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A Linking/S^1-equivariant Variational Argument in the Space of Dual Legendrian Curves and the Proof of the Weinstein Conjecture on S^3 "in the Large"

To Jalila Ben Othman, in loving memory

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Abstract

Let α be a contact form on S^3, let ξ be its Reeb vector-field and let v be a non-singular vector-field in $ker\ \alpha$. Let C_β be the space of curves x on S^3 such $\dot{x} = a\xi + bv, \dot{a} = 0, a \geq 0$. Let L^+, respectively L^-, be the set of curves in C_β such that $b \geq 0$, respectively $b \leq 0$. Let, for $x \in C_\beta$, $J(x) = \int_0^1 \alpha_x(\dot{x})dt$. The framework of the present paper has been introduced previously in eg [3].

We establish in this paper that some cycles (an infinite number of them, indexed by odd integers, tending to ∞) in the S^1-equivariant homology of C_β, **relative** to $L^+ \cup L^-$ and to some specially designed "bottom set", see section 4, are achieved in the Morse complex of (J, C_β) by unions of unstable manifolds of critical points (at infinity)which must include periodic orbits of ξ; ie unions of unstable manifolds of critical points at infinity alone cannot achieve these cycles. At the odd indexes $(2k-1) = 1+(2k-2)$, 1 for the linking, $(2k-2)$ for the S^1-equivariance, we find that the equivariant contributions of a critical point at infinity to L^+ and to L^- are fundamentally asymmetric when compared to those of a periodic orbit [5]. The topological argument of existence of a periodic orbit for ξ turns out therefore to be surprisingly close, in spirit, to the linking/equivariant argument of P. Rabinowitz in [12]; e.g. the definition of the "bottom sets" of section 4 can be related in part to the linking part in the argument of [12]. The objects and the frameworks are strikingly different, but the original proof of [12] can be recognized in our proof, which uses degree theory, the Fadell-Rabinowitz index [8] and the fact that $\pi_{n+1}(S^n) = \mathbb{Z}_2, n \geq 3$. We need of course to prove, in our framework, that these topological classes cannot be achieved by critical points at infinity only, periodic orbits of ξ excluded, and this is the fundamental difficulty.

The arguments hold under the basic assumption that no periodic orbit of index 1 connects L^+ and L^-. It therefore follows from the present work that either a periodic orbit of index 1 connects L^+ and L^- (as is probably the case for all three dimensional over-twisted [8] contact forms, see the work of H. Hofer [10], the periodic orbit found in [10] should be of index 1 in the present framework); or (with a flavor of exclusion in either/or) a linking/equivariant variational argument a la P. Rabinowitz [12] can be put to work. Existence of (possibly multiple) periodic orbits of ξ, maybe of high Morse index, follows then.
Therefore, to a certain extent, the present result runs, especially in the case of three-dimensional over-twisted [8] contact forms, against the existence of non-trivial algebraic invariants defined by the periodic orbits of ξ and independent of what $ker\ \alpha$ and/or α are.

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1 Introduction

Preliminaries, Notations.

The variational problem (J, C_β) has been defined above and is studied in details throughout [1], [3] and other references.

In what follows, ∂ is the intersection operator for the variational problem (J, C_β). At a periodic orbit z of ξ, ∂ can be decomposed into a sum $\partial = \partial_{per} + \partial^\infty$, where ∂_{per} is the intersection operator relative to the periodic orbits of ξ and where ∂^∞ is the intersection operator at z with the critical points at infinity of (J, C_β). Given a critical point (at infinity)

$z^{(\infty)}$ of (J, C_β) of Morse index l, we denote in the sequel by $\partial z^{(\infty)}$ the union of the unstable manifolds of the critical points (at infinity) $y^{(\infty)}$ of index $(l-1)$, each counted with a multiplicity equal to the intersection number $i(z^{(\infty)}, y^{(\infty)})$. This is the algebraic number of flow-lines from $z^{(\infty)}$ to $y^{(\infty)}$. The notation $W_u(\partial z^{(\infty)})$ is also used and refers to the same object. In section 9 below, $W_u((\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j})$ designates the union of the unstable manifolds of the critical points at infinity in $(\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j}$, counted with their respective multiplicities. See section 9, Definition 1 and what follows for more details.

The basic Morse relation.

Let us consider the Morse relation:

$$(*) \qquad \partial c^{(\infty)}_{2k} = c_{2k-1} + h_{2k-1,\infty},$$

see [1], Lemma 2.14, p126, where $h_{2k-1,\infty}$ is the closure of a collection of unstable manifolds of critical points at infinity, of Morse index $(2k-1)$, dominated by a collection of periodic orbits of the Reeb vector-field of α, ξ, of index $2k$, y_{2k}s (they can be reduced to a single one, we do not use this here) and where c_{2k-1} denotes the closure of a collection of unstable manifolds of periodic orbits of ξ, of Morse index $(2k-1)$, satisfying the relation $\partial_{per} c_{2k-1} = 0$. ∂_{per} is, as stated above, the intersection operator related to the periodic orbits of ξ. c_{2k-1} is assumed here to be a minimal cycle (see [1]), which means that c_{2k-1} cannot be decomposed into smaller cycles for ∂_{per}. Let

$$\Gamma_{2s} = \{\text{set of curves made of } s \pm v\text{-jumps alternating with } s \text{ pieces of } \xi\text{-orbits}\}.$$

Let L^+ be the set of curves in $\cup\Gamma_{2s}$ having all their $\pm v$-jumps along $+v$ and let L^- be the set of curves in $\cup\Gamma_{2s}$ having all their $\pm v$-jumps oriented along $-v$. Let D_1 be an appropriate neighborhood of the critical points (at infinity) of index 1 of J, derived by flowing down along the unstable manifolds of these critical points small neighborhoods of zero in their stable manifolds. See section 4 and the figures there for more precise information.

The first result of this paper states, in a first and rough formulation, that the Fadell-Rabinowitz index [8] of the intersection $h_{2k-1,\infty} \cap (J^{-1}([\epsilon,\infty)) \setminus D_1)$, is at most $(k-2)$[1]. The removal of D_1 from $J^{-1}([\epsilon,\infty))$ is needed in order to warrant that the "bottom set" of X, which is $X \cap (J^{-1}(\epsilon) \cup \partial D_1)$, is connected in dimension $(2k-2)$, since there are no critical points of index 1 in the Morse complex of X. We will need to modify this later.

We then find that the proof of the estimate from above on the Fadell-Rabinowitz index of $h_{2k-1,\infty} \cap (J^{-1}([\epsilon,\infty)) \setminus D_1)$ derives from a more general argument: considering a stratified set $\tilde{X}$, of top dimension $2k$, we assume that $\tilde{X}$ is a manifold in dimensions $2k, (2k-1)$ and that S^1 acts effectively on $\tilde{X}$ and freely on its cells of dimension $2k$ and $(2k-1)$ and $(2k-2)$. We also assume that we are given an S^1-invariant functional J_∞ on $\tilde{X}$ and a corresponding

[1]Observe that, unlike in [12] and also in [1], we take here for the definition of the Fadell-Rabinowitz index of a topological set X with a free or effective S^1-action and classifying map f, the maximal power m to which the cohomological Chern class $[x]$ of $\mathbb{PC}^\infty$ can be raised and $f^*([x]^m)$ is non zero in the rational cohomology of X as in [8]. The Fadell-Rabinowitz index of $\mathbb{PC}^m$ is therefore m. Compare this to Lemma 1.13 of [12] where the Fadell-Rabinowitz index of S^{2m-1} for P. Rabinowitz in [12] is normalized to be m, one more than we would find with the present definition-which is also the definition in [8]-for the quotient $\mathbb{PC}^{m-1}$ of S^{2m-1} by the action of S^1. We find this definition to be more convenient for our purpose.

S^1-invariant flow such that $\tilde{X}$ is the closure of the closure of the union of unstable manifolds for this flow. We assume that the Palais-Smale condition holds and that $\tilde{X}$ does not contain any critical point of index 1.

Under the above assumptions, we claim that that $X = \tilde{X}/S^1$ is of Fadell-Rabinowitz index $(k-2)$ at most and that there is a classifying map for $\tilde{X} \longrightarrow X$ valued into $S^{2k-3} \longrightarrow \mathbb{PC}^{k-2}$.

Although our argument will contain the proof of the more general claim above, we will provide this proof within the framework of Contact Form Geometry [1], [3], [4] and we will discuss mainly the case when $X = h_{2k-1,\infty} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1)$, that is $X = \overline{\cup W_u(y_{2k-1,\infty})} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1)$ using the notation of [1]. In section 10 of this paper, we will show how the definition of X can be modified in our specific case in order to derive the verification of the assumptions above.

However, this result does not suffice to impede the above Morse relation since the same conclusion holds true for the collection of periodic orbits c_{2k-1} as well, i.e. $\overline{W_u(c_{2k-1})} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1)$ is also of Fadell-Rabinowitz index $(k-2)$ at most.

For $(*)$ above to be impossible, we need a more involved estimate on the Fadell-Rabinowitz index of the Morse complexes of dimension $(2k-1)$ **relative** to the values of the classifying maps on the topological boundary of these Morse complexes as deformation occurs from a collection of periodic orbits c_{2k-1} to $h_{2k-1,\infty} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1)$. See the Morse relation above and see section 11, below, of this paper.

Indeed, the main difference between the case of the periodic orbits $\overline{W_u(c_{2k-1})}$ and the case of critical points at infinity $\overline{W_u(h_{2k-1,\infty})}$ stems from the fact that the periodic orbits "link" the set L^+ of curves in $\cup\Gamma_{2s}$ having only positive v-jumps (no H_0^1-index if critical at infinity) with the set L^- of curves in $\cup\Gamma_{2s}$ having only negative $-v$-jumps (again no H_0^1-index if critical at infinity). This "linking" occurs because of the first eigenfunction of the linearized operator at a periodic orbit, see [3], [5].

On the other hand, whereas "linking" of L^- and L^+ occurs as a result of the existence of periodic orbits, at the "bottom level", in $J^{-1}(\epsilon)$, this linking does not occur and $J^{-1}(\epsilon) \cap L^+$ and $J^{-1}(\epsilon) \cap L^-$ are disconnected. They are connected through critical points of index 1.

Let W_1 be the union of their unstable manifolds (of dimension 1). The "linking" induced by the periodic orbits can be recognized on the classifying maps. Namely, using the map "b" of [5], it is proven in [5] that that the pair (A, B), where

$$A = \overline{W_u(c_{2k-1}) \setminus (L^+ \cup L^-)}$$

and

$$B = (\overline{W_u(c_{2k-1}) \setminus (L^+ \cup L^-)}) \cap [(\partial L^+ \cup \partial L^-) \cup J_\infty^{-1}(\epsilon) \cup W_1] \cup (\overline{\partial_\infty(c_{2k-1} \setminus (L^+ \cup L^-))})$$

maps through the pair

$$(C_\beta \setminus (L^+ \cup L^-), (C_\beta - (L^+ \cup L^-)) \cap (\partial(L^+ \cup L^-) \cup J^{-1}(\epsilon)) \cup W_1 \cup A_{2k-2}),$$

where A_{2k-2} is the set of curves in C_β such that v-component b has at least two zeros and at most $(2k-2)$ zeros, into the pair

$$(\mathbb{PC}^\infty \times [-1, 1], \mathbb{PC}^\infty \times \{-1, 0, 1\} \cup \mathbb{PC}^{k-2} \times [-1, 1])$$

and the composition is onto one of the generators of the homology of dimension $(2k-1)$ in the target. There are two such generators since $[-1,1]/\{-1,0,1\}$ has two generators in its homology at order 1. The sets L^+ and L^- are to be thought in the formulae above as small attracting (for the decreasing pseudo-gradient flow) neighborhoods of themselves.

On the other hand, each critical point at infinity $h_{2k-1,\infty,j}$ in the collection $h_{2k-1,\infty}$ introduces a basic asymmetry between L^+ and L^-, namely $\overline{W_u(h_{2k-1,\infty,j})}\cap L^+$ and $\overline{W_u(h_{2k-1,\infty,j})}\cap L^-$. One of them (and maybe both) is of Fadell-Rabinowitz index $(k-2)$ at most. See section 7, Lemma 3 below.

