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Borderline Variational Problems Involving Fractional Laplacians and Critical Singularities

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Abstract

We consider the problem of attainability of the best constant $C > 0$ in the following critical fractional Hardy-Sobolev inequality: For all $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$C\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx,$$

where $0 \leq s < \alpha < 2$, $n > \alpha$, $2^*_\alpha(s) := \frac{2(n-s)}{n-\alpha}$ and $\gamma \in \mathbb{R}$. This allows us to establish the existence of nontrivial weak solutions for the following doubly critical problem on $\mathbb{R}^n$,

$$(-\Delta)^{\frac{\alpha}{2}} u - \gamma \frac{u}{|x|^\alpha} = |u|^{2^*_\alpha - 2} u + \frac{|u|^{2^*_\alpha(s)-2} u}{|x|^s} \quad \text{in } \mathbb{R}^n,$$

where $2^*_\alpha := \frac{2n}{n-\alpha}$ is the critical α-fractional Sobolev exponent, and $\gamma < \gamma_H := 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$, the latter being the best fractional Hardy constant on $\mathbb{R}^n$.

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1 Introduction

We consider the problem of existence of nontrivial weak solutions to the following doubly critical problem on $\mathbb{R}^n$ involving the Fractional Laplacian:

$$(-\Delta)^{\frac{\alpha}{2}}u-\gamma\frac{u}{|x|^{\alpha}}=|u|^{2^*_\alpha-2}u+\frac{|u|^{2^*_\alpha(s)-2}u}{|x|^s}\quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $0\leq s<\alpha<2$, $n>\alpha$, $2^*_\alpha:=\frac{2n}{n-\alpha}$, $2^*_\alpha(s):=\frac{2(n-s)}{n-\alpha}$ and $\gamma\in\mathbb{R}$. The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined on the Schwartz class (space of rapidly decaying C^∞ functions in $\mathbb{R}^n$) through the Fourier transform,

$$(-\Delta)^{\frac{\alpha}{2}}u=\mathcal{F}^{-1}(|\xi|^{\alpha}(\mathcal{F}u))\quad \forall\xi\in\mathbb{R}^n,$$

where $\mathcal{F}u$ denotes the Fourier transform of u, $\mathcal{F}u(\xi)=\int_{\mathbb{R}^n}e^{-2\pi ix.\xi}u(x)dx$. See [11] and references therein for the basics on the fractional Laplacian.

Problems involving two non-linearities have been studied in the case of local operators such as the Laplacian $-\Delta$, the p-Laplacian $-\Delta_p$ and the Biharmonic operator Δ^2 (See [4], [17], [25] and [36]). Problem (1.1) above is the non-local counterpart of the one studied by Filippucci-Pucci-Robert in [17], who treated the case of the p-Laplacian in an equation involving both the Sobolev and the Hardy-Sobolev critical exponents.

Questions of existence and non-existence of solutions for fractional elliptic equations with singular potentials were recently studied by several authors. All studies focus, however, on problems with only one critical exponent –mostly the non-linearity $u^{2^*_\alpha-1}$– and to a lesser extent the critical Hardy-Sobolev singular term $\frac{u^{2^*_\alpha(s)-1}}{|x|^s}$ (see [10], [16], [37] and the references therein). These cases were also studied on smooth bounded domains (see for example [2], [3], [5], [15], [34] and the references therein). In general, the case of two critical exponents involve more subtleties and difficulties, even for local differential operators.

The variational approach that we adopt here, relies on the following fractional Hardy-Sobolev type inequality:

$$C\Big(\int_{\mathbb{R}^n}\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*_\alpha(s)}}\leq\int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx-\gamma\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^{\alpha}}dx\quad\text{for all } u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{1.2}$$

where $\gamma<\gamma_H:=2^{\alpha}\frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$ is the best fractional Hardy constant on $\mathbb{R}^n$. The fractional space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}|2\pi\xi|^{\alpha}|\mathcal{F}u(\xi)|^2d\xi=\int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx.$$

The best constant in the above fractional Hardy-Sobolev inequality is defined as:

$$\mu_{\gamma,s}(\mathbb{R}^n):=\inf_{u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}|(-\Delta)^{\frac{\alpha}{4}}u|^2dx-\gamma\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^{\alpha}}dx}{\Big(\int_{\mathbb{R}^n}\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx\Big)^{\frac{2}{2^*_\alpha(s)}}}. \tag{1.3}$$

One step towards addressing Problem (1.1) consists of proving the existence of extremals for $\mu_{\gamma,s}(\mathbb{R}^n)$, when $s\in[0,\alpha)$ and $\gamma\in(-\infty,\gamma_H)$. Note that the Euler-Lagrange equation corresponding to the minimization problem for $\mu_{\gamma,s}(\mathbb{R}^n)$ is –up to a constant factor– the following:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u - \gamma\frac{u}{|x|^\alpha} = \frac{u^{2^*_\alpha(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

When $\alpha = 2$, i.e., in the case of the standard Laplacian, the above minimization problem (1.3) has been extensively studied. See for example [8], [9], [17], [19], [21] and [22]. The non-local case has also been the subject of several studies, but in the absence of the Hardy term, i.e., when $\gamma = 0$. In [16], Fall, Minlend and Thiam proved the existence of extremals for $\mu_{0,s}(\mathbb{R}^n)$ in the case $\alpha = 1$. Recently, J. Yang in [37] proved that there exists a positive, radially symmetric and non-increasing extremal for $\mu_{0,s}(\mathbb{R}^n)$ when $\alpha \in (0,2)$. Asymptotic properties of the positive solutions were given by Y. Lei [26], Lu and Zhu [31], and Yang and Yu [38].

In section 3, we consider the remaining cases in the problem of deciding whether the best constant in the fractional Hardy-Sobolev inequality $\mu_{\gamma,s}(\mathbb{R}^n)$ is attained. We use Ekeland's variational principle to show the following.

Theorem 1.1. *Suppose* $0 < \alpha < 2$, $0 \le s < \alpha < n$ *and* $\gamma < \gamma_H := 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$.

1. *If either* $\{s > 0\}$ *or* $\{s = 0$ *and* $\gamma \ge 0\}$*, then* $\mu_{\gamma,s}(\mathbb{R}^n)$ *is attained.*
2. *If* $s = 0$ *and* $\gamma < 0$*, then there are no extremals for* $\mu_{\gamma,s}(\mathbb{R}^n)$.
3. *If either* $\{0 < \gamma < \gamma_H\}$ *or* $\{0 < s < \alpha$ *and* $\gamma = 0\}$*, then any non-negative minimizer for* $\mu_{\gamma,s}(\mathbb{R}^n)$ *is positive, radially symmetric, radially decreasing, and approaches zero as* $|x| \to \infty$.

In section 4, we consider problem (1.1) and use the mountain pass lemma to establish the following result.

Theorem 1.2. *Let* $0 < \alpha < 2$, $0 < s < \alpha < n$ *and* $0 \le \gamma < \gamma_H$*. Then, there exists a nontrivial weak solution of (1.1).*

Recall that $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a weak solution of (1.1), if we have for all $\varphi \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}}u(-\Delta)^{\frac{\alpha}{4}}\varphi dx = \int_{\mathbb{R}^n} \gamma\frac{u}{|x|^\alpha}\varphi dx + \int_{\mathbb{R}^n} |u|^{2^*_\alpha-2}u\varphi dx + \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)-2}u}{|x|^s}\varphi dx.$$

The standard strategy to construct weak solutions of (1.1) is to find critical points of the corresponding functional on $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. However, (1.1) is invariant under the following conformal one parameter transformation group,

$$T_r : H^{\frac{\alpha}{2}}(\mathbb{R}^n) \to H^{\frac{\alpha}{2}}(\mathbb{R}^n); \qquad u(x) \to T_r[u](x) = r^{\frac{n-\alpha}{2}}u(rx) \quad r > 0, \tag{1.5}$$

which means that the convergence of Palais-Smale sequences is not a given. As it was argued in [17], there is an asymptotic competition between the energy carried by the two critical nonlinearities. Hence, the crucial step here is to balance the competition to avoid the domination of one term over another. Otherwise, there is vanishing of the weakest one, leading to a solution for the same equation but with only one critical nonlinearity. In order to deal with this issue, we choose a suitable minimax energy level, in such a way that after a careful analysis of the concentration phenomena, we could eliminate the possibility of a vanishing weak limit for these well chosen Palais-Smale sequences, while ensuring that none of the two nonlinearities dominate the other.

2 Preliminaries and a description of the functional setting

We start by recalling and introducing suitable function spaces for the variational principles that will be needed in the sequel. We first recall the following useful representation given in [2] and [5] for the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ as a trace class operator, as well as for the space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$.

