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# Dynamics of a Class of Advective-diffusive Equations in Ecology *

*This paper is dedicated to Professor Carlos Fernández-Pérez for his retirement with our most profound gratitude for all the PDE's that we learned from him and our best wishes for the future*

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**Abstract**

In this paper we characterize the dynamics of a generalized version of the model of Belgacem and Cosner [5] within the range of values of the parameters where the trivial steady state is unstable and the problem cannot admit a positive steady state. In such regimes, the dynamics is governed by the metasolutions.

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# 1 Introduction

This paper studies the dynamics of the parabolic problem

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda m - a u^{p-1}) & \text{in } \Omega, \quad t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain (open and connected set) of $\mathbb{R}^N$, $N \geq 1$, $D > 0$, $\alpha > 0$, $\lambda \in \mathbb{R}$, $p \geq 2$, $a \in C(\bar{\Omega})$ satisfies $a > 0$, in the sense that $a \geq 0$ and $a \neq 0$, $\nu$ stands for the outward unit normal along the boundary of $\Omega$, $\partial\Omega$, and $m \in C^2(\bar{\Omega})$ is a function such that $m(x_+) > 0$ for some $x_+ \in \Omega$. Thus, either $m \geq 0$, $m \neq 0$, in $\Omega$, or else $m$ changes sign in $\Omega$. The initial data $u_0$ are in $L^\infty(\Omega)$. In our analysis, $\lambda \in \mathbb{R}$ is regarded as a parameter. So, the dynamics might depend on $\lambda$.

Under these conditions, there exists $T > 0$ such that (1.1) admits a unique classical solution, $u_\lambda(x, t; u_0)$, in $[0, T]$ (see, e.g., Henry [10], Daners and Koch [6] and Lunardi [17]), and it is unique if it exists. According to the parabolic maximum principle (see Nirenberg [18]), $u_\lambda(\cdot, t; u_0) \gg 0$ in $\Omega$, in the sense that

$$u_\lambda(x, t; u_0) > 0 \quad \text{for all} \quad x \in \bar{\Omega} \quad \text{and} \quad t \in (0, T].$$

Thus, since $a > 0$ in $\Omega$,

$$\partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda m - a u^{p-1}) \leq \nabla \cdot (D\nabla u - \alpha u \nabla m) + \lambda m u$$

and hence, thanks again to the parabolic maximum principle,

$$u_\lambda(\cdot, t; u_0) \ll z_\lambda(\cdot, t; u_0) \quad \text{for all} \quad t \in (0, T],$$

where $z_\lambda(x, t; u_0)$ stands for the unique solution of the linear parabolic problem

$$\begin{cases} \partial_t z = \nabla \cdot (D\nabla z - \alpha z \nabla m) + \lambda m z & \text{in } \Omega, \quad t > 0, \\ D\partial_\nu z - \alpha z \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t > 0, \\ z(\cdot, 0) = u_0 > 0 & \text{in } \Omega. \end{cases}$$

As $z_\lambda$ is globally defined in time, $u_\lambda(x, t; u_0)$ cannot blow up in a finite time and hence, also is defined for all $t > 0$. The main goal of this paper is to ascertain the limiting behavior of $u_\lambda(x, t; u_0)$ as $t \uparrow \infty$.

The problem (1.1) is a generalized version of a former model, with $\lambda = 1$, $p = 2$ and $a = 1$, introduced by Belgacem and Cosner [5], which has been previously analyzed by the authors in [1] and [2], where the effect of varying the advection term $\alpha > 0$ on the *classical dynamics* of the model was studied. So, this paper can be regarded as a natural continuation of [1] and [2].

According to [1], if $a(x) > 0$ for all $x \in \bar{\Omega}$ then the dynamics of (1.1) is governed by its non-negative steady-states (see Theorem 2.2 in Section 2), which are the non-negative solutions of the semilinear elliptic boundary value problem

$$\begin{cases} \nabla \cdot (D\nabla \theta - \alpha \theta \nabla m) + \theta(\lambda m - a\theta^{p-1}) = 0 & \text{in } \Omega, \\ D\partial_\nu \theta - \alpha \theta \partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this paper we will focus our attention on the degenerate case where $a^{-1}(0)$ is the closure of some smooth nonempty subdomain of $\Omega$,

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{with} \quad \bar{\Omega}_0 \subset \Omega. \tag{1.3}$$

According to [1], if $\theta = 0$ is a linearly stable steady state of (1.1), then it is a global attractor with respect to (1.1). Similarly, if $\theta = 0$ is linearly unstable and (1.2) possesses a positive solution, $\theta_\lambda$, then $\theta_\lambda$ is unique and it attracts to all solutions of (1.1) as $t \uparrow \infty$. The main result of this paper characterizes the limiting behavior of $u_\lambda(x, t; u_0)$ as $t \uparrow \infty$ when $\theta = 0$ is linearly unstable and (1.2) does not admit a positive solution. Essentially, this occurs for sufficiently large $\lambda$ provided $m(x_+) > 0$ for some $x_+ \in \Omega_0$, [1]. Under these circumstances, the main result of this paper establishes that if $a \in C^2(\bar{\Omega})$, then

$$\lim_{t\uparrow\infty} u_\lambda(\cdot, t; u_0) = +\infty \quad \text{uniformly in } \bar{\Omega}_0, \tag{1.4}$$

whereas

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \leq \liminf_{t\to\infty} u_\lambda(\cdot, t; u_0) \leq \limsup_{t\to\infty} u_\lambda(\cdot, t; u_0) \leq L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0, \tag{1.5}$$

where $L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ and $L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ stand for the minimal and maximal solutions, respectively, of the singular boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \theta(\lambda m - a\theta^{p-1}) = 0 & \text{in } \Omega \setminus \bar{\Omega}_0, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega, \\ \theta = +\infty & \text{on } \partial\Omega_0, \end{cases} \tag{1.6}$$

whose existence will be shown in Section 5. This is the first result of this nature available for non-self-adjoint differential operators like the ones dealt with in this paper. One of the most novel parts of the proof consists in establishing (1.4) for the case when $m(x)$ changes sign. All the previous available results for degenerate diffusive logistic boundary value problems were established for the the Laplace operator without advection terms (see Gómez-Reñasco and López-Gómez [9], Gómez–Reñasco [8], López-Gómez [12] and Du and Huang [7]). In [13, Section 8] and [15] a rather complete account of historical bibliographic details is given.

The distribution of this paper is as follows. In section 2 we give some extensions of the previous findings of [1] which are going to be used in this paper. In Section 3 we generalize the Hadamard formula of López-Gómez and Sabina de Lis [16] to the non-self-adjoint context of this paper. In Section 4 we will use that Hadamard formula to establish that

$$\lim_{\lambda\uparrow\lambda^*} \theta_\lambda = +\infty \quad \text{uniformly in } \bar{\Omega}_0, \tag{1.7}$$

where $\lambda^* > 0$ is the limiting value of $\lambda$ for which (1.2) admits a positive solution. These are the main ingredients to get (1.4) in Section 6 when $m > 0$ in $\Omega$. In Section 5 we establish the existence of the minimal and the maximal large solutions $L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ and $L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$. Finally, in Section 7 we get (1.4) in the general case when $m$ changes of sign in $\Omega$ by perturbing the weight function $m$, instead of the parameter $\lambda$, as it is usual in the available literature. This technical device should have a huge number of applications to deal with spatially heterogeneous Reaction Diffusion equations.

## 2 Notations and preliminaries

In this section, instead of (1.1), we consider the following generalized parabolic problem

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda n - a u^{p-1}) & \text{in } \Omega, \quad t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $n \in C(\bar{\Omega})$ is arbitrary. Naturally, in the special case $n = m$, (2.1) provides us with (1.1). Throughout this paper we denote by $u_{[\lambda,n]}(x, t; u_0)$ the unique (positive) solution of (2.1). Also, we set

$$u_\lambda(x, t; u_0) := u_{[\lambda,m]}(x, t; u_0).$$

So, $u_\lambda(x, t; u_0)$ stands for the unique (positive) solution of (1.1). The main goal of this section is adapting the abstract theory of the authors in [1] to the problem (2.1). As the proofs of these results are straightforward modifications of those of [1], they will be omitted here.

The dynamic of (2.1) is regulated by its non-negative steady-states, if they exist, which are the non-negative solutions, $\theta$, of the semi linear elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla \theta - \alpha \theta \nabla m) - \theta(\lambda n - a\theta^{p-1}) = 0 & \text{in } \Omega, \\ D\partial_\nu \theta - \alpha \theta \partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Subsequently, given a linear second order uniformly elliptic operators in $\Omega$,

$$\mathfrak{L} := -\text{div}\,(A(x)\nabla \cdot) + \langle b(x), \nabla \cdot \rangle + c, \qquad A = (a_{ij})_{1\leq i,j\leq N}, \quad b = (b_j)_{1\leq j\leq N},$$

with $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$, $b_j, c \in L^\infty(\Omega)$, $1 \leq i, j \leq N$, a smooth subdomain $\mathcal{O} \subset \Omega$, two nice disjoint pieces of the boundary of $\mathcal{O}$, $\Gamma_0$ and $\Gamma_1$, such that $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$, and a boundary operator

$$\mathfrak{B} \;:\; C(\Gamma_0) \otimes C^1(\mathcal{O} \cup \Gamma_1) \to C(\partial\mathcal{O})$$

of the general mixed type

$$\mathfrak{B}\psi := \begin{cases} \psi & \text{on } \Gamma_0, \\ \partial_\nu \psi + \beta\psi & \text{on } \Gamma_1, \end{cases} \qquad \psi \in C(\Gamma_0) \otimes C^1(\mathcal{O} \cup \Gamma_1),$$

where $\nu = A\mathbf{n}$ is the co-normal vector field and $\beta \in C(\Gamma_1)$, we will denote by $\sigma[\mathfrak{L}, \mathfrak{B}, \mathcal{O}]$ the principal eigenvalue of $(\mathfrak{L}, \mathfrak{B}, \mathcal{O})$, i.e., the unique value of $\tau$ for which the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \tau\varphi & \text{in } \mathcal{O}, \\ \mathfrak{B}\varphi = 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

admits a positive eigenfunction $\varphi > 0$. Naturally, if $\Gamma_1 = \emptyset$, we will simply denote $\mathfrak{D} := \mathfrak{B}$ (Dirichlet), and if $\Gamma_0 = \emptyset$ and $\beta = 0$, we will write $\mathfrak{N} := \mathfrak{B}$ (Neumann).

The principal eigenvalue

$$\sigma[-\nabla \cdot (D\nabla - \alpha\nabla m) - \lambda n, D\partial_\nu - \alpha\partial_\nu m, \Omega], \tag{2.3}$$

plays a significant role to describe the dynamic of (2.1). Let denote by $\psi_0 > 0$ its associated principal eigenfunction normalized so that $\|\psi_0\|_\infty = 1$. By performing the change of variable

$$\phi_0 := e^{-\alpha m/D}\psi_0, \tag{2.4}$$

differentiating and rearranging terms in the $\psi_0$-equation, it is easily seen that $\psi_0 > 0$ provides us with a principal eigenfunction of

$$\sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega], \tag{2.5}$$

and that, actually,

$$\sigma[-\nabla \cdot (D\nabla - \alpha\nabla m) - \lambda n, D\partial_\nu - \alpha\partial_\nu m, \Omega] = \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega] \tag{2.6}$$

for all $\lambda \in \mathbb{R}$ and $\alpha > 0$. The next result extends [1, Th. 2.1] to cover the general case when $n \in \mathcal{C}(\bar{\Omega})$ is arbitrary. For the sake of completeness we are including a short self-contained proof of it.

