\mu}\ln(y) + C_1 + o(1), \\ l_2(y) = \dfrac{1}{2\mu}\ln(y) + C_2 + o(1) \end{cases} \tag{2.40}$$

for some constants C_1,C_2. Lemma 2.7 then follows directly.

2.5 The moving plane procedure

In this subsection, we shall use the moving plane method to finish the proof of Theorem 1.1. Due to the fact that the asymptotical behavior of u is not homogeneous near infinity, in particular, there is a transition layer along Γ, the classic moving plane method has to be carefully modified. Indeed, we have to use the exact asymptotical formulas of the 0-level sets $x = k_i(y), i = 1, 2$ near infinity as well the asymptotical behavior of u along these curves.

Define $u_\lambda(x, y) := u(2\lambda - x, y)$ and $w_\lambda := u_\lambda - u$ in $D_\lambda := \{(x, y) : x \ge \lambda,\ y \in \mathbb{R}\}$.

Lemma 2.9 *When λ is sufficiently large, there holds $w_\lambda > 0$ in D_λ.*

Proof. When $\lambda > \lambda_0$ is sufficiently large, by Lemma 2.7 we know that

$$k_1^\lambda(y) := 2\lambda - k_1(y) \ge k_2(y), \quad \forall y \ge Y_1.$$

By Lemma 2.3 and Lemma 2.4, we see that there exist constants $K > 0, Y_2 > Y_1$ and λ_1 sufficiently large such that when $\lambda > \lambda_1$, there hold $w_\lambda > 0$ in $D_{K,Y_2,\lambda} = \{(x, y) \in D_\lambda : x < k_1^\lambda(y) + K,\ y \ge Y_2\}$ and $u < \alpha^-$ in $D^c_{K,Y_2,\lambda} := \{(x, y) \in D_\lambda : x > k_1^\lambda(y) + K,\ y \ge Y_2$ or $\forall x \ge \lambda,\ y \le Y_2\}$. Note that $F''(s) > \mu_0 > 0$ for $s \in (-1, \alpha^-]$ by the definition of α^-.

We claim that $w_\lambda \ge 0$ in D_λ for $\lambda > \lambda_1$. If it is not true, there exists a sequence of points $\{(x_m, y_m)\}_{m=1}^\infty \in D^c_{K,Y_2,\lambda}$ such that

$$\lim_{m\to\infty} w_\lambda(x_m, y_m) = \lim_{m\to\infty} (u_\lambda(x_m, y_m) - u(x_m, y_m)) = \inf_{D^c_{K,Y_2,\lambda}} w_\lambda(x, y) < 0.$$

It can be seen that $u_\lambda(x_m, y_m) < \alpha^-$ when m is large enough. Then we can follow the standard translating arguments to obtain a contradiction. Define $w_\lambda^m(x, y) := w_\lambda(x + x_m, y + y_m)$ in $D^c_{K,Y_2,\lambda} - (x_m, y_m)$. Then w_λ^m converges to $w_\lambda^\infty(x, y)$ in $C^3_{loc}(D^\infty)$ for some piecewise

Lipschitz domain D^∞ in $\mathbb{R}^2$ which contains a small ball centered at the origin. Furthermore, w_λ^∞ attains its negative minimum at the origin and satisfies a linearized equation

$$w_{xx} + w_y - F''(\xi(x,y))w = 0, \quad (x,y) \in D^\infty \tag{2.41}$$

where $\xi(x,y) = su(x,y) + (1-s)u_\lambda(x,y)$ for some $s \in (0,1)$ and $F''(\xi(0,0)) > \mu_0 > 0$. This is a contradiction, which leads to the claim. Then the lemma follows from the strong maximum principle (or the Harnack inequality) applied to a parabolic equation similar to (2.41) which is satisfied by w_λ. ■

Now we define

$$\Lambda = \inf\{\lambda : u_\lambda(x,y) > u(x,y), (x,y) \in D_\lambda\}.$$

Lemma 2.10 *There holds*

$$\Lambda = (C_1 + C_2)/2$$

where C_1, C_2 are as in Lemma 2.7.

Proof. We shall prove this lemma by contradiction. Suppose the lemma does not hold. By Lemma 2.3 and Lemma 2.7, we can easily see that $\Lambda > (C_1 + C_2)/2$ and $w_\Lambda > 0, \forall (x,y) \in D_\Lambda$. Then there exists a sequence of numbers $\{\lambda_m\}$ such that $\lambda_m < \Lambda$, and $\lim_{m\to\infty} \lambda_m = \Lambda$ and the infimum of w_{λ_m} in D_{λ_m} is negative. Using Lemma 2.3, Lemma exponetial1, Lemma 2.7 and the translating arguments in the proof of Lemma 2.9, we can show that the infimum of w_{λ_m} in D_{λ_m} is achieved at a point (x_m, y_m), i.e.,

$$w_{\lambda_m}(x_m, y_m) = \inf_{D_{\lambda_m}} w_{\lambda_m} < 0. \tag{2.42}$$

Since w_{λ_m} satisfies a parabolic equation similar to (2.41) with $\xi(x_m, y_m) = su(x_m, y_m) + (1-s)u_{\lambda_m}(x_m, y_m)$ for some $s \in (0,1)$, by the strong maximum principle we know that $u(x_m, y_m) > \alpha^-$ and hence $y_m > -K_1$ and $x_m - k_1(y_m) < K$ if $y_m > Y_1$ for some constant $K, K_1 > 0$ independent of m. By Lemma 2.3, Lemma 2.7 and the assumption $\Lambda > (C_1 + C_2)/2$, we know $y_m < K_2$ for some constant K_2 independent of m. Therefore there exists a subsequence of $\{m\}$ (still denoted by the same) such that (x_m, y_m) converges to $(x_0, y_0) \in D_\Lambda$ and w_{λ_m} converges to w_Λ in $C^3_{loc}(D_\Lambda)$ as well as in $C^3(B_1(x_0,y_0) \cap \bar{D_\Lambda})$. It is easy to see that $\frac{\partial}{\partial x} w_\Lambda(x_0, y_0) = 0$. Furthermore, w_Λ satisfies a parabolic equation similar to (2.41) in D_Λ. Then by the Hopf Lemma, we have $\frac{\partial}{\partial x} w_\Lambda(x_0, y_0) < 0$. This is a contradiction, which proves the lemma. ■