For a function $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, let $w = E_\alpha(u)$ be its α-harmonic extension to the upper half-space, $\mathbb{R}^{n+1}_+$, that is the solution to the following problem:

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w = u & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases}$$

Define the space $X^\alpha(\mathbb{R}^{n+1}_+)$ as the closure of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ for the norm

$$\|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} := \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy\right)^{\frac{1}{2}},$$

where $k_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$ is a normalization constant chosen in such a way that the extension operator $E_\alpha(u) : H^{\frac{\alpha}{2}}(\mathbb{R}^n) \to X^\alpha(\mathbb{R}^{n+1}_+)$ is an isometry, that is, for any $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, we have

$$\|E_\alpha(u)\|_{X^\alpha(\mathbb{R}^{n+1}_+)} = \|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|(-\Delta)^{\frac{\alpha}{4}} u\|_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

Conversely, for a function $w \in X^\alpha(\mathbb{R}^{n+1}_+)$, we denote its trace on $\mathbb{R}^n \times \{y = 0\}$ as $\mathrm{Tr}(w) := w(.,0)$. This trace operator is also well defined and satisfies

$$\|w(.,0)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} \le \|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)}. \tag{2.2}$$

We shall frequently use the following useful fact: Since $\alpha \in (0,2)$, the weight $y^{1-\alpha}$ belongs to the Muckenhoupt class A_2; [32], which consists of all non-negative functions w on $\mathbb{R}^n$ satisfying for some constant C, the estimate

$$\sup_B\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{-1} dx\right) \le C, \tag{2.3}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

If $\Omega \subset \mathbb{R}^{n+1}$ is an open domain, we denote by $L^2(\Omega, |y|^{1-\alpha})$ the space of all measurable functions on Ω such that $\|w\|^2_{L^2(\Omega,|y|^{1-\alpha})} = \int_\Omega |y|^{1-\alpha}|w|^2 dxdy < \infty$, and by $H^1(\Omega, |y|^{1-\alpha})$ the weighted Sobolev space

$$H^1(\Omega, |y|^{1-\alpha}) = \left\{w \in L^2(\Omega, |y|^{1-\alpha}) : \nabla w \in L^2(\Omega, |y|^{1-\alpha})\right\}.$$

It is remarkable that most of the properties of classical Sobolev spaces, including the embedding theorems have a weighted counterpart as long as the weight is in the Muckenhoupt class A_2; see [14] and [23]. Note that $H^1(\mathbb{R}^{n+1}_+, y^{1-\alpha})$ – up to a normalization factor– is also isometric to $X^\alpha(\mathbb{R}^{n+1}_+)$.

In [7], Caffarelli and Silvestre showed that the extension function $E_\alpha(u)$ is related to the fractional Laplacian of the original function u in the following way:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{\partial w}{\partial \nu^\alpha} := -k_\alpha \lim_{y\to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y).$$

With this representation, the non-local problem (1.1) can then be written as the following local problem:

$$\begin{cases} -\mathrm{div}\,(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial w}{\partial \nu^\alpha} = \gamma \frac{w(.,0)}{|x|^\alpha} + w(.,0)^{2^*_\alpha - 1} + \frac{w(.,0)^{2^*_\alpha(s)-1}}{|x|^s} & \text{on } \mathbb{R}^n. \end{cases} \tag{2.4}$$

A function $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ is said to be a weak solution to (2.4), if for all $\varphi \in X^\alpha(\mathbb{R}^{n+1}_+)$,

$$\begin{aligned} k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle dxdy &= \gamma \int_{\mathbb{R}^n} \frac{w(x,0)}{|x|^\alpha}\varphi dx + \int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha-2} w(x,0)\varphi dx \\ &\quad + \int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)-2} w(x,0)}{|x|^s}\varphi dx. \end{aligned}$$

Note that for any weak solution w in $X^\alpha(\mathbb{R}^{n+1}_+)$ to (2.4), the function $u = w(.,0)$ defined in the sense of traces, is in $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and is a weak solution to problem (1.1). The energy functional corresponding to (2.4) is

$$\begin{aligned} \Phi(w) = \frac{1}{2}\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \frac{\gamma}{2}\int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha}dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha}\,dx \\ - \frac{1}{2^*_\alpha(s)}\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx. \end{aligned}$$

Hence the associated trace of any critical point w of Φ in $X^\alpha(\mathbb{R}^{n+1}_+)$ is a weak solution for (1.1).

The starting point of the study of existence of weak solutions of the above problems is therefore the following fractional trace inequalities which will guarantee that the above functionals are well defined and bounded below on the right function spaces. We start with the fractional Sobolev inequality [10], which asserts that for $n > \alpha$ and $0 < \alpha < 2$, there exists a constant $C(n,\alpha) > 0$ such that

$$\Big(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq C(n,\alpha)\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \text{for all } u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{2.5}$$

where $2^*_\alpha = \frac{2n}{n-\alpha}$. Another important inequality is the fractional Hardy inequality (see [18] and [24]), which states that under the same conditions on n and α, we have

$$\gamma_H \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \text{for all } u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n), \tag{2.6}$$

where γ_H is the best constant in the above inequality on $\mathbb{R}^n$, that is

$$\gamma_H = \gamma_H(\alpha) := \inf\left\{ \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx};\ u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\} \right\}. \tag{2.7}$$

It has also been shown there that $\gamma_H(\alpha) = 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$. Note that $\gamma_H(\alpha)$ converges to the best classical Hardy constant $\gamma_H(2) = \frac{(n-2)^2}{4}$ when $\alpha \to 2$.

By interpolating these inequalities via Hölder's inequality, one gets the following fractional Hardy-Sobolev inequalities.

Lemma 2.1 (Fractional Hardy-Sobolev Inequalities)**.** *Assume that* $0 < \alpha < 2$ *and* $0 \le s \le \alpha < n$*. Then, there exist positive constants c and C, such that*

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}} \le c \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \quad \textit{for all } u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n). \tag{2.8}$$

Moreover, if $\gamma < \gamma_H := 2^\alpha \frac{\Gamma^2(\frac{n+\alpha}{4})}{\Gamma^2(\frac{n-\alpha}{4})}$*, then*

$$C\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}} \le \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx \quad \textit{for all } u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n). \tag{2.9}$$

Proof. Note that for s = 0 (resp., s = α) the first inequality is just the fractional Sobolev (resp., the fractional Hardy) inequality. We therefore have to only consider the case where $0 < s < \alpha$ in which case $2^*_\alpha(s) > 2$. By applying Hölder's inequality, then the fractional Hardy and the fractional Sobolev inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx &= \int_{\mathbb{R}^n} \frac{|u|^{\frac{2s}{\alpha}}}{|x|^s} |u|^{2^*_\alpha(s) - \frac{2s}{\alpha}} dx \\
&\le \Big(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx\Big)^{\frac{s}{\alpha}} \Big(\int_{\mathbb{R}^n} |u|^{(2^*_\alpha(s) - \frac{2s}{\alpha})\frac{\alpha}{\alpha - s}} dx\Big)^{\frac{\alpha - s}{\alpha}} \\
&= \Big(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx\Big)^{\frac{s}{\alpha}} \Big(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx\Big)^{\frac{\alpha - s}{\alpha}} \\
&\le C_1\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx\Big)^{\frac{s}{\alpha}} C_2\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx\Big)^{\frac{2^*_\alpha}{2} \cdot \frac{\alpha - s}{\alpha}} \\
&\le c\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx\Big)^{\frac{n-s}{n-\alpha}} \\
&= c\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx\Big)^{\frac{2^*_\alpha(s)}{2}}.
\end{aligned}$$

From the definition of γ_H, it follows that for all $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$\frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}}} \ge \Big(1 - \frac{\gamma}{\gamma_H}\Big) \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx}{\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}}}.$$

Hence (2.8) implies (2.9) whenever $\gamma < \gamma_H$.

Remark 2.1. One can use (2.1) to rewrite inequalities (2.6), (2.8) and (2.9) as the following trace class inequalities:

$$\gamma_H \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha}\, dx \leq \ \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}, \tag{2.10}$$

$$\Big(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}\, dx\Big)^{\frac{2}{2^*_\alpha(s)}} \leq c\, \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}, \tag{2.11}$$

$$C\Big(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s}\, dx\Big)^{\frac{2}{2^*_\alpha(s)}} \leq \ \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx. \tag{2.12}$$

The best constant $\mu_{\gamma,s}(\mathbb{R}^n)$ in inequality (2.9), can also be written as:

$$S(n,\alpha,\gamma,s) = \inf_{w\in X^\alpha(\mathbb{R}^{n+1}_+)\setminus\{0\}} \frac{k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx}{\Big(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}}}.$$

We shall therefore investigate whether there exist extremal functions where this best constant is attained. Theorems 1.1 and 1.2 can therefore be stated in the following way:

Theorem 2.1. *Suppose* $0 < \alpha < 2$, $0 \leq s < \alpha < n$ *and* $\gamma < \gamma_H$. *We then have the following:*

1. *If* $\{s > 0\}$ *or* $\{s = 0$ *and* $\gamma \geq 0\}$, *then* $S(n,\alpha,\gamma,s)$ *is attained in* $X^\alpha(\mathbb{R}^{n+1}_+)$.
2. *If* $s = 0$ *and* $\gamma < 0$, *then there are no extremals for* $S(n,\alpha,\gamma,s)$ *in* $X^\alpha(\mathbb{R}^{n+1}_+)$.

Theorem 2.2. *Let* $0 < \alpha < 2$, $0 < s < \alpha < n$ *and* $0 \leq \gamma < \gamma_H$. *Then, there exists a non-trivial weak solution to (2.4) in* $X^\alpha(\mathbb{R}^{n+1}_+)$.

3 Proof of Theorem 1.1

We shall minimize the functional

$$I_{\gamma,s}(w) = \frac{k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx}{\Big(\int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}}}$$

on the space $X^\alpha(\mathbb{R}^{n+1}_+)$. Whenever $S(n,\alpha,\gamma,s)$ is attained at some $w \in X^\alpha(\mathbb{R}^{n+1}_+)$, then it is clear that $u = \mathrm{Tr}(w) := w(.,0)$ will be a function in $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, where $\mu_{\gamma,s}(\mathbb{R}^n)$ is attained.

Note first that inequality (2.10) asserts that $X^\alpha(\mathbb{R}^{n+1}_+)$ is embedded in the weighted space $L^2(\mathbb{R}^n, |x|^{-\alpha})$ and that this embeding is continuous. If $\gamma < \gamma_H$, it follows from (2.10) that

$$\|w\| := \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx\right)^{\frac{1}{2}}$$

is well-defined on $X^\alpha(\mathbb{R}^{n+1}_+)$. Set $\gamma_+ = \max\{\gamma, 0\}$ and $\gamma_- = -\max\{\gamma, 0\}$. The following inequalities then hold for any $u \in X^\alpha(\mathbb{R}^{n+1}_+)$,

$$(1 - \frac{\gamma_+}{\gamma_H})\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} \le \|w\|^2 \le (1 + \frac{\gamma_-}{\gamma_H})\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}. \tag{3.1}$$

Thus, $\| \,.\, \|$ is equivalent to the norm $\| \,.\, \|_{X^\alpha(\mathbb{R}^{n+1}_+)}$.