**Theorem 2.1** *For every $\alpha > 0$ the map*

$$\lambda \mapsto \Sigma(\lambda) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega]$$

*is real analytic and strictly concave if $n \neq 0$. Moreover:*

(a) *If $n \equiv 0$, then, for any $\lambda \in \mathbb{R}$,*

$$\Sigma(\lambda) = \sigma[-D\Delta - \alpha\nabla m \cdot \nabla, \mathfrak{N}, \Omega] = 0.$$

(b) *If $n > 0$ ($n \geq 0$ but $n \neq 0$), then $\Sigma(\lambda)\lambda < 0$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Moreover, by Part (a), $\Sigma(0) = 0$.*

(c) *If $\int_\Omega n \geq 0$, there exists $x_- \in \Omega$ such that $n(x_-) < 0$, $n \leq m$ and $n = m$ if $n \leq 0$, then there exists $\lambda_- := \lambda_-(\alpha, n) < 0$ such that*

$$\Sigma(\lambda) \begin{cases} < 0 & \text{if } \lambda \in (-\infty, \lambda_-) \cup (0, \infty), \\ = 0 & \text{if } \lambda \in \{\lambda_-, 0\}, \\ > 0 & \text{if } \lambda \in (\lambda_-, 0). \end{cases} \tag{2.7}$$

(d) *If $\int_\Omega n < 0$ and $n(x_+) > 0$ for some $x_+ \in \Omega$, $n \leq m$ and $n = m$ if $n \leq 0$, then there exists $\alpha_0 := \alpha_0(n) > 0$ such that if $0 < \alpha < \alpha_0$ there exists $\lambda_+ := \lambda_+(\alpha, n) > 0$ such that*

$$\Sigma(\lambda) \begin{cases} < 0 & \text{if } \lambda \in (-\infty, 0) \cup (\lambda_+, \infty), \\ = 0 & \text{if } \lambda \in \{0, \lambda_+\}, \\ > 0 & \text{if } \lambda \in (0, \lambda_+), \end{cases}$$

*if $\alpha = \alpha_0$ then $\Sigma(\lambda) < 0$ for each $\lambda \in \mathbb{R} \setminus \{0\}$, while $\Sigma(0) = 0$, and if $\alpha > \alpha_0$ then there exists $\lambda_- := \lambda_-(\alpha, n) < 0$ such that* (2.7) *holds.*

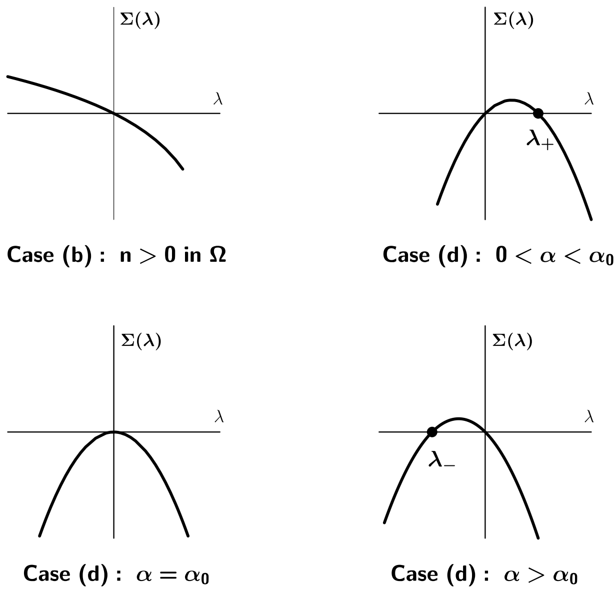


Figure 1: Some possible graphs of $\Sigma(\lambda)$

Figure 1 shows $\Sigma(\lambda)$ in each of the cases (b) and (d) of Theorem 2.1. In case (c), $\Sigma(\lambda)$ has the same graph as in case (d) for $\alpha > \alpha_0$.

*Proof.* According to [14, Th. 9.1], for every $\alpha \in \mathbb{R}$ the map $\lambda \mapsto \Sigma(\lambda)$ is real analytic and strictly concave if $n \neq 0$. In case $n \equiv 0$, it is obvious that the constant 1 is an eigenfunction associated to the zero eigenvalue of $-D\Delta - \alpha\nabla m \cdot \nabla$ in $\Omega$ under Neumann boundary conditions. Hence, $\Sigma(\lambda) = 0$ and Part (a) holds.

Part (b) is a direct consequence from $\Sigma(0) = 0$ taking into account that $\lambda \mapsto \Sigma(\lambda)$ is decreasing in $\lambda$, because $n > 0$.

In the remaining cases, (c) and (d), the function $n$ changes sign in $\Omega$. Thus, by [14, Th. 9.1],

$$\lim_{\lambda \to \pm\infty} \Sigma(\lambda) = -\infty.$$

Next we will show that

$$\Sigma'(0) = -\int_\Omega n(x) e^{\alpha m(x)/D}\, dx \Big/ \int_\Omega e^{\alpha m(x)/D}\, dx, \tag{2.8}$$

which is reminiscent from [1, (14)]. According to Kato [11, Th. 2.6 on p. 377] and [11, Rem. 2.9 on p. 379], the perturbed eigenfunction $\varphi(\lambda)$ from the constant $\varphi(0) = 1$

associated to the principal eigenvalue $\Sigma(\lambda)$ as $\lambda$ perturbs from 0 is real analytic as a function of $\lambda$. Thus, differentiating with respect to $\lambda$ the identities

$$\begin{cases} -D\Delta\varphi(\lambda) - \alpha\nabla m \cdot \nabla\varphi(\lambda) - \lambda n\varphi(\lambda) = \Sigma(\lambda)\varphi(\lambda) & \text{in } \Omega, \\ \partial_\nu \varphi(\lambda) = 0 & \text{on } \partial\Omega, \end{cases}$$

yields

$$-D\Delta\varphi'(\lambda) - \alpha\nabla m \cdot \nabla\varphi'(\lambda) - n\varphi(\lambda) - \lambda n\varphi'(\lambda) = \Sigma'(\lambda)\varphi(\lambda) + \Sigma(\lambda)\varphi'(\lambda).$$

and particularizing at $\lambda = 0$, we are driven to

$$\mathfrak{L}_0\varphi'(0) := -D\Delta\varphi'(0) - \alpha\nabla m \cdot \nabla\varphi'(0) = n + \Sigma'(0).$$

Consequently, as the adjoint operator of $\mathfrak{L}_0$ subject to Neumann boundary conditions admits the following realization

$$\mathfrak{L}_0^* v := -\nabla \cdot (D\nabla v - \alpha v \nabla m)$$

for all $v \in C^2(\bar{\Omega})$ such that

$$D\partial_\nu v - \alpha v \partial_\nu m = 0 \qquad \text{on } \partial\Omega,$$

and $\varphi_0^* := e^{\alpha m/D}$ satisfies

$$\mathfrak{L}_0^*\varphi_0^* = 0 \text{ in } \Omega \quad \text{and} \quad D\partial_\nu\varphi_0^* - \alpha\varphi_0^*\partial_\nu m = 0 \text{ on } \partial\Omega,$$

we conclude that

$$\langle n + \Sigma'(0), \varphi_0^* \rangle = \langle \mathfrak{L}_0\varphi'(0), \varphi_0^* \rangle = \langle \varphi'(0), \mathfrak{L}_0^*\varphi_0^* \rangle = 0$$

and therefore, (2.8) holds. As a byproduct, we find that

$$\operatorname{sign}\Sigma'(0) = -\operatorname{sign} f(\alpha) \qquad \text{for all } \alpha \geq 0,$$

where

$$f(\alpha) := \int_\Omega n(x) e^{\alpha m(x)/D}\, dx.$$

As

$$\begin{aligned} f'(\alpha) &= \frac{1}{D}\int_\Omega n(x)m(x)e^{\alpha m(x)/D}\, dx \\ &= \frac{1}{D}\int_{n\leq 0} n^2(x)e^{\alpha m(x)/D}\, dx + \frac{1}{D}\int_{n\geq 0} n(x)m(x)e^{\alpha m(x)/D}\, dx \\ &\geq \frac{1}{D}\int_{n\leq 0} n^2(x)e^{\alpha m(x)/D}\, dx + \frac{1}{D}\int_{n\geq 0} n^2(x)e^{\alpha m(x)/D}\, dx \\ &= \frac{1}{D}\int_\Omega n^2(x)e^{\alpha m(x)/D}\, dx > 0, \end{aligned}$$

because $n \neq 0$, the function $f$ is strictly increasing. Moreover, in the region where $n > 0$ we have that $m \geq n > 0$, whereas $m = n$ if $n \leq 0$. Therefore,

$$\lim_{\alpha \to \infty} f(\alpha) = \infty.$$

Lastly, note that $f(0) = \int_\Omega n$. Therefore, we have that $f(\alpha) > 0$ for all $\alpha > 0$ if $\int_\Omega n \geq 0$, while in case $\int_\Omega n < 0$, there exists a unique $\alpha_0 > 0$ such that $f(\alpha) < 0$ if $\alpha < \alpha_0$ and $f(\alpha) > 0$ if $\alpha > \alpha_0$. The proof is complete. □

According to Belgacem and Cosner [5], it is well known that in the special case when $a(x) > 0$ for all $x \in \bar{\Omega}$, the dynamics of (2.1) is determined by the following theorem:

**Theorem 2.2** *Suppose $\alpha > 0$ and $a(x) > 0$ for all $x \in \bar{\Omega}$. Then,* (2.2) *admits a positive solution, $\theta_{[\lambda,n]}$, if, and only if, $\Sigma(\lambda) < 0$. Moreover, it is unique if it exists and it is a global attractor for* (2.1). *Furthermore,*

$$\lim_{t \to \infty} \|u_{[\lambda,n]}(\cdot, t; u_0)\|_\infty = 0$$

*if $\Sigma(\lambda) \geq 0$.*

As this paper focuses attention in the more general degenerate case when

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{is a smooth subdomain with} \quad \bar{\Omega}_0 \subset \Omega, \tag{2.9}$$

the principal eigenvalue,

$$\Sigma_0(\lambda) := \sigma[-D\Delta - \alpha \nabla m \cdot \nabla - \lambda n, \mathfrak{D}, \Omega_0], \qquad \alpha > 0, \quad \lambda \in \mathbb{R},$$

will also play a significant role to characterize the dynamic of (2.1). The next theorem collects some of the main properties of $\Sigma_0(\lambda)$.