We use this fact and prove that the Morse relation $(*)$ is impossible.

The argument requires some further technical adjustments, which can be completed only under the basic assumption that there are no periodic orbit of index 1.

Under this assumption, we may arrange so that no critical point of index 1 connects $J^{-1}(\epsilon)\cap L^+$ and $J^{-1}(\epsilon)\cap L^-$, see section 3, below.

The removal of D_1 from the "X" sets above ignores the fact that the periodic orbits link L^+ and L^-, whereas these two sets are not linked in $J^{-1}(\epsilon)$. In order to restore this information, we modify the "bottom set " $J^{-1}(\epsilon)\cup\partial D_1$: we "open up" one "side of the bottom set", connecting $J^{-1}(\epsilon)\cap L^+$ and $J_0^{-1}(\epsilon)$ (the component of $J^{-1}(\epsilon)$ close to "small" back and forth runs along v) and we create in this way a new "bottom set" D_1^+. Now $D_1^+\cup(J^{-1}(\epsilon)\cap L^-)$ may be viewed, after a re-parametrization of flow-lines and after a related definition of a new functional $\tilde{J}$ as $\tilde{J}^{-1}(\epsilon)$. See J. Milnor [11], Theorem 4.1, pp37-38. We now have a disconnected "bottom set" $\tilde{J}^{-1}(\epsilon)$, where L^+ and L^- are not connected anymore, but L^+ and $J_0^{-1}(\epsilon)$ are connected.

Let W_1^- be the part of W_1 related to L^-, ie connecting the various components of $J^{-1}(\epsilon)\cap L^-$ exclusively.

Replacing J by $\tilde{J}$ in the pairs above and W_1 by W_1^-, we find now a classifying map with values in $(\mathbb{PC}^\infty\times[-1,1],\mathbb{PC}^\infty\times(\{-1\}\cup[0,1])\cup\mathbb{PC}^{k-2}\times[-1,1])$. This pair has the advantage, when compared to the former one, that it has only one generator in dimension $(2k-1)$.

We can now use the asymmetry of L^+ and L^- for $h_{2k-1,\infty}$ as described above and we prove, after careful modifications that are embedded in an isotopy of decreasing pseudo-gradients for the functional, that the Fadell-Rabinowitz index of $\overline{W_u(h_{2k-1,\infty})}\cap\tilde{J}^{-1}([\epsilon,\infty))$ is at most $(k-2)$ **relative** to the value of the classifying map on the "bottom set" $B_0 = D_1^+\cup(J^{-1}(\epsilon)\cap L^-)\cup W_1^-$, which is constrained to take values in $\mathbb{PC}^\infty\times\{-1\}\cup[0,1])\cup\mathbb{PC}^{k-2}\times[-1,1]$.

The same conclusion cannot hold for $\overline{W_u(c_{2k-1})}\cap\tilde{J}^{-1}([\epsilon,\infty))$ and the contradiction argument follows.

As stated above, a basic assumption is used in this argument: namely, it is assumed that the sets $J^{-1}(\epsilon)\cap L^+$ and $J^{-1}(\epsilon)\cap L^-$ are not connected by a periodic orbit of index 1.

We conjecture that, in the framework of over-twisted contact forms [7], the periodic orbit found by H. Hofer [10] is of index 1 (when viewed in our framework).

In some regards, our present paper indicates that, for the existence of periodic orbits, either an equivariant/linking argument "a la P. Rabinowitz" [12] works, yielding a sequence of periodic orbits of odd Morse index $(2k-1)$ for k large, or this argument does not work and a periodic orbit of index 1 is found, as in H.Hofer [10].

This is not established rigorously, but strongly indicated by the proof. This is emphasized in the last section of this paper.

Theorem 1.3,(i) of [1], the proof of which was not complete, see [2], follows from the claims above:

Theorem 1 *Assume that α is a contact form on S^3 and that the Reeb vector-field of α has no periodic orbit of Morse index* 1. *Then, (*) is impossible for k large enough and J has a sequence of critical values corresponding to periodic orbits of index* $(2k-1)$.

Let us recall that the existence of one periodic orbit for the contact forms of the tight contact structure of S^3 is a theorem by P. Rabinowitz [12], established without dimension restrictions, whereas the existence of one periodic orbit for the contact forms of all overtwisted [7] contact structures on a closed contact three dimensional manifold is a theorem by H. Hofer [10].

Theorem 1 above gives a new proof for the Weinstein conjecture on S^3. This new proof combines the case of the tight contact structure on S^3 and the case of all the other overtwisted ones [7] and, therefore, could lead to a better understanding of the existence process for periodic orbits of ξ. This new proof could also possibly lead to multiplicity results, on all three dimensional closed contact manifolds with a finite fundamental group.

The present paper and the corresponding topological argument for existence show also how to overcome the non-compactness of the variational problem associated to the periodic orbits problem for the Reeb vector-field ξ of a given contact form α on a three dimensional closed contact manifold which either has a finite fundamental group or which has a finite second homotopy group or both. The Weinstein conjecture on such manifolds M^3 now follows from this result as soon as we know that the S^1-equivariant classifying map, eg "b" as above, from $(C_\beta, L^+\cup L^-)$ into the pair $(\mathbb{PC}^\infty\times[-1,1], \mathbb{PC}^\infty\times\{-1,1\})$ is onto in rational homology. Observe that for the contact structures such that $ker\ \alpha \longrightarrow M^3$ is trivial, we can define a differentiable path of non-singular vector-fields $v_s, s\in[0,1]$ of $ker\ \alpha$, starting at v and ending at $-v$. There is a related deformation of the sets $L_s^+, s\in[0,1]$. Observe that $L_0^+ = L^+$, whereas $L_1^+ = L^-$. It follows that L^+ and L^- are "cobordant" (these are cobordisms of stratified sets) in C_β. Assuming that eg L^+ has infinite Fadell-Rabinowitz index, then the topological conclusion on the classifying map "b" which we have sought holds and the conjecture holds. If L^+ does not have infinite Fadell-Rabinowitz index, other arguments should apply, see e.g [1]. This line of proof requires several additional verifications. For this reason, we do not claim the proof of the three-dimensional Weinstein conjecture in full generality here, but only on S^3. On S^3, a direct verification of the topological argument described above can be completed . Taking into account the above observations, the proof should work for a large class of contact structures on more general closed three-dimensional manifolds as well. The assumption that β is a contact form with the same orientation as α, which we use in our work, see e.g. [3], can be overcome as indicated in [1] and other works.

2. The Fadell-Rabinowitz index of $X = \overline{\cup W_u(y_{2k-1,\infty})}\cap(J^{-1}([\epsilon,\infty))\setminus D_1)$

By assumption, the set X can be written as a union of closures of unstable manifolds of critical points at infinity of index $(2k-1)$:

$$X = \overline{\cup W_u(y_{2k-1,\infty})} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1).$$

As stated above, D_1 is derived after flowing down along the unstable manifolds of dimension 1 of the critical points (at infinity) of index 1 of J small neighborhoods of zero (transverse to the flow) in their stable manifolds. See section 4 in order to understand this construction with the help of a drawing.

Let us assume that X is a manifold in dimensions $(2k-1)$ and $(2k-2)$, see section 10 for the verification of these assumptions. It follows from these assumptions that each $y_{2k-1,\infty}$ is simple and that there cannot be more than one flow-line from each to $y_{2k-1,\infty}$ to each $y_{2k-2,\infty}$. This observation helps understand this setting. We then claim:

Lemma 1 *The Fadell-Rabinowitz index of X is at most $(k-2)$ and there is a classifying map for the S^1-action on X with values in $S^{2k-3}/\mathbb{PC}^{k-2}$.*

Proof of Lemma 1. Consider the topological boundary of each $\overline{W_u(y_{2k-1,\infty})} \cap (J^{-1}([\epsilon, \infty)) \setminus D_1)$, which we denote by $Z_{2k-2,\infty}$. It is a chain of dimension $(2k-2)$. Let

$$f : Z_{2k-2,\infty} \to \mathbb{PC}^\infty$$

be any classifying map for the S^1-action on $\tilde{Z}_{2k-2,\infty} \to Z_{2k-2,\infty}$, where $\tilde{Z}_{2k-2,\infty}$ is the set of S^1-invariant curves over $Z_{2k-2,\infty}$, $\tilde{Z}_{2k-2,\infty} = S^1 * Z_{2k-2,\infty}$.

We may assume f to be C^∞ so that by general position arguments and after deformation, its image may be assumed, after deformation, to have values in $\mathbb{PC}^{k-1}$:

$$f : Z_{2k-2,\infty} \to \mathbb{PC}^{k-1}.$$

Then using degree theory, we may assume that $deg f = 0$, since $Z_{2k-2,\infty}$ is a boundary. Observe that $Z_{2k-2,\infty}$ is connected, being the image through the time 1-map of the decreasing pseudo-gradient acting on an unstable sphere S^{2k-2} for $y_{2k-1,\infty}$.

In the special framework of [1], [3], [4] and [5], with

$$C_\beta = \{x \in H^1(S^1, M) \mid \beta(\dot{x}) = d\alpha(\dot{x}, v) = 0, \alpha(\dot{x}) = \text{C, C not prescribed and positive}\},$$

v non singular in $ker\ \alpha$ and with $\cup\Gamma_{2s}$, we may also assume that f is given on the set of curves x such that the v-component b of their tangent vector $\dot{x}$ has at least two zeros and at most $(2k-2)$ zeros and is equal to the map "b" of [5], which, after deformation, has values in $\mathbb{PC}^{k-2}$. This is done in the specific case as in [3], [4], [5] etc. In other cases, f might be given on some other set that maps into $\mathbb{PC}^{k-2}$. For simplicity and in order to make our arguments more transparent, we assume in the sequel that we are in the specific framework of [1], [3], [4], [5]. The generalization is clear.

Let us pick up a point x_0, which is a regular value for f in $\mathbb{PC}^{k-1}$, x_0 not in $\mathbb{PC}^{k-2}$ and let us consider $f^{-1}\{x_0\}$. If there are no points in $f^{-1}\{x_0\}$, our argument is complete, see below. Otherwise, we assume for the sake of simplicity that $f^{-1}\{x_0\} = \{z_1, z_2\}$, i.e. it is made of exactly two points where f has Jacobians with opposite signs (a natural orientation is available on $Z_{2k-2,\infty}$. This follows immediately from the definition of this set).

We then consider a generic path from z_1 to z_2. We can choose x_0 so that z_1 and z_2 are not in the stable manifold of any critical point (at infinity) of X. Using the decreasing pseudo-gradient vector field that defines X, we can deform this path into a path in $W_{2k} \cup L^+ \cup L^- \cup J_0^{-1}(\epsilon) \cup D_1$. Here W_{2k} is the set of curves such that the v-component b of their tangent vector has $2k$-sign changes and not less. Indeed we may assume that if z_i has $2k$ zeros, then these $2k$ zeros survive all along the decreasing flow-line, until the "bottom set" is reached. This is not essential in the argument but it is rather a side remark. Next L^+ is the set of curves where b is positive, L^- is the set with b negative. The set $J_0^{-1}(\epsilon)$ is the component of $J^{-1}(\epsilon)$ made of "small" curves in $J^{-1}([0,\epsilon])$, close to back and forth or forth and back runs (one or several) along v. Lastly D_1 is a small neighborhood of the unstable manifolds of the critical points (at infinity) of J of index 1 deleted from a small neighborhood of its trace in $L^+ \cup L^-$. For this reason, all the curves of $\partial D_1 \setminus (L^+ \cup L^-)$ are such that their v-component b has at least two zeros. See Lemma 7 for more precise information for this specific case. The argument extends to the general case when the classifying map is not "b" anymore. Since the Morse index of these critical points (at infinity) is 1, we find that the union of the unstable manifolds of these critical points at (infinity)is a compact set. In addition we find that b on this neighborhood can be deformed to a function $\tilde{b}$ having a finite, a priori bounded number of zeros, given by the projection of b onto the unstable directions, so that $\partial D_1 \setminus (L^+ \cup L^-)$ can be mapped through a modification of the map b into $\mathbb{PC}^r$, for a fixed r independent from k.