We start by considering the case when $s > 0$. Ekeland's variational principle [12] applied to the functional $I(w) := I_{\gamma,s}(w)$ yields the existence of a minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $S(n,\alpha,\gamma,s)$ such that as $k \to \infty$,

$$\int_{\mathbb{R}^n} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1, \tag{3.2}$$

$$I(w_k) \longrightarrow S(n,\alpha,\gamma,s), \tag{3.3}$$

and

$$I'(w_k) \to 0 \text{ in } (X^\alpha(\mathbb{R}^{n+1}_+))', \tag{3.4}$$

where $(X^\alpha(\mathbb{R}^{n+1}_+))'$ denotes the dual of $X^\alpha(\mathbb{R}^{n+1}_+)$. Consider the functionals $J, K : X^\alpha(\mathbb{R}^{n+1}_+) \longrightarrow \mathbb{R}$ by

$$J(w) := \frac{1}{2}\|w\|^2 = \frac{k_\alpha}{2} \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla w|^2 dxdy - \frac{\gamma}{2} \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx,$$

and

$$K(w) := \frac{1}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|w(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx.$$

Straightforward computations yield that as $k \to \infty$,

$$J(w_k) \longrightarrow \frac{1}{2} S(n,\alpha,\gamma,s),$$

and

$$J'(w_k) - S(n,\alpha,\gamma,s)K'(w_k) \longrightarrow 0 \text{ in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{3.5}$$

Consider now the Levy concentration functions Q of $\frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}$, defined as

$$Q(r) = \int_{B_r} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \quad \text{for} \quad r > 0,$$

where B_r is the ball of radius r in $\mathbb{R}^n$. Since $\int_{\mathbb{R}^n} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1$ for all $k \in \mathbb{N}$, then by continuity, and up to considering a subsequence, there exists $r_k > 0$ such that

$$Q(r_k) = \int_{B_{r_k}} \frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \frac{1}{2} \quad \text{for all } k \in \mathbb{N}.$$

Define the rescaled sequence $v_k(x,y) := r_k^{\frac{n-\alpha}{2}} w_k(r_k x, r_k y)$ for $k \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^{n+1}_+$, in such a way that $(v_k)_{k\in\mathbb{N}}$ is also a minimizing sequence for $S(n,\alpha,\gamma,s)$. Indeed, it is easy to check that $v_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ and that $\|w_k\|^2 = \|v_k\|^2$,

$$\lim_{k\to\infty}\left(k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla v_k|^2 dxdy - \gamma\int_{\mathbb{R}^n}\frac{|v_k(x,0)|^2}{|x|^\alpha}dx\right) = S(n,\alpha,\gamma,s) \tag{3.6}$$

and

$$\int_{\mathbb{R}^n}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = \int_{\mathbb{R}^n}\frac{|w_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = 1.$$

Moreover, we have that

$$\int_{B_1}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = \frac{1}{2} \quad \text{for all } k\in\mathbb{N}. \tag{3.7}$$

In addition, $\|v_k\|^2 = S(n,\alpha,\gamma,s)+o(1)$ as $k\to\infty$, so (3.1) yields that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$. Therefore, without loss of generality, there exists a subsequence -still denoted v_k- such that

$$v_k \rightharpoonup v \text{ in } X^\alpha(\mathbb{R}^{n+1}_+) \text{ and } v_k(.,0)\to v(.,0) \text{ in } L^q_{loc}(\mathbb{R}^n) \quad \text{for every } 1\le q<2^*_\alpha. \tag{3.8}$$

We shall show that the weak limit of the minimizing sequence is not identically zero, that is $v\not\equiv 0$. Indeed, suppose $v\equiv 0$. It follows from (3.8) that

$$v_k \rightharpoonup 0 \text{ in } X^\alpha(\mathbb{R}^{n+1}_+) \text{ and } v_k(.,0)\to 0 \text{ in } L^q_{loc}(\mathbb{R}^n) \quad \text{for every } 1\le q<2^*_\alpha. \tag{3.9}$$

For $\delta>0$, define $B^+_\delta := \{(x,y)\in\mathbb{R}^{n+1}_+ : |(x,y)|<\delta\}$, $B_\delta := \{x\in\mathbb{R}^n : |x|<\delta\}$ and let $\eta\in C^\infty_0(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\eta\equiv 1$ in $B^+_{\frac{1}{2}}$ and $0\le\eta\le 1$ in $\mathbb{R}^{n+1}_+$.
We use $\eta^2 v_k$ as test function in (3.5) to get that

$$\begin{aligned} k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\nabla v_k.\nabla(\eta^2 v_k)dxdy - \gamma\int_{\mathbb{R}^n}\frac{v_k(x,0)(\eta^2 v_k(x,0))}{|x|^\alpha}dx \\ = S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|v_k(x,0)|^{2^*_\alpha(s)-1}(\eta^2 v_k(x,0))}{|x|^s}dx + o(1). \end{aligned} \tag{3.10}$$

Simple computations yield $|\nabla(\eta v_k)|^2 = |v_k\nabla\eta|^2 + \nabla v_k.\nabla(\eta^2 v_k)$, so that we have

$$\begin{aligned} & k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta v_k)|^2 dxdy - k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\nabla v_k.\nabla(\eta^2 v_k)dxdy \\ &= k_\alpha\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|v_k\nabla\eta|^2 dxdy \\ &= k_\alpha\int_E y^{1-\alpha}|\nabla\eta|^2|v_k|^2 dxdy, \end{aligned}$$

where $E := \text{Supp}(|\nabla\eta|)$. Since $\alpha\in(0,2)$, $y^{1-\alpha}$ is an A_2-weight, and since E is bounded, we have that the embedding $H^1(E,y^{1-\alpha})\hookrightarrow L^2(E,y^{1-\alpha})$ is compact (See [2] and [23]). It follows from $(3.9)_1$ that

$$k_\alpha\int_E y^{1-\alpha}|\nabla\eta|^2|v_k|^2 dxdy \to 0 \quad \text{as } k\to\infty.$$

Therefore,

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta v_k)|^2 dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\nabla v_k.\nabla(\eta^2 w_k)dxdy + o(1).$$

By plugging the above estimate into (3.10), and using (3.7), we get that

$$\begin{aligned}
\|\eta v_k\|^2 &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta v_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^2}{|x|^\alpha} dx \\
&= S(n,\alpha,\gamma,s) \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2}(|\eta v_k(x,0)|^2)}{|x|^s} dx + o(1) \\
&\leq S(n,\alpha,\gamma,s) \int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx + o(1) \\
&= \frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} \left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{2}{2^*_\alpha(s)}} + o(1).
\end{aligned} \tag{3.11}$$

By straightforward computations and Hölder's inequality, we get that

$$\begin{aligned}
\left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{1}{2^*_\alpha(s)}} &= \left(\int_{B_1} \frac{|\eta v_k(x,0) + (1-\eta)v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{1}{2^*_\alpha(s)}} \\
&\leq \left(\int_{B_1} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{1}{2^*_\alpha(s)}} + \left(\int_{B_1} \frac{|(1-\eta)v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{1}{2^*_\alpha(s)}} \\
&\leq \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{1}{2^*_\alpha(s)}} + C \left(\int_{B_1} |v_k(x,0)|^{2^*_\alpha(s)} dx \right)^{\frac{1}{2^*_\alpha(s)}}.
\end{aligned}$$

From $(3.9)_2$, and the fact that $2^*_\alpha(s) < 2^*_\alpha$, we obtain

$$\int_{B_1} |v_k(x,0)|^{2^*_\alpha(s)} dx \to 0 \quad \text{as } k \to \infty.$$

Therefore,

$$\left(\int_{B_1} \frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{2}{2^*_\alpha(s)}} \leq \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{2}{2^*_\alpha(s)}} + o(1). \tag{3.12}$$

Plugging the above inequality into (3.11), we get that

$$\begin{aligned}
\|\eta v_k\|^2 &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta v_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^2}{|x|^\alpha} dx \\
&\leq \frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} \left(\int_{\mathbb{R}^n} \frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \right)^{\frac{2}{2^*_\alpha(s)}} + o(1).
\end{aligned}$$

On the other hand, it follows from the definition of $S(n,\alpha,\gamma,s)$ that

$$S(n,\alpha,\gamma,s)\left(\int_{\mathbb{R}^n}\frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} \leq \|\eta v_k\|^2$$
$$\leq \frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}}\left(\int_{\mathbb{R}^n}\frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} + o(1).$$

Note that $\frac{S(n,\alpha,\gamma,s)}{2^{1-\frac{2}{2^*_\alpha(s)}}} < S(n,\alpha,\gamma,s)$ for $s \in (0,\alpha)$, hence (3.12) yields that

$$o(1) = \int_{\mathbb{R}^n}\frac{|\eta v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = \int_{B_1}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx + o(1).$$

This contradicts (3.7) and therefore $v \not\equiv 0$.