**Theorem 2.3** *Suppose* (2.9) *and $\alpha > 0$. Then, $\Sigma_0(\lambda)$ is real analytic, strictly concave if $n \not\equiv 0$ in $\Omega_0$, and*

$$\Sigma(\lambda) < \Sigma_0(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

*Moreover, the following assertions are true:*

(a) *There exists $\lambda_1 := \lambda_1(\alpha, n) < 0$ such that $\Sigma_0(\lambda_1) = 0$ if, and only if, $n(x_-) < 0$ for some $x_- \in \Omega_0$.*

(b) *There exists $\lambda_2 := \lambda_2(\alpha, n) > 0$ such that $\Sigma_0(\lambda_2) = 0$ if, and only if, $n(x_+) > 0$ for some $x_+ \in \Omega_0$.*

Figure 2 shows each of the possible graphs of $\Sigma_0(\lambda)$ according to the sign of $n(x)$ in $\Omega_0$. In all cases, we have superimposed the graphs of $\Sigma(\lambda)$, with a dashed line, and the graph of $\Sigma_0(\lambda)$, with a continuous line. It should be noted that we have chosen a particular type of $\Sigma(\lambda)$ for all cases.

The next theorem provides us with the dynamic of (2.1). It generalizes the theory of [1] to cover the general case when $n \neq m$.

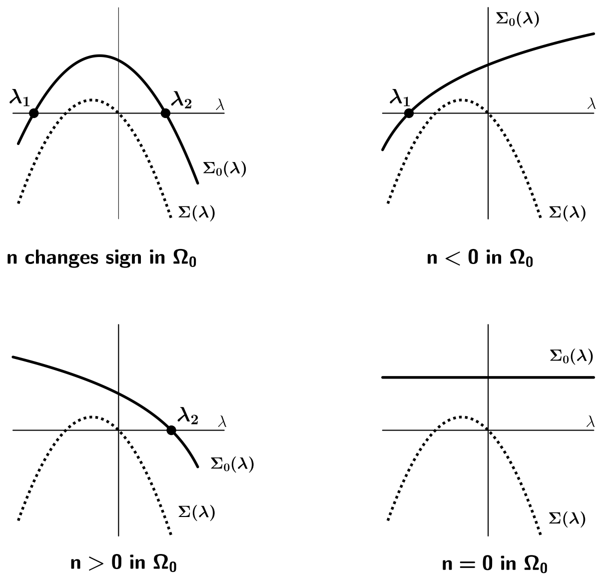


Figure 2: The four possible graphs of $\Sigma_0(\lambda)$

**Theorem 2.4** *Suppose* (2.9) *and* $\alpha > 0$. *Then,* (2.2) *admits a positive solution,* $\theta_{[\lambda,n]}$, *if and only if* $\Sigma_0(\lambda) > 0$ *and* $\Sigma(\lambda) < 0$. *Moreover, is unique if it exists and in such case* $\theta_{[\lambda,n]}$ *is a global attractor for* (2.1). *Furthermore, u is driven to extinction in* $L^\infty(\Omega)$ *if* $\Sigma(\lambda) \geq 0$, *whereas*

$$\lim_{t\to\infty} \|u_{[\lambda,n]}(\cdot,t;u_0)\|_\infty = \infty \qquad \textit{if} \quad \Sigma_0(\lambda) \leq 0.$$

One of the main goals of this paper is to show that, actually, in case $\Sigma_0(\lambda) \leq 0$ the solution $u_{[\lambda,n]}(\cdot,t;u_0)$ can approximate the minimal metasolution supported in $\bar{\Omega} \setminus \bar{\Omega}_0$ of (2.2) as $t \uparrow \infty$.

# 3 First variations of the principal eigenvalues $\Sigma_0(\lambda)$

Throughout this section the domain $\Omega_0$ is assumed to be of class $C^1$ and consider $T[\delta] : \bar{\Omega}_0 \to \bar{\Omega}_\delta$, with $\delta \simeq 0$ and $\Omega_\delta := T[\delta](\Omega_0)$, an holomorphic family of $C^2$-diffeormorphisms that can be expressed in the form

$$T[\delta](x) = x + \sum_{k=1}^{\infty} \delta^k T_k(x) \quad \text{for all} \quad x \in \bar{\Omega}_0, \tag{3.1}$$

where $T_k \in C^2(\bar{\Omega}_0; \mathbb{R}^N)$ for each $k \geq 1$ and

$$\limsup_{k\to\infty} \left\{ \|T_k\|_{\infty,\Omega_0} + \|D_x T_k\|_{\infty,\Omega_0} + \|D_x^2 T_k\|_{\infty,\Omega_0} \right\}^{1/k} < +\infty. \tag{3.2}$$

According to T. Kato [11, Ch. VII], it is easily seen that the family of eigenvalue problems

$$\begin{cases} -D\Delta\Phi - \alpha\nabla m \cdot \nabla\Phi - \lambda n\Phi = \tau\Phi & \text{in } \Omega_\delta, \\ \Phi = 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{3.3}$$

is real holomorphic in $\delta$. Thus, setting

$$S(\delta) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, \Omega_\delta]$$

for $\lambda \in \mathbb{R}$ and $\alpha > 0$ fixed, $S(\delta)$ admits the next expansion as $\delta \downarrow 0$

$$S(\delta) = \sum_{k=0}^{\infty} \delta^k S_k, \qquad S_k = S^{k)}(0), \tag{3.4}$$

where $S^{k)}(0)$ stands for the $k$–th derivative of $S(\delta)$ with respect to $\delta$ at $\delta = 0$. Similarly, the associated perturbed eigenfunction, $\phi[\delta]$, admits the expansion

$$\phi[\delta] = \sum_{k=0}^{\infty} \delta^k \phi_k, \qquad \phi_k := \phi^{k)}[0],$$

where $\phi_0$ if the principal eigenfunction associated to $S(0) = \Sigma_0(\lambda)$ normalized so that

$$\int_{\Omega_0} e^{\frac{\alpha}{D}m} \phi_0^2 = 1. \tag{3.5}$$

The next result provides us with an extension of Theorem 2.1 of [16] which provides us with the value of $S_1$ in (3.4).

**Theorem 3.1** *Let $\Omega_0$ be a bounded domain of class $C^1$ and $\Omega_\delta$, $\delta \simeq 0$, a perturbed family of domains from $\Omega_0$ given by a family of $C^2$-diffeomorphisms $T[\delta]$ satisfying* (3.1) *and* (3.2). *Then, for every $\lambda \in \mathbb{R}$ and $\alpha > 0$, the eigenvalue problem* (3.3) *is real holomorphic in $\delta$ and the first variation of the principal eigenvalue, $S(\delta)$, is given by*

$$S_1 := S'(0) = -D \int_{\partial\Omega_0} e^{\frac{\alpha}{D}m} \left( \frac{\partial\phi_0}{\partial\nu} \right)^2 \langle T_1, \nu \rangle, \tag{3.6}$$

*where $\phi_0$ is the principal eigenfunction associated with $S(0) = \Sigma_0(\lambda)$, normalized so that* (3.5) *holds.*

*Proof.* Subsequently, for each $\delta \simeq 0$ we set

$$\varphi[\delta] := e^{\frac{\alpha}{2D}m} \phi[\delta] \quad \text{in } \Omega_\delta.$$

This function is positive and satisfies

$$\begin{cases} -D\Delta\varphi[\delta] - W\varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{3.7}$$

with

$$W := \lambda n - \frac{\alpha}{2}(\Delta m + \frac{\alpha}{2D}|\nabla m|^2).$$

By (3.5),

$$\int_{\Omega_0} \varphi^2[0] = 1.$$

Next, we consider

$$\psi[\delta](x) := \varphi[\delta](y), \qquad Q[\delta](x) := W(y) \qquad \text{with } T[\delta](x) = y.$$

Then, setting

$$(T[\delta])^{-1}(y) = (x_1(y), x_2(y), ..., x_N(y)).$$

and substituting in (3.7) yields

$$\begin{cases} -D\sum_{k,l=1}^{N}\langle\nabla_y x_k, \nabla_y x_l\rangle \frac{\partial^2\psi[\delta]}{\partial x_k \partial x_l} - D\sum_{l=1}^{N}\Delta_y x_l \frac{\partial\psi[\delta]}{\partial x_l} - Q[\delta]\psi[\delta] = S(\delta)\psi[\delta] & \text{in } \Omega_0, \\ \psi[\delta] = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{3.8}$$

Now, arguing as in Section 2 of [16] and setting $T_1 = (T_{1,1}, T_{1,2}, ..., T_{1,N})$ we can conclude from (3.8) that

$$-D\Delta_x\psi[\delta] + 2D\delta \sum_{k,l=1}^{N} \frac{\partial T_{1,k}}{\partial x_l} \frac{\partial^2\psi[\delta]}{\partial x_k \partial x_l} + D\delta \sum_{l=1}^{N} \Delta_x T_{1,l} \frac{\partial\psi[\delta]}{\partial x_l} - Q[\delta]\psi[\delta] = S(\delta)\psi[\delta] + O(\delta^2). \tag{3.9}$$

Setting

$$\psi[\delta] = \sum_{k=0}^{\infty} \delta^k \psi_k, \quad \psi_k = \frac{d^k\psi}{d\delta^k}[0], \tag{3.10}$$

and

$$Q[\delta] = \sum_{k=0}^{\infty} \delta^k Q_k, \quad Q_k = \frac{d^k Q}{d\delta^k}[0], \tag{3.11}$$

with

$$\psi_0 = \psi[0] = \varphi[0], \quad Q_0 = W \quad \text{and} \quad Q_1 = \langle\nabla W, T_1\rangle = \langle\nabla Q_0, T_1\rangle, \tag{3.12}$$

substituting (3.4), (3.10) and (3.11) in (3.9), dividing by $\delta$ and letting $\delta \to 0$ gives

$$-D\Delta_x\psi_1 + 2D \sum_{k,l=1}^{N} \frac{\partial T_{1,k}}{\partial x_l} \frac{\partial^2\psi_0}{\partial x_k \partial x_l} + D \sum_{l=1}^{N} \Delta_x T_{1,l} \frac{\partial\psi_0}{\partial x_l} - Q_0\psi_1 - Q_1\psi_0 = S_0\psi_1 + S_1\psi_0, \tag{3.13}$$

where we have used that

$$-D\Delta_x\psi_0 - Q_0\psi_0 = S_0\psi_0. \tag{3.14}$$

Hence, multiplying (3.13) by $\psi_0$, integrating by parts in the first factor and using (3.14), we find that

$$S_1 = 2D\sum_{k,l=1}^N \int_{\Omega_0} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial^2\psi_0}{\partial x_k\partial x_l}\psi_0 + D\sum_{l=1}^N \int_{\Omega_0} \Delta_x T_{1,l}\frac{\partial \psi_0}{\partial x_l}\psi_0 - \int_{\Omega_0} Q_1\psi_0^2. \tag{3.15}$$

A further integration by parts in the first two terms of (3.15), complete in the second term and partial in the first one after a cancelation of its half, gives

$$S_1 = -2D\sum_{k,l=1}^N \int_{\Omega_0} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial\psi_0}{\partial x_k}\frac{\partial\psi_0}{\partial x_l} - D\sum_{k,l=1}^N \int_{\Omega_0} \frac{\partial^2 T_{1,k}}{\partial x_k\partial x_l}\frac{\partial \psi_0}{\partial x_l}\psi_0 - \int_{\Omega_0} Q_1\psi_0^2. \tag{3.16}$$