Let $B_1 = J_0^{-1}(\epsilon) \cup [J^{-1}(\epsilon) \cap (L^+ \cup L^-)] \cup \partial D_1$

We use this path and standard methods, see M. Hirsch [9], pp 126-127 to modify the map f near the path and make it have values in $\mathbb{PC}^{k-1} \setminus \{x_0\} \cong \mathbb{PC}^{k-2}$. Let us outline the argument in more detail:

Let p be the path as above. After deformation, we may assume that this path takes the following form: p starts at z_1 with a decreasing flow-line in the corresponding $W_u(y_{2k-2,\infty})$. This flow-line will, using a general position argument, reach the "bottom set" B_1. The same reasoning applies to z_2, and this happens whereas the flow-lines do not leave their respective sets, $W_u(y_{2k-2,\infty})$. There are no critical points of index 1 above B_1 by construction and therefore we may assume that the remainder of the path p is in a subset Z which is a manifold in dimension $(2k-2)$ and in dimension $(2k-3)$, so that p does not cross any singularity in $W_u(y_{2k-1,\infty})$. The cancellation procedure of section 1 and of [9], pp126-127, may be applied. By a general position argument, we can assume, for a given copy of $\mathbb{PC}^{k-2}$, that $f(p) \cap \mathbb{PC}^{k-2} = \emptyset$. Thus, we may assume that $f(p)$ and in fact $f(D^{2k-2})$ is contained in a disk D_2^{2k-2} around x_0.

As a map, from D^{2k-2} into D_2^{2k-2}, using a degree argument, f is then homotopic relative to its boundary value (from ∂D^{2k-2} into ∂D_2^{2k-2}) to a map with values in ∂D_2^{2k-2}. Using an equivariant family of small sections to the S^1-action in S^{2k-1} and using the lift $\hat{f}$ of f, we can lift this homotopy into a homotopy of S^1-equivariant maps above. Since $\hat{f}(\tau * x) = e^{ip\tau} f(x)$, the same relation will hold for all lifts along the homotopy and, at the end of this homotopy, the classifying map for $\overline{\partial(W_u(y_{2k-1,\infty}) \cap J^{-1}([\epsilon,\infty))}$ will have values in $S^{2k-1} \setminus S^1 * \{x_0\}$. Thus it will have values in S^{2k-3}, $\mathbb{PC}^{k-2}$ as claimed, with the map unchanged on the set of curves where b has at least one sign-change and at most $(2k-2)$ zeros, again as claimed.

We find then a new map $\tilde{f}$, **equal** to f on the set where b has at least one zero (with a sign change) and at most $(2k-2)$ zeros.

We extend now this map, or rather some power of this map to all of $\overline{W_u(y_{2k-1,\infty})} \cap \tilde{J}^{-1}([\epsilon,\infty))$. The map $\tilde{f}$ can be assumed to be defined in fact on all of $\overline{W_u(S^{2k-2})} \cap (J^{-1}([\epsilon,\infty)\setminus D_1))$, since this set retracts by deformation on $\partial\overline{W_u(y_{2k-1,\infty})}\cap(J^{-1}([\epsilon,\infty)\setminus D_1))$. Restricting, it follows that $\tilde{f}$ is defined from S^{2k-2} into $\mathbb{PC}^{k-2}$. Lifting, we find an equivariant map $\hat{f}: S^{2k-2} \to S^{2k-3}$. The map $\hat{f}$ is equivariant in that $\hat{f}(e^{i\tau} * x) = e^{pi\tau}\hat{f}(x)$, for a given integer p. This is completed with an appropriate modification of the map b: this modification elevates the various components of b on the functions, $sin\,(2j\pi t)$ and $cos\,(2j\pi t)$ in its Fourier decomposition, to appropriate powers (which depend on j) so that the modified map, with the introduction of these powers, satisfies the equivariant law as written above. See [5] for the transformation of b into its L^2-projection on the appropriate Fourier modes.

Restricting the map $\hat{f}$ to $S^{2k-2} \times \{1\}$, we find a map $g: S^{2k-2} \to S^{2k-3}$. We know that the homotopy group of order $(2k-2)$ of S^{2k-3} is Z_2 for $k \geq 3$. Therefore, if we knew that g was a double-meaning that its homotopy class would be zero-, we could extend it to D^{2k-1}, thereby extending $\tilde{f}$, with values in $\mathbb{PC}^{k-2}$, to all of $\overline{W_u(y_{2k-1,\infty})} \cap \tilde{J}^{-1}([\epsilon,\infty))$.

In order to be sure that g is a double, we need to be able to compose it with a map of degree 2, or a map of even degree from S^{2k-3} into itself. There are such maps and, thinking in terms of the covering map $h: S^1 * \overline{W_u(y_{2k-1,\infty})} \cap (J^{-1}([\epsilon,\infty) \setminus D_1)) \to S^{2k-3}$ over $\tilde{f}$, in order to define h, we can assume that we have composed its original value with a map from S^{2k-3} into itself and we have raised each (complex) component to the power 2 and re-normalized thereafter so that the norm stays 1. The resulting map is equivariant: it does satisfy the law $h(e^{i\tau} * x) = e^{pi\tau}h(x)$ with a suitable h, for which there is a suitable p. After this composition, the map g that we find is equal to the previous value for g composed with a map of even degree from S^{2k-3} in itself and it follows that the new map g is a double and the extension can be completed.

In this way, we find that the map "b", defined on the set of curves having at least one sign-change and at most $(2k-2)$ zeros, appropriately modified by (i) reducing it to its orthogonal L^2-projection on the basis of functions $sin\,(2j\pi t), cos\,(2j\pi t), 1 \leq j \leq (k-1)$, (ii) by raising these components to the appropriate powers and (iii) by taking only "part" of this map on the U_1 as above (i.e. changing b on U_1 into its projection on the corresponding negative eigenfunction(s) thereby finding a function with values in a finite dimensional fixed $\mathbb{C}^{r+1}$), extends to all of $h_{2k-1,\infty}$ into a map which is equivariant with the use of an $e^{ip\tau}$ factor of covariance in lieu of $e^{i\tau}$. As noted above, this procedure is not available only for the map "b". It can be completed for every classifying map (which we may assume to be valued into S^{2k-3}), with the use of an S^1-equivariant map h of even degree, from S^{2k-3} into S^{2k-3}, $h(e^{i\tau} * x) = e^{pi\tau}h(x)$, for a given integer p. The claim follows. ■

3. $h_{2k-1,\infty}$ and c_{2k-1}, splitting of the argument above and introduction of a basic assumption

The above argument is insensitive to the fact that the $y_{2k-1,\infty}$ are periodic orbits or critical points at infinity. This is essentially due to the fact that the "bottom set" B_1 is "above" any critical point of index 1, so that L^+ and L^- can be connected through this "bottom set". In order to distinguish between the case of the periodic orbits and the case of the critical points at infinity, we need to keep L^+ and L^- separated in the "bottom set".

We are therefore led to introduce the following basic assumption, (A), in our work:

(A) L^+ and L^- are not connected by a periodic orbit of index 1.

One of the connected components of $J^{-1}(\epsilon)$ is made of "small curves": these are the curves of C_β close to one or several back and forth or forth and back runs along v, they are contractible in a given, small neighborhood of eg their base point. We have denoted this component $J_0^{-1}(\epsilon)$ above.

In addition to (A), we also assume that each of $L^+ \cap J^{-1}(\epsilon)$ and $L^- \cap J^{-1}(\epsilon)$ is connected to $J_0^{-1}(\epsilon)$ by a critical point of index 1, respectively $x_+^{1,\infty}$ and $x_-^{1,\infty}$. After re-parametrization of the flow-lines of a pseudo-gradient for J which modifies this functional, but leaves the flow-lines of the pseudo-gradient unchanged, see J.Milnor [11], Theorem 4.1, pp37-38, and after tangencies between critical points of index 1, we may assume that these are the only critical points of index 1 connecting the "small" contractible curves (as above) of C_β to L^+ and to L^-. Using this re-parametrization procedure [11] and again tangencies, we may also assume then that L^+ and L^- are not connected by critical points at infinity of index 1. The unstable manifold of such a critical point at infinity, $\bar{x}^\infty$, on the side going to L^+ or on the side going to L^-, is made of curves having changes in the orientations of their $\pm v$-jumps. This change of sign allows us, without disturbing the flow-lines in L^+ and in L^-, to complete a tangency with $x_+^{1,\infty}$ or with $x_-^{1,\infty}$ possibly after re-parametrizing the flow-lines and changing the functional as in J. Milnor [11]. This removes the direct connection between L^+ and L^-. Then L^+ and L^--we may need to change J into $\tilde{J}$- are then no longer directly connected by critical points of index 1. They are connected through the "small" contractible curves of C_β.

All the re-parametrizations and tangencies completed above do not perturb the flow-lines in L^- and in L^+.

The most general form of our basic assumption is that we do not have a periodic orbit of index 1 connecting curves of L^+ with curves of L^- whereas there would be at the same time critical points at infinity of index 1 connecting $L^+ \cap J^{-1}(\epsilon)$ and the "small" contractible (as above, in a given small neighborhood of e.g. their base point) curves of C_β and connecting $L^- \cap J^{-1}(\epsilon)$ and the "small" contractible curves of C_β. If this assumption does not hold, we would find a "circle" of critical points of index 1 between L^+, L^- and the "small" contractible curves and our arguments then collapse. As long as some separation occurs along this circle, it appears that the above arguments go through.

4. Bottom sets

Our "bottom set" B_1 above, which is $J^{-1}(\epsilon) \cup \partial D_1$, is connected. This does not allow us to recognize the contribution of the periodic orbits, as described above. We therefore define below another "bottom set" B_0. In its manifold part (outside the unstable manifold of the critical point $x_-^{1,\infty}$), it disconnects $L^+ \cup J_0^{-1}(\epsilon)$ and L^-. This of course destroys an essential feature of our argument above about the Fadell-Rabinowitz index of X, namely that the flow-lines out of z_1 and z_2 can be connected in the "bottom set". We cannot assert this anymore with B_0. We will see how to overcome this difficulty.

For the purpose of our argument below, we need in fact to define two distinct "bottom sets", D_1^+ and D_1^- which are built from the same principle, but are different and not symmetric in their definition.

The basic pieces for the definition of D_1^+ are $J^{-1}(\epsilon) \cap L^+$ and $J_0^{-1}(\epsilon)$, where $J_0^{-1}(\epsilon)$ is the component of $J^{-1}(\epsilon)$ made of "small" contractible curves of C_β (near back and forth or forth and back runs along v). These various pieces are glued to boundaries of neighborhoods of unstable manifolds of the various critical points at infinity of index 1 connecting the various components of $J^{-1}(\epsilon) \cap L^+$ and connecting a component of this latter set to $J_0^{-1}(\epsilon)$. Flowing down the boundary (transverse to the flow) of a small neighborhood of 0 in the the stable manifold of each such critical point of index 1 on each side of its unstable manifold and glueing it to the corresponding bottom components of $J^{-1}(\epsilon)$ (which requires deletion of a neighborhood of the trace of this unstable manifold on the bottom component and glueing them, see the two figures below), we find for D_1^+ a manifold which acts exactly like a level surface for J, i.e. the flow of a decreasing pseudo-gradient vector field is transverse to D_1^+.

For D_1^-, we carry out the same construction **only** with $J^{-1}(\epsilon) \cap L^-$; that is we do not add $J_0^{-1}(\epsilon)$ and do not connect it through the unstable manifold of $x_-^{1,\infty}$ to $J_0^{-1}(\epsilon)$.

