

## Levinson's Problem on Affine-Periodic Solutions\*

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### Abstract

In this paper, Levinson's problem is introduced to affine-periodic systems. It is proved that every affine-dissipative-repulsive system admits an affine-periodic solution, which extends previous well-known results for dissipative systems.

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## 1 Introduction

The qualitative theory of differential equations was established by Poincaré and Lyapunov in the late 19th century. One major subject of this theory is to seek regular solutions with properties such as periodicity, quasi-periodicity, almost periodicity, almost automorphicity and so on, according to the “time-periodicity” of the systems under consideration. Many physical systems exhibit dissipativeness, namely, the system energy dissipates due to friction, loss of heat etc, and hence all the orbits of the differential equations that describe these physical systems eventually enter and remain in a bounded set of the phase space. The systems are said to be “dissipative” mathematically. In 1944, Norman Levinson [13] posed a conjecture on the existence of periodic solutions for dissipative systems, which may be formulated as follows:

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Consider the ODE

$$x' = f(t, x), \tag{1.1}$$

where  $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^n$  and  $f(t + \theta, x) = f(t, x)$  for some  $\theta > 0$ .

**Problem 1.** *Prove that if system (1.1) is dissipative, equivalently, if solutions of (1.1) are ultimately bounded (UB), then (1.1) has a  $\theta$ -periodic solution.*

The problem was solved for  $n = 2$  independently by Cartwright ([4]) and Massera [17] in 1950 via Brouwer’s fixed point theorem to 2-dimensional orientation-preserving homomorphism, and for general  $n$  by Yoshizawa [19] in 1960s on basis of the asymptotic fixed point theory.

Consider the functional differential system (FDS for short)

$$x' = F(t, x_t), \tag{1.2}$$

where  $x_t(s) = x(t + s)$  for  $s \in [-r, 0]$ ,  $r > 0$  is a constant,  $F(t, \varphi)$  is a functional on  $\mathbb{R}^1 \times C([-r, 0], \mathbb{R}^n)$  and  $F(t + \theta, \varphi) = F(t, \varphi)$ . Furthermore, one asks Levinson’s problem for functional differential equations:

**Problem 2.** *Show that Problem 1 is true for 1.2.*

Hale and Lopes [10], Arino et al. [1], Makay [16] and Burton and Zhang [3] studied Problem 2; Burton and Zhang [2] developed Problem 2 to infinite-delay FDS; Li et al. [14, 15] extended Problem 2 to differential inclusions of the type

$$x' \in F(t, x_t).$$

Some systems are neither dissipative nor conservative, but they possess the dissipative-repulsive property, which can accurately describe some competitive ecosystems. More precisely, a subsystem of the system is dissipative and the rest part is repulsive; in other words, the components of the solutions corresponding to the dissipative subsystem eventually enter and remain in a bounded set, the remaining components of the solutions become expanded as time goes by. This leads to the extension of Levinson’s problem to dissipative-repulsive systems.

**Problem 3.** *Show that Problem 1 is true for 1.1 or 1.2 when they are dissipative-repulsive.*

Obviously, asymptotic fixed point theory doesn’t work for Problem 3. Since 1970s, topological degree theory has been applied to the study of periodic solutions for differential equations. This method is based on priori bounds of the periodic solutions of a family companion equations associated to the system under consideration. Particularly, we mention works of Hale and Mawhin [11], Wu et al. [18] and Erbe et al. [8, 9]. Problem 3 was completely solved by Küpper et al. [12] without any companion conditions, because all the companion conditions certainly add somewhat additional assumptions on the systems.

Movements of the universe obey certain natural laws. As expressions of the beauty of nature, differential equations often exhibit certain symmetry rather than periodicity, for example, the following affine symmetry:

$$\begin{aligned} f(t + \theta, x) &= Qf(t, Q^{-1}x) \quad \text{in (1.1),} \\ F(t + \theta, \varphi) &= QF(t, Q^{-1}\varphi) \quad \text{in (1.2),} \end{aligned} \tag{1.3}$$

where  $Q \in GL(\mathbb{R}^n)$ . We introduce  $(Q, \theta)$ -affine-periodic systems and  $(Q, \theta)$ -affine-periodic solutions in the following definitions:

**Definition 1.1** The system (1.1) or (1.2) is said to be a  $(Q, \theta)$ -affine-periodic system (APS for short), if (1.3) holds.