We now conclude by proving that v_k converges weakly in $\mathbb{R}^{n+1}_+$ to v, and that

$$\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = 1.$$

Indeed, for $k \in \mathbb{N}$, let $\theta_k = v_k - v$, and use the Brezis-Lieb Lemma (see [6] and [37]) to deduce that

$$1 = \int_{\mathbb{R}^n}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx = \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx + \int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx + o(1),$$

which yields that both

$$\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx \text{ and } \int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx \text{ are in the interval } [0,1]. \tag{3.13}$$

The weak convergence $\theta_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ implies that

$$\|v_k\|^2 = \|v+\theta_k\|^2 = \|v\|^2 + \|\theta_k\|^2 + o(1).$$

By using (3.5) and the definition of $S(n,\alpha,\gamma,s)$, we get that

$$\begin{aligned}
o(1) &= \|v_k\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|v_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\\
&= \left(\|v\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)\\
&+ \left(\|\theta_k\|^2 - S(n,\alpha,\gamma,s)\int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right) + o(1)\\
&\geq S(n,\alpha,\gamma,s)\left[\left(\int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right]\\
&+ S(n,\alpha,\gamma,s)\left[\left(\int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n}\frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s}dx\right] + o(1).
\end{aligned}$$

Set now

$$A := \left(\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx,$$

and

$$B := \left(\int_{\mathbb{R}^n} \frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} - \int_{\mathbb{R}^n} \frac{|\theta_k(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx.$$

Note that since $2^*_\alpha(s) > 2$, we have $a^{\frac{2}{2^*_\alpha(s)}} \geq a$ for every $a \in [0,1]$, and equality holds if and only if $a = 0$ or $a = 1$. It then follows from (3.13) that both A and B are non-negative. On the other hand, the last inequality implies that $A + B = o(1)$, which means that $A = 0$ and $B = o(1)$, that is

$$\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = \left(\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}},$$

hence

$$\text{either} \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 0 \text{ or } \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1.$$

The fact that $v \not\equiv 0$ yields $\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx \neq 0$, and $\int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)}}{|x|^s} dx = 1$, which yields that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\nabla v|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|v(x,0)|^2}{|x|^\alpha} dx = S(n,\alpha,\gamma,s).$$

Without loss of generality we may assume $v \geq 0$ (otherwise we take $|v|$ instead of v), and we then obtain a positive extremal for $S(n,\alpha,\gamma,s)$ in the case $s \in (0,\alpha)$.

Suppose now that $s = 0$ and $\gamma \geq 0$. By a result in [10], extremals exist for $S(n,\alpha,\gamma,s)$ whenever $s = 0$ and $\gamma = 0$. Hence, we only need to show that there exists an extremal for $S(n,\alpha,\gamma,0)$ in the case $\gamma > 0$. First note that in this case, we have that

$$S(n,\alpha,\gamma,0) < S(n,\alpha,0,0). \tag{3.14}$$

Indeed, if $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ is an extremal for $S(n,\alpha,0,0)$, then by estimating the functional at w, and using the fact that $\gamma > 0$, we obtain

$$\begin{aligned} S(n,\alpha,\gamma,0) &= \inf_{u \in X^\alpha(\mathbb{R}^{n+1}_+)\setminus\{0\}} \frac{\|u\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|u(x,0)|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |u(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} \\ &\leq \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} \\ &< \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}} = S(n,\alpha,0,0). \end{aligned}$$

Now we show that $S(n,\alpha,\gamma,0)$ is attained whenever $S(n,\alpha,\gamma,0) < S(n,\alpha,0,0)$. Indeed, let $(w_k)_{k\in\mathbb{N}} \subset X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ be a minimizing sequence for $S(n,\alpha,\gamma,0)$. Up to multiplying by a positive constant, we assume that

$$\lim_{k\to\infty}\left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla w_k|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx\right) = S(n,\alpha,\gamma,0) \tag{3.15}$$

and

$$\int_{\mathbb{R}^n} |w_k(x,0)|^{2^*_\alpha} dx = 1. \tag{3.16}$$

The sequence $\left(\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)}\right)_{k\in\mathbb{N}}$ is therefore bounded, and there exists a subsequence - still denoted w_k- such that $w_k \rightharpoonup w$ weakly in $X^\alpha(\mathbb{R}^{n+1}_+)$. The weak convergence implies that

$$\|w_k\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} = \|w_k - w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + o(1)$$

and

$$\int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx = \int_{\mathbb{R}^n} \frac{|(w-w_k)(x,0)|^2}{|x|^\alpha} dx + \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx + o(1).$$

The Brezis-Lieb Lemma ([6, Theorem 1]) and (3.16) yield that

$$\int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \le 1,$$

for large k, hence

$$\begin{aligned}
S(n,\alpha,\gamma,0) &= \|w_k\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w_k(x,0)|^2}{|x|^\alpha} dx + o(1) \\
&\ge \|w_k - w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} + \|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x|^\alpha} dx + o(1) \\
&\ge S(n,\alpha,0,0)\left(\int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \\
&\quad + S(n,\alpha,\gamma,0)\left(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} + o(1) \\
&\ge S(n,\alpha,0,0) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&\quad + S(n,\alpha,\gamma,0) \int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx + o(1).
\end{aligned}$$

Use the Brezis-Lieb Lemma again to get that

$$\begin{aligned}
S(n,\alpha,\gamma,0) &\ge (S(n,\alpha,0,0) - S(n,\alpha,\gamma,0)) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&\quad + S(n,\alpha,\gamma,0) \int_{\mathbb{R}^n} |w_k(x,0)|^{2^*_\alpha} dx + o(1) \\
&= (S(n,\alpha,0,0) - S(n,\alpha,\gamma,0)) \int_{\mathbb{R}^n} |(w_k - w)(x,0)|^{2^*_\alpha} dx \\
&\quad + S(n,\alpha,\gamma,0) + o(1).
\end{aligned}$$

Since $S(n,\alpha,\gamma,0) < S(n,\alpha,0,0)$, we get that $w_k(.,0) \to w(.,0)$ in $L^{2^*_\alpha}(\mathbb{R}^n)$, that is

$$\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx = 1.$$

The lower semi-continuity of I then implies that w is a minimizer for $S(n,\alpha,\gamma,0)$. Note that $|w|$ is also an extremal in $X^\alpha(\mathbb{R}^{n+1}_+)$ for $S(n,\alpha,\gamma,0)$, therefore there exists a non-negative extremal for $S(n,\alpha,\gamma,s)$ in the case $\gamma > 0$ and $s = 0$, and this completes the proof of the case when $s = 0$ and $\gamma \geq 0$.

Now we consider the case when $\gamma < 0$.

Claim 3.1. *If $\gamma \leq 0$, then $S(n,\alpha,\gamma,0) = S(n,\alpha,0,0)$, hence, there are no extremals for $S(n,\alpha,\gamma,0)$ whenever $\gamma < 0$.*

Indeed, we first note that for $\gamma \leq 0$, we have $S(n,\alpha,\gamma,0) \geq S(n,\alpha,0,0)$. On the other hand, if we consider $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ to be an extremal for $S(n,\alpha,0,0)$ and define for $\delta \in \mathbb{R}$, and $\bar{x} \in \mathbb{R}^n$, the function $w_\delta := w(x - \delta\bar{x}, y)$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$, then by a change of variable, we get

$$\begin{aligned} S(n,\alpha,\gamma,0) \leq I_\delta : &= \frac{\|w_\delta\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w_\delta(x,0)|^2}{|x|^\alpha} dx}{(\int_{\mathbb{R}^n} |w_\delta(x,0)|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}} \\ &= \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)} - \gamma \int_{\mathbb{R}^n} \frac{|w(x,0)|^2}{|x+\delta\bar{x}|^\alpha} dx}{(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}}, \end{aligned}$$

so that

$$S(n,\alpha,\gamma,0) \leq \lim_{\delta\to\infty} I_\delta = \frac{\|w\|^2_{X^\alpha(\mathbb{R}^{n+1}_+)}}{(\int_{\mathbb{R}^n} |w(x,0)|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}} = S(n,\alpha,0,0).$$

Therefore, $S(n,\alpha,\gamma,0) = S(n,\alpha,0,0)$. Since there are extremals for $S(n,\alpha,0,0)$ (see [10]), there is none for $S(n,\alpha,\gamma,0)$ whenever $\gamma < 0$. This establishes (2) and completes the proof of Theorem 2.1.

Back to Theorem 1.1, since the non-negative α-harmonic function w is a minimizer for $S(n,\alpha,\gamma,s)$ in $X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$, which exists from Theorem 2.1, then $u := \mathrm{Tr}(w) = w(.,0) \in H^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}$ and by (2.1), u is a minimizer for $\mu_{\gamma,s}(\mathbb{R}^n)$ in $H^{\frac{\alpha}{2}}(\mathbb{R}^n) \setminus \{0\}$. Therefore, (1) and (2) of Theorem 1.1 hold. For (3), let u^* be the Schwarz symmetrization of u. By the fractional Polya-Szegö inequality [33], we have

$$\|(-\Delta)^{\frac{\alpha}{2}} u^*\|^2_{L^2(\mathbb{R}^n)} \leq \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)}.$$

Furthermore, it is clear (Theorem 3.4. of [28]) that

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx \leq \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx \leq \int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx.$$

Combining the above inequalities and the fact that $\gamma \geq 0$, we get that

$$\mu_{\gamma,s}(\mathbb{R}^n) \leq \frac{\|(-\Delta)^{\frac{\alpha}{2}} u^*\|^2_{L^2(\mathbb{R}^n)} - \gamma \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}}$$
$$\leq \frac{\|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^n)} - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}}} = \mu_{\gamma,s}(\mathbb{R}^n).$$

This implies that u^* is also a minimizer and achieves the infimum of $\mu_{\gamma,s}(\mathbb{R}^n)$. Therefore the equality sign holds in all the inequalities above, that is

$$\gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^\alpha} dx = \gamma \int_{\mathbb{R}^n} \frac{|u^*|^2}{|x|^\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx = \int_{\mathbb{R}^n} \frac{|u^*|^{2^*_\alpha(s)}}{|x|^s} dx.$$

From Theorem 3.4. of [28], in the case of equality, it follows that $u = |u| = u^*$ if either $\gamma \neq 0$ or if $s \neq 0$. In particular, u is positive, radially symmetric and decreasing about origin. Hence u must approach a limit as $|x| \to \infty$, which must be zero.