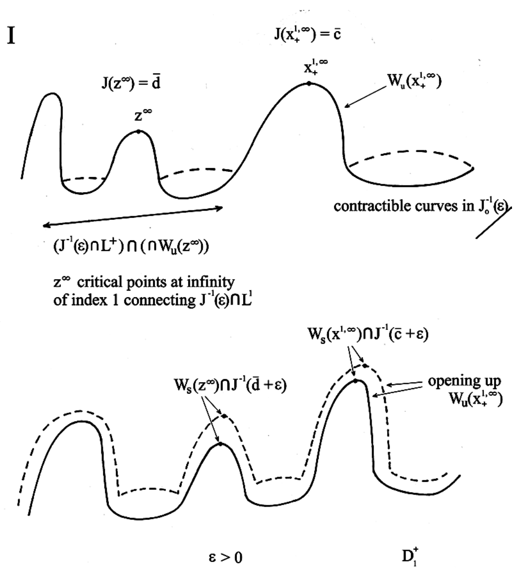

Figure 1

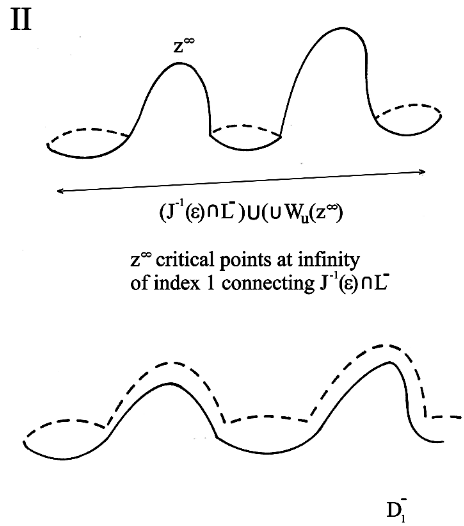

Figure 2

There is a fundamental asymmetry between the definition of D_1^+ and the definition of D_1^-.

For the purpose of our argument, we will denote by U_1 the part of D_1^+ which has been built using the stable manifold of the critical points of index 1 connecting $L^+ \cap J^{-1}(\epsilon)$ and $J_0^{-1}(\epsilon)$ on one hand and connecting the various components of $J^{-1}(\epsilon) \cap L^+$ between themselves on the other hand. We will denote by B_0 the union $D_1^+ \cup D_1^- \cup W_u(x_-^{1,\infty})$. The manifold part of B_0 is $D_1^+ \cup D_1^-$, which disconnects $L^+ \cup J_0^{-1}(\epsilon)$ and L^-. This is what we have sought.

As noted above, we may assume that we did re-parametrize the flow-lines of a/the decreasing pseudo-gradient just as in J.Milnor [11], Theorem 4.1, pp37-38 and that we thus have derived a new functional $\tilde{J}$ that has the same critical points (at infinity) as J, with the same stable and unstable manifolds for each of these critical points (at infinity) and for which $D_1^+ \cup D_1^-$ is $\tilde{J}^{-1}(\epsilon)$.

5. Splitting of the critical points at infinity of $h_{2k-1,\infty}$ into two groups

We split the critical points at infinity decomposing $h_{2k-1,\infty}$ into two groups. In the first group, the $y_{2k-1,j}^\infty$ are such that one of their large $\pm v$-jumps is along $+v$, whereas, in the second group, all the large $\pm v$-jumps of the critical points at infinity are along $-v$. Completing tangencies, we may assume that the second group is reduced to a single $z_{2k-1}^{\infty,-}$. In section 9, we will have to recall that we reached this single $z_{2k-1}^{\infty,-}$ out of several such critical points at infinity, all of which have their large $\pm v$-jumps along $-v$.

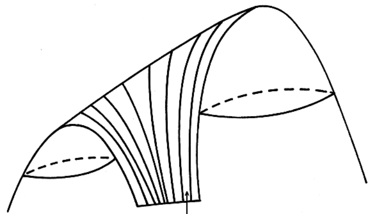

Figure 3

6. Requirements for the application of the arguments of section 2 after the definition of a new "bottom set" B_0

In order to apply the arguments of section 2, we now need to know that the traces of $W_u(z^\infty_{2k-1,-})$ and the trace of each $W_u(y^\infty_{2k-1,j})$ on the components D_1^+ and D_1^- of the bottom set B_0 are connected, see section 7 and section 8 below. On the other hand, we also need to know that the trace of $W_u(z^\infty_{2k-1,-})$ on D_1^- is connected. These results are established in the next section, after appropriate modifications of the pseudo-gradient vector field.

7. Preliminary technical results

We start with:

The classifying map on $h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)$.

Let $J_0^{-1}(\epsilon)$ be the component of $J^{-1}(\epsilon)$ corresponding to curves close to back and forth or forth and back runs along v, which we have also have been referring to as the component of $J^{-1}(\epsilon)$ made of "small" contractible curves.

We first modify $W_u(y^\infty_{2k-1,j})$ with the addition of "bridges" in order to render $W_u(h_{2k-1,\infty})\cap J_0^{-1}(\epsilon)$ connected. This is done by the introduction of additional critical points $z^\infty_{2k-1,j}$s, of critical value eg 2ϵ, which have their boundaries made of flow-lines all abutting to "small" contractible curves. Each $z^\infty_{2k-1,j}$ has in its boundary two companion critical points at infinity of index $(2k-2)$, z^i_{2k-2}, $i = 1, 2$, which, together with $z^\infty_{2k-1,j}$ help build the "bridge. The critical values of these latter points are e.g. $3\epsilon/2$. The functional J is again slightly perturbed but we employ the same notation J or $\tilde{J}$. With these "bridges", $W_u(h_{2k-1,\infty}) \cap J_0^{-1}(\epsilon)$ is now connected in dimension $(2k-2)$. We then claim that:

Lemma 2 *The Fadell-Rabinowitz index of $h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)$ is $(k-2)$ at most. After possible addition of "bridges", the classifying map for the S^1-action on $\overline{W_u(h_{2k-1,\infty})} \cap J_0^{-1}(\epsilon)$ may* *be assumed to have values in* $S^{2k-3}/\mathbb{PC}^{k-2}$.

Proof of lemma 2. Here $J_0^{-1}(\epsilon)$ designates the level surface ϵ of the functional J, in the connected component corresponding to contractible curves.

The proof of the Lemma starts with the relation:

$$\partial c_{2k}^{(\infty)} = c_{2k-1} + h_{2k-1,\infty}$$

where c_{2k-1} and $h_{2k-1,\infty}$, as well as $c_{2k}^{(\infty)}$ designate the collection of unstable manifolds (with closures) of the various critical points (at infinity) involved in the definition of each piece. It then follows that:

$$\partial c_{2k}^{(\infty)} \cap J_0^{-1}(\epsilon) = \partial(c_{2k}^{(\infty)} \cap J_0^{-1}(\epsilon)) = c_{2k-1} \cap J_0^{-1}(\epsilon) + h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)$$

Since $c_{2k-1} \cap J_0^{-1}(\epsilon)$ has a classifying map with values in S^{2k-3}, the same can be inferred of $h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)$ if this set is connected. If not, we have resolved this connectedness issue with the addition of a finite family of paths, with tubular neighborhoods (following appropriate constructions).

In all, after some required modifications, we may assume that the classifying map for every trace on $J_0^{-1}(\epsilon)$ of the closure of a collection of unstable manifolds of dimension $(2k-1)$, which we assume to be a manifold in dimensions $(2k-1)$ and $(2k-2)$, cobordant to c_{2k-1} has values in $S^{2k-3}/\mathbb{PC}^{k-2}$.

We may add the unstable manifolds of the critical points (at infinity) of index 1 to $c_{2k-1} \cap J_0^{-1}(\epsilon)$ and to $h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)$ and also $J^{-1}(\epsilon) \cap (L^+ \cup L^-)$. Since this latter set is of low Fadell-Rabinowitz index, we can assert that the Fadell-Rabinowitz index of the union is at most $(k-2)$.

Now recalling our construction above when we were defining the "bottom sets", we take the "side" of L^+ and "open-up" the unstable manifolds of dimension 1 connecting $J_0^{-1}(\epsilon)$ to $J^{-1}(\epsilon) \cap L^+$ and we also connect the various components of $J^{-1}(\epsilon) \cap L^+$ between themselves, in order to create a "level surface" D_1^+ transverse to the flow.

The "opening-up" is completed with the use of the Morse Lemma at $x_+^{1,\infty}$ and the various other critical points at infinity of index 1 related to $J^{-1}(\epsilon \cap L^+$. The "top" of D_1^+ at $x_+^{1(\infty)}$ is made of the trace of $W_s(x_+^{1(\infty)})$, the stable manifold of $x_+^{1(\infty)}$, on a level surface just above $x_+^{1(\infty)}$. A neighborhood of this "top" is subjected to the downward pseudo-gradient flow on both "sides" of $x_+^{1(\infty)}$ and connects $J_0^{-1}(\epsilon)$ and $J^{-1}(\epsilon) \cap L^+$. The set D_1^+ is the union of the three pieces $J_0^{-1}(\epsilon)$, $J^{-1}(\epsilon) \cap L^+$ and the piece related to these unstable manifolds of dimension 1.

It is then clear that the Fadell-Rabinowitz index of $c_{2k-1} \cap D_1^+$ and of $h_{2k-1,\infty} \cap D_1^+$, as well as that of their union, is at most $(k-2)$ since these sets can be equivariantly mapped into $(c_{2k-1} \cap J_0^{-1}(\epsilon)) \cup W_u(x_+^{1,(\infty)}) \cup (c_{2k-1} \cap J^{-1}(\epsilon) \cap L^+)$ and into $(h_{2k-1,\infty} \cap J_0^{-1}(\epsilon)) \cup W_u(x_+^{1,(\infty)}) \cup (h_{2k-1,\infty} \cap J^{-1}(\epsilon) \cap L^+)$ as well as into their union. ■

Lemma 3 *Let* z_{2k-1}^{∞} *be a critical point at infinity of index* $(2k-1)$*. Let* ∂ *be the intersection operator. Then* $\partial z_{2k-1}^{\infty} \cap L^+$ *or* $\partial z^{\infty} \cap L^-$ *is empty for a suitable globally defined, admissible (i.e. leaving* L^+ *and* L^- *invariant) decreasing pseudo-gradient vector field. In fact, the classifying map for the* S^1*-action on either* $\overline{W_u(z_{2k-1}^{\infty})} \cap L^+$ *or on* $\overline{W_u(z_{2k-1}^{\infty})} \cap L^-$*, or on both* *can be assumed to be have values in $S^{2k-3}/\mathbb{PC}^{k-2}$.*

Proof of Lemma 3. Assume that z^{∞}_{2k-1} has e.g. at least one large positive v-jump. We then claim that, for a suitable pseudo-gradient vector field, $\partial z^{\infty}_{2k-1} \cap L^-$ is empty for a large enough index.

We will use several arguments from [3]. Assume that z^{∞}_{2k-1} dominates z^{∞}_{2k-2}, of index $(2k-2)$ and that $W_u(z^{\infty}_{2k-2})$ is entirely contained in L^-. It follows that z^{∞}_{2k-2} has an H^1_0-index [3], p7, see also p77, equal to zero. For k large, by [3], Lemma 11, p96, z^{∞}_{2k-2} must have, after a C^2-perturbation of the contact form, some characteristic (see eg [3], p101) ξ-pieces. We may assume that no decreasing pseudo-gradient vector field may be created at z^{∞}_{2k-2} with the introduction of a small negative v-jump anywhere, so that all the characteristic ξ-pieces of z^{∞}_{2k-2} have decreasing normals [4] with the positive orientation along $+v$.

We then introduce a small negative v-jump as a companion to the now small positive v-jump inherited from z^{∞}_{2k-1}. Together, these small negative and positive v-jumps can travel across the large negative v-jumps of z^{∞}_{2k-2}, until the small positive v-jump reaches the position of a decreasing normal along a characteristic ξ-piece of z^{∞}_{2k-2} so that the flow-line continues past z^{∞}_{2k-2}, not in L^-. This characteristic ξ-piece must exist for k large enough after adjustment of v-rotation along z^{∞}_{2k-2}, see [3], Lemma 11, p96. The claim follows and extends with the introduction of additional pairs of tiny positive and negative $\pm v$-jumps (this does not affect L^+ and this does not affect L^-) to all flow-lines from z^{∞}_{2k-1} to z^{∞}_{2k-2}. This corresponds to a modification of the pseudo-gradient flow, from z^{∞}_{2k-1}, as it reaches z^{∞}_{2k-2}.

We then claim that $H = \bigcup_{z^{\infty}_{2k-2} \in \partial z^{\infty}_{2k-1}} \overline{W_u(z^{\infty}_{2k-2}) \cap W_s(L^- \setminus \tilde{J}^{-1}(0,\epsilon))}$ can be deformed on a CW-complex of top dimension $(2k-3)$. This follows from the fact that, above the level ϵ, the only critical point (at infinity) of $\tilde{J}$ of index 1 is $x^{1,\infty}_-$ and all its other critical points (at infinity) are of index 2 or more. Since the z^{∞}_{2k-2}s are of index $(2k-2)$, we can use the reverse flow to the decreasing pseudo-gradient flow on H and deform it to a CW-complex of dimension $(2k-3)$.