We note that $u_\Lambda \geq u$ in D_Λ and $u_x(\lambda, y) = -\frac{1}{2}\frac{\partial}{\partial x} w_\lambda(\lambda, y) > 0, \forall y \in \mathbb{R}$ when $\lambda > \Lambda$. Similarly, we can use the moving plane method from the left, i.e., repeating the above procedure for w_λ in $D_\lambda^- := \{(x,y) : x < \lambda\}$, and conclude $u_\Lambda \geq u$ in D_Λ^-. Therefore, Theorem 1.1 is proven.

3 Classification of the entire solutions for the unbalanced Allen-Cahn equation in $\mathbb{R}$

In this section, we assume that the double well potential F in the Allen-Cahn equation (1.1) is unbalanced, i.e., F satisfies (1.2) and $F(1) > F(-1) = 0$. In this case, one dimensional traveling wave solution g to (1.7) exists for a unique $c_0 > 0$ which only depends on F, and g is unique up to translation. It is easy to see that there is an entire solution of (1.4) from a rotation of the trivial extension of g to two dimensional plane. Indeed, $u(x, y) = g(c_0 y \pm x)$ is such a solution. In addition to the one dimensional traveling wave solutions, the so called V-shaped two dimensional entire solutions are shown to exist in [26], [37]. These solutions are monotone in y and even with respect to x after a proper translation. The 0-level set of such solutions are asymptotically two straight rays forming a shape of V.

To prove the classification result Theorem 1.2 for unbalanced potentials, we shall first show that 0-level set of u must be global Lipchitz, then follow the argument of [27] for the classification of two dimensional traveling wave solutions to the Allen-Cahn equation.

3.1 Lipchitz 0-level set

Lemma 3.1 *Assume that u is a solution to* (1.4)*,* (1.5) *and* (1.6)*, and the graph of $y = k(x)$ is the* 0*-level set of u. Then $k(x) \in C^3(\mathbb{R})$ and $|k'(x)| \leq C, x \in \mathbb{R}$ for some constant $C > 0$.*

Proof. By (1.5) and (1.6), the 0-level set $y = k(x)$ is well defined. u is a $C^3(\mathbb{R}^2)$ function from standard parabolic estimate, so $k(x)$ is in $C^3(\mathbb{R})$ by the implicit function theorem.

We shall prove the global Lipschitz property by contradiction. Suppose that there exists a sequence $\{x_m\}$ such that $k'(x_m) \to \infty$ as m tends to infinity. Since $u_x(x, k(x)) + u_y(x, k(x))k'(x) = 0, \forall x \in \mathbb{R}$ and ∇u is bounded in $\mathbb{R}^2$, we derive $u_y(x_m, k(x_m)) \to 0$ as m goes to infinity. We shall investigate the translation of u along $(x_m, k(x_m))$. Define $u^m(x, y) := u(x + x_m, y + y_m)$. Since u is bounded in $C^{3,\beta}(\mathbb{R}^2)$ for some $\beta \in (0, 1)$, it is easy to see that u^m (up to a subsequence) converges to u^* in $C^3_{loc}(\mathbb{R}^2)$, and u^* satisfies (1.4). Hence $u^*_y(x, y)$ satisfies the linearized equation

$$w_{xx} + w_y - F''(u^*)w = 0, \quad (x, y) \in \mathbb{R}^2. \tag{3.43}$$

By (1.5), we know that $u^*_y(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^2$. Since

$$u^*_y(0, 0) = \lim_{m\to\infty} u_y(x_m, k(x_m)) = 0,$$

by the strong maximum principle for parabolic equations we obtain $u^*_y \equiv 0$ in $\mathbb{R}^2$. Therefore, $u^*(x, y) = u^*(x)$ satisfies the one dimensional stationary Allen-Cahn equation (2.14) with $|u^*(x)| \leq 1, x \in \mathbb{R}$ and $u^*(0) = 0$. Then we have $u^*(x) = g_\alpha(x \pm K_\alpha), x \in \mathbb{R}$, where g_α is a periodic solution of the one dimensional Allen-Cahn equation

$$\begin{cases} g''_\alpha(x) - F'(g_\alpha(x)) = 0, \quad x \in \mathbb{R}, \\ g'_\alpha(0) = 0, \quad g_\alpha(0) = \alpha, \end{cases} \tag{3.44}$$

with $\alpha = \max_{\mathbb{R}} u^*(x) \geq 0$ and K_α is the smallest positive zero of g_α if $\alpha > 0$. However, the periodicity of u^* obviously contradicts with monotonicity condition (1.5) of u. The lemma is proved. ■

3.2 Proof of Theorem 1.2

We will prove a classification theorem for entire solutions with Lipchitz level sets here, then Theorem 1.2 will follow naturally from Lemma 3.1. We state the classification theorem as follows:

Theorem 3.1 *Let u be a bounded non-constant solution of* (1.4) *that satisfies*

$$\begin{cases} \lim\limits_{A\to+\infty} \inf\limits_{y\geq A+\phi(x)} u > 0, \\ \lim\limits_{A\to-\infty} \sup\limits_{y\leq A+\phi(x)} u < 0 \end{cases} \tag{3.45}$$

for some globly Lipchitz continous function ϕ. Then u is either planar, i.e. $u(x,y) = g(c_0 y \pm x)$, or the unique V-shaped solution up to translation.