A further integration by parts in the first term of (3.16) yields

$$\begin{aligned}-2D\sum_{k,l=1}^N \int_{\Omega_0} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial\psi_0}{\partial x_k}\frac{\partial\psi_0}{\partial x_l} &= -2D\int_{\partial\Omega_0}\langle\nabla_x\psi_0, T_1\rangle\langle\nabla_x\psi_0,\nu\rangle\\ &\quad + 2D\sum_{k,l=1}^N \int_{\Omega_0} T_{1,k}\frac{\partial\psi_0}{\partial x_l}\frac{\partial^2\psi_0}{\partial x_l\partial x_k} + 2D\int_{\Omega_0}\Delta_x\psi_0\langle\nabla_x\psi_0, T_1\rangle.\end{aligned}$$

Similarly, integrating by parts with respect to $x_l$ and then with respect to $x_k$, the second term of (3.16) can be expressed as

$$\begin{aligned}-D\sum_{k,l=1}^N \int_{\Omega_0} \frac{\partial^2 T_{1,k}}{\partial x_k\partial x_l}\frac{\partial\psi_0}{\partial x_l}\psi_0 &= D\int_{\partial\Omega_0}|\nabla_x\psi_0|^2\langle T_1,\nu\rangle\\ &\quad - 2D\sum_{k,l=1}^N \int_{\Omega_0} T_{1,k}\frac{\partial\psi_0}{\partial x_l}\frac{\partial^2\psi_0}{\partial x_k\partial x_l} + D\sum_{k=1}^N \int_{\Omega_0}\Delta_x\psi_0\frac{\partial T_{1,k}}{\partial x_k}\psi_0.\end{aligned}$$

Consequently, substituting these identities into (3.16) we obtain that

$$\begin{aligned}S_1 =& -2D\int_{\partial\Omega_0}\langle\nabla_x\psi_0, T_1\rangle\langle\nabla_x\psi_0,\nu\rangle + D\int_{\partial\Omega_0}|\nabla_x\psi_0|^2\langle T_1,\nu\rangle\\ & -2\int_{\Omega_0}(Q_0\psi_0 + S_0\psi_0)\langle\nabla_x\psi_0, T_1\rangle - \sum_{k=1}^N\int_{\Omega_0}(Q_0\psi_0 + S_0\psi_0)\frac{\partial T_{1,k}}{\partial x_k}\psi_0 - \int_{\Omega_0} Q_1\psi_0^2\end{aligned}$$

where we have used (3.14). Finally, thanks to (3.12) and

$$\begin{aligned}&-S_0\sum_{k=1}^N\int_{\Omega_0}\frac{\partial T_{1,k}}{\partial x_k}\psi_0^2 = 2S_0\int_{\Omega_0}\langle\nabla_x\psi_0, T_1\rangle\psi_0,\\ &-\sum_{k=1}^N\int_{\Omega_0} Q_0\frac{\partial T_{1,k}}{\partial x_k}\psi_0^2 = 2\int_{\Omega_0} Q_0\langle\nabla_x\psi_0, T_1\rangle\psi_0 + \int_{\Omega_0}\langle\nabla_x Q_0, T_1\rangle\psi_0^2,\end{aligned}$$

and

$$\nabla_x \varphi[0] = \frac{\partial \varphi[0]}{\partial \nu} \nu \quad \text{on} \quad \partial\Omega_0,$$

we may infer that

$$S_1 = -D \int_{\partial\Omega_0} \left( \frac{\partial \varphi[0]}{\partial \nu} \right)^2 \langle T_1, \nu \rangle.$$

Finally, taking into account that

$$\varphi[0] = e^{\frac{\alpha}{2D} m} \phi[0] = e^{\frac{\alpha}{2D} m} \phi_0,$$

(3.6) holds. □

A particular class of perturbations $\Omega_\delta$ of $\Omega_0$ is given by

$$\Omega_\delta := \Omega_0 \cup \left\{ x \in \mathbb{R}^N \setminus \Omega_0 \ : \ \text{dist}\,(x, \partial\Omega_0) < \delta \right\}. \tag{3.17}$$

The associated holomorphic family $T[\delta]$ with $\delta \simeq 0$ can be defined through the next theorem going back to [16, Th. 3.1],

**Theorem 3.2** *Assume that $\Omega_0$ is a bounded domain of $\mathbb{R}^N$ of class $C^3$. If $\Omega_\delta$ is given by* (3.17)*, then for each $\delta > 0$ sufficiently small, there exists a mapping $T[\delta] : \bar{\Omega}_0 \to \mathbb{R}^N$, such that*

(i) *$T[\delta] \in C^2(\bar{\Omega}_0; \mathbb{R}^N)$ and $T[\delta] : \bar{\Omega}_0 \to \bar{\Omega}_\delta$ is a bijection.*

(ii) *$T[\delta]$ is real holomorphic in $\delta$ for $\delta \simeq 0$, in the sense that* (3.1) *and* (3.2) *hold.*

(iii) *$T_1|_{\partial\Omega_0} = \nu$ where $\nu$ is the outward unit normal along $\partial\Omega_0$.*

For (3.17), we obtain from (3.6) that

$$S_1 := S'(0) = -D \int_{\partial\Omega_0} e^{\frac{\alpha}{D} m} \left( \frac{\partial \phi_0}{\partial \nu} \right)^2 < 0. \tag{3.18}$$

In particular, the decay of the principal eigenvalue with respect to the domain is always linear independently of the size of the advection, $\alpha > 0$, though the linear decay rate is affected by the size of $\alpha$ and the nodal behavior of $m(x)$, of course.

# 4 Limiting behavior of the positive steady state solution in $\bar{\Omega}_0$

Throughout this section we will assume that $\lambda > 0$ and that

$$\text{there exists} \quad x_+ \in \Omega_0 \quad \text{such that} \quad m(x_+) > 0. \tag{4.1}$$

Should it not be the case, i.e. $m \leq 0$ in $\Omega_0$, then $\Sigma_0(\lambda) > 0$ for all $\lambda > 0$. Hence, by Theorems 2.1 and 2.4, there exists $\lambda_+(\alpha) \geq 0$ such that (1.2) admits a (unique) positive solution, $\theta_\lambda$, if and only if $\lambda > \lambda_+(\alpha)$. Moreover, thanks to Theorem 2.4, $\theta_\lambda$ is a global attractor for the positive solutions of (1.1). Contrarily, under condition (4.1), owing to Theorem 2.3, there exists a $\lambda^*$,

$$0 \le \lambda_+(\alpha) < \lambda^* := \lambda_2(\alpha, m),$$

such that (1.2) has a positive solution if and only if $\lambda \in (\lambda_+, \lambda^*)$. Moreover, by Theorem 2.1, we already know that $\lambda_+ > 0$ if and only if $\int_\Omega m < 0$ and $\alpha \in (0, \alpha_0)$. The main goal of this section is to establish the next result.

**Theorem 4.1** *Suppose* (4.1) *and* $a(x)$ *is of class* $C^1$ *in a neighborhood of* $\partial\Omega_0$. *Then,*

$$\lim_{\lambda \uparrow \lambda_2} \theta_\lambda(x) = \infty \quad \text{uniformly in } \bar{\Omega}_0. \tag{4.2}$$

*Proof.* For sufficiently small $\delta > 0$, we consider $\Omega_\delta$ as in (3.17). By Theorem 3.2, $\Omega_\delta$ is an holomorphic perturbation from $\Omega_0$. Now, consider the principal eigenvalues

$$S(\delta) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda^* m, \mathfrak{D}, \Omega_\delta]$$

with associated principal eigenfunctions $\phi[\delta] > 0$. By (3.4) and (3.18),

$$S(\delta) = S_1\delta + O(\delta^2) \quad \text{as} \quad \delta \downarrow 0 \quad \text{with} \quad S_1 < 0. \tag{4.3}$$

The function $\varphi[\delta]$ defined by

$$\varphi[\delta] := e^{\alpha m/D}\phi[\delta] > 0,$$

where $\phi[\delta]$ stands for the (unique) principal eigenfunction of $S(\delta)$ normalized so that

$$\|\varphi[\delta]\|_{\infty,\Omega_\delta} = 1,$$

satisfies

$$\begin{cases} -\nabla \cdot (D\nabla\varphi[\delta] - \alpha\varphi[\delta]\nabla m) - \lambda^* m\varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \partial\Omega_\delta. \end{cases} \tag{4.4}$$

Moreover, there exists $\lambda \in (\lambda_+, \lambda^*)$ sufficiently close $\lambda^*$ such that

$$S(\delta) < S(\delta/2) < (\lambda - \lambda^*) \max_{\bar{\Omega}_\delta} m < 0. \tag{4.5}$$

Let $\vartheta_\delta \in C(\bar{\Omega})$ be the function defined by

$$\vartheta_\delta(y) = \begin{cases} C\varphi[\delta](y) & \text{for } y \in \bar{\Omega}_\delta, \\ 0 & \text{for } y \notin \Omega_\delta, \end{cases}$$

where $C > 0$ is a constant to be chosen later. If we suppose that

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) \le S(\delta/2) - S(\delta) \quad \text{for all } y \in \Omega_\delta \tag{4.6}$$

then $\vartheta_\delta$ is a subsolution of problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \lambda m\theta = -a\theta^p & \text{in } \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

Indeed, by (4.6) and (4.5), we have that

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) < (\lambda - \lambda^*)m(y) - S(\delta) \quad \text{for each } y \in \Omega_\delta.$$

So, multiplying by $C\varphi[\delta]$, we find that

$$S(\delta)\vartheta_\delta + (\lambda^* - \lambda)m\vartheta_\delta < -a\vartheta_\delta^p \qquad \text{in} \qquad \Omega_\delta.$$

Hence, owing to (4.4), we find that

$$-\nabla \cdot (D\nabla\vartheta_\delta - \alpha\vartheta_\delta\nabla m) - \lambda^* m\vartheta_\delta + (\lambda^* - \lambda)m\vartheta_\delta < -a\vartheta_\delta^p \qquad \text{in} \qquad \Omega_\delta$$

which proves the previous assertion. Now, thanks to (4.3), it is apparent that (4.6) holds provided

$$C = C(\delta) := \frac{1}{\mathrm{Sup}_{\Omega_\delta\setminus\Omega_0}\varphi[\delta]} \left( \frac{-S_1/2 + O(\delta)}{\mathrm{Sup}_{\Omega_\delta\setminus\Omega_0} a/\delta} \right)^{1/(p-1)}. \tag{4.8}$$

According to (4.12) and (4.13) of [16], one can easily infer that

$$\mathrm{Sup}_{\Omega_\delta\setminus\Omega_0}\varphi[\delta] = C_0\delta + o(\delta) \ \text{ with } \ C_0 > 0 \quad \text{and} \quad \lim_{\delta\downarrow 0} \mathrm{Sup}_{\Omega_\delta\setminus\Omega_0} a/\delta = 0,$$

by the Hopf boundary lemma. Since $S_1 < 0$, we find that

$$\lim_{\delta\downarrow 0}(\delta C(\delta)) = +\infty. \tag{4.9}$$

By (4.9),

$$\lim_{\delta\downarrow 0} \vartheta_\delta(y) = +\infty \quad \text{for all } y \in \Omega_0, \tag{4.10}$$

uniformly on compact subsets of $\Omega_0$. To complete the proof of the theorem it suffices to show that

$$\lim_{\delta\downarrow 0} \vartheta_\delta(y) = +\infty \quad \text{for all } y \in \partial\Omega_0. \tag{4.11}$$

We use the proof of [16, Th. 4.3] and we obtain that

$$\mathrm{Inf}_{\partial\Omega_0}\varphi[\delta] = C_1\delta + o(\delta) \quad \text{with } C_1 > 0$$

as $\delta \downarrow 0$. Therefore, for every $y \in \partial\Omega_0$,

$$\vartheta_\delta(y) = C(\delta)\varphi[\delta](y) \geq C(\delta)\,\mathrm{Inf}_{\partial\Omega_0}\varphi[\delta] = \delta C(\delta)(C_1 + o(1)).$$

Lastly, by (4.9), (4.11) holds.