4 Proof of Theorem 1.2

We shall now use the existence of extremals for the fractional Hardy-Sobolev type inequalities, established in Section 3, to prove that there exists a nontrivial weak solution for (2.4). The energy functional Ψ associated to (2.4) is defined as follows:

$$\Psi(w) = \frac{1}{2}\|w\|^2 - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx - \frac{1}{2^*_\alpha(s)} \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx \quad \text{for } w \in X^\alpha(\mathbb{R}^{n+1}_+), \tag{4.1}$$

where again $u := \mathrm{Tr}(w) = w(.,0)$. Fractional trace Hardy, Sobolev and Hardy-Sobolev inequalities yield that $\Psi \in C^1(X^\alpha(\mathbb{R}^{n+1}_+))$. Note that a weak solution to (2.4) is a nontrivial critical point of Ψ.

Throughout this section, we use the following notation for any sequence $(w_k)_{k\in\mathbb{N}} \in X^\alpha(\mathbb{R}^{n+1}_+)$:

$$u_k := \mathrm{Tr}(w_k) = w_k(.,0) \quad \text{for all } k \in \mathbb{N}.$$

We split the proof in three parts:

4.1 Existence of a suitable Palais-Smale sequence

We first verify that the energy functional Ψ satisfies the conditions of the Mountain Pass Lemma leading to a minimax energy level that is below a suitable threshold. The following is standard.

Lemma 4.1 (Ambrosetti and Rabinowitz [1]). *Let $(V, \|\,\|)$ be a Banach space and $\Psi : V \to \mathbb{R}$ a C^1–functional satisfying the following conditions:*
(a) $\Psi(0) = 0$,
(b) There exist $\rho, R > 0$ such that $\Psi(u) \geq \rho$ for all $u \in V$, with $\|u\| = R$,

(c) There exists $v_0 \in V$ *such that* $\limsup_{t\to\infty} \Psi(tv_0) < 0$.
Let $t_0 > 0$ *be such that* $\|t_0 v_0\| > R$ *and* $\Psi(t_0 v_0) < 0$, *and define*

$$c_{v_0}(\Psi) := \inf_{\sigma\in\Gamma} \sup_{t\in[0,1]} \Psi(\sigma(t)),$$

where

$$\Gamma := \{\sigma \in C([0,1], V) : \sigma(0) = 0 \text{ and } \sigma(1) = t_0 v_0\}.$$

Then, $c_{v_0}(\Psi) \geq \rho > 0$, *and there exists a Palais-Smale sequence at level* $c_{v_0}(\Psi)$, *that is there exists a sequence* $(w_k)_{k\in\mathbb{N}} \in V$ *such that*

$$\lim_{k\to\infty} \Psi(w_k) = c_{v_0}(\Psi) \text{ and } \lim_{k\to\infty} \Psi'(w_k) = 0 \quad \text{strongly in } V'.$$

We now prove the following.

Proposition 4.1. *Suppose* $0 \leq \gamma < \gamma_H$ *and* $0 \leq s < \alpha$, *and consider* Ψ *defined in (4.1) on the Banach space* $X^\alpha(\mathbb{R}^{n+1}_+)$. *Then, there exists* $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ *such that* $w \geq 0$ *and* $0 < c_w(\Psi) < c^\star$, *where*

$$c^\star = \min\left\{\frac{\alpha}{2n} S(n,\alpha,\gamma,0)^{\frac{n}{\alpha}}, \frac{\alpha - s}{2(n-s)} S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}}\right\}, \tag{4.2}$$

and a Palais-Smale sequence $(w_k)_{k\in\mathbb{N}}$ *in* $X^\alpha(\mathbb{R}^{n+1}_+)$ *at energy level* $c_w(\Psi)$, *that is,*

$$\lim_{k\to\infty} \Psi(w_k) = c_w(\Psi) \text{ and } \lim_{k\to\infty} \Psi'(w_k) = 0 \quad \text{strongly in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{4.3}$$

Proof. [Proof of Proposition 4.1] In the sequel, we will use freely the following elementary identities involving $2^*_\alpha(s)$:

$$\tfrac{1}{2} - \tfrac{1}{2^*_\alpha} = \tfrac{\alpha}{2n}, \quad \tfrac{2^*_\alpha}{2^*_\alpha - 2} = \tfrac{n}{\alpha}, \quad \tfrac{1}{2} - \tfrac{1}{2^*_\alpha(s)} = \tfrac{\alpha - s}{2(n-s)} \quad \text{and} \quad \tfrac{2^*_\alpha(s)}{2^*_\alpha(s) - 2} = \tfrac{n-s}{\alpha - s}.$$

First, we note that the functional Ψ satisfies the hypotheses of Lemma 4.1, and that condition (c) is satisfied for any $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$. Indeed, it is standard to show that $\Psi \in C^1(X^\alpha(\mathbb{R}^{n+1}_+))$ and clearly $\Psi(0) = 0$, so that (a) of Lemma 4.1 is satisfied. For (b) note that by the definition of $S(n,\alpha,\gamma,s)$, we have that

$$S(n,\alpha,\gamma,0)\Big(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}} \leq \|w\|^2 \text{ and } S(n,\alpha,\gamma,s)\Big(\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx\Big)^{\frac{2}{2^*_\alpha(s)}} \leq \|w\|^2.$$

Hence,

$$\begin{aligned}\Psi(w) &\geq \frac{1}{2}\|w\|^2 - \frac{1}{2^*_\alpha} S(n,\alpha,\gamma,0)^{-\frac{2^*_\alpha}{2}} \|w\|^{2^*_\alpha} - \frac{1}{2^*_\alpha(s)} S(n,\alpha,\gamma,s)^{-\frac{2^*_\alpha(s)}{2}} \|w\|^{2^*_\alpha(s)} \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\alpha} S(n,\alpha,\gamma,0)^{-\frac{2^*_\alpha}{2}} \|w\|^{2^*_\alpha - 2} - \frac{1}{2^*_\alpha(s)} S(n,\alpha,\gamma,s)^{-\frac{2^*_\alpha(s)}{2}} \|w\|^{2^*_\alpha(s) - 2}\right) \|w\|^2.\end{aligned} \tag{4.4}$$

Since $s \in [0, \alpha)$, we have that $2^*_\alpha - 2 > 0$ and $2^*_\alpha(s) - 2 > 0$. Thus, by (3.1), we can find $R > 0$ such that $\Psi(w) \geq \rho$ for all $w \in X^\alpha(\mathbb{R}^{n+1}_+)$ with $\|w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} = R$. Regarding (c), note that

$$\Psi(tw) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)}\int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha(s)}}{|x|^s} dx,$$

hence $\lim_{t\to\infty} \Psi(tw) = -\infty$ for any $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$, which means that there exists $t_w > 0$ such that $\|t_w w\|_{X^\alpha(\mathbb{R}^{n+1}_+)} > R$ and $\Psi(tw) < 0$, for $t \geq t_w$.

Now we show that there exists $w \in X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ such that $w \geq 0$ and

$$c_w(\Psi) < \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}. \tag{4.5}$$

From Theorem 2.1, we know that there exists a non-negative extremal w in $X^\alpha(\mathbb{R}^{n+1}_+)$ for $S(n, \alpha, \gamma, 0)$ whenever $\gamma \geq 0$. By the definition of t_w, and the fact that $c_w(\Psi) > 0$, we obtain

$$c_w(\Psi) \leq \sup_{t\geq 0} \Psi(tw) \leq \sup_{t\geq 0} f(t), \qquad \text{where } f(t) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx \quad \forall t > 0.$$

Simple computations yield that $f(t)$ attains its maximum at the point $\tilde{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx}\right)^{\frac{1}{2^*_\alpha - 2}}$. It then follows that

$$\sup_{t\geq 0} f(t) = (\frac{1}{2} - \frac{1}{2^*_\alpha})\left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}}\right)^{\frac{2^*_\alpha}{2^*_\alpha - 2}} = \frac{\alpha}{2n}\left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n} |u|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}}\right)^{\frac{n}{\alpha}}.$$

Since w is an extremal for $S(n, \alpha, \gamma, 0)$, we get that

$$c_w(\Psi) \leq \sup_{t\geq 0} f(t) = \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}.$$

We now need to show that equality does not hold in (4.5). Indeed, otherwise we would have that $0 < c_w(\Psi) = \sup_{t\geq 0} \Psi(tw) = \sup_{t\geq 0} f(t)$. Consider $t_1 > 0$ (resp. $t_2 > 0$) where $\sup_{t\geq 0} \Psi(tw)$ (resp., $\sup_{t\geq 0} f(t)$) is attained. We get that

$$f(t_1) - \frac{t_1^{2^*_\alpha(s)}}{2^*_\alpha(s)}\int_{\mathbb{R}^n} \frac{|w(x, 0)|^{2^*_\alpha(s)}}{|x|^s} dx = f(t_2),$$

which means that $f(t_1) > f(t_2)$ since $t_1 > 0$. This contradicts the fact that t_2 is a maximum point of $f(t)$, hence the strict inequality in (4.5) holds.