To prove Theorem 3.1, we need the following comparison theorem and proposition.

Theorem 3.2 *(Comparison principle in $\mathbb{R}^2$) Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the Lipchitz continuous function in* (3.45) *and set*

$$\begin{cases} \Omega^+(y_0) = \{y > y_0 + \phi(x)\}, \\ \Omega^+(y_0) = \{y > y_0 + \phi(x)\}, \\ \Gamma(y_0) = \{y = y_0 + \phi(x)\}. \end{cases}$$

Let $\bar{u}$ and $\underline{u}$ be two Lipchitz-continuous functions, respectively super- and subsolution for (1.4), (3.45)*, namely:*

$$\begin{cases} S(\bar{u}) \leq 0 \text{ in } \mathbb{R}^2 \text{ and } \lim\limits_{y_0\to+\infty} \inf\limits_{\Omega^+(y_0)} \bar{u} = 1, \\ S(\underline{u}) \geq 0 \text{ in } \mathbb{R}^2 \text{ and } \lim\limits_{y_0\to-\infty} \sup\limits_{\Omega^-(y_0)} \underline{u} = -1, \end{cases}$$

the inequalities $S(\bar{u}) \leq 0 \leq S(\underline{u})$ holding in the distribution sense. For any $t \in \mathbb{R}$, set $w^t(x,y) = w(x, y+t)$. Then the set $I = \{t \in \mathbb{R}, \forall s \geq t, \bar{u}^s \geq \underline{u} \text{ in } \mathbb{R}^2\}$ is not empty. Let $t^* := \inf I$. If $t^* > -\infty$, we have $\bar{u}^{t^*} \geq \underline{u}$ in $\mathbb{R}^2$ and $\inf_{\Gamma(y_0)}(\bar{u}^{t^*} - \underline{u}) = 0$ for any $y_0 \in \mathbb{R}$.

Proposition 3.1 *Let v be a bounded non-constant solution of* (1.4) *and* (3.45)*. Then $-1 < v < 1$ in $\mathbb{R}^2$ and each level set $\{v = \lambda\}$ is a globally Lipchitz function ϕ_λ whose Lipchitz norm is $\frac{1}{c_0}$. Moreover v is increasing in any unit direction (τ_x, τ_y) such that $\tau_y > \frac{1}{\sqrt{1+c_0^2}}$, and*

$$\begin{cases} \lim\limits_{A\to+\infty} \inf\limits_{y\geq A+\phi(x)} u(x,y) - 1 = 0, \\ \lim\limits_{A\to-\infty} \sup\limits_{y\leq A+\phi(x)} u(x,y) + 1 = 0. \end{cases}$$

Next we shall use Theorem 3.2 and Proposition 3.1 to prove Theorem 3.1 first, and leave Theorem 3.2 and Proposition 3.1 to be proven later.

Proof of Theorem 3.1. Let v be a bounded non-constant solution of (1.4) and (3.45). Then Proposition 3.1 applies. Let $u(x,y) = u(-x,y)$ be the unique V-shaped solution. The comparison principle Theorem 3.2 can then be applied to $\bar{u} = u$ and $\underline{u} = v$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $u(x, y + t_0) \geq v(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and

$$\inf_{y=B+\frac{1}{c_0}|x|} (u(x, y + t_0) - v(x, y)) = 0 \tag{3.46}$$

for all $B \in \mathbb{R}$. By the properties of u and v, we know

$$\begin{cases} u(x, B + \frac{1}{c_0}|x| + t_0) \to g(\frac{c_0}{\sqrt{1+c_0^2}}(B + t_0)) < 1, \\ v(x, B + \frac{1}{c_0}|x|) \to v_\pm(B), \end{cases} \tag{3.47}$$

as $x \to \pm\infty$. We note that $v_\pm(B)$ exist because v is non-decreasing in both directions $(\pm\frac{c_0}{\sqrt{1+c_0^2}}, \frac{1}{\sqrt{1+c_0^2}})$ and $v_\pm(B) \in (-1, 1]$. From now on, let us fix $B \in \mathbb{R}$, and define $v_\pm = v_\pm(B)$. According to the values of $v_\pm$, four cases may occur:

Case 1: $v_- = v_+ = 1$. It follows from (3.46) and (3.47) that $u(x_0, y_0 + t_0) = v(x_0, y_0)$ for some $(x_0, y_0) \in \mathbb{R}^2$. Since both functions u^{t_0} and v satisfy the same equation, the strong maximun principle yields $u_{t_0} \equiv v$. This is impossible due to (3.47).

Case 2: $-1 < v_- < 1$ and $v_+ = 1$. Choose any real number ρ_0 and call $w(x, y) = u(x + \rho_0, y + \frac{1}{c_0}\rho_0)$. By the comparison theorem, there exists a real number $t = t(\rho_0)$ such that $v(x, y) \leq w(x, y + t)$ for all $(x, y) \in \mathbb{R}^2$ and such that (3.46) holds, with w instead of u. Since $v(x, y) \not\equiv w(x, y + t)$ and because of the different asymptotic behaviors as $x \to \infty$, it follows that

$$v(x, B + \frac{1}{c_0}|x|) - w(x, B + \frac{1}{c_0}|x| + t) \to 0 \text{ as } x \to -\infty,$$

whence $g(\frac{c_0}{\sqrt{1+c_0^2}}(B + t)) = v_- = g(\frac{c_0}{\sqrt{1+c_0^2}}(B + t_0))$, i.e. $t = t_0$.