Finally, since (1.2) admits a unique positive solution $\theta_\lambda$ for each $\lambda \in (\lambda_+, \lambda^*)$, we find that

$$\vartheta_\delta(y) \leq \theta_\lambda(y) \quad \text{for all } y \in \bar{\Omega}$$

for $\lambda$ sufficiently close to $\lambda^*$ and for sufficiently small $\delta > 0$ satisfying (4.6). Therefore, the growth to infinity of $\vartheta_\delta$ leads to the corresponding behavior for $\theta_\lambda$ and (4.2) holds. □

# 5 Existence of large solutions

Throughout this section we consider a smooth subdomain $\mathcal{O} \subset \Omega \setminus \bar{\Omega}_0$ such that, for some $\eta > 0$,

$$\{x \in \Omega \ : \ \ \mathrm{dist}\,(x, \partial\Omega) < \eta\} \subset \mathcal{O}. \tag{5.1}$$

Note that this entails

$$\mathrm{dist}\,(\partial\Omega, \Omega \cap \partial\mathcal{O}) > 0. \tag{5.2}$$

Also, for every $M \in (0, \infty)$, we consider the next family of parabolic problems

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda n - a u^{p-1}) & \text{in } \mathcal{O},\ t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega,\ t > 0, \\ u = M & \text{on } \partial\mathcal{O} \setminus \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \mathcal{O}, \end{cases} \tag{5.3}$$

whose unique (positive) solution is denoted by $u := u_{[\lambda,n,M,\mathcal{O}]}(x, t; u_0)$. The dynamics of (5.3) is regulated by the non-negative solutions, $\theta_{[\lambda,n,M,\mathcal{O}]}$, of the semilinear elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \theta(\lambda n - a\theta^{p-1}) = 0 & \text{in } \mathcal{O}, \\ D\partial_\nu \theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega, \\ \theta = M & \text{on } \partial\mathcal{O} \setminus \partial\Omega. \end{cases} \tag{5.4}$$

The following result characterizes the existence of positive solutions of (5.4).

**Theorem 5.1** *Suppose $\mathcal{O} \subset \Omega \setminus \bar{\Omega}_0$ is sufficiently smooth, $n \in \mathcal{C}(\bar{\Omega})$, $\lambda \in \mathbb{R}$ and $M > 0$. Then,* (5.4) *has a unique positive solution, $\theta_{[\lambda,n,M,\mathcal{O}]}$. Moreover, for every $x \in \bar{\mathcal{O}}$, $\theta_{[\lambda,n,M,\mathcal{O}]}(x) > 0$ and*

$$\theta_{[\lambda,n,M_1,\mathcal{O}]}(x) < \theta_{[\lambda,n,M_2,\mathcal{O}]}(x) \quad \textit{if} \ \ M_1 < M_2. \tag{5.5}$$

*Furthermore, $\theta_{[\lambda,n,M,\mathcal{O}]}$ is a global attractor for* (5.3).

*Proof.* First, we introduce the change of variable

$$\theta = e^{\alpha m/D} w$$

in order to transform the problem (5.4) in

$$\begin{cases} -D\Delta w - \alpha\nabla m \cdot \nabla w - \lambda n w = -a e^{\alpha(p-1)m/D} w^p & \text{in } \mathcal{O}, \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \\ w = M e^{-\alpha m/D} & \text{on } \partial\mathcal{O} \setminus \partial\Omega. \end{cases} \tag{5.6}$$

To establish the existence of arbitrarily large supersolutions of (5.6) we proceed as follows. For sufficiently large $k \in \mathbb{N}$, we will set

$$A_k := (\partial\mathcal{O} \setminus \partial\Omega) + B_{\frac{1}{k}}(0) = \left\{ x \in \Omega : \mathrm{dist}(x, \partial\mathcal{O} \setminus \partial\Omega) < \frac{1}{k} \right\}.$$

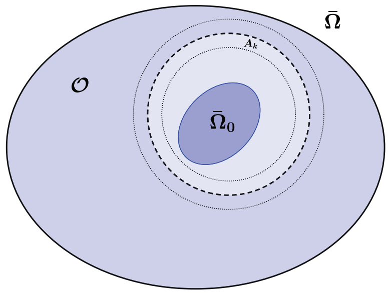


Figure 3: The neighborhoods $A_k$ around $\partial\mathcal{O} \setminus \partial\Omega$

Figure 3 shows an scheme of their construction in the special case when $\Omega\setminus\bar{\Omega}_0$ is connected. In such case also $A_k$ is connected.

As the Lebesgue measure of $A_k$ goes to zero as $k \uparrow \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_k\right] > 0 \quad \text{for all} \;\; k \geq k_0. \tag{5.7}$$

Let $\varphi_{k_0}$ be any a principal eigenfunction associated to

$$\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right]$$

and consider any smooth function $\phi : \bar{\mathcal{O}} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} \varphi_{k_0}(x) & \text{in} \;\; A_{2k_0} \cap \bar{\mathcal{O}}, \\ g(x) & \text{in} \;\; \bar{\mathcal{O}} \setminus A_{2k_0}, \end{cases}$$

where $g$ is any smooth function such that

$$\operatorname{Inf}_{\bar{\mathcal{O}}\setminus A_{2k_0}} g > 0 \quad \text{and} \quad \partial_\nu g = 0 \quad \text{on} \;\; \partial\Omega.$$

In the general case when $A_k$ is not connected, it must possess finitely many components, because $\Omega_0$ is smooth. In such case, one should take the corresponding principal eigenfunction on each of these components.

Subsequently, we consider $\bar{w} := C\phi$, where $C > 0$ is sufficiently large so that

$$C^{p-1} a e^{\alpha(p-1)m/D} g^p > D\Delta g + \alpha\nabla m \cdot \nabla g + \lambda n g \quad \text{in} \;\; \mathcal{O} \setminus A_{2k_0} \tag{5.8}$$

and

$$C \geq \frac{M e^{-\alpha m/D}}{\min\limits_{\partial\mathcal{O}\setminus\partial\Omega} \varphi_{k_0}} \qquad \text{for all} \;\; x \in \partial\mathcal{O} \setminus \partial\Omega. \tag{5.9}$$

We claim that $\bar{w} = C\phi$ is a supersolution of (5.6). Indeed, let $x \in A_{2k_0} \cap \mathcal{O}$. Then, by (5.7), we have that

$$-\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right] < 0 \leq C^{p-1}a(x)e^{\alpha(p-1)m(x)/D}\varphi_{k_0}^{p-1}(x).$$

Hence, multiplying by $C\varphi_{k_0}$ yields

$$-\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right] C\varphi_{k_0}(x) \leq C^p a(x)e^{\alpha(p-1)m(x)/D}\varphi_{k_0}^{p}(x)$$

or, equivalently,

$$-D\Delta\bar{w} - \alpha\nabla m \cdot \nabla\bar{w} - \lambda n\bar{w} \geq -ae^{\alpha(p-1)m/D}\bar{w}^p \quad \text{in } A_{2k_0} \cap \mathcal{O}. \tag{5.10}$$

Moreover, thanks to (5.8), (5.10) also holds in $\mathcal{O} \setminus A_{2k_0}$, and, due to (5.9),

$$\bar{w}(x) \geq Me^{-\alpha m/D} \qquad \text{for all } x \in \partial\mathcal{O} \setminus \partial\Omega.$$

Note that, by the choice of $g$, $\bar{w}$ satisfies the boundary condition on $\partial\Omega$. Finally, we choose $\underline{w} = 0$ as subsolution of (5.6). As $\underline{w} \leq \bar{w}$, the existence of a positive solution, $w_{[\lambda,n,M,\mathcal{O}]}$ can be inferred from the main theorem of Amann [3]. Necessarily,

$$0 \leq w_{[\lambda,n,M,\mathcal{O}]} \leq \overline{w}.$$

Moreover, $w_{[\lambda,n,M,\mathcal{O}]} \neq 0$ because $w_{[\lambda,n,M,\mathcal{O}]} = Me^{-\alpha m/D} > 0$ in $\partial\mathcal{O} \setminus \partial\Omega$. Therefore, the function

$$\theta_{[\lambda,n,M,\mathcal{O}]} := e^{\alpha m/D}w_{[\lambda,n,M,\mathcal{O}]}$$

provides us with a positive solution of (5.4). The remaining assertions of the theorem are a direct consequence from the maximum principle, arguing as in the proof of [1, Th. 4.1]. The global attractiveness is a consequence from the uniqueness of the positive solution, by the abstract theory of D. Sattinger [19]. So, we will omit the technical details here. □

As a consequence from Theorem 5.1, the following limit is well defined

$$\theta_{[\lambda,n,\infty,\mathcal{O}]}(x) := \lim_{M\uparrow\infty} \theta_{[\lambda,n,M,\mathcal{O}]}(x) \quad \text{for each } x \in \bar{\mathcal{O}} \tag{5.11}$$

though it might be everywhere infinity. If we can show that $\theta_{[\lambda,n,M,\mathcal{O}]}(x)$ is bounded above by some function $U_{[\lambda,n,\mathcal{O}]}(x)$, uniformly bounded on compact subsets of $\mathcal{O} \cup \partial\Omega$ and independent of $M$, then, by a simple regularity and compactness argument, $\theta_{[\lambda,n,\infty,\mathcal{O}]}$ should be a solution of

$$\begin{cases} -\nabla\cdot(D\nabla\theta - \alpha\theta\nabla m) - \theta(\lambda n - a\theta^{p-1}) = 0 & \text{in } \mathcal{O}, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega, \\ \theta = +\infty & \text{on } \partial\mathcal{O} \setminus \partial\Omega. \end{cases} \tag{5.12}$$

The existence of $U_{[\lambda,n,\mathcal{O}]}(x)$ is guaranteed by the next theorem.

**Theorem 5.2** *Suppose $\mathcal{O} \subset \Omega\setminus\bar{\Omega}_0$ is a smooth subdomain satisfying* (5.1)*, $n \in \mathcal{C}(\bar{\Omega})$, $\lambda \in \mathbb{R}$ and $a \in \mathcal{C}^2(\bar{\Omega})$. Then, there exists a function $U_{[\lambda,n,\mathcal{O}]} \in \mathcal{C}^2(\mathcal{O} \cup \partial\Omega)$ such that*

$$\theta_{[\lambda,n,M,\mathcal{O}]}(x) \leq U_{[\lambda,n,\mathcal{O}]}(x) \quad \textit{for each } x \in \mathcal{O} \cup \partial\Omega \quad \textit{and} \quad \textit{for all } M > 0.$$

*Proof.* The change of variable

$$\theta = e^{\frac{\alpha}{2D}m} v \tag{5.13}$$

transforms the problem (5.4) in

$$\begin{cases} -D\Delta v = W(\lambda, n)v - a e^{\frac{\alpha(p-1)}{2D}m} v^p & \text{in } \mathcal{O}, \\ 2D\partial_\nu v - \alpha v \partial_\nu m = 0 & \text{on } \partial\Omega, \\ v = e^{-\frac{\alpha}{2D}m} M & \text{on } \partial\mathcal{O} \setminus \partial\Omega, \end{cases} \tag{5.14}$$

where

$$W(\lambda, n) := \lambda n - \frac{\alpha}{2}\left(\Delta m + \frac{\alpha}{2D}|\nabla m|^2\right).$$

By Theorem 5.1, for each $M > 0$

$$v_{[\lambda,n,M,\mathcal{O}]}(x) := e^{-\frac{\alpha}{2D}m} \theta_{[\lambda,n,M,\mathcal{O}]}(x).$$

is the unique positive solution of (5.14).