To finish the proof of Proposition 4.1, we can assume without loss that

$$\frac{\alpha - s}{2(n - s)} S(n, \alpha, \gamma, s)^{\frac{n-s}{\alpha-s}} < \frac{\alpha}{2n} S(n, \alpha, \gamma, 0)^{\frac{n}{\alpha}}.$$

Let now w in $X^\alpha(\mathbb{R}^{n+1}_+) \setminus \{0\}$ be a positive minimizer for $S(n,\alpha,\gamma,s)$, whose existence was established in Section 3, and set

$$\bar{f}(t) = \frac{t^2}{2}\|w\|^2 - \frac{t^{2^*_\alpha(s)}}{2^*_\alpha(s)}\int_{\mathbb{R}^n}\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx.$$

As above, we have

$$\begin{aligned} c_w(\Psi) \leq \sup_{t\geq 0} f(t) &= (\frac{1}{2} - \frac{1}{2^*_\alpha(s)})\left(\frac{\|w\|^2}{(\int_{\mathbb{R}^n}\frac{|u|^{2^*_\alpha(s)}}{|x|^s}dx)^{\frac{2}{2^*_\alpha(s)}}}\right)^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-2}} \\ &= \frac{\alpha - s}{2(n-s)}S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}}. \end{aligned}$$

Again, if equality holds, then $0 < c_w(\Psi) \leq \sup_{t\geq 0}\Psi(tw) = \sup_{t\geq 0}\bar{f}(t)$, and if $t_1, t_2 > 0$ are points where the respective suprema are attained, then a contradiction is reached since

$$\bar{f}(t_1) - \frac{t_1^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^n}|u|^{2^*_\alpha}dx = \bar{f}(t_2).$$

Therefore,

$$0 < c_w(\Psi) < c^\star = \min\left\{\frac{\alpha}{2n}S(n,\alpha,\gamma,0)^{\frac{n}{\alpha}}, \frac{\alpha - s}{2(n-s)}S(n,\alpha,\gamma,s)^{\frac{n-s}{\alpha-s}}\right\}.$$

Finally, the existence of a Palais-Smale sequence at that level follows immediately from Lemma 4.1.

4.2 Analysis of the Palais-Smale sequences

We now study the concentration properties of weakly null Palais-Smale sequences. For $\delta > 0$, we shall write $B_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$.

Proposition 4.2. *Let $0 \leq \gamma < \gamma_H$ and $0 < s < \alpha$. Assume that $(w_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence of Ψ at energy level $c \in (0, c^\star)$. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, then there exists a positive constant $\epsilon_0 = \epsilon_0(n,\alpha,\gamma,c,s) > 0$ such that for every $\delta > 0$, one of the following holds:*

1. $\limsup_{k\to\infty}\int_{B_\delta}|u_k|^{2^*_\alpha}dx = \limsup_{k\to\infty}\int_{B_\delta}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx = 0$;
2. $\limsup_{k\to\infty}\int_{B_\delta}|u_k|^{2^*_\alpha}dx$ *and* $\limsup_{k\to\infty}\int_{B_\delta}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx \geq \epsilon_0$.

The proof of Proposition 4.2 requires the following two lemmas.

Lemma 4.2. *Let $(w_k)_{k\in\mathbb{N}}$ be a Palais-Smale sequence as in Proposition 4.2. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, then for any $D \subset\subset \mathbb{R}^n \setminus \{0\}$, there exists a subsequence of $(w_k)_{k\in\mathbb{N}}$, still denoted by $(w_k)_{k\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty}\int_D \frac{|u_k|^2}{|x|^\alpha}dx = \lim_{k\to\infty}\int_D \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx = 0 \tag{4.6}$$

and

$$\lim_{k\to\infty}\int_D |u_k|^{2^*_\alpha}dx = \lim_{k\to\infty}\int_D |(-\Delta)^{\frac{\alpha}{4}}u_k|^2dx = 0, \tag{4.7}$$

where $u_k := w_k(.,0)$ *for all* $k \in \mathbb{N}$.

Proof. [Proof of Lemma 4.2] Fix $D \subset\subset \mathbb{R}^n \setminus \{0\}$, and note that the following fractional Sobolev embedding is compact:

$$H^{\frac{\alpha}{2}}(\mathbb{R}^n) \hookrightarrow L^q(D) \quad \text{for every } 1 \le q < 2^*_\alpha.$$

Using the trace inequality (2.2), and the assumption that $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, we get that

$$u_k \to 0 \quad \text{strongly for every } 1 \le q < 2^*_\alpha.$$

On the other hand, the fact that $|x|^{-1}$ is bounded on $D \subset\subset \mathbb{R}^n \setminus \{0\}$ implies that there exist constants $C_1, C_2 > 0$ such that

$$0 \le \lim_{k\to\infty}\int_D \frac{|u_k|^2}{|x|^\alpha}dx \le C_1 \lim_{k\to\infty}\int_D |u_k|^2dx$$

and

$$0 \le \lim_{k\to\infty}\int_D \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx \le C_2 \lim_{k\to\infty}\int_D |u_k|^{2^*_\alpha(s)}dx.$$

Since $s \in (0,\alpha)$, we have that $1 \le 2, 2^*_\alpha(s) < 2^*_\alpha$. Thus, (4.6) holds.

To show (4.7), we let $\eta \in C_0^\infty(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\eta_* := \eta(.,0) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\eta_* \equiv 1$ in D and $0 \le \eta \le 1$ in $\mathbb{R}^{n+1}_+$. We first note that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2dxdy + o(1). \tag{4.8}$$

Indeed, apply the following elementary inequality for vectors X, Y in $\mathbb{R}^{n+1}$,

$$\left||X+Y|^2 - |X|^2\right| \le C(|X||Y| + |Y|^2),$$

with $X = y^{\frac{1-\alpha}{2}}\eta\nabla w_k$ and $Y = y^{\frac{1-\alpha}{2}}w_k\nabla\eta$, to get for all $k \in \mathbb{N}$, that

$$\left|y^{1-\alpha}|\nabla(\eta w_k)|^2 - y^{1-\alpha}|\eta\nabla w_k|^2\right| \le C\left(y^{1-\alpha}|\eta\nabla w_k||w_k\nabla\eta| + y^{1-\alpha}|w_k\nabla\eta|^2\right).$$

By Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2dxdy - \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2dxdy\right| \\
&\le\ C_3\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)}\Big(\int_{\mathrm{Supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2dxdy\Big)^{\frac{1}{2}} + C_3\int_{\mathrm{Supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2dxdy \qquad (4.9)\\
&\le C_4\left[\Big(\int_{\mathrm{Supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2dxdy\Big)^{\frac{1}{2}} + \int_{\mathrm{Supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2dxdy\right].
\end{aligned}$$

Since the embedding $H^1(\mathrm{Supp}(\nabla\eta), y^{1-\alpha}) \hookrightarrow L^2(\mathrm{Supp}(\nabla\eta), y^{1-\alpha})$ is compact, and $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$, we get that

$$\int_{\mathrm{Supp}(\nabla\eta)} y^{1-\alpha}|w_k|^2 dxdy = o(1),$$

which gives

$$\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy = \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy + o(1).$$

Thus, (4.8) holds.
Now recall that the sequence $(w_k)_{k\in\mathbb{N}}$ has the following property:

$$\lim_{k\to\infty} \Psi'(w_k) = 0 \quad \text{strongly in } (X^\alpha(\mathbb{R}^{n+1}_+))'. \tag{4.10}$$

Since $\eta^2 w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ for all $k \in \mathbb{N}$, we can use it as a test function in (4.10) to get that

$$\begin{aligned} o(1) &= \langle \Psi'(w_k), \eta^2 w_k\rangle \\ &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle\nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy - \gamma\int_{\mathbb{R}^n} \frac{\eta_*^2|u_k|^2}{|x|^\alpha}dx \\ &\quad - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha}dx - \int_{\mathbb{R}^n} \frac{\eta_*^2|u_k|^{2^*_\alpha(s)}}{|x|^s}dx. \end{aligned}$$

Regarding the first term, we have

$$\begin{aligned} k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle\nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy \\ &\quad + k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k\langle\nabla(\eta^2), \nabla w_k\rangle dxdy. \end{aligned}$$

From Hölder's inequality, and the fact that $w_k \to 0$ in $L^2(\mathrm{Supp}(|\nabla\eta|), y^{1-\alpha})$, it follows that as $k \to \infty$,

$$\begin{aligned} &\left| k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle\nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy - k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy \right| \\ &\qquad = \left| k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k\langle\nabla(\eta^2), \nabla w_k\rangle dxdy \right| \\ &\qquad \le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|w_k||\nabla(\eta^2)||\nabla w_k| dxdy \\ &\qquad \le C \int_{\mathrm{Supp}(|\nabla\eta|)} y^{1-\alpha}|w_k||\nabla w_k| dxdy \\ &\qquad \le C\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)} \left(\int_{\mathrm{Supp}(|\nabla\eta|)} y^{1-\alpha}|w_k|^2 dxdy\right)^{\frac{1}{2}} \\ &\qquad = o(1). \end{aligned}$$

Thus, we have proved that

$$k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}\langle \nabla w_k, \nabla(\eta^2 w_k)\rangle dxdy = k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy + o(1).$$

Using the above estimate coupled with (4.8), we obtain

$$\begin{aligned} o(1) &= \langle \Psi'(w_k), \eta^2 w_k\rangle \\ &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\eta\nabla w_k|^2 dxdy - \gamma\int_K \frac{\eta_*^2|u_k|^2}{|x|^\alpha}dx \\ &\quad - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha}dx - \int_K \frac{\eta_*^2|u_k|^{2^*_\alpha(s)}}{|x|^s}dx + o(1) \\ &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha}dx + o(1) \\ &\geq \|\eta w_k\|^2 - \int_{\mathbb{R}^n} \eta_*^2|u_k|^{2^*_\alpha}dx + o(1) \quad \text{as } k \to \infty, \end{aligned} \tag{4.11}$$

where $K = \mathrm{Supp}(\eta_*)$. Therefore,

$$\|\eta w_k\|^2 \leq \int_{\mathbb{R}^n} |\eta_* u_k|^2|u_k|^{2^*_\alpha-2}dx + o(1) \quad \text{as } k \to \infty. \tag{4.12}$$

By Hölder's inequality, and using the definition of $S(n, \alpha, \gamma, 0)$, we then get that

$$\begin{aligned} \|\eta w_k\|^2 &\leq \left(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha}dx\right)^{\frac{2}{2^*_\alpha}} \left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha}dx\right)^{\frac{2^*_\alpha-2}{2^*_\alpha}} + o(1) \\ &\leq S(n,\alpha,\gamma,0)^{-1}\|\eta w_k\|^2 \left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha}dx\right)^{\frac{2^*_\alpha-2}{2^*_\alpha}} + o(1). \end{aligned} \tag{4.13}$$