As a consequence, $t = t_0$ does not depend on ρ_0 and

$$v(x, y) \leq u(x + \rho_0, y + \frac{1}{c_0}\rho_0 + t_0) \tag{3.48}$$

for all $(x, y) \in \mathbb{R}^2$ and $\rho_0 \in \mathbb{R}$. Passing to the limit as $\rho_0 \to -\infty$ implies that $v(x, y) \leq g(-x\frac{1}{\sqrt{1+c_0^2}} + (y + t_0)\frac{c_0}{\sqrt{1+c_0^2}})$ for all $(x, y) \in \mathbb{R}^2$.

On the other hand, since v is non-decreasing in the direction $(-\frac{c_0}{\sqrt{1+c_0^2}}, \frac{1}{\sqrt{1+c_0^2}})$, we have

$$v(x, y) \leq \lim_{r_n \to -\infty} v(x + c_0 r_n, y - r_n) = V(x + c_0 y).$$

Because V satisfies $V'' + c_0 V' - F'(V) = 0, V(+\infty) = 1, V(-\infty) = -1$, we deduce that $V(x + c_0 y) = g(x + c_0 y + t_1)$ for some $t_1 \in \mathbb{R}$. We then redefine the coordinates $Y = x\cos\alpha + y\sin\alpha$ and $X = -x\sin\alpha + y\cos\alpha$, where $\tan\alpha = -\frac{1}{c_0}$, and denote $v(x, y)$ by $v(X, Y)$(with a slight abuse of notations), we remark that by (3.48), v satisfies

$$\begin{cases} \lim_{A\to+\infty} \sup_{Y>A} |v(X, Y) - 1| = 0, \\ \lim_{A\to-\infty} \sup_{Y<A} |v(X, Y) + 1| = 0. \end{cases}$$

Then by the virtue of Theorem 2 in [8], we get that $v(X, Y)$ only depends on Y, and then conclude that

$$v(x, y) = g(x + c_0 y + t_1), \quad \forall (x, y) \in \mathbb{R}^2.$$

Case 3: $v_- = 1, -1 < v_+ < 1$. The same argument as in Case 2 yields the existence of $t_2 \in \mathbb{R}$ such that $v(x, y) = g(-x + c_0 y + t_2)$ for all $(x, y) \in \mathbb{R}^2$.

Case 4: $-1 < v_\pm < 1$. It then follows that

$$\sup_{x\in\mathbb{R}} |\phi_\lambda(x) - \frac{1}{c_0}|x|| < \infty$$

for all $\lambda \in (0, 1)$. Hence $\phi_\lambda(x)/|x| \to \frac{1}{c_0}$. By Theorem 2 of [13], v is unique, and then equals (up to a shift) to the solution u. ∎

3.3 Proof of Theorem 3.2

Instead of the whole space $\mathbb{R}^2$, we consider first a subset $\Omega \subset \mathbb{R}^2$. Assume $-1 < a' < b' < 1$ such $f(u) = -F'(u)$ is nonincreasing in u for u in $[-1, a']$ or $[b', 1]$. We have the following lemmas.

Lemma 3.2 *Under the same setting of Theorem 3.2, let $\Omega = \Omega^-(y_1)$ for some $y_1 \in \mathbb{R}$ and assume that*

$$\begin{cases} \underline{u} \le a' \text{ in } \Omega^-(y_1) \text{ and } \lim\limits_{y\to-\infty} \sup\limits_{\Omega^-(y)} \underline{u} = -1, \\ \underline{u} \le \bar{u} \text{ on } \Gamma(y_1). \end{cases} \tag{3.49}$$

Let $I_1 = \{t \in \mathbb{R}^-, \forall s \in [t, 0], \bar{u}^s \ge \underline{u} \text{ on } \Gamma(y_1)\}$. We have $0 \in I_1$ and $\forall t \in I_1$, $\bar{u}^t \ge \underline{u}$ in Ω^-. Let $t^* = \inf I_1$. It is the case that $\bar{u}^{t^*} \ge \underline{u}$ in $\Omega^-(y_1)$. Furthermore, if $t^* \ne -\infty$, then $\inf_{\Gamma(y_1)}(\bar{u}^{t^*} - \underline{u}) = 0$.

Lemma 3.3 *Let $\Omega = \Omega^+(y_2)$ for some $y_2 \in \mathbb{R}$ and assume that*

$$\begin{cases} \bar{u} \ge b' \text{ in } \Omega^+(y_2) \text{ and } \lim\limits_{y\to\infty} \inf\limits_{\Omega^+(y)} \bar{u} = 1, \\ \bar{u} \ge \underline{u} \text{ on } \Gamma(y_2). \end{cases} \tag{3.50}$$

Let $I_2 = \{t \in \mathbb{R}^+, \forall s \in [t, 0], \bar{u} \ge \underline{u}^s \text{ on } \Gamma(y_2)\}$. We have $0 \in I_2$ and $\forall t \in I_2$, $\bar{u}^t \ge \underline{u}$ in Ω^+. Let $t^* = \sup I_2$. It is the case that $\bar{u} \ge \underline{u}^{t^*}$ in $\Omega^+(y_2)$. Furthermore, if $t^* \ne \infty$, then $\inf_{\Gamma(y_2)}(\bar{u}^{t^*} - \underline{u}) = 0$.