It follows that we can assume that the classifying map for the S^1-action on H has values in $\mathbb{PC}^{k-2}$. The claim of Lemma 3 is established since the additional pieces that we can find in $\overline{W_u(z^{\infty}_{2k-1})} \cap L^-$, outside of H, are of top dimension $(2k-3)$.

The above proof requires some further work if z^{∞}_{2k-2} is in $\partial^{\infty} c_{2k-1}$: Indeed, let us consider, for a given z^{∞}_{2k-2}, $\overline{W_u(z^{\infty}_{2k-2}) \cap W_s(x^{1,\infty}_-)}$. This latter set divides the set F of flow-lines originating at z^{∞}_{2k-2} and abutting to $J_0^{-1}(\epsilon)$ from the set of flow-lines originating at z^{∞}_{2k-2} and abutting to $B_0 \cap L^-$.

When z^{∞}_{2k-2} is part of $\partial^{\infty} c_{2k-1}$, the classifying map is given by the map "b" of [5] on $F \setminus z^{\infty}_{2k-2}$. The above argument is insensitive to this. Therefore in this case, we need a slightly more involved argument to understand better the set H introduced above, see below. ■

8. Isotopy of decreasing pseudo-gradients

We recall that we have split the critical points at infinity decomposing $h_{2k-1,\infty}$ into two subsets. Those in the first subset have some large positive v-jump, whereas those in the second subset only have large negative $-v$-jumps. Observe that if z^{∞}_{2k-1} has some positive large v-jump and k is large, then according to Lemma 3 above, $\overline{W_u(z^{\infty}_{2k-1})} \cap L^-$ has a classifying map with values in $S^{2k-3}/\mathbb{PC}^{k-2}$. If z^{∞}_{2k-1} has some negative large $-v$-jump, we can take L^- to be L^+ in the above statement.

Thus, applying Lemma 3 above to our set of specific critical points at infinity, the $y^{\infty}_{2k-1,j}$ of the first group are such that $\overline{W_u(y^{\infty}_{2k-1,j})} \cap L^-$ has a classifying map taking its values in $S^{2k-3}/\mathbb{PC}^{k-2}$; whereas for the second group, it is the classifying map for $\overline{W_u(z^{\infty}_{2k-1,j})} \cap L^+$ that takes values in a low $S^{2k-3}/\mathbb{PC}^{k-2}$. Completing tangencies as stated above, we may assume that the second group is reduced to a single $z^{\infty,-}_{2k-1}$.

We then claim that under our basic assumption, which we use here in an essential way, we can complete an isotopy of the decreasing pseudo-gradient flow which leaves the flow-lines in L^+ and L^- undisturbed and such that the following claims hold true:

Lemma 4 *$W_u(z^{\infty,-}_{2k-1}) \cap D^+_1$ and $W_u(z^{\infty,-}_{2k-1}) \cap D^-_1$ are connected.*

Lemma 5 *(i) $W_u(y^{\infty}_{2k-1,j}) \cap D^+_1$ is connected.*

(ii) The classifying map on $\overline{W_u(y^{\infty}_{2k-1,j}) \cap L^-}$ may be assumed to be have values in $S^{2k-3}/\mathbb{PC}^{k-2}$.

Proof of Lemma 4. The arguments for this proof are strongly inspired by J.Milnor's proof of the h-cobordism theorem, see Theorem 6.4, p70 of [11].

We recall that we make the basic assumption that there are no critical points (at infinity) of index 1, $\tilde{x}^{1(\infty)}_{\pm}$ connecting curves of $J^{-1}(\epsilon)\cap L^+$ and $J^{-1}(\epsilon)\cap L^-$. Under our basic assumption, after completing tangencies that leave L^- invariant, we may assume that there is only one critical point at infinity of index 1, $x^{1\infty}_{-}$, connecting L^- and the "small" contractible curves (as above) of C_β as well as one critical point at infinity of index 1, $x^{1\infty}_{+}$, connecting L^+ and the "small" contractible curves of C_β (as above), whereas there is no critical point (at infinity) $x^{1(\infty)}$ connecting L^- and L^+.

We consider the critical point at infinity $z^{\infty,-}_{2k-1}$ from $h_{2k-1,\infty}$, as above, such that its larger $\pm v$-jumps are all negative and we consider a level c just below $J(z^{\infty,-}_{2k-1})$.

$W_s(L^-) \cap J^{-1}(c)$ is an open connected set with a boundary $(\partial W_s(L^-)) \cap J^{-1}(c)$ that is connected in its top dimension.

We claim that, for such a critical point at infinity $z^{\infty,-}_{2k-1}$ with only negative large $(-v)$-jumps, we can arrange so that, for each $c \lesssim J(z^{\infty,-}_{2k-1})$, c close to $J(z^{\infty,-}_{2k-1})$, $(W_u(z^{\infty,-}_{2k-1}) \setminus W_s(L^-)) \cap J^{-1}(c)$ is connected. To obtain this conclusion, it suffices that each connected component of $W_s(L^-) \cap W_u(z^{\infty,-}_{2k-1}) \cap J^{-1}(c)$ has a connected boundary.

If a connected component, an open set in $W_s(L^-) \cap W_u(z^{\infty,-}_{2k-1}) \cap J^{-1}(c)$, has a boundary made of two or more distinct connected components C_1 and C_2, we need to modify the flow, keeping the curves of L^- in L^-, so that, for this modified flow, C_1 and C_2 are changed and define the same connected component of the boundary of the intersection set.

The level c is very close to $J(z^{\infty,-}_{2k-1})$ and therefore, C_1 and C_2 may be assumed to be contained in $W_s(x^{1,\infty}_{-})$, where $x^{1,\infty}_{-}$ is the only critical point at infinity of index 1 connecting

L^- and the small contractible curves of C_β. We now connect C_1 and C_2 with two paths p_1 and p_2, one in $W_s(L^-) \cap W_u(z_{2k-1}^{\infty,-}) \cap J^{-1}(c)$, the other one in $W_s(x_-^{1,\infty}) \cap J^{-1}(c)$. Assuming that M^3 is S^3, or assuming that $J^{-1}(c)$ is connected and simply connected, we may find a surface Σ in $J^{-1}(c)$ connecting p_1 and p_2. The argument extends to the case when the second homotopy group of M^3 is finite. We are working with rational homology, which allows us to take multiples of our topological classes. Along such multiples, the loop defined by the composition of p_1 and of p_2 is iterated accordingly and we may therefore assume that this iterated loop is contractible in $J^{-1}(c)$, so that the argument generalizes.

The curves of $W_u(z_{2k-1}^{\infty,-}) \cap J^{-1}(c)$ that are in L^- define, for c close to $z_{2k-1}^{\infty,-}$, an open ball with a connected boundary. We may define our pseudo-gradient vector field so that a small open neighborhood of this closed ball flows into L^-. We may then assume that p_1 does not intersect the closure of this open ball. In addition, Σ may be assumed to be embedded in $J^{-1}(c)$, using a general position argument. Again, using such a general position argument, Σ defines the trace of a deformation along which p_1 of $W_s(L^-) \cap W_u(z_{2k-1}^{\infty,-}) \cap J^{-1}(c)$ is brought to p_2. Observe that p_1 and p_2 do not intersect L^-. After perturbation, Σ also may be assumed not to intersect L^-. Indeed Σ may be assumed to be in some Γ_{2m} for m large. We may add to the curves of Σ $4m$ tiny positive v-jumps that are brought to be zero when reaching p_1 and p_2. The curves are not in $J^{-1}(c)$ anymore, but they are at a nearby level and we can flow them back to this level since none of the curves of Σ was critical to begin with. Then, Σ does not intersect L^-. This simple deformation can now be "opened up" and transformed into an isotopy of decreasing pseudo-gradient flow. At the time 1 of the deformation, the two modified components, C_1 and C_2 are now connected, whereas the evolution of the curves of L^- is not disturbed.

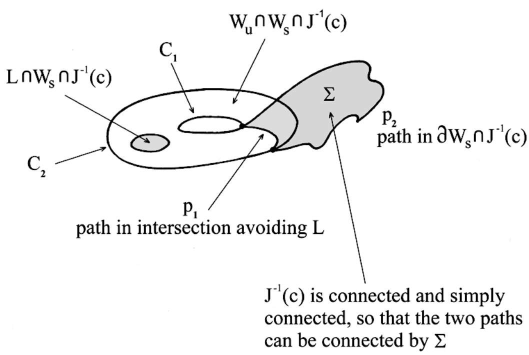

Figure 4

A similar construction/deformation may be carried out in order to connect all the various components of $W_u(z_{2k-1}^{\infty,-})$ going into L^-. Once these modifications are performed, we can complete tangencies between various $z_{2k-1}^{\infty,-}$s. As long as the tangencies occur as described in the figure above, without involving flow-lines abutting in L^-, the recomposition of the unstable manifolds obeys the rule that each connected component of curves attracted by L^- has a connected boundary, so that the complement of $W_s(L^-)$ in $W_u(z^{\infty,-}_{2k-1})$ (after tangencies) is connected (in its top dimension).

The conclusion follows for the first claim of Lemma 4. The proof of the second claim follows from the same argument, slightly modified. ■

Proof of Lemma 5. The only statement that requires additional proof is the claim about the classifying map. The addition of the various surfaces, Σ, constructed as above does not change the Fadell-Rabinowitz index since these are contractible pieces and they may be assumed not to dominate any critical point above D_1^- (after re-parametrization, see above and J. Milnor [11]). Then, after "opening up Σ" as above, we find that $\overline{W_u(y^{\infty}_{2k-1,j}) \cap L^-}$ is contained in a set having a classifying map with values in $S^{2k-3}/\mathbb{PC}^{k-2}$ as claimed. ■

The arguments collapse if $\partial W_s(L^-)$ is not connected.

9. The extension of Lemma 1 to $\overline{W_u(h_{2k-1,\infty})} \cap \tilde{J}^{-1}[\epsilon, \infty)$

Proposition 1 *(i) Lemma* 1 *extends to* $\overline{W_u(h_{2k-1,\infty})} \cap \tilde{J}^{-1}[\epsilon, \infty)$. *The classifying map after deformation has values in* $S^{2k-3}, \mathbb{PC}^{k-2}$.

(ii) Along this deformation, the classifying map restricted to $\overline{W_u(h_{2k-1,\infty})} \cap (\tilde{J}^{-1}(\epsilon) \cup W_u(x_-^{1,\infty})) = \overline{W_u(h_{2k-1,\infty})} \cap B_0$ *has values in* $(\mathbb{PC}^{k-1} \times [0,1] \cup \mathbb{PC}^{k-2} \times [-1,1] \cup \mathbb{PC}^{k-1} \times \{-1\})$.

Proof of Proposition 1.

Extending Lemma 3 to $\partial^\infty c_{2k-1}$*, with the "b" pre-assigned value [5] of the classifying map when the v-component of the tangent vector to the curves has at least one sign-change.*

As a first step, we extend Lemma 3 and prove that, if y^{∞}_{2k-2} is in $\partial^\infty c_{2k-1} \cap \partial y^{\infty}_{2k-1,j}$, then the classifying map "b" of [5] can be **extended** to $\overline{W_u(y^{\infty}_{2k-1,j}) \cap L^-}$ with values in $S^{2k-3}/\mathbb{PC}^{k-2}$ on this latter set.

We need a few preliminary definitions, Lemmas etc. for this. We start with:

Definition 1 Let $(\partial^\infty c_{2k-1})_-$ be the **critical points at infinity** in $\partial^\infty c_{2k-1}$ having all their large v-jumps oriented along $-v$ and having a non-zero H^1_0-index.

Requirements on decreasing flow-lines. *We require that our decreasing pseudo-gradient flow leaves the sets,* L^+ *and* L^-*, invariant (respectively) and that it never increases the number of zeros of the v-component b of the curves under decreasing deformation, the latter* **solely** *for the closure of the set of flow-lines originating at any periodic orbit of index* $(2k-1)$.