As $a \in C^2(\bar{\Omega})$, there exists $b \in C^2(\bar{\Omega})$ such that

$$b \equiv 0 \quad \text{in } \Omega \setminus \mathcal{O} \supseteq \bar{\Omega}_0 \quad \text{and} \quad 0 < b \le a \quad \text{in } \mathcal{O} \cup \partial\Omega. \tag{5.15}$$

In case $\mathcal{O} = \Omega \setminus \bar{\Omega}_0$, we can choose $b(x) = a(x)$, of course.

Now, we fix a $M_0 > 0$ and consider the function $\tilde{V}_{[\lambda,n,\mathcal{O}]} \in C^2(\mathcal{O} \cup \partial\Omega)$ defined by

(i) $\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) = v_{[\lambda,n,M_0,\mathcal{O}]}(x)$ for $x \in \bar{\Omega}$ and near $\partial\Omega$.

(ii) $\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) = b^\beta(x)$ with $\beta = 3/(1-p) < 0$ for $x \in \mathcal{O}$ near $\partial\mathcal{O} \setminus \partial\Omega$.

(iii) $\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) > 0$ for $x \in \mathcal{O} \cup \partial\Omega$.

We claim that, for any sufficiently large positive constant $C > 0$,

$$V_{[\lambda,n,\mathcal{O}]} := C\tilde{V}_{[\lambda,n,\mathcal{O}]}$$

is a supersolution of (5.14) for all $M > M_0$. Therefore,

$$\theta_{[\lambda,n,M,\mathcal{O}]}(x) = e^{\frac{\alpha}{2D}m(x)} v_{[\lambda,n,M,\mathcal{O}]}(x) \le e^{\frac{\alpha}{2D}m(x)} V_{[\lambda,n,\mathcal{O}]}(x) =: U_{[\lambda,n,\mathcal{O}]}(x)$$

for each $x \in \mathcal{O} \cup \partial\Omega$, which ends the proof of the theorem. Indeed, since $\tilde{V}_{[\lambda,n,\mathcal{O}]} = v_{[\lambda,n,M_0,\mathcal{O}]}$ near $\partial\Omega$, we have that

$$2D\partial_\nu V_{[\lambda,n,\mathcal{O}]} - \alpha V_{[\lambda,n,\mathcal{O}]} \partial_\nu m = 0 \quad \text{on } \partial\Omega,$$

and

$$\begin{aligned} -D\Delta V_{[\lambda,n,\mathcal{O}]} &= -CD\Delta\tilde{V}_{[\lambda,n,\mathcal{O}]} = CW(\lambda,n)\tilde{V}_{[\lambda,n,\mathcal{O}]} - aCe^{\frac{\alpha(p-1)}{2D}m}\tilde{V}^p_{[\lambda,n,\mathcal{O}]} \\ &= W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - aC^{-(p-1)}e^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]} \\ &\ge W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]} \end{aligned}$$

in a neighborhood of $\partial\Omega$, provided $C > 1$.

On the other hand, for each $M > M_0$, if $x \in \mathcal{O}$ is sufficiently close to $\partial\mathcal{O} \setminus \partial\Omega$, we have that

$$V_{[\lambda,n,\mathcal{O}]}(x) = Cb^{\beta}(x) > e^{-\frac{\alpha}{2D}m(x)}M,$$

because $b = 0$ on $\partial\mathcal{O} \setminus \partial\Omega$ and $\beta < 0$. Using (5.15), a direct calculation shows that in a neighborhood of $\partial\mathcal{O} \setminus \partial\Omega$,

$$\begin{aligned}
-D\Delta V_{[\lambda,n,\mathcal{O}]} &- W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} + ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]} \\
&= Cb^{p\beta+1}\left[-D\beta b\Delta b - D\beta(\beta-1)|\nabla b|^2 - W(\lambda,n)b^2 + \frac{a}{b}e^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\right] \\
&\geq Cb^{p\beta+1}\left[-D\beta b\Delta b - D\beta(\beta-1)|\nabla b|^2 - W(\lambda,n)b^2 + e^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\right],
\end{aligned}$$

since $a/b \geq 1$ in $\mathcal{O}$. Thus, there exists $C_0 > 1$ such that

$$-D\Delta V_{[\lambda,n,\mathcal{O}]} \geq W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}$$

in a neighborhood of $\partial\mathcal{O} \setminus \partial\Omega$, for all $C > C_0$.

Finally, for any $x \in \mathcal{O}$ separated away from $\partial\mathcal{O}$, we have that

$$\begin{aligned}
-D\Delta V_{[\lambda,n,\mathcal{O}]}(x) &- W(\lambda,n)(x)V_{[\lambda,n,\mathcal{O}]}(x) + ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}(x) \\
&= C\left[-D\Delta\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) - W(\lambda,n)(x)\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) + ae^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\tilde{V}^p_{[\lambda,n,\mathcal{O}]}(x)\right] \geq 0
\end{aligned}$$

for sufficiently large $C > C_0$. Therefore, $V_{[\lambda,n,\mathcal{O}]}$ is a supersolution of (5.14) and the proof is complete. □

In the previous proof we have used a technical device of Y. Du and Q. Huang [7]. Owing to Theorem 5.2, the limit (5.11) is finite in $\mathcal{O} \cup \partial\Omega$ and $\theta_{[\lambda,n,\infty,\mathcal{O}]}$ is a solution of (5.12). The following result characterizes the existence of solutions of the singular problem (5.12).

**Theorem 5.3** *Under the same conditions of Theorem 5.2,* (5.12) *possesses a minimal and a maximal positive solution, denoted by* $L^{\min}_{[\lambda,n,\mathcal{O}]}$ *and* $L^{\max}_{[\lambda,n,\mathcal{O}]}$*, respectively, in the sense that any other positive solution* $L$ *of* (5.12) *must satisfy*

$$L^{\min}_{[\lambda,n,\mathcal{O}]} \leq L \leq L^{\max}_{[\lambda,n,\mathcal{O}]} \qquad \textit{in } \mathcal{O} \cup \partial\Omega. \tag{5.16}$$

*Proof.* Subsequently, for each $k \in \mathbb{N}$, we consider

$$\mathcal{O}_k := \{x \in \mathcal{O} \ : \ \operatorname{dist}(x, \partial\mathcal{O} \setminus \partial\Omega) > 1/k\}.$$

For $k$ large, $\mathcal{O}_k$ satisfies (5.1) and we can consider (5.12) in each of these $\mathcal{O}_k$'s, instead of $\mathcal{O}$. By our previous analysis, (5.12) in $\mathcal{O}_k$ admits a positive solution, $\theta_{[\lambda,n,\infty,\mathcal{O}_k]}$, for sufficiently large $k$. Then, the minimal and maximal solutions of (5.12) in $\mathcal{O}$ are given by

$$L^{\min}_{[\lambda,n,\mathcal{O}]} := \theta_{[\lambda,n,\infty,\mathcal{O}]} \quad \text{and} \quad L^{\max}_{[\lambda,n,\mathcal{O}]} := \lim_{k\to\infty} \theta_{[\lambda,n,\infty,\mathcal{O}_k]}.$$

Indeed, let $L$ be a positive solution of (5.12). By the maximum principle

$$\theta_{[\lambda,n,M,O]} \leq L \quad \text{for all} \;\; M > 0$$

and letting $M \to \infty$ yields

$$L^{\min}_{[\lambda,n,O]} := \theta_{[\lambda,n,\infty,O]} := \lim_{M\to\infty} \theta_{[\lambda,n,M,O]} \leq L.$$

Similarly, for sufficiently large $k > 0$,

$$L \leq \theta_{[\lambda,n,\infty,O_k]} \quad \text{in} \;\; O_k \cup \partial\Omega$$

and letting $k \to \infty$ shows that

$$L \leq \lim_{k\to\infty} \theta_{[\lambda,n,\infty,O_k]} =: L^{\max}_{[\lambda,n,O]} \quad \text{in} \;\; O \cup \partial\Omega.$$

This ends the proof. □

# 6 Dynamics of (1.1) when $m > 0$

In this section, we suppose (2.9) and $m > 0$, in the sense that $m \geq 0$ but $m \neq 0$. Our main goal is to ascertain the dynamics of (1.1) in this particular case.

As in the special case when $m = 0$ in $\Omega_0$, the dynamics of (1.1) it has been already described by Theorem 2.4, throughout this section we will assume that $m(x_+) > 0$ for some $x_+ \in \Omega_0$. Then, according to Theorems 2.1 and 2.3, there exists $\lambda^* := \lambda_2(\alpha, m) > 0$ such that

$$\Sigma(\lambda) = \sigma[-D\Delta - \alpha\nabla m\cdot\nabla - \lambda m, \mathfrak{N}, \Omega] \begin{cases} > 0 & \text{if } \; \lambda < 0, \\ = 0 & \text{if } \; \lambda = 0, \\ < 0 & \text{if } \; \lambda > 0. \end{cases}$$

and

$$\Sigma_0(\lambda) = \sigma[-D\Delta - \alpha\nabla m\cdot\nabla - \lambda m, \mathfrak{D}, \Omega_0] \begin{cases} > 0 & \text{if } \; \lambda < \lambda^*, \\ = 0 & \text{if } \; \lambda = \lambda^*, \\ < 0 & \text{if } \; \lambda > \lambda^*. \end{cases}$$

Then, thanks to Theorem 2.4, for every $\lambda \in (0, \lambda^*)$ the problem (1.2) possesses a unique positive solution, $\theta_\lambda := \theta_{\lambda,m}$. Moreover, owing to Theorem 4.1, if, in addition, $a$ is of class $C^1$ in a neighborhood of $\partial\Omega_0$, then

$$\lim_{\lambda\uparrow\lambda^*} \theta_\lambda = \infty \quad \text{uniformly in} \;\; \bar{\Omega}_0. \tag{6.1}$$

The next result provides us the convergence to $\infty$ of the solution of (1.1), $u_\lambda(x, t; u_0)$, as $t \uparrow \infty$, for all $x \in \bar{\Omega}_0$ and $\lambda \geq \lambda^*$.