Proof. We only prove Lemma 3.3. Noticing that $\inf_{\Gamma(y_2)}(\bar{u}^{t^*} - \underline{u}) = 0$ by Lipchitz-continuity, the question reduces to proving $0 \in I_2$, or $\bar{u} \ge \underline{u}$ in $\Omega^+(y_2)$.

Let us set $\epsilon = \sup\{\underline{u} - \bar{u}\}$ and suppose that $\epsilon > 0$. We can then find a sequence of points (x_k, y_k) such that $\underline{u}(x_k, y_k) - \bar{u}(x_k, y_k) \to \epsilon$. For each index k, we define

$$\begin{aligned} &\Omega_k = \Omega - (x_k, y_k), \\ &\phi_k(x) = \phi(x + x_k) - y_k, \\ &\Gamma_k = \{(x, y) : y = \phi_k(x)\}, \\ &\bar{u}_k(x, y) = \bar{u}(x + x_k, y + y_k), \\ &\underline{u}_k(x, y) = \underline{u}(x + x_k, y + y_k). \end{aligned}$$

Then we have

$$\begin{cases} \bar{u}_k \geq b' \text{ in } \Omega_k \text{ and } \lim_{y\to\infty} \inf_{\Omega_k(y)} \bar{u}_k = 1, \\ \bar{u}_k \geq \underline{u}_k \text{ on } \Gamma_k. \end{cases} \tag{3.51}$$

By the Lipchitz-continuity of ϕ and Ascoli-Arzela theorem, up to a subsequence, ϕ_k converges uniformly on compact subsets of $\mathbb{R}$ to a Lipchitz-continuous function ϕ_∞. In this sense, we assume $\Omega_k \to \Omega_\infty$. Similarly, we can assume $\bar{u}_k \to \bar{u}_\infty$, $\underline{u}_k \to \underline{u}_\infty$ locally in Ω_∞ for two Lipchitz-continuous functions $\bar{u}_\infty, \underline{u}_\infty$. We can also assume $S(\bar{u}_k) \to S(\bar{u}_\infty) \leq 0$, $S(\underline{u}_k) \to S(\underline{u}_\infty) \geq 0$ in the distribution sense. Combining these results, we have

$$S(\bar{u}_\infty) - S(\underline{u}_\infty) \leq 0 \text{ in } \Omega_\infty,$$

i.e.

$$(\bar{u}_\infty - \underline{u}_\infty)_{xx} + (\bar{u}_\infty - \underline{u}_\infty)_y \leq f(\underline{u}_\infty) - f(\bar{u}_\infty).$$

We study the above inequality at $(0,0)$. Notice $(0,0) \in \Omega_\infty$ but $\notin \Gamma_\infty$, by the facts $\underline{u}(x_k, y_k) - \bar{u}(x_k, y_k) \to \epsilon > 0$ and $\bar{u} \geq \underline{u}$ on $\Gamma(y_2)$. So $(0,0)$ is an interior point of Ω_∞. $\bar{u}_\infty - \underline{u}_\infty$ achieves its absolute minimum at $(0,0)$, which gives

$$[(\bar{u}_\infty - \underline{u}_\infty)_{xx} + (\bar{u}_\infty - \underline{u}_\infty)_y]|_{(0,0)} \geq 0.$$

On the other hand, $\underline{u}_\infty(0,0) > \bar{u}_\infty(0,0) \geq b'$ and $f(u)$ is strictly decreasing on $[b', b]$, so

$$[f(\underline{u}_\infty) - f(\bar{u}_\infty)]_{(0,0)} < 0.$$

This is a contradiction. So $\epsilon \leq 0$, i.e. $\bar{u} \geq \underline{u}$ in $\Omega^+(y_2)$. ■

Proof of Theorem 3.2 By the asymptotic conditions of the theorem, there exist $y_1 \leq y_2 \in \mathbb{R}$ such that

$$\bar{u} \geq b' \text{ in } \Omega^+(y_2),$$
$$\underline{u} \leq a' \text{ in } \Omega^-(y_1).$$

Let $t_0 = y_2 - y_1$. For any $t \geq t_0$, we have $\bar{u}^t \geq b' > a' \geq \underline{u}$ on $\Gamma(y_1)$. From Lemmas 3.2 and 3.3, we get $\bar{u}^t > \underline{u}$ in $\mathbb{R}^2$. So the set I is not empty.

Define $t^* = \inf I$. It comes that $\bar{u}^{t^*} \geq \underline{u}$ in $\mathbb{R}^2$. We consider the case $t^* > -\infty$. Suppose that

$$\exists y_0 \in \mathbb{R}, \inf_{\Gamma(y_0)} (\bar{u}^{t^*} - \underline{u}) \geq \delta > 0.$$

On the other hand, there exist $y_1^* < y_0 < y_2^*$ such that

$$\begin{cases} \bar{u}^{t^*} \geq (1 + b')/2 \text{ in } \Omega^+(y_2^*), \\ \underline{u} \leq a' \text{ in } \Omega^-(y_1^*). \end{cases}$$