Thus,

$$\left[1 - S(n,\alpha,\gamma,0)^{-1}\left(\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha}dx\right)^{\frac{2^*_\alpha-2}{2^*_\alpha}}\right]\|\eta w_k\|^2 \leq o(1). \tag{4.14}$$

In addition, it follows from (4.3) that

$$\Psi(w_k) - \frac{1}{2}\langle \Psi'(w_k), w_k\rangle = c + o(1),$$

that is,

$$\left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha}dx + \left(\frac{1}{2} - \frac{1}{2^*_\alpha(s)}\right)\int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx = c + o(1), \tag{4.15}$$

from which follows that

$$\int_{\mathbb{R}^n} |u_k|^{2^*_\alpha}dx \leq \frac{2n}{\alpha}c + o(1) \quad \text{as } k \to \infty. \tag{4.16}$$

Plugging (4.16) into (4.14), we obtain that

$$\left[1 - S(n,\alpha,\gamma,0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}}\right]\|\eta w_k\|^2 \le o(1) \quad \text{as } k \to \infty.$$

On the other hand, by the upper bound (4.2) on c, we have that

$$c < \frac{\alpha}{2n} S(n,\alpha,\gamma,0)^{\frac{n}{\alpha}}.$$

This yields that $1 - S(n,\alpha,\gamma,0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}} > 0$, and therefore, $\lim_{k\to\infty} \|\eta w_k\|^2 = 0$.
Using (2.1) and (3.1), we obtain that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}}(\eta_* u_k)|^2 dx = \lim_{k\to\infty} k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy = 0.$$

It also follows from the definition of $S(n,\alpha,\gamma,0)$ that $\lim_{k\to\infty} \int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx = 0$, hence,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{4}}(\eta_* u_k)|^2 dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx = 0.$$

Since $\eta_{*|_D} \equiv 1$, the last equality yields (4.7).

Lemma 4.3. *Let $(w_k)_{k\in\mathbb{N}}$ be Palais-Smale sequence as in Proposition 4.2. For any $\delta > 0$, set*

$$\begin{aligned} &\theta := \limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx; \qquad \zeta := \limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \text{ and} \\ &\mu := \limsup_{k\to\infty} \int_{B_\delta} \left(|(-\Delta)^{\frac{\alpha}{4}} u_k|^2 dx - \gamma \frac{|u_k|^2}{|x|^\alpha} \right) dx, \end{aligned} \tag{4.17}$$

where $u_k := Tr(w_k) = w_k(.,0)$ for all $k \in \mathbb{N}$. If $w_k \rightharpoonup 0$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, then the following hold:

1. $\theta^{\frac{2}{2^*_\alpha}} \le S(n,\alpha,\gamma,0)^{-1}\mu$ *and* $\zeta^{\frac{2}{2^*_\alpha(s)}} \le S(n,\alpha,\gamma,s)^{-1}\mu$.
2. $\mu \le \theta + \zeta$.

Proof. [Proof of Lemma 4.3] First note that it follows from Lemma 4.2 that θ, ζ and μ are well-defined and are independent of the choice of $\delta > 0$. Let now $\eta \in C_0^\infty(\mathbb{R}^{n+1}_+)$ be a cut-off function such that $\eta_* := \eta(.,0) \equiv 1$ in B_δ, and $0 \le \eta \le 1$ in $\mathbb{R}^{n+1}_+$.

1. Since $\eta w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$, we get from the definition of $S(n,\alpha,\gamma,s)$ that

$$S(n,\alpha,\gamma,0)\left(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx. \tag{4.18}$$

On the other hand, from the definition of η and (2.1), it follows that

$$\begin{aligned}\|\eta w_k\|^2 &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \\ &= \int_{\mathbb{R}^n} \left(|(-\Delta)^{\frac{\alpha}{4}}(\eta_* u_k)|^2 - \gamma \frac{|\eta_* u_k|^2}{|x|^\alpha} \right) dx \\ &= \int_{B_\delta} \left(|(-\Delta)^{\frac{\alpha}{4}} u_k|^2 - \gamma \frac{|u_k|^2}{|x|^\alpha} \right) dx + \int_{\mathrm{Supp}(\eta_*)\setminus B_\delta} \left(|(-\Delta)^{\frac{\alpha}{4}}(\eta_* u_k)|^2 - \gamma \frac{|\eta_* u_k|^2}{|x|^\alpha} \right) dx,\end{aligned}$$

and

$$\left(\int_{B_\delta} |u_k|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \le \left(\int_{\mathbb{R}^n} |\eta_* u_k|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}}.$$

Note that $\mathrm{Supp}(\eta_*) \setminus B_\delta \subset\subset \mathbb{R}^n \setminus \{0\}$. Therefore, taking the upper limits at both sides of (4.18), and using Lemma 4.2, we get that

$$S(n,\alpha,\gamma,0)\left(\int_{B_\delta} |u_k|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \le \int_{B_\delta} \left(|(-\Delta)^{\frac{\alpha}{4}} u_k|^2 dx - \gamma \frac{|u_k|^2}{|x|^\alpha} \right) dx + o(1) \quad \text{as } k \to \infty,$$

which gives

$$\theta^{\frac{2}{2^*_\alpha}} \le S(n,\alpha,\gamma,0)^{-1}\mu.$$

Similarly, we can prove that

$$\zeta^{\frac{2}{2^*_\alpha(s)}} \le S(n,\alpha,\gamma,s)^{-1}\mu.$$

2. Since $\eta^2 w_k \in X^\alpha(\mathbb{R}^{n+1}_+)$ and $\langle \Psi'(w_k), \eta^2 w_k \rangle = o(1)$ as $k \to \infty$, we have

$$\begin{aligned} o(1) &= \langle \Psi'(w_k), \eta^2 w_k \rangle \\ &= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla w_k, \nabla(\eta^2 w_k) \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \\ &\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\ &= \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |\eta \nabla w_k|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx \right) \\ &\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx + k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k \rangle dxdy. \end{aligned} \tag{4.19}$$

By Hölder's inequality, and the fact that $w_k \to 0$ in $L^2(\mathrm{Supp}(|\nabla \eta|), y^{1-\alpha})$, we obtain that

$$
\begin{aligned}
\left|k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} w_k \langle \nabla(\eta^2), \nabla w_k\rangle dxdy\right| &\le k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} |w_k||\nabla(\eta^2)||\nabla w_k| dxdy \\
&\le C \int_{\mathrm{Supp}(|\nabla\eta|)} y^{1-\alpha}|w_k||\nabla w_k| dxdy \\
&\le C\|w_k\|_{X^\alpha(\mathbb{R}^{n+1}_+)}\|w_k\|_{L^2(\mathrm{Supp}(|\nabla\eta|), y^{1-\alpha})} \\
&\le o(1) \quad \text{as } k\to\infty.
\end{aligned}
$$

Plugging the above estimate into (4.19), and using (2.1), we get that

$$
\begin{aligned}
o(1) &= \langle \Psi'(w_k), \eta^2 w_k\rangle \\
&= \left(k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha}|\nabla(\eta w_k)|^2 dxdy - \gamma \int_{\mathbb{R}^n} \frac{|\eta_* u_k|^2}{|x|^\alpha} dx\right) \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\
&= \int_{\mathbb{R}^n} \left(|(-\Delta)^{\frac{\alpha}{4}}(\eta_* u_k)|^2 - \gamma \frac{|\eta_* u_k|^2}{|x|^\alpha}\right) dx \\
&\quad - \int_{\mathbb{R}^n} \eta_*^2 |u_k|^{2^*_\alpha} dx - \int_{\mathbb{R}^n} \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\
&\ge \int_{B_\delta} \left(|(-\Delta)^{\frac{\alpha}{4}} u_k|^2 - \gamma \frac{|u_k|^2}{|x|^\alpha}\right) dx - \int_{B_\delta} |u_k|^{2^*_\alpha} dx - \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \\
&\quad - \int_{\mathrm{Supp}(\eta_*)\setminus B_\delta} \left(\gamma \frac{|\eta_* u_k|^2}{|x|^\alpha} dx + \eta_*^2 |u_k|^{2^*_\alpha} dx + \frac{\eta_*^2 |u_k|^{2^*_\alpha(s)}}{|x|^s}\right) dx + o(1).
\end{aligned}
$$

Noting that $\mathrm{Supp}(\eta_*)\setminus B_\delta \subset\subset \mathbb{R}^n\setminus\{0\}$, and taking the upper limits on both sides, we get that $\mu \le \theta + \zeta$.