**Theorem 6.1** *Suppose $m \geq 0$, $m(x_+) > 0$ for some $x_+ \in \Omega_0$, and $a(x)$ is of class $C^1$ in a neighborhood of $\partial\Omega_0$. Then, for every $\lambda \geq \lambda^*$,*

$$\lim_{t\to\infty} u_\lambda(\cdot, t; u_0) = \infty \quad \textit{uniformly in} \;\; \bar{\Omega}_0. \tag{6.2}$$

*Proof.* As $m>0$, by the parabolic maximum principle, for each $\varepsilon>0$ and $t\geq 0$,

$$u_\lambda(\cdot,t;u_0)\geq u_{\lambda^*-\varepsilon}(\cdot,t;u_0)\quad \text{in}\ \ \Omega$$

since $\lambda>\lambda^*-\varepsilon$. Hence, thanks to Theorem 2.4,

$$\liminf_{t\to\infty} u_\lambda(\cdot,t;u_0)\geq \lim_{t\to\infty} u_{\lambda^*-\varepsilon}(\cdot,t;u_0)=\theta_{\lambda^*-\varepsilon}\quad \text{in}\ \ \Omega.$$

Consequently, by Theorem 4.1,

$$\liminf_{t\to\infty} u_\lambda(\cdot,t;u_0)\geq \lim_{\varepsilon\downarrow 0}\theta_{\lambda^*-\varepsilon}=\infty\quad \text{uniformly in}\ \ \bar\Omega_0.$$

The proof is complete. □

The next result provides us with the dynamics of (1.1).

**Theorem 6.2** *Suppose $m\geq 0$, $m(x_+)>0$ for some $x_+\in\Omega_0$, and $a\in C^2(\bar\Omega)$. Then, the following assertions are true:*

(a) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0)=0$ *in* $C(\bar\Omega)$ *if* $\lambda\leq 0$.

(b) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0)=\theta_\lambda$ *in* $C(\bar\Omega)$ *if* $0<\lambda<\lambda^*$.

(c) *In case $\lambda\geq\lambda^*$, we have that:*

  (i) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0)=\infty$ *uniformly in* $\bar\Omega_0$.

  (ii) *In $\bar\Omega\setminus\bar\Omega_0$ the next estimates hold*

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]}\leq \liminf_{t\to\infty} u_\lambda(\cdot,t;u_0)\leq \limsup_{t\to\infty} u_\lambda(\cdot,t;u_0)\leq L^{\max}_{[\lambda,m,\Omega\setminus\bar\Omega_0]}.$$

  (iii) *If, in addition, $u_0$ is a subsolution of* (1.2)*, then*

$$\lim_{t\to\infty} u_\lambda(\cdot,t;u_0)=L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]}\quad \text{in}\ \ \bar\Omega\setminus\bar\Omega_0.$$

*Proof.* Parts (a) and (b) follow as a direct consequence from Theorem 2.4 in case $n=m>0$. Part (c)(i) is given by Theorem 6.1. So, it remains to prove Parts (c)(ii) and (c)(iii). Suppose $\lambda\geq\lambda^*$. By Theorem 6.1, for each $M>0$ there exists a constant $T_M>0$ such that

$$u_\lambda(x,t;u_0)\geq M\quad \text{for each}\ \ (x,t)\in\partial\Omega_0\times[T_M,\infty).$$

By the parabolic maximum principle, for each $(x,t)\in\bar\Omega\setminus\bar\Omega_0\times(0,\infty)$, we find that

$$u_\lambda(x,t+T_M;u_0)\geq u_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x,t;u_\lambda(\cdot,T_M;u_0)).$$

Therefore, for each $x\in\bar\Omega\setminus\bar\Omega_0$,

$$\liminf_{t\to\infty} u_\lambda(x,t;u_0)\geq \lim_{t\to\infty} u_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x,t;u_\lambda(\cdot,T_M;u_0))=\theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x). \tag{6.3}$$

Hence, owing to Theorem 5.3 and letting $M \to \infty$ in (6.3) yields

$$\liminf_{t\to\infty} u_\lambda(\cdot, t; u_0) \geq L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

which ends the proof of one of the inequalities.

Next, we will asume that $u_0$ is a subsolution of (1.2). Then, for each $t > 0$ the function $u_\lambda(\cdot, t; u_0)$ is a subsolution of (1.2) in $\Omega$, since $t \to u_\lambda(\cdot, t; u_0)$ is increasing. Fix $t > 0$ and set

$$M_t := \max_{\partial\Omega_0} u_\lambda(\cdot, t; u_0).$$

Then, for every $M \geq M_t$,

$$u_\lambda(\cdot, t; u_0) \leq \theta_{[\lambda,m,M,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

because $u_\lambda(\cdot, t; u_0)$ is a subsolution of the problem (5.4). Therefore,

$$u_\lambda(\cdot, t; u_0) \leq \lim_{M\to\infty} \theta_{[\lambda,m,M,\Omega\setminus\bar{\Omega}_0]} = L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

and letting $t \to \infty$,

$$\limsup_{t\to\infty} u_\lambda(\cdot, t; u_0) \leq L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0.$$

Therefore, Part (c)(iii) get proven.

To conclude the proof, it remains to obtain the upper estimates for an arbitrary $u_0 > 0$. The strategy adopted here to get these estimates consists in obtaining bounds in $\bar{\Omega} \setminus \bar{\Omega}_0$ for $u_\lambda(\cdot, t; u_0)$. Those bounds can be derived as follows. Fix $\lambda \geq \lambda^*$, set $m_M := \max_{\bar{\Omega}} m > 0$, and let $\varphi_\lambda$ denote the principal eigenfunction associated with the eigenvalue problem

$$\begin{cases} -\nabla \cdot (D\nabla\psi - \alpha\psi\nabla m) - \lambda m_M \psi = \sigma\psi & \text{in } \Omega, \\ D\partial_\nu\psi - \alpha\psi\partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized so that $\|\varphi_\lambda\|_\infty = 1$. Since $\varphi_\lambda(x) > 0$ for all $x \in \bar{\Omega}$, there exists $\kappa > 1$ such that

$$u_0 < \kappa\varphi_\lambda. \tag{6.4}$$

Subsequently, we denote

$$\Sigma_{m_M}(\lambda) := \sigma[-\nabla \cdot (D\nabla - \alpha\nabla m) - \lambda m_M, D\partial_\nu - \alpha\partial_\nu m, \Omega].$$

Thanks to (2.6), $\Sigma_{m_M}(\lambda) < 0$ because $m_M > 0$ and $\lambda \geq \lambda^* > 0$. Let $\Lambda > \lambda$ be such that

$$\|a\|_\infty \kappa^{p-1} + \Sigma_{m_M}(\lambda) \leq (\Lambda - \lambda) m_M.$$

For this choice, we find that for each $x \in \Omega$

$$\Sigma_{m_M}(\lambda)\kappa\varphi_\lambda + (\lambda - \Lambda) m_M \kappa\varphi_\lambda \leq -a(x)\kappa^p\varphi_\lambda \leq -a(x)\kappa^p\varphi_\lambda^p$$

because $\varphi_\lambda \leq 1$. Equivalently,

$$-\nabla \cdot [D\nabla(\kappa\varphi_\lambda) - \alpha\kappa\varphi_\lambda\nabla m] - \Lambda m_M \kappa\varphi_\lambda \leq -a(x)\kappa^p\varphi_\lambda^p.$$

Moreover, $\kappa\varphi_\lambda$ satisfies

$$D\partial_\nu(\kappa\varphi_\lambda) - \alpha\kappa\varphi_\lambda\partial_\nu m = 0 \quad \text{on} \ \partial\Omega.$$

Therefore, $\kappa\varphi_\lambda$ provides us with a subsolution of the problem

$$\begin{cases} -\nabla\cdot(D\nabla\theta - \alpha\theta\nabla m) - \theta(\Lambda m_M - a\theta^{p-1}) = 0 & \text{in} \ \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on} \ \partial\Omega. \end{cases}$$

Thus, by the parabolic maximum principle, it follows from (6.4) that

$$u_\lambda(x,t;u_0) \le u_\lambda(x,t;\kappa\varphi_\lambda) \tag{6.5}$$

for all $(x,t) \in \Omega\times(0,\infty)$. Similarly, since $\lambda m \le \Lambda m_M$,

$$u_\lambda(x,t;\kappa\varphi_\lambda) \le u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda) \quad \text{for each} \ (x,t)\in\Omega\times(0,\infty). \tag{6.6}$$

Therefore, by (6.5) and (6.6), for each $(x,t) \in \Omega\times(0,\infty)$, we have that

$$\lim_{t\to\infty} u_\lambda(x,t;u_0) \le \lim_{t\to\infty} u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda).$$

Lastly, since $m \le m_M$,

$$\lambda_2(\alpha,m_M) \le \lambda_2(\alpha,m) = \lambda^* \le \lambda < \Lambda$$

and, according to Part (c)(iii), we find that

$$\lim_{t\to\infty} u_\lambda(x,t;u_0) \le \lim_{t\to\infty} u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda) = L^{\min}_{[\Lambda,m_M,\Omega\setminus\bar\Omega_0]} \quad \text{in} \ \bar\Omega\setminus\bar\Omega_0. \tag{6.7}$$

Consequently, $u_\lambda(x,t;u_0)$ is uniformly bounded above in any compact subset of $\bar\Omega\setminus\bar\Omega_0$ for each $t>0$, which provides us with the necessary a priori bounds to complete the proof of the theorem.

Subsequently, for sufficiently large $k\in\mathbb{N}$ we consider

$$\Omega_k := \left\{x\in\Omega\setminus\bar\Omega_0 \ : \ d(x,\Omega_0) < 1/k\right\}.$$

Fix one of these values of $k$. Since $\partial\Omega_k \subset \Omega\setminus\bar\Omega_0$, it follows from (6.7) that there exists a constant $M_0>0$ such that, for each $M\ge M_0$ and $t>0$,

$$u_\lambda(\cdot,t;u_0) \le M \quad \text{on} \ \partial\Omega_k,$$

and hence, the parabolic maximum principle shows that

$$u_\lambda(\cdot,t;u_0) \le u_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]}(\cdot,t;u_0) \quad \text{in} \ \bar\Omega\setminus\Omega_k \tag{6.8}$$

for all $t>0$. By Theorem 5.1, (5.4) has a unique positive solution, $\theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]}$, which is a global attractor of (5.3). Letting $t\to\infty$ in (6.8) yields

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \le \theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]} \quad \text{in} \ \bar\Omega\setminus\Omega_k.$$

Consequently, taking limits as $M\to\infty$ gives

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \le L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_k]} \quad \text{in} \ \bar\Omega\setminus\Omega_k. \tag{6.9}$$

Finally, thanks to the proof of Theorem 5.3, letting $k\to\infty$ in (6.9) yields

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \le L^{\max}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \text{in} \ \bar\Omega\setminus\bar\Omega_0$$

and the upper estimate is proven. □

# 7 Dynamics of (1.1) when $m$ changes sign

Throughout this section, we suppose that $m$ changes sign in $\Omega$, i.e., there are $x_- \in \Omega$ and $x_+ \in \Omega$ such that

$$m(x_-) < 0 \quad \text{and} \quad m(x_+) > 0.$$

The main goal of this section is to obtain the dynamics of (1.1) for all $\lambda > 0$. In the special case when that $m \leq 0$ in $\Omega_0$, according to Theorems 2.1 and 2.4, the existence of $\theta_\lambda$ is guaranteed for $\lambda > \lambda_+ \geq 0$ and it is a global attractor for (1.1). So, we suppose that

$$x_+ \in \Omega_0.$$

By Theorem 2.3, there exists $\lambda^* := \lambda_2(\alpha, m) > 0$ such that

$$\Sigma_0(\lambda) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m, \mathfrak{D}, \Omega_0] \begin{cases} > 0 & \text{if } \lambda \in [0, \lambda^*), \\ = 0 & \text{if } \lambda = \lambda^*, \\ < 0 & \text{if } \lambda > \lambda^*. \end{cases}$$