If $m := \inf_{\Omega^-(y_2^*) \cap \Omega^+(y_1^*)} (u^{t^*} - \underline{u}) = 0$, then there exists a sequence (x_k, y_k) in $\Omega^-(y_2^*) \cap \Omega^+(y_1^*)$ such that $u^{t^*}(x_k, y_k) - \underline{u}(x_k, y_k) \to 0$. By the same argument and notation of Lemma 3.3, we have $\bar{u}(x + x_k, y + y_k) \to \bar{u}_\infty$, $\underline{u}(x + x_k, y + y_k) \to \underline{u}_\infty$ locally for some functions $\bar{u}_\infty$, $\underline{u}_\infty$, which satisfy

$$\begin{cases} S(\bar{u}_\infty) \leq 0, \\ S(\underline{u}_\infty) \geq 0 \end{cases}$$

in distribution sense. Let $u_\infty = \bar{u}_\infty - \underline{u}_\infty$. Then

$$\begin{cases} Lu_\infty + du_\infty \leq 0 \text{ in } \mathbb{R}^2, \\ u \geq 0 \text{ in } \mathbb{R}^2, u(0,0) = 0, \end{cases}$$

where

$$d = \frac{f(\bar{u}_\infty) - f(\underline{u})_\infty}{u_\infty}$$

is a bounded function. The strong maximum principle implies $u_\infty \equiv 0$. Note that on $\Gamma(y_0)$, $\bar{u}^{t^*} - \underline{u} \geq \delta > 0$, so $u_k(0, \phi(x_k) + y_0 - y_k) \geq \delta > 0$. Note also that $\phi(x_k) - y_k$ is bounded, there exists a y' such that $u_\infty(0, y') \geq \delta$. This is impossible.
Consequently $m > 0$, that is to say:

$$\inf_{\Omega^-(y_2^*) \cap \Omega^-(y_1^*)} (\bar{u}^{t^*} - \underline{u}) > 0.$$

Since both $\bar{u}$ and $\underline{u}$ are Lipchitz-continuous, it is still true with $t^* - \eta$ replacing t^* for any $\eta \in [0, \eta_0]$, $\eta_0 > 0$ small enough. From our choice of y_2^*, we can also choose η_0 such that $\bar{u}^{t^*-\eta} \geq b'$ in $\Omega^+(y_2^*)$ for any $\eta \in [0, \eta_0]$. By Lemma 3.2 and 3.3, we deduce that $\bar{u}^{t^*-\eta} - \underline{u} \geq 0$ for any $\eta \in [0, \eta_0]$. This contradicts the definition of t^*. ■

Theorem 3.3 *Under the assumptions of Theorem 3.2, if u is a solution of* (1.4) *and* (3.45)*, then u is increasing in y.*

Proof. Let u be such a solution. By the standard parabolic estimates, u is C^1 in $\mathbb{R}^2$.

Applying Theorem 3.2, there exists a $t^* > -\infty$, such that for any fixed $y_0 \in \mathbb{R}$, $\inf_{\Gamma(y_0)}(u^{t^*} - u) = 0$. There exists then a sequence $(x_k, y_k) \in \Gamma(y_0)$ such that $u^{t^*}(x_k, y_k) - u(x_k, y_k) \to 0$. As in the proof of Theorem 3.2, the functions $\phi(x_k + x) - y_k \to \phi_\infty(x)$, $u(x_k + x, y_k + y) \to u_\infty(x, y)$ locally in $\mathbb{R}$ and $\mathbb{R}^2$. We have $u_\infty^{t^*}(0,0) \geq u_\infty$ in $\mathbb{R}^2$ and $u_\infty^{t^*}(0,0) = u_\infty(0,0)$. We conclude similarly that $u_\infty^{t^*} \equiv u_\infty$ in $\mathbb{R}^2$. This implies t^* is a period of u_∞. By the uniform limiting behavior of u, u_∞ is not a periodic function, so t^* has to be 0. By the definition of t^*, for any $t > 0$, $u^t \geq u$ in $\mathbb{R}^2$. With the strong maximum principle, we conclude that $u^t > u$ in $\mathbb{R}^2$. In other words, u is increasing in y. ■

3.4 Proof of Proposition 3.1

We shall prove this propostion with several lemmas.

Lemma 3.4 *Under the assumptions of Proposition 3.1, the function u satisfies* $-1 < u < 1$ *in* $\mathbb{R}^2$ *and for some global Lipchitz function* $\phi : \mathbb{R} \to \mathbb{R}$*,*

$$\begin{cases} \lim\limits_{A \to +\infty} \inf\limits_{y > A + \phi(x)} u(x, y) = 1, \\ \lim\limits_{A \to -\infty} \sup\limits_{y < A + \phi(x)} u(x, y) = -1. \end{cases}$$

Proof. Let $M = \sup_{\mathbb{R}^2} u$. By the asymptotic conditions, $M \geq 0$. Let (x_k, y_k) be a sequence in $\mathbb{R}^2$ such that $u(x_k, y_k) \to M$ as $k \to \infty$. Denote $u_k(x, y) := u(x_k, y_k)$. Up to extraction of a subsequence, the functions u_k converge in $C^2_{loc}(\mathbb{R}^2)$ to a classical solution u_∞ of (1.4), namely

$$(u_\infty)_{xx} + c_0(u_\infty)_y + f(u_\infty) = 0 \text{ in } \mathbb{R}^2,$$

and $u_\infty(0,0) = M = \max_{\mathbb{R}^2} u_\infty$. Therefore, $f(M) \geq 0$. Note that one may extend $F(u)$ outside $[-1, 1]$ so that $f(u) = -F'(u)$ is negative in $(1, \infty)$. It follows that $M \leq 1$.

Similarly, one can prove that $m := \inf_{\mathbb{R}^2} u \geq -1$. Hence, $-1 \leq u \leq 1$ in $\mathbb{R}^2$. The strong maximun principle then yields $-1 < u < 1$ in $\mathbb{R}^2$.