**Definition 1.2** The solution  $x(t)$  to the system (1.1) or (1.2) is said to be a  $(Q, \theta)$ -affine-periodic solution, if  $x(t)$  satisfies

$$x(t + \theta) = Qx(t) \text{ for all } t \in \mathbb{R}^1. \tag{1.4}$$

Notice that when  $Q = I$  (identity matrix),  $Q = -I$ ,  $Q^N = I$ ,  $Q \in SO(n)$ , a  $(Q, \theta)$ -affine-periodic solution  $x(t)$  is just  $\theta$ -periodic, anti-periodic, harmonic and quasi-periodic respectively. For some relative studies, we refer to [5, 6, 7, 21] and references therein. Moreover, a  $(Q, \theta)$ -affine-periodic solution can also be unbounded if it has the form of  $e^{at} \cos(bt)$  or  $e^{-at} \sin(bt)$ , where  $Q \in GL(n) \setminus SO(n)$ . Now a basic problem is to investigate the existence of  $(Q, \theta)$ -affine-periodic solutions  $x(t)$  for APS (1.1) or (1.2). Naturally, we should pose an analogous Levinson's problem for APS:

**Problem 4.** *Show that Levinson's problem holds when APS (1.1) or (1.2) is dissipative-repulsive.*

It is well known that dissipativeness doesn't necessarily imply the existence of almost periodic solutions, and hence Problem 4 is also unknown when  $Q \in SO(n)$ , i.e. quasi-periodic solutions. It should be emphasized that when  $Q \in SO(n)$ , our quasi-periodic solutions are a kind of special quasi-periodic ones, whose orbits are invariant under  $Q$ , which means that the solutions lie in  $S^n$  (the  $n$ -dimensional sphere).

After introducing Levinson's problem to  $(Q, \theta)$ -affine-periodic systems, we state the main aim of the present paper: It is just to prove the existence of affine-periodic solutions of affine-periodic dissipative-repulsive systems by setting up a degree theoretic formulism. Under certain assumptions of  $Q$ , we can conclude that every  $(Q, \theta)$ -affine-periodic dissipative-repulsive system admits a  $(Q, \theta)$ -affine periodic solution, which asserts affirmatively Levinson's problem for  $(Q, \theta)$ -affine-periodic systems.

The paper is organized as follows. In Section 2, we first study APS described by ordinary differential equations. In Section 3, we deal with the case of functional differential systems. In section 4, we give some examples to illustrate our results and make some comments.

## 2 Approach to APS

Consider the differential equation (1.1)

$$x' = f(t, x),$$

where  $f(t, x) : \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz continuous with respect to the second variable. Assume that  $f(t, x)$  is  $(Q, \theta)$ -affine-periodic, i.e.

$$f(t + \theta, x) = Qf(t, Q^{-1}x) \text{ for all } (t, x),$$

where  $\theta > 0$  and  $Q \in GL(\mathbb{R}^n)$ . Denote by  $x(t, x_0) = x(t, 0, x_0)$  the solution of (1.1) with the initial value  $x(0) = x_0$ . Let  $J_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^l$  be the projections such that  $J_1x = y$  and  $J_2x = z$ , where  $x = (y, z)$ ,  $y \in \mathbb{R}^m$  and  $z \in \mathbb{R}^l$  with  $m + l = n$ . We give the exact definition on the affine-dissipative-repulsive system as follows.

**Definition 2.1** The system (1.1) is said to be affine-dissipative-repulsive (ADR for short) if there exist  $B, b$  such that for any  $a, b' > 0$ , there is  $T = T(a, b') > 0$  such that the following hold for all solutions  $x(t, x_0)$  with initial value  $x_0 = (y_0, z_0)$  satisfying  $|y_0| \leq a$ :

- i)  $|J_1 Q^{-k} x(t + k\theta, x_0)| \leq B$ , whenever  $t + k\theta \geq T$  and  $|z_0| \leq b$ ,
- ii)  $|J_2 Q^{-k} x(t + k\theta, x_0)| > 0$ , whenever  $0 \leq t + k\theta \leq T$  and  $b \leq |z_0| \leq b + b'$ ;  
 $|J_2 Q^{-k} x(t + k\theta, x_0)| > b$ , whenever  $t + k\theta \geq T$  and  $b \leq |z_0| \leq b + b'$ ,

where  $k \in \mathbb{Z}_+^1$  and  $t \geq 0$ .

**Hypothesis 2.1** For each  $k \in \mathbb{Z}_+^1$ ,  $Q^{-k}$  has the form of

$$\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}_{n \times n},$$

where the matrices  $Q_1 \in \mathbb{R}^{m \times m}$ ,  $Q_2 \in \mathbb{R}^{m \times l}$ ,  $Q_3 \in \mathbb{R}^{l \times m}$ ,  $Q_4 \in \mathbb{R}^{l \times l}$  and  $Q_4$  is nonsingular.

We have the following result on the existence of affine-periodic solutions.