Therefore, starting from $y^{\infty}_{2k-1,j}$ as above, which has at least one large positive v-jump, and reaching to a critical point at infinity of $(\partial^\infty c_{2k-1})_-$, we find curves that have a mixture of positive and of negative steady $\pm v$-jumps. On such curves, we can add additional negative or positive $\pm v$-jumps as we please. We are not bound by any requirement since the flow-line does not originate at a periodic orbit of index $(2k-1)$. We then claim:

Lemma 6 *Any critical point at infinity in $(\partial^\infty c_{2k-1})_- \cap \partial z^\infty_{2k-1,j}$ has no characteristic ξ-piece. After a C^2-bounded, C^1-small perturbation of the contact form α in the vicinity of this critical point at infinity, we may assume that the maximal number of sign-changes for b on its unstable manifold is atmost* $(2k-4)$.

Remark 1 Lemma 6 is not absolutely required in our proof of Theorem 1, but it is a convenient result.

Proof of Lemma 6. Following our requirements and observation above, this critical point at infinity cannot have any characteristic ξ-piece, since we can then, on flow-lines out of $z^\infty_{2k-1,j}$ and reaching this critical point at infinity, introduce a decreasing normal [4] along this characteristic ξ-piece and bypass this critical point at infinity. We may assume that it has some non-zero H^1_0-index for k large enough. Indeed, otherwise, we can use Lemma 3 and Proposition 15 of [3]. There is enough v-rotation on the various ξ-pieces and we can transport it in a given ξ-piece, thereby creating a non-zero H^1_0-index on this ξ-piece.

Since c_{2k-1} dominates this critical point at infinity, the maximal number of zeros on its unstable manifold is $(2k-2)$ at most. Since it has a non-zero H^1_0-index, we can use Lemma 3 of [3] and modify this maximal number of zeros at least by 2. The claim follows.■

Lemma 7 $x^{1,\infty}_-$ *may be assumed to have at least one large positive and one large negative v-jump.*

Proof of Lemma 7. $x^{1,\infty}_-$ introduces a genuine difference of topology in the level sets of the functional J. It cannot therefore have a characteristic ξ-piece. If it had e.g. only negative large v-jumps, then its H^1_0-index cannot be zero: $x^{1,\infty}_-$ connects $J_0^{-1}(\epsilon)$ and $J^{-1}(\epsilon) \cap L^-$ and this cannot be achieved with a Morse index totally at infinity.

Since the H^1_0-index of $x^{1,\infty}_-$ is non-zero, we can modify it using again Lemma 3 of [3]. It cannot become 2, this would be too high. Thus, it has to become zero; the index of $x^{1,\infty}_-$ is totally at infinity and this is a contradiction as pointed out above. ■

It follows that there exists a neighborhood of $W_u(x^{1,\infty}_-) \cap J^{-1}([\epsilon,\infty))$ where the classifying map for the S^1-action may be assumed to be given by the map "b" of [5], since the v-component of $\dot{x}$ has at least two zeros.

The classifying map on $\overline{\cup_{z_l^{2,\infty}} W_u((\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j}) \cap W_s(z_l^{2,\infty})}$ *and nearby:*

By what has been shown so far, the classifying map is given on part of

$$\overline{W_u((\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j}) \cap W_s(x^{1,\infty}_-)}$$

and now we need to extend this map to a set that retracts by deformation on

$$\overline{\bigcup_{z_l^{2,\infty}} W_u((\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j}) \cap W_s(z_l^{2,\infty})}$$

where the $z_l^{2,\infty}$ are all the critical points at infinity of index 2 dominating $x^{1,\infty}_-$. This is a stratified set of top dimension $(2k-4)$. Its classifying map may be assumed, by a general position argument, to have values in $\mathbb{PC}^{k-2}$. A homotopy of this classifying map may also be assumed, using the same general position argument, to have values in $\mathbb{PC}^{k-2}$.

The critical points at infinity of this stratified set are of two types: there are those which contain a sign-change in their large $\pm v$-jumps. The map "b" of [5] is well-defined on a full neighborhood of these critical points at infinity.

Then, there are those having all negative large $(-v)$-jumps. Their H_0^1-index cannot be zero since they dominate $x_-^{1,\infty}$ which has a sign-change in its large $\pm v$-jumps. In a neighborhood of these critical points at infinity, we define a "b"-map which is slightly different from the map "b" of [5]: there is a connected region, diffeomorphic to a cone, in the unstable manifold of such a critical point at infinity made of curves such all possible $\pm v$-jumps are non-zero and negative. On the boundary of this region, some of these negative v-jumps are zero. All of these correspond to H_0^1-directions near the dominating critical point at infinity.

Along this boundary, turning one of the zero v-jumps corresponding to H_0^1-index directions into positive tiny v-jumps defines a convex entering set of normal directions into the curves of the unstable manifold where b changes sign. Furthermore, if this critical point at infinity dominates another critical point at infinity of the same family with a non-zero H_0^1-index, then, since all $\pm v$-jumps that are non-zero on this boundary are negatively oriented, we deduce that this H_0^1-position must have existed above, in the dominating critical point at infinity and must have survived all along the flow-lines connecting these two critical points at infinity of the same family. It follows that the set of entering normals is well-defined. Since the regions where all possible $\pm v$-jumps are negative cannot dominate $x^{1,\infty}$, we find that we can use this set of entering normals and define the map "b" all over our stratified set, except for the periodic orbits. Observe that, if on some flow-lines originating at one of the critical points at infinity as above, with all large negatively oriented $\pm v$-jumps, there is a positive v-jump due to the use of an H_0^1-direction and that this positive v-jump cancels with a negative $-v$-jump as we approach a lower critical points of the same family, then the map "b" of [5] is defined on the flow-lines, originating and ending critical points at infinity excluded. Using Lemma 6 above, it can be glued with the map "b" as defined above, with values in $S^{2k-3}/\mathbb{PC}^{k-2}$. Observe in addition that if, starting from $y^\infty_{2k-1,j}$, we end up at a critical point at infinity of $\partial^\infty c_{2k-1}$ with all its $\pm v$-jumps oriented along $+v$, then the map "b" of [5] will be defined in the vicinity of the flow-lines starting at this critical point at infinity and ending into L^-, with at least two zeros and at most $(2k-4)$ zeros and we can again glue this map with the other map "b" as defined above, with a resulting map with values in $S^{2k-3}/\mathbb{PC}^{k-2}$. We could use a weaker statement than the statement of Lemma 6, with $(2k-2)$ zeros in lieu of $(2k-4)$.

The periodic orbits are of top index $(2k-3)$, with a maximal number of zeros of b on their unstable manifold equal to $(2k-4)$. In order to define the map b, we need b to have at least two zeros. The map b is identically zero at the periodic orbit, but we can perturb the unstable manifold so that b is non-zero at the top perturbed critical point and has $(2k-2)$ zeros, with a maximal number of zeros for b on this perturbed unstable manifold equal to $(2k-2)$ near the top, $(2k-4)$ below; this, if the periodic orbit is of index $(2k-3)$; $(2k-4)$ otherwise, in lieu of $(2k-2)$. The flow-lines that dominate $x^{1,\infty}$ in this unstable manifold must be such that b has at least two zeros on their curves. There could be other periodic orbits/critical points at infinity in their closure, for which we proceed as above.

The resulting map "b" extends to this stratified set, valued into $\mathbb{PC}^{k-2}$.

Resolving the multiplicity of $\overline{\underset{z_l^{2,\infty}}{\cup} W_u((\partial^\infty c_{2k-1})_- \cap \partial y^\infty_{2k-1,j}) \cap W_s(z_l^{2,\infty})}$ *at the critical points (at infinity) that it contains.*

We now resolve the "multiplicity" of this stratified set of decreasing flow-lines at each critical point (at infinity), thereby creating a stratified set T_{2k-4}, which is a section to the decreasing flow abutting into L^-.

Indeed, the original set is a closed invariant set of decreasing flow-lines. Far away from the critical points (at infinity), it can be perturbed into a section to a decreasing flow abutting into L^-. Close to the critical points (at infinity), we find possibly several "leaves" for this stratified set, intersecting at the critical point (at infinity). The "leaves" define components, some of them abutting to L^-, the other ones to e.g. $J_0^{-1}(\epsilon)$. We can resolve them also into sections to a decreasing flow.

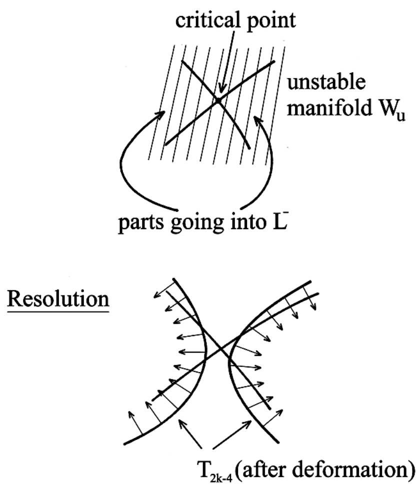

Figure 5

On T_{2k-4}, two classifying maps are now defined: the map "b" as above and the map Ψ, mapping T_{2k-4} into its limit set at infinity L^-_∞ and from there, to $\mathbb{PC}^\infty$. Since L^-_∞ is, after deformation, of top dimension $(2k-4)$, Ψ may be assumed to have values in $\mathbb{PC}^{k-2}$ (top dimension $(2k-3)$ would lead to the same conclusion).

The homotopy between these two maps "b" and Ψ, restricted to T_{2k-4} may be assumed to have values in $\mathbb{PC}^{k-2}$ as well.

We now conclude the argument. The figures of reference are as follows:

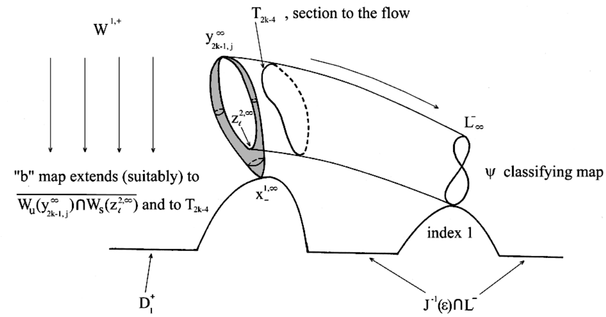

Figure 6

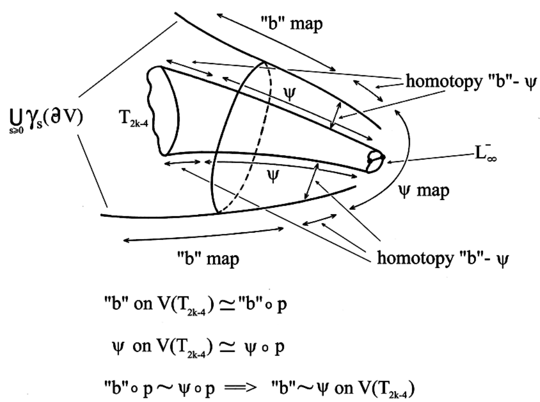

Figure 7

Taking a small neighborhood V of T_{2k-4} in a section to the flow, we follow it using the decreasing flow γ_s. $\underset{s\geq 0}{\cup}\gamma_s(V)$ has $\underset{s\geq 0}{\cup}\gamma_s(\partial V)$ as a boundary. There is a projection p : $\underset{s\geq 0}{\cup}\gamma_s(V) \longrightarrow \underset{s\geq 0}{\cup}\gamma_s(T_{2k-4})$ and the map "b" and the map Ψ which were defined above on T_{2k-4} thereby extend, using p, to $\underset{s\geq 0}{\cup}\gamma_s(V)$. Observe that the standard map "b" of [5] is homotopic to the map "b" defined above, the homotopy having values in $\mathbb{PC}^{k-2}$, and observe that, on $\underset{s\geq 0}{\cup}\gamma_s(\partial V)$, this map maybe viewed after deformation as constant on each $\underset{s\geq 0}{\cup}\gamma_s(z)$, equal to its value at $p(z)$, since on each flow-line the changes of sign of the function b can be recorded unchanged as the time-parameter s increases; it is only the sizes of the function b and its shapes, on the various intervals between zeros that we can track, which change.

The ω-limit set of each $\underset{s\geq 0}{\cup}\gamma_s(V)$ and $\underset{s\geq 0}{\cup}\gamma_s(\partial V)$ is the same: it is L^-_∞, with its map Ψ.