Now let $\lim_{A\to+\infty} \inf_{y>A+\phi(x)} u(x, y) = \epsilon$ for some $\epsilon > \theta$. Then there exists a sequence $(x_n, y_n) \in \mathbb{R}^2$ such that $y_n - \phi(x_n) \to +\infty$ and $u(x_n, y_n) \to \epsilon$. Up to extraction of a subsequence, the functions $u_n(x, y) := u(x + x_n, y + y_n)$ converge in $C^2_{loc}(\mathbb{R}^2)$ to a solution $\tilde{u}_\infty$ of (1.4). Notice $(0, 0)$ is a minimum point of $\tilde{u}_\infty(x, y)$, $f(\tilde{u}_\infty(0, 0))$ has to be 0. Consequently $\tilde{u}_\infty(0, 0)$ has to be 1 and $\lim_{A\to+\infty} \inf_{y>A+\phi(x)} u(x, y) = 1$. Similarly we have $\lim_{A\to-\infty} \sup_{y<A+\phi(x)} u(x, y) = -1$. ∎

Lemma 3.5 *Under the assumptions of Proposition 3.1, the function u is increasing in any unit direction $\tau = (\tau_x, \tau_y)$ such that $\tau_y > \cos\alpha_0$, where $\alpha_0 \in (0, \pi/2)$ is an angle with $\cot\alpha_0$ being the Lipchitz norm of ϕ.*

Proof. By Theorem 3.3, u is increasing in direction (0,1).

If τ satisfies the condition of the lemma, we introduce the rotated coordinates:

$$X = -\tau_y x + \tau_x y; \qquad Y = -\tau_x x - \tau_y y.$$

In the new system (X, Y), the function $v(X, Y) = u(x, y)$ satisfies

$$\tau_y^2 v_{XX} + \tau_x^2 v_{YY} + 2\tau_x\tau_y v_{XY} - c_0\tau_x v_X + c_0\tau_y v_Y + f(v) = -1.$$

The second order operator is not positive definite, but we can still use similar arguments for parabolic equations to get the same results. Noticing that the curve $y = \cot|x|$ is a globally Lipchitz graph in frame (X, Y), it follows that

$$\lim_{A\to+\infty} \inf_{Y\geq A+\psi(X)} v(X, Y) = 1, \qquad \lim_{A\to-\infty} \sup_{Y\leq A+\psi(X)} v(X, Y) = 0$$

for some Lipchitz function $\psi(X)$. Once again, we get $u_Y > 0$ in $\mathbb{R}^2$. In other words, $\tau \cdot \nabla u < 0$. ∎

Lemma 3.6 *Under the assumptions of the Proposition 3.1, there are two real numbers $t_\pm$ such that $u(x + x_k, y + \frac{1}{c_0}|x_k|) \to g(\pm x + c_0 y + t_\pm)$ in $C^2_{loc}(\mathbb{R}^2)$ for any sequence $x_k \to \pm\infty$.*

Proof. By Lemma 3.5 and continuity, u is non-increasing in directions $(\pm\sin\alpha, \cos\alpha)$. Thus for any sequence $x_k \to \pm\infty$, there exist the following limits

$$\lim_{k\to\infty} u(x + x_k, y + \cot\alpha|x_k|) = v_\pm(x, y)$$

for some functions $v_\pm(x,y)$. Noticing that $v_\pm(x,y)$ just depend on $\pm x + c_0 y$, we can write

$$v_\pm(x,y) = U_\pm(\pm x + c_0 y)$$

for some functions $U_\pm$. It follows that $U_\pm$ satisfy one-dimensional traveling wave equation

$$U''_\pm + c_0 U'_\pm + f(U_\pm) = 0.$$

Furthermore, by the asymptotic conditions, $U_\pm(-\infty) = 0, U_\pm(\infty) = 1$. From the uniqueness result of traveling wave solutions, there exist two real numbers $t_\pm$ such that $U_\pm(s) = g(s+t_\pm)$ for all $s \in \mathbb{R}$. ■

Proposition 3.1 then follows from Lemma 3.4-3.6. This proves Theorem 3.1 and therefore Theorem 1.2.

References

[1] Stanley Alama, Lia Bronsard and Changfeng Gui, *Stationary layered solutions in R^2 for an Allen-Cahn systems with multiple-well potentials*, Calculus of Variation and PDE **5** (1997), 359-390.

[2] G. Alberti, L. Ambrosio and X. Cabre, *On a long standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property* , Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. Acta Appl. Math. **65** (2001), no. 1-3, 9–33.

[3] L. Ambrosio and X. Cabre, *Entire solutions of seminlinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), no. 4, 725–739.

[4] M. Barlow, R. Bass and C. Gui, *The Liouville property and a conjecture of de Giorgi*, Communications on Pure and Applied Mathematics **53** (2000), 1007-1038.

[5] H. Berestycki, L. Caffarelli and L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1089–1111.

[6] H. Berestycki, F. Hamel, *Generalized travelling waves for reaction-diffusion equations*, In: Perspectives in Nonlinear Partial Differential Equations. In honor of H. Brezis, Amer. Math. Soc., Contemp. Math. **446** (2007), 101-123.

[7] H. Berestycki, F. Hamel, Reaction-diffusion Equations and Propagation Phenomena, Applied Mathematical Sciences, Springer-Verlag, to appear.

[8] H. Berestycki, F. Hamel, R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations,* Duke Math. J. **103** (2000), no. 3, 375–396.

[9] A. Bonnet, F. Hamel, *Existence of non-planar solutions of a simple model of premixed Bunsen flames*, SIAM J. Math. Anal. **31** (1999), 80–118.

[10] G. Chapuisat, *Existence and nonexistence of curved front solution of a biological equation*, J. Diff. Equations **236** (2007), 237-279.