Proof. [Proof of Proposition 4.2] It follows from Lemma 4.3 that

$$
\theta^{\frac{2}{2^*_\alpha}} \le S(n,\alpha,\gamma,0)^{-1}\mu \le S(n,\alpha,\gamma,0)^{-1}\theta + S(n,\alpha,\gamma,0)^{-1}\zeta,
$$

which gives

$$
\theta^{\frac{2}{2^*_\alpha}}(1 - S(n,\alpha,\gamma,0)^{-1}\theta^{\frac{2^*_\alpha-2}{2^*_\alpha}}) \le S(n,\alpha,\gamma,0)^{-1}\zeta. \tag{4.20}
$$

On the other hand, by (4.15), we have

$$
\theta \le \frac{2n}{\alpha} c.
$$

Substituting the last inequality into (4.20), we get that

$$
(1 - S(n,\alpha,\gamma,0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}})\theta^{\frac{2}{2^*_\alpha}} \le S(n,\alpha,\gamma,0)^{-1}\zeta.
$$

Recall that the upper bounded (4.2) on c implies that

$$1 - S(n, \alpha, \gamma, 0)^{-1}(\frac{2n}{\alpha}c)^{\frac{\alpha}{n}} > 0.$$

Therefore, there exists $\delta_1 = \delta_1(n, \alpha, \gamma, c) > 0$ such that $\theta^{\frac{2}{2^*_\alpha}} \leq \delta_1 \zeta$. Similarly, there exists $\delta_2 = \delta_2(n, \alpha, \gamma, c, s) > 0$ such that $\zeta^{\frac{2}{2^*_\alpha(s)}} \leq \delta_2 \theta$. These two inequalities yield that there exists $\epsilon_0 = \epsilon_0(n, \alpha, \gamma, c, s) > 0$ such that

$$\text{either} \quad \theta = \zeta = 0 \quad \text{or} \quad \{\theta \geq \epsilon_0 \text{ and } \zeta \geq \epsilon_0\}. \tag{4.21}$$

It follows from the definition of θ and ζ that

$$\text{either } \limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx = \limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx = 0;$$
$$\text{or} \quad \limsup_{k\to\infty} \int_{B_\delta} |u_k|^{2^*_\alpha} dx \geq \epsilon_0 \quad \text{and} \quad \limsup_{k\to\infty} \int_{B_\delta} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx \geq \epsilon_0.$$

4.3 End of proof of Theorem 2.2

We shall first eliminate the possibility of a zero weak limit for the Palais-Smale sequence of Ψ, then we prove that the nontrivial weak limit is indeed a weak solution of Problem (2.4). In the sequel $(w_k)_{k\in\mathbb{N}}$ will denote the Palais-Smale sequence for Ψ obtained in Proposition 4.2.

First we show that

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx > 0. \tag{4.22}$$

Indeed, otherwise $\lim_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^{2^*_\alpha} dx = 0$, which once combined with the fact that $\langle \Psi'(w_k), w_k \rangle \to 0$ yields that

$$\|w_k\|^2 = \int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx + o(1) \quad \text{as } k \to \infty.$$

By combining this estimate with the definition of $S(n, \alpha, \gamma, s)$, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} &\leq S(n, \alpha, \gamma, s)^{-1} \|w_k\|^2 \\ &\leq S(n, \alpha, \gamma, s)^{-1} \int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx + o(1), \end{aligned}$$

which implies that

$$\left(\int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx\right)^{\frac{2}{2^*_\alpha(s)}} \left[1 - S(n, \alpha, \gamma, s)^{-1} (\int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx)^{\frac{2^*_\alpha(s)-2}{2^*_\alpha(s)}}\right] \leq o(1).$$

It follows from (4.2) and (4.15) that as $k \to \infty$,

$$\int_{\mathbb{R}^n} \frac{|u_k|^{2^*_\alpha(s)}}{|x|^s} dx = 2c\frac{n-s}{\alpha - s} + o(1) \quad \text{and} \quad (1 - S(n, \alpha, \gamma, s)^{-1} (2c\frac{n-s}{\alpha-s})^{\frac{\alpha-s}{n-s}}) > 0.$$

Hence,

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{|u_k|^{2^*_\alpha(s)}}{|x|^s}dx=0. \tag{4.23}$$

Using that $\lim_{k\to\infty}\int_{\mathbb{R}^n}|u_k|^{2^*_\alpha}dx=0$, in conjunction with (4.23) and (4.15), we get that $c+o(1)=0$, which contradicts the fact that $c>0$. This completes the proof of (4.22).

Now, we show that for small enough $\epsilon>0$, there exists another Palais-Smale sequence $(v_k)_{k\in\mathbb{N}}$ for Ψ satisfying the properties of Proposition 4.2, which is also bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$ and satisfies

$$\int_{B_1}|v_k(x,0)|^{2^*_\alpha}dx=\epsilon \quad \text{for all } k\in\mathbb{N}. \tag{4.24}$$

For that, consider ϵ_0 as given in Proposition 4.2. Let $\beta=\limsup_{k\to\infty}\int_{\mathbb{R}^n}|u_k|^{2^*_\alpha}dx$, which is positive by (4.22). Set $\epsilon_1:=\min\{\beta,\frac{\epsilon_0}{2}\}$ and fix $\epsilon\in(0,\epsilon_1)$. Up to a subsequence, there exists by continuity a sequence of radii $(r_k)_{k\in\mathbb{N}}$ such that $\int_{B_{r_k}}|u_k|^{2^*_\alpha}dx=\epsilon$ for each $k\in\mathbb{N}$. Let now

$$v_k(x,y):=r_k^{\frac{n-\alpha}{2}}w_k(r_kx,r_ky) \quad \text{for } x\in\mathbb{R}^n \text{ and } y\in\mathbb{R}_+.$$

It is clear that

$$\int_{B_1}|v_k(x,0)|^{2^*_\alpha}dx=\int_{B_{r_k}}|u_k|^{2^*_\alpha}dx=\epsilon \quad \text{for all } k\in\mathbb{N}. \tag{4.25}$$

It is easy to check that $(v_k)_{k\in\mathbb{N}}$ is also a Palais-Smale sequence for Ψ that satisfies the properties of Proposition 4.2.

We now show that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$. Indeed, since $(v_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence, there exist positive constants $C_1,C_2>0$ such that

$$\begin{aligned}C_1+C_2\|v_k\| &\ge \Psi(v_k)-\frac{1}{2^*_\alpha(s)}\langle\Psi'(v_k),v_k\rangle\\ &\ge\left(\frac12-\frac{1}{2^*_\alpha(s)}\right)\|v_k\|^2+\left(\frac{1}{2^*_\alpha}-\frac{1}{2^*_\alpha(s)}\right)\int_{\mathbb{R}^n}|v_k(x,0)|^{2^*_\alpha}dx\\ &\ge\left(\frac12-\frac{1}{2^*_\alpha(s)}\right)\|v_k\|^2.\end{aligned} \tag{4.26}$$

The last inequality holds since $2<2^*_\alpha(s)<2^*_\alpha$. Combining (4.26) with (3.1), we obtain that $(v_k)_{k\in\mathbb{N}}$ is bounded in $X^\alpha(\mathbb{R}^{n+1}_+)$.

It follows that there exists a subsequence – still denoted by v_k – such that $v_k\rightharpoonup v$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k\to\infty$. We claim that v is a nontrivial weak solution of (2.4). Indeed, if $v\equiv0$, then Proposition 4.2 yields that

$$\text{either } \limsup_{k\to\infty}\int_{B_1}|v_k(x,0)|^{2^*_\alpha}dx=0 \text{ or } \limsup_{k\to\infty}\int_{B_1}|v_k(x,0)|^{2^*_\alpha}dx\ge\epsilon_0.$$

Since $\epsilon\in(0,\frac{\epsilon_0}{2})$, this is in contradiction with (4.25), thus, $v\not\equiv0$.
To show that $v\in X^\alpha(\mathbb{R}^{n+1}_+)$ is a weak solution of (2.4), consider any $\varphi\in C_0^\infty(\mathbb{R}^{n+1}_+)$, and

write

$$
\begin{aligned}
o(1) &= \langle \Psi'(v_k), \varphi \rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v_k, \nabla \varphi \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{v_k(x,0)\varphi}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} |v_k(x,0)|^{2^*_\alpha - 2} v_k(x,0)\varphi dx - \int_{\mathbb{R}^n} \frac{|v_k(x,0)|^{2^*_\alpha(s)-2} v_k(x,0)\varphi}{|x|^s} dx.
\end{aligned}
\tag{4.27}
$$

Since $v_k \rightharpoonup v$ in $X^\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$, we have that

$$
\int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v_k, \nabla \varphi \rangle dxdy \to \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v, \nabla \varphi \rangle dxdy, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^{n+1}_+).
$$

In addition, the boundedness of v_k in $X^\alpha(\mathbb{R}^{n+1}_+)$ yields that $v_k(.,0), |v_k(.,0)|^{2^*_\alpha-2}v_k(.,0)$ and $|v_k(.,0)|^{2^*_\alpha(s)-2}v_k(.,0)$ are bounded in $L^2(\mathbb{R}^n, |x|^{-\alpha}), L^{\frac{2^*_\alpha}{2^*_\alpha-1}}(\mathbb{R}^n)$ and $L^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-1}}(\mathbb{R}^n, |x|^{-s})$, respectively. Therefore, we have the following weak convergence:

$$
\begin{aligned}
&v_k(.,0) \rightharpoonup v(.,0) \quad \text{in } L^2(\mathbb{R}^n, |x|^{-\alpha}) \\
&|v_k(.,0)|^{2^*_\alpha-2}v_k(.,0) \rightharpoonup |v(.,0)|^{2^*_\alpha-2}v(.,0) \quad \text{in } L^{\frac{2^*_\alpha}{2^*_\alpha-1}}(\mathbb{R}^n) \\
&|v_k(.,0)|^{2^*_\alpha(s)-2}v_k(.,0) \rightharpoonup |v(.,0)|^{2^*_\alpha(s)-2}v(.,0) \quad \text{in } L^{\frac{2^*_\alpha(s)}{2^*_\alpha(s)-1}}(\mathbb{R}^n, |x|^{-s}).
\end{aligned}
$$

Thus, taking limits as $k \to \infty$ in (4.27), we obtain that

$$
\begin{aligned}
0 &= \langle \Psi'(v), \varphi \rangle \\
&= k_\alpha \int_{\mathbb{R}^{n+1}_+} y^{1-\alpha} \langle \nabla v, \nabla \varphi \rangle dxdy - \gamma \int_{\mathbb{R}^n} \frac{v(x,0)\varphi}{|x|^\alpha} dx \\
&\quad - \int_{\mathbb{R}^n} |v(x,0)|^{2^*_\alpha-2} v(x,0)\varphi dx - \int_{\mathbb{R}^n} \frac{|v(x,0)|^{2^*_\alpha(s)-2} v(x,0)\varphi}{|x|^s} dx.
\end{aligned}
$$

Hence v is a weak solution of (2.4).

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