Since the map "b" and the map Ψ are homotopic when restricted to T_{2k-4}, with a homotopy with values in $\mathbb{P}\mathbb{C}^{k-2}$ and since their values on $\underset{s\geq 0}{\cup}\gamma_s(V)$ are derived with the use of p, they are homotopic as maps defined on this larger set, with the same target value set $\mathbb{P}\mathbb{C}^{k-2}$.

As we reach to L^-_∞, starting with ∂V and flowing down, we may gradually use this homotopy and insert the map Ψ, so that the classifying map takes the well-defined value Ψ on L^-_∞. Going deeper into $\underset{s\geq 0}{\cup}\gamma_s(V)$, we use the map Ψ more and more on the flow-lines. When we reach T_{2k-4}, the map is Ψ all along the decreasing flow-lines. Of course, we have used an interval $[-\epsilon, 0]$ of times s to replace "b" by Ψ as we start in T_{2k-4}.

We have therefore extended the map "b" on $\partial^\infty c_{2k-1} \cap \partial z^\infty_{2k-1,j}$ to the flow-lines abutting in L^- and the extension has values in $\mathbb{P}\mathbb{C}^{k-2}$. Using the fact that the "bottom set" D_1^+ is connected, we may now apply, without perturbing the topological arguments of section 11 below, the procedure of Lemma 1 above to the topological boundary of $W_u(y^\infty_{2k-1,j}) \cap J^{-1}([\epsilon,\infty))$. We find a classifying map with values in $\mathbb{P}\mathbb{C}^{k-2}$ on $\overline{W_u(y^\infty_{2k-1,j}) \cap J^{-1}([\epsilon,\infty))}$. We will use this later.

Conclusion for the extension of Lemma 3.

We complete the modifications described in the first part of this paper, for all sets $W_u(y^\infty_{2k-1,j})$ such that $\partial y^\infty_{2k-1,j} \cap L^-$ has a classifying map with values in $\mathbb{P}\mathbb{C}^{k-2}$. The modifications do not occur on flow-lines abutting in L^- then since, by Lemma 3, the classifying map on $\overline{W_u(y^\infty_{2k-1,j})} \cap L^-$, and even on $\overline{\underset{m}{\cup} W_u(y^\infty_{2k-1,m})} \cap L^-$, may be assumed to be given with values in $S^{2k-3}, \mathbb{P}\mathbb{C}^{k-2}$. These modifications occur on flow-lines abutting in D_1^+. We know that each $\partial W_u(y^\infty_{2k-1,j})$ is connected. By Lemma 5, we know that $W_u(y^\infty_{2k-1,j}) \cap D_1^+$ is connected and, according to the construction of D_1^+, see section 4, no critical point (at infinity) of index 1 dominates D_1^+, aside from $x_-^{1,\infty}$.

The arguments for Lemma 1 can then be applied to each of these sets $W_u(y^\infty_{2k-1,j})$.

Once the classifying map is defined on these unstable manifolds in $h_{2k-1,\infty}$, we are left with the $z^\infty_{2k-1,j}$ of $h_{2k-1,\infty}$ such that their large $\pm v$-jumps are along $-v$. We have reduced them to a single $z^\infty_{2k-1,-}$, which we denote z^∞ in the sequel.

The conclusion for the proof of Proposition 1.

Let now $\overline{W^{1,+}}$ be the closure of the set of decreasing flow-lines abutting to the "bottom set" D_1^+.

Arguing as above, but using $z^{\infty,-}_{2k-1}$ in lieu of $y^\infty_{2k-1,j}$, we may assume that the classifying map on $\overline{\partial^\infty W_u(c_{2k-1}) \cap \partial W_u(z^\infty_{2k-1,-}) \cap W^{1,+}}$ also has values in $\mathbb{P}\mathbb{C}^{k-2}$: this involves extending as above a variant of the map "b" of [5] into L^+. The reasoning is identical to the case for $y^\infty_{2k-1,j}$, only now L^- is replaced by L^+.

There is however no global reduction of the classifying map on all of $\overline{W_u(z^{\infty,-}_{2k-1})}$ as above for $\overline{W_u(y^\infty_{2k-1,j})}$ since the "bottom set" is not connected now. The argument is different. It goes as follows:

After our reasoning above, also Lemma 1 and Proposition 1, we know that the classifying map has values in $\mathbb{PC}^{k-2}$ on $\overline{W_u(y^{\infty}_{2k-1,j})}$, on the trace of h^{∞}_{2k-1} and c_{2k-1} on the bottom set D^+_1 and also on $\overline{\partial^{\infty} W_u(c_{2k-1}) \cap \partial W_u(z^{\infty}_{2k-1,-})} \cap \overline{W^{1,+}}$. Since $\partial y^{\infty}_{2k-1,j} + \partial z^{\infty}_{2k-1,-} + \partial^{\infty} c_{2k-1} = 0$, we deduce from the claims above that we may assume that the classifying map on $\overline{\partial W_u(z^{\infty}_{2k-1,-}) \cap W^{1,+}}$ has values in $\mathbb{PC}^{k-2}$. Using the proof of Lemma 4 and the proof of Lemma 5 and the connectedness of $\overline{W_u(z^{\infty}_{2k-1,-}) \cap \partial \overline{W^{1,+}}}$, we find since this set and $\overline{\partial W_u(z^{\infty}_{2k-1,-}) \cap \overline{W^{1,+}}}$ add up to a boundary of top dimension $(2k-2)$, that $\overline{W_u(z^{\infty}_{2k-1,-}) \cap \partial \overline{W^{1,+}}}$ has also a classifying map with values in $\mathbb{PC}^{k-2}$.

Through our previous modifications, the classifying map is given on $(W_u(c_{2k-1}) \cup W_u(h_{2k-1,\infty} \setminus z^{\infty})) \cap D^+_1$, with values in $\mathbb{PC}^{k-2}$.

This classifying map can be extended to $(W_u(c_{2k-1}) \cup W_u(h_{2k-1,\infty})) \cap D^+_1$, with values in $\mathbb{PC}^{k-1}$. By Lemma 2, it is of degree zero. Since this map restricted to $(W_u(c_{2k-1}) \cup W_u(h_{2k-1,\infty} \setminus z^{\infty})) \cap D^+_1$ has values in $\mathbb{PC}^{k-2}$ and since $W_u(z^{\infty}) \cap D^+_1$ is connected, we can modify the classifying map relative to this preassigned value on $(W_u(c_{2k-1}) \cup W_u(h_{2k-1,\infty} \setminus z^{\infty})) \cap D^+_1$ so that it now has values in $\mathbb{PC}^{k-2}$.

It thus follows that the **topological** boundary $\partial W_u(z^{\infty}) \setminus (\partial W_u(z^{\infty}) \cap L^-)$ is of the Fadell-Rabinowitz index $(k-2)$ at most and therefore, the **topological** boundary $(\partial W_u(z^{\infty}) \cap L^-)$ is also of Fadell-Rabinowitz index $(k-2)$ at most. By Lemma 4, it is a connected set if we attach to it, without increasing its index, boundaries of appropriate neighborhoods (see section 4, above) of unstable manifolds of critical points at infinity of index 1 connecting the various components of $J^{-1}(\epsilon) \cap L^-$. These neighborhoods were used in section 4 in order to define the appropriate "bottom set" D^-_1 in L^-, formed essentially of $J^{-1}(\epsilon) \cap L^-$ and of these unstable manifolds, glued together so that this defines a "level surface" (ie a "bottom set" transverse to the flow), see section 4.

We may therefore assume that, on all of $\partial W_u(z^{\infty})$ as well as on the trace of $W_u(z^{\infty})$ on $B_0 = D^+_1 \cup D^-_1 \cup W_u(x^{1,\infty}_-)$, the classifying map is given, extending the one previously defined on $\overline{W_u(h_{2k-1,\infty} \setminus z^{\infty})}$ with values in $S^{2k-3}, \mathbb{PC}^{k-2}$.

Using the arguments of Lemma 1, this map can now be extended to $\overline{W_u(z^{\infty})}$, so that the modifications of Lemma 1 have now been completed on all of $\overline{W_u(h_{2k-1,\infty})}$, with a trace on the bottom set B_0 with values in $(\mathbb{PC}^{k-1} \times \{-1\} \cup \mathbb{PC}^{k-1} \times [0,1] \cup \mathbb{PC}^{k-2} \times [-1,1])$.

Summarizing, the scheme of proof of Theorem 1 is as follows, supported by the following figure:

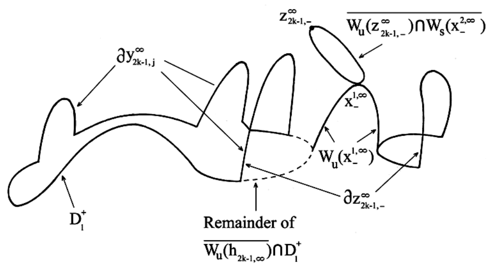

Figure 8

Step1: The classifying map has values in $\mathbb{PC}^{k-2}$ on $\partial y^{\infty}_{2k-1,j} \cup (\overline{W_u(y^{\infty}_{2k-1,j}) \cap D^+_1})$ (Lemma 1).

Step2: The classifying map can be extended to the trace of $\overline{W_u(h_{2k-1,\infty})}$ on D^+_1, with values in $\mathbb{PC}^{k-2}$. Therefore, the classifying map on $\overline{W_u(z^{\infty}_{2k-1,-}) \cap W_s(x^{1,\infty}_{-})}$ has values in $\mathbb{PC}^{k-1}$, with degree zero.

Step3: We know that $\overline{\partial z^{\infty}_{2k-1,-} \cap W_s(L^-) \cup W_u(z^{\infty}_{2k-1,-}) \cap D^-_1}$ is of dimension $(2k-2)$ and connected. Now Step 2 implies that the classifying map on this set is of degree zero and the conclusion follows. ■

10. Multiplicity of domination in dimension $(2k-1)$ and $(2k-2)$; algebraic intersection numbers and flow-lines

If a $y^{\infty}_{2k-1,j}$ appears multiple times in the definition of $h_{2k-1,\infty}$, or if $z^{\infty,-}_{2k-1}$ appears a number of times, we may resolve this multiplicity and introduce several distinct critical points, as many as needed, with very close unstable manifolds. The functional is slightly changed and its critical points as well, but the arguments are essentially the same.

We need now to resolve the multiplicities of $\overline{W_u(h_{2k-1,\infty})}$ at the order $(2k-2)$.

The case for the $y^{\infty}_{2k-1,j}$s.

Following the technique introduced above, we claim that:

Lemma 8 The decreasing flow can be modified so that the algebraic intersection numbers $y^{\infty}_{2k-1,j} - z^{(\infty)}_{2k-2}$ are equal in absolute value to the number of actual flow-lines from $y^{\infty}_{2k-1,j}$ to $z^{(\infty)}_{2k-2}$. L^+ and L^- remain invariant under this flow.

Proof of Lemma 8. We need to complete cancellations of flow-lines from a $y^{\infty}_{2k-1,j}$ to a $z^{(\infty)}_{2k-2}$ with opposite intersection numbers +1 and −1. Between $y^{\infty}_{2k-1,j}$ and $z^{(\infty)}_{2k-2}$, for $(2k-2) \geq 2$, we may assume that we do not find any critical point (at infinity) of index 1. After reparametrization of the flow-lines as in [11], Theorem 4.1, pp37-38, there is no loss of generality in this assumption. Then, the traces of the unstable manifold of $y^{\infty}_{2k-1,j}$ and of the stable manifold of $z^{(\infty)}_{2k-2}$ on an intermediate level surface $J^{-1}(c)$ may be assumed to be connected. if $M^3 = S^3$, we may also assume, without loss of generality, that this level surface is simply connected. If M is not S^3, but its second homotopy group is finite, we may assume that we have taken appropriate multiples of our topological classes, so that a similar argument applies.

We then join two intersection points with opposite intersection numbers in $W_u(y^{\infty}_{2k-1,j}) \cap J^{-1}(c)$ and in $W_s(z^{(\infty)}_{2k-2}) \cap J^{-1}(c)$ with two paths p_1 and p_2. We connect p_1 and p_2 along a surface Σ, as above, in $J^{-1}(c)$. As above, we "slide" $W_u(z^{(\infty)}_{2k-2})$ along Σ, modifying it in this way. At the end of the process, the cancellation of the two intersection points is performed. The argument follows the work of J.Milnor (Proof of the h-cobordism theorem) [11], Theorem 6.1, p70. The remaining various boundaries between the various critical points at infinity of index $(2k-1)$ can be pieced together so that there is no singularity in dimension $(2k-2)$ and the argument can proceed.