[11] X. Chen, *Generation and propagation of interfaces in reaction -diffusion equations*, J. Diff. Eqns. **96** (1992) , 116–141.

[12] X. Chen, J.-S. Guo, F. Hamel, H. Ninomiya, J.-M. Roquejoffre, *Traveling waves with paraboloid like interfaces for balanced bistable dynamics,* Ann. Inst. H. Poincare, Analyse Non Lineaire **24** (2007), 369-393.

[13] X. Chen, J.-S. Guo, *Existence and uniqueness of entire solutions for a reaction-diffusion equation,* J. Diff. Eqns. **212** (2005), 62-84.

[14] X. Chen, J.-S. Guo, H. Ninomiya, *Entire solutions of reaction-diffusion equations with balanced bistable nonlinearities*, Proceedings of the Royal Society of Edinburgh **136A** (2006), 1207-1237.

[15] E. de Giorgi, Convergence Problems for Functionals and Operators, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, Rome, 1978, E. de Giorgi et al (eds), Pitagora, Bologna, 1979.

[16] P.C. Fife, Dynamics of Internal Layers and Diffusive Interfaces, CBMS-NSF Regional Conference, Series in Applied Mathematics **53**, 1988.

[17] N. Ghoussoub and C. Gui, *On a conjecture of de Giorgi and some related problems*, Math. Ann. **311** (1998), 481-491.

[18] N. Ghoussoub and C. Gui, *About De Giorgi's conjecture in dimensions 4 and 5,* Annals of Mathematics **157** (2003), 313-334.

[19] C. Gui, *Hamiltonian identity for elliptic partial differential equations*, Journal of Functional Analysis **254** , no. 4 (2008), 904–933.

[20] C. Gui, *Symmetry of Some Entire Solutions to the Allen-Cahn Equation in Two Dimensions,* Journal of Differential Equations **252**, 11 (2012), 5853-874.

[21] C. Gui, *Axial Symmetry of some steady state solutions to nonlinear Schrodinger equations,* Proceedings of AMS **139** (2011), 1023-1032.

[22] C. Gui, *Symmetry of traveling wave solutions to the Allen-Cahn equation in R^2,* Archive for Rational Mechanics and Analysis **203**, number 3 (2012), 1037-1065.

[23] F. Hamel, R. Monneau, *Solutions of semilinear elliptic equations in $\mathbb{R}^n$ with conical shaped level sets*, Comm. Part. Diff. Equations **25** (2000), 769-819.

[24] F. Hamel, R. Monneau, Conical-shaped Travelling Fronts applied to the Mathematical Analysis of the Shape of Premixed Bunsen Flames, In: Nonlinear PDE's in Condensed Matter and Reactive Flows, H. Berestycki and Y. Pomeau eds., Kluwer Academic Publ.,(2002), 169-187.

[25] F. Hamel, R. Monneau, J.-M. Roquejoffre, *stability of travelling waves in a model for conical flames in two space dimensions,* Ann. Scient. Ecole Norm. Sup. **37** (2004), 469-506.

[26] F. Hamel, R. Monneau, J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Disc. Cont. Dyn. Systems **13** (2005), 1069-1096.

[27] F. Hamel, R. Monneau, J.-M. Roquejoffre, *Asymptotic properties and classification of bistable fronts with Lipschitz level sets,* Disc. Cont. Dyn. Systems **14** (2006), 75-92.

[28] F. Hamel, N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math. **52** (1999), 1255-1276.

[29] F. Hamel, N. Nadirashvili, *Travelling waves and entire solutions of the Fisher-KPP equation in R^n,* Arch. Ration. Mech. Anal. **157** (2001), 91-163.

[30] F. Hamel, J.-M. Roquejoffre, *Heteroclinic connections for multidimensional bistable reaction-diffusion equations,* Disc. Cont. Dyn. Systems, to appear.

[31] M. Haragus, A. Scheel, *Corner defects in almost planar interface propagation,* Ann. Inst. H. Poincare, Analyse Non Lineaire **23** (2006), 283-329.

[32] T. Kapitula, *Multidimensional stability of planar traveling waves*, Trans. Amer. Math. Soc. **349** (1997), 257–69.

[33] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Etude de lequation de la diffusion avec croissance de la quantite de mati'ere et son application 'a un probl'eme biologique*, Bull. Univ. etat Moscou, Ser. Intern. **A 1** (1937), 1-26.

[34] H. Matano, M. Nara, M. Taniguchi, *Stability of planar waves in the Allen-Cahn equation*, preprint.

[35] L. Modica, *A gradient bound and a Liouville theorem for non linear Poisson eqautions,* Comm. Pure. Appl. Math. **38** (1985), 679-684.

[36] Y. Morita, H. Ninomiya, *Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space*, Bull. Inst. Math. Acad. Sin. (N.S.) **3** (2008), 567-584.

[37] H. Ninomiya, M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations,* J. Diff. Eq. **213** (2005), 204-233.

[38] H. Ninomiya, M. Taniguchi, *Global stability of traveling curved fronts in the Allen-Cahn equations*, Dis. Cont. Dyn. Syst. **15** (2006), 819-832.

[39] J.-M. Roquejoffre, V. Roussier-Michon, *Nontrivial large-time behaviour in bistable reaction-diffusion equations*, Ann. Mat. Pura Appl. **188** (2009), 207-233.

[40] M. Taniguchi, *Traveling fronts of pyramidal shapes in the Allen-Cahn equation*, SIAM J. Math. Anal. **39** (2007), 319-344.

[41] M. Taniguchi, *The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations,* J. Diff. Equations **246** (2009), 2103-2130.