Subsequently, for sufficiently small $\varepsilon > 0$, we introduce the truncated functions

$$m_\varepsilon(x) := \begin{cases} \varepsilon & \text{if } m(x) \geq \varepsilon, \\ m(x) & \text{if } m(x) < \varepsilon. \end{cases}$$

By Theorem 2.4, the limiting behavior of $u_{[\lambda, m_\varepsilon]}(x, t; u_0)$ as $t \to \infty$ is regulated by the positive solution $\theta_{[\lambda, m_\varepsilon]}$ whenever

$$0 \leq \tilde{\lambda}_+(\alpha, m_\varepsilon) < \lambda < \lambda_2(\alpha, m_\varepsilon) \equiv \lambda^*_\varepsilon \tag{7.1}$$

where $\tilde{\lambda}_+(\alpha, m_\varepsilon) := \lambda_+(\alpha, m_\varepsilon)$ if $\lambda_+(\alpha, m_\varepsilon)$ exists, while it equals zero if not. It should be remembered that $\lambda_+(\alpha, m_\varepsilon)$ is the unique positive zero of the principal eigenvalue

$$\Sigma(\lambda, m_\varepsilon) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m_\varepsilon, \mathfrak{N}, \Omega]$$

if it exists. Note that

$$\lim_{\varepsilon \to 0} \tilde{\lambda}_+(\alpha, m_\varepsilon) = \lim_{\varepsilon \to 0} \lambda_+(\alpha, m_\varepsilon) = \infty.$$

Fix $\hat{\lambda} \geq \lambda^*$. By the continuity of the principal eigenvalue with respect to $\varepsilon$, there exists $\varepsilon_0 := \varepsilon_0(\hat{\lambda}) > 0$ such that

$$\lambda_+(\alpha, m_{\varepsilon_0}) = \hat{\lambda}.$$

Similarly, since

$$\lim_{\varepsilon \to 0} \lambda_2(\alpha, m_\varepsilon) = \infty \quad \text{and} \quad \hat{\lambda} \geq \lambda^*,$$

there exists $\varepsilon_1 := \varepsilon_1(\hat{\lambda}) > \varepsilon_0$ such that $\lambda^*_{\varepsilon_1} = \hat{\lambda}$. Note that $\varepsilon_1 = \max m$, or, equivalently, $m_{\varepsilon_1} = m$, if $\hat{\lambda} = \lambda^*$. According to (7.1), for each $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, the problem (2.2) possesses a unique positive solution, $\theta_{[\hat{\lambda}, m_\varepsilon]}$. Figure 4 shows the graphs of $\Sigma(\lambda, m_\varepsilon)$ and $\Sigma_0(\lambda, m_\varepsilon)$ for $\varepsilon = 0$ and some $\varepsilon \in (\varepsilon_0, \varepsilon_1)$.

The next result provides us with the limiting behavior of $\theta_{[\hat{\lambda}, m_\varepsilon]}$ in $\bar{\Omega}_0$ as $\varepsilon \uparrow \varepsilon_1$.

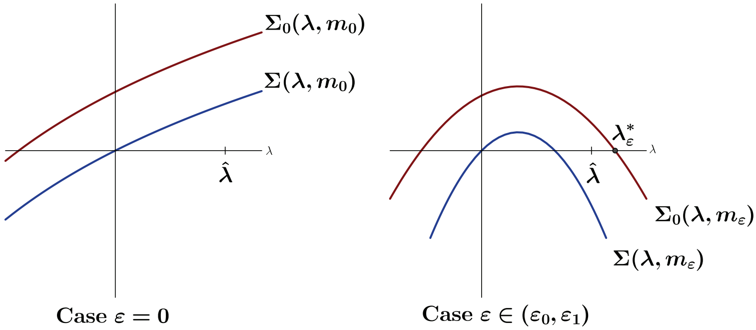


Figure 4: The graphs of $\Sigma(\lambda, m_\varepsilon)$ and $\Sigma_0(\lambda, m_\varepsilon)$

**Theorem 7.1** *Suppose $a(x)$ is of class $C^1$ in a neighborhood of $\partial\Omega_0$. Then*

$$\lim_{\varepsilon\uparrow\varepsilon_1} \theta_{[\hat\lambda, m_\varepsilon]}(x) = +\infty \quad \textit{uniformly in} \;\; x \in \bar\Omega_0.$$

*Proof.* We will argue as in the proof of the Theorem 4.1. For $\delta > 0$ with $\delta \simeq 0$, consider the holomorphic perturbation from $\Omega_0$, $\Omega_\delta$, defined in (3.17), as well as the principal eigenvalue

$$\begin{aligned} S(\delta) &:= \sigma[-D\Delta - \alpha\nabla m\cdot\nabla - \hat\lambda m_{\varepsilon_1}, \mathfrak{D}, \Omega_\delta] \\ &= \sigma[-\nabla\cdot(D\nabla - \alpha\nabla m) - \hat\lambda m_{\varepsilon_1}, \mathfrak{D}, \Omega_\delta]. \end{aligned}$$

Let $\varphi[\delta] > 0$ be the unique positive solution of

$$\begin{cases} -\nabla\cdot(D\nabla\varphi[\delta] - \alpha\varphi[\delta]\nabla m) - \hat\lambda m_{\varepsilon_1}\varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \; \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \; \partial\Omega_\delta, \end{cases} \tag{7.2}$$

satisfying

$$\|\varphi[\delta]\|_{\infty,\Omega_\delta} = 1.$$

Applying (3.4) and (3.18) to the weight function $n = m_{\varepsilon_1}$ yields

$$S(\delta) = S_1\delta + O(\delta^2) \quad \text{with} \;\; S_1 < 0.$$

Let $\varepsilon \in (\varepsilon_0, \varepsilon_1)$ such that

$$S(\delta) < S(\delta/2) < \hat\lambda(\varepsilon - \varepsilon_1) < 0 \tag{7.3}$$

and consider the function $\vartheta_\delta \in C(\bar\Omega)$ defined by

$$\vartheta_\delta(y) = \begin{cases} C\varphi[\delta](y) & \text{for } \; y \in \bar\Omega_\delta, \\ 0 & \text{for } \; y \notin \Omega_\delta, \end{cases}$$

where $C > 0$ is a constant to be chosen later. If we suppose that

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) \le S(\delta/2) - S(\delta) \quad \text{for all } \; y \in \Omega_\delta \tag{7.4}$$

then $\vartheta_\delta$ is a subsolution of problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \hat{\lambda} m_\varepsilon \theta = -a\theta^p & \text{in } \Omega, \\ D\partial_\nu \theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Indeed, by (7.3), for every $y \in \Omega_\delta$

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) < \hat{\lambda}(\varepsilon - \varepsilon_1) - S(\delta).$$

Hence, since $\varepsilon - \varepsilon_1 \leq m_\varepsilon - m_{\varepsilon_1}$,

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) < \hat{\lambda}(m_\varepsilon - m_{\varepsilon_1}) - S(\delta).$$

Multiplying by $C\varphi[\delta]$, it follows from (7.2) that $\vartheta_\delta$ is a subsolution of problem (7.5).

As in the proof of Theorem 4.1, (7.4) holds if $C = C(\delta)$ is given by (4.8). Therefore,

$$\lim_{\delta \downarrow 0} \vartheta_\delta(y) = +\infty \quad \text{for all } y \in \bar{\Omega}_0. \tag{7.6}$$

Finally, by the maximum principle, for each $\delta > 0$, $\delta \simeq 0$, satisfying (7.3), we have that

$$\vartheta_\delta(y) \leq u_{[\hat{\lambda}, m_\varepsilon]}(y) \quad \text{for all } y \in \bar{\Omega}.$$

Consequently, (7.6) ends the proof. □

The next result, provides us the behavior of $u_{\hat{\lambda}}(x,t;u_0)$ in $\bar{\Omega}_0$ as $t \uparrow \infty$. It is a counterpart of Theorem 6.1 for the case dealt with in this section.

**Theorem 7.2** *If $a(x)$ is of class $C^1$ in a neighborhood of $\partial\Omega_0$, then*

$$\lim_{t\to\infty} u_{\hat{\lambda}}(x,t;u_0) = +\infty \quad \textit{uniformly in } \bar{\Omega}_0.$$

*Proof.* Let $\varepsilon \in (\varepsilon_0, \varepsilon_1)$. Since $\hat{\lambda} > 0$ and $m \geq m_\varepsilon$ in $\Omega$, by the parabolic maximum principle, we find that

$$u_{\hat{\lambda}}(\cdot,t;u_0) \geq u_{[\hat{\lambda}, m_\varepsilon]}(\cdot,t;u_0) \quad \text{in } \Omega$$

for all $t > 0$. Thus, letting $t \to \infty$ yields

$$\liminf_{t\to\infty} u_{\hat{\lambda}}(\cdot,t;u_0) \geq \lim_{t\to\infty} u_{[\hat{\lambda}, m_\varepsilon]}(\cdot,t;u_0) = \theta_{[\hat{\lambda}, m_\varepsilon]} \quad \text{in } \Omega.$$

As this holds for all $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, from Theorem 7.1 it becomes apparent that

$$\liminf_{t\to\infty} u_{\hat{\lambda}}(\cdot,t;u_0) \geq \lim_{\varepsilon \uparrow \varepsilon_1} \theta_{[\hat{\lambda}, m_\varepsilon]} = \infty \quad \text{uniformly in } \bar{\Omega}_0$$

and the proof of the theorem is complete. □

Finally, the next theorem provides us with the dynamics of (1.1) for the class of $m$'s dealt with in this section. As the proof follows the general patterns of the proof of Theorem 6.2, we will omit the technical details here. It should be emphasized that Theorem 7.2 holds true for all $\hat{\lambda} \geq \lambda^*$.

**Theorem 7.3** *If $a \in C^2(\bar{\Omega})$, then:*

(a) $\lim_{t\to\infty} u_\lambda(\cdot, t; u_0) = 0$ *in* $C(\bar{\Omega})$ *if* $0 \le \lambda \le \tilde{\lambda}_+(\alpha, m)$.

(b) $\lim_{t\to\infty} u_\lambda(\cdot, t; u_0) = \theta_\lambda$ *in* $C(\bar{\Omega})$ *if* $\tilde{\lambda}_+(\alpha, m) < \lambda < \lambda^*$.

(c) *In case $\lambda \ge \lambda^*$, the following assertions are true:*

(i) $\lim_{t\to\infty} u_\lambda(\cdot, t; u_0) = +\infty$ *uniformly in* $\bar{\Omega}_0$.

(ii) *In $\bar{\Omega} \setminus \bar{\Omega}_0$ the following estimate holds*

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \le \liminf_{t\to\infty} u_\lambda(\cdot, t; u_0) \le \limsup_{t\to\infty} u_\lambda(\cdot, t; u_0) \le L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}.$$

(iii) *If, in addition, $u_0$ is a subsolution of* (1.2)*, then*

$$\lim_{t\to\infty} u_\lambda(\cdot, t; u_0) = L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad in \ \ \bar{\Omega} \setminus \bar{\Omega}_0.$$

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