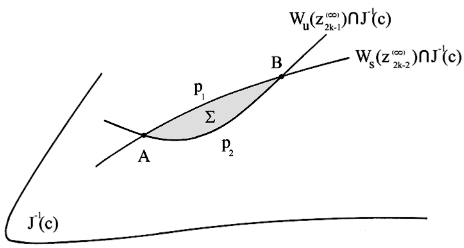

Figure 9

Of course, we need to check that this does not perturb the flow-lines in L^-. This is quite clear for the $y^{\infty}_{2k-1,j}$s as above. ■

The case for $z^{\infty}_{2k-1,-}$.

For $z^{\infty}_{2k-1,-}$, some additional care is required. However, we can then modify the argument here: if $z^{\infty}_{2k-1,-}$ dominates a critical point at infinity of L^- of index $(2k-2)$ with an algebraic number of intersection equal to 0 with two flow-lines of opposite intersection numbers $+1$ and -1, we can introduce an additional critical point of index $(2k-1)$ and, with the help of this additional critical point, resolve this multiple domination into simple dominations of distinct critical points for a modified functional:

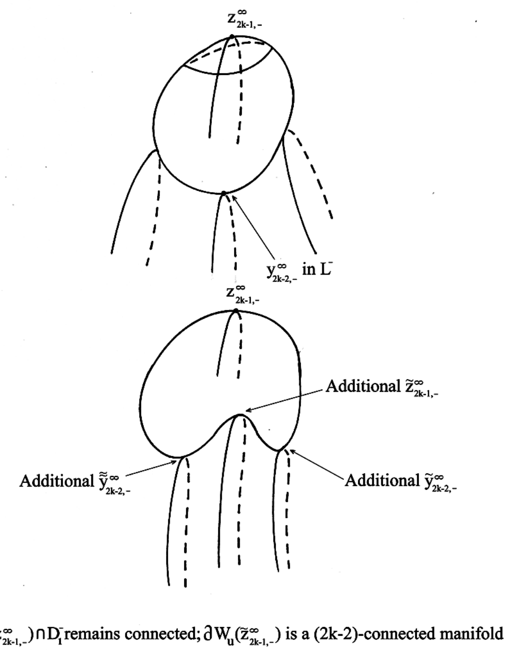

Figure 10

It is important to note that the bottom set for the modified $W'_u(z^\infty_{2k-1,-})$, $W'_u(z^\infty_{2k-1,-}) \cap D_1^-$ remains connected since there are only points in the unstable sphere of $z^\infty_{2k-1,-}$ which are attracted to the critical points at infinity of L^- of index $(2k-2)$. The contradiction argument above can therefore be carried out unchanged.

Deleting neighborhoods of periodic orbits in c_{2k-1}.

For each periodic orbit z_i dominated by c_{2k-1}, we choose a neighborhood W_i which we delete from c_{2k-1}. Using Proposition 7.24, p608 of [6], which provides an understanding of the behavior of the flow-lines of c_{2k-1} near z_i, we see that the "b"-map of [5] has values in $\partial W_i \cap c_{2k-1}$ in $\mathbb{PC}^{k-1} \times \{-1, 1\} \cup \mathbb{PC}^{k-2} \times [-1, 1]$. We therefore delete in the pairs of section 2, the sets, W_i, from the first sets of our pairs and we add the sets, ∂W_i, to the second sets of the pairs, leaving the reasoning and the arguments unchanged.

11. The proof of Theorem 1.3 (i) of [1], of Theorem 1 of the present paper and the proof of the Weinstein Conjecture on S^3, "in the large"

We recall that we have modified in sections 1, 3 and 4 our original functional into the functional $\tilde{J}$. $\tilde{J}^{-1}(\epsilon)$ is $D_1^+ \cup D_1^-$. The sets L^+ and L^- are to be thought in what follows as small attracting (for the decreasing pseudo-gradient flow) neighborhoods of themselves.

From our results in [5], Propositions 4 and 5, we know that the map "b" of pairs in homology of dimension $(2k-1)$:

$$H_{2k-1}(\overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)},$$
$$(\overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)}) \cap [(\partial L^+ \cup \partial L^-) \cup \tilde{J}_\infty^{-1}(\epsilon)] \cup \overline{\partial_\infty (c_{2k-1} \smallsetminus (L^+ \cup L^-))}) \overset{\text{"b"}_*}{\longrightarrow}$$
$$H_{2k-1}(\mathbb{PC}^{k-1} \times [-1,1], \mathbb{PC}^{k-2} \times [-1,1] \cup \mathbb{PC}^{k-1} \times \{-1,[0,1]\})$$

is onto.

On the other hand, we know that we have the excision isomorphism (also between pairs):

$$H_{2k-1}(\overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)},$$
$$(\overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)}) \cap [(\partial L^+ \cup \partial L^-) \cup \tilde{J}_\infty^{-1}(\epsilon)] \cup (\overline{\partial_\infty (c_{2k-1} \smallsetminus (L^+ \cup L^-))})) \overset{\text{exc}}{\cong}$$

$$H_{2k-1}(\overline{W_u(c_{2k-1} + h_{2k-1,\infty}) \smallsetminus (L^\pm)},$$
$$(\overline{W_u(c_{2k-1} + h_{2k-1,\infty}) \smallsetminus (L^\pm)}) \cap [(\partial(L^\pm) \cup \tilde{J}_\infty^{-1}(\epsilon))] \cup (\overline{W_u(h_{2k-1,\infty} \smallsetminus (L^\pm)})))$$

Consider the map "b", appropriately modified as indicated above. We know-this is a key point-that this map extends as an equivariant map to $\overline{W_u(h_{2k-1,\infty}) \smallsetminus (L^+ \cup L^-)}$ and that the restriction of the extension to this set has values in $\mathbb{PC}^{k-2} \times [-1,1]$. We modify slightly our pairs above by introducing in the second sets of the pairs, the additional set B_0 of section. The functional J is modified into $\tilde{J}$, the set $\tilde{J}^{-1}(\epsilon) \cup B_0$ is alternatively $D_1^+ \cup D_1^- \cup W_u(x_-^{1,\infty})$. We then find the two pairs of sets (A, B) and (C, D), where:

$$A = \overline{W_u(c_{2k-1} + h_{2k-1,\infty}) \smallsetminus (L^+ \cup L^-)}$$

$$B = \overline{W_u(c_{2k-1} + h_{2k-1,\infty}) \smallsetminus (L^+ \cup L^-)}) \cap [(\partial L^+ \cup \partial L^-) \cup \tilde{J}_\infty^{-1}(\epsilon) \cup B_0]$$

$$C = \overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)}$$
$$D = (\overline{W_u(c_{2k-1}) \smallsetminus (L^+ \cup L^-)}) \cap [(\partial L^+ \cup \partial L^-) \cup \tilde{J}_\infty^{-1}(\epsilon) \cup B_0] \cup (\overline{\partial_\infty (c_{2k-1} \smallsetminus (L^+ \cup L^-))})$$

The homomorphism:

$$H_{2k-1}(A, B) \overset{n_*}{\longrightarrow} H_{2k-1}(C, D)$$

is onto. This follows from the fact that the excision homomorphism above is onto and from the fact that c_{2k-1} is assumed to be a minimal cycle (see [1]) of ∂_{per}, i.e. we assume that c_{2k-1} cannot be decomposed into smaller cycles for ∂_{per}. Observe that A is a cycle of dimension $(2k-1)$ relative to B. This follows from the relation $(*)$ which we assume to hold.

Let us also consider the three following maps:

$$H_{2k-1}(A, B) \overset{l_*}{\longrightarrow} H_{2k-1}(\mathbb{PC}^{k-1} \times [-1,1], \mathbb{PC}^r \times [-1,1] \cup \mathbb{PC}^{k-1} \times \{-1,[0,1]\})$$

$$H_{2k-1}(C,D)\overset{"b"_*}{\longrightarrow}H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,[0,1]\})$$

$$H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-1}\times\{-1,[0,1]\}\cup\mathbb{PC}^{r}\times[-1,1])\overset{m_*}{\longrightarrow}$$

$$H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,[0,1]\})$$

The two homomorphisms above, m_* and "b_*", are onto in dimension $(2k-1)$ (the addition of D_1^+ in the second factor of the pairs (A,B) and (C,D) does not change much to the surjectivity of "b_*" since U_1 maps into a fixed $\mathbb{PC}^r\times[-1,1]$) and the commutation relation "b"$_*\circ n_* = m_*\circ l_*$ holds. It follows that l_* is non-zero. On the other hand, we have the inclusion map

$$i:(A,B)\overset{i}{\longrightarrow}(C_\beta\smallsetminus(L^+\cup L^-),(C_\beta-(L^+\cup L^-))\cap(\partial(L^+\cup L^-)\cup\tilde{J}^{-1}(\epsilon)\cup B_0)$$

The map "b" extends then in a natural way into a map:

$$(C_\beta\smallsetminus(L^+\cup L^-),(C_\beta-(L^+\cup L^-))\cap(\partial(L^+\cup L^-)\cup\tilde{J}^{-1}(\epsilon)\cup B_0))$$

$$\longrightarrow(\mathbb{PC}^\infty\times[-1,1],\mathbb{PC}^\infty\times\{-1,[0,1]\}\cup\mathbb{PC}^r\times[-1,1])$$

Justification of this fact this requires the use of a general position argument in order to remove the periodic orbits. Also the equivariance of the map is as above, on compact sets, with a p in the $e^{ip\tau}$ that may tend to ∞ with the compact sets getting larger and also appropriate powers are taken. This new map implies that

$$(\overline{W_u(c_{2k-1}+h_{2k-1,\infty})\smallsetminus(L^+\cup L^-)},(\overline{W_u(c_{2k-1}+h_{2k-1,\infty})\smallsetminus(L^+\cup L^-)})\cap[(\partial L^+\cup\partial L^-)\cup\tilde{J}_\infty^{-1}(\epsilon)\cup B_0])$$

is not a boundary in

$$(C_\beta\smallsetminus(L^+\cup L^-),(C_\beta-(L^+\cup L^-))\cap[\partial(L^+\cup L^-)\cup\tilde{J}^{-1}(\epsilon)\cup B_0]),$$

i.e. that the relation:

$$\partial c_{2k}^{(\infty)}=c_{2k-1}+h_{2k-1,\infty}$$

is not possible. The argument is complete.

12. Existence argument without the basic assumption

Along a deformation of contact forms, L^+ and L^- might change with the addition or substraction of critical points at infinity z_j^∞ of index j, typically of index $(2k-1)$. The Morse complex of e.g. L^+ then changes with the addition or the substraction of a smaller Morse complex. Using the arguments of Lemma 3, section 6, this smaller Morse complex maps through the "global" equivariant map "b", see section 11 above, into $\mathbb{PC}^\infty\times[0,1]\cup\mathbb{PC}^r\times$

$[-1,1]$, with r small when compared to j or k. The target value of the classifying map l_* of section 11 is then unchanged.

The conclusion is that, either using these equivariant/linking classes, we find a periodic orbit (maybe an iterate) of index $(2k-1)$, for k large; or there is a periodic orbit of index 1 connecting L^+ and L^-. If there is no such periodic orbit and these latter sets are connected directly by a critical point at infinity of index 1, then, after some reasoning, we find that we can complete tangencies with other critical points of index 1 connecting $J_0^{-1}(\epsilon)$ and each of these two sets (we might need to re-parametrize the flow-lines as in J. Milnor [11], Theorem 4.1 ,pp 37-38, thereby modifying the functional but not the flow-lines) and completely disconnect these two sets. The existence argument then proceeds "a la P. Rabinowitz [12]".

To a certain extent, the arguments of this paper indicate that either we can use the existence argument of H. Hofer [10] and find a periodic orbit of index 1 or the equivariant/linking argument of P. Rabinowitz [12] can be used, one line of proof excluding the other one. Of course, this is only an indication and not a proof of a rigorous statement.

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