**Theorem 2.1** Assume that  $f(t, x) : \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz continuous with respect to the second variable. Under Hypothesis 2.1, if system (1.1) is  $(Q, \theta)$ -affine-periodic and ADR, then it admits a  $(Q, \theta)$ -affine-periodic solution  $x(t)$ , i.e.,  $x(t + \theta) = Qx(t)$  for all  $t \geq 0$ .

*Proof.* Denote by  $S_1(\sigma)$  and  $S_2(\sigma)$  the open balls centered at the origin with radius  $\sigma$ , respectively in  $\mathbb{R}^m$  and  $\mathbb{R}^l$ . Due to the arbitrariness of  $a$  and  $b'$  in Definition 2.1, put  $a = B + 1$ ,  $b' = h$  and set  $D = S_1(a) \times S_2(b)$ , where  $B$  and  $b$  are determined by Definition 2.1 and  $h > 0$  is a constant satisfying

$$|x(t, x_0)| < h, \quad \forall t \in [0, \theta] \text{ and } x_0 \in \bar{D}.$$

Define a Poincaré map  $P(x_0) = Q^{-1}x(\theta, x_0)$ . Note that

$$P^j(x_0) = Q^{-j}x(j\theta, x_0).$$

Take a prime number  $N$  such that  $N\theta \geq T(a, b') + \theta$ . For any  $x_0 \in \partial D$ , we have  $|J_1 x_0| = a$  or  $|J_2 x_0| = b$ . Then it follows from i) and ii) that

$$P^N(x_0) \neq x_0, \quad \forall x_0 \in \partial D.$$

We claim that for each fixed point  $x_0$  of  $P^N$  in  $\bar{D}$ ,

$$P(x_0) \in D.$$

Suppose to the contrary that there exists a fixed point  $x_0 \in \bar{D}$  of  $P^N$  such that  $P(x_0) \notin D$ . Then

$$P(x_0) = P^{N+1}(x_0) = Q^{-(N+1)}x((N+1)\theta, x_0) = q = (y_1, z_1).$$

Since  $(N+1)\theta > T(a, b')$  and  $a > B$ , by the definitions of  $a$  and  $D$ , we obtain that  $y_1 \in S_1(a)$ , and  $b \leq |z_1| \leq b + b'$ . Notice that  $(N-1)\theta \geq T(a, b')$ . Thus,

$$J_2 Q^{-(N-1)}x((N-1)\theta, q) \notin \bar{S}_2(b),$$

and hence

$$x_0 = P^N(x_0) = P^{(N-1)}(q) \notin \bar{D},$$

which is a contradiction. By a modular degree theorem ([20]),

$$\deg(\text{id} - P, D, 0) = \deg(\text{id} - P^N, D, 0) \pmod{N}. \tag{2.1}$$

If we can prove

$$\deg(\text{id} - P^N, D, 0) \neq 0,$$

then from (2.1) it follows that  $\deg(\text{id} - P, D, 0) \neq 0$ . Hence  $P$  has a fixed point  $x_*$  in  $D$ , and  $x(t, x_*)$  is a  $(Q, \theta)$ -affine-periodic solution of (1.1). Indeed, consider the homotopy:

$$\begin{aligned} H_1(y_0, z_0, \mu) = & (y_0 - \mu J_1 Q^{-N} x(N\theta, (y_* - \mu y_* + \mu y_0, z_0)), \\ & \mu z_0 - J_2 Q^{-N} x(N\theta, (y_* - \mu y_* + \mu y_0, z_0))), \end{aligned}$$

where  $y_* \in S_1(a)$  is fixed and  $\mu \in [0, 1]$ . By i) and ii), we obtain

$$0 \notin H_1(\partial D \times [0, 1]).$$

Thereby,

$$\begin{aligned} \deg(\text{id} - P^N, D, 0) &= \deg(H_1(\cdot, 1), D, 0) = \deg(H_1(\cdot, 0), D, 0) \\ &= \deg(-J_2 Q^{-N} x(N\theta, (y_*, \cdot)), S_2(b), 0). \end{aligned}$$

Furthermore, consider another homotopy

$$H_2(z_0, \mu) = J_2 Q^{-N} x(\mu N\theta, (y_*, z_0)),$$

where  $\mu \in [0, 1]$ . By ii) we have

$$0 \notin H_2(\partial S_2(b) \times [0, 1]),$$

which together with the homotopy invariance and the nonsingularity of  $Q_4$  yields

$$\deg(J_2 Q^{-N} x(N\theta, (y_*, \cdot)), S_2(b), 0) = \deg(J_2 Q^{-N} (y_*, \cdot)^T, S_2(b), 0) = \pm 1.$$

Then

$$\deg(\text{id} - P^N, D, 0) \neq 0,$$

which implies

$$\deg(\text{id} - P, D, 0) \neq 0.$$

Hence, there is an  $x_* \in D$  such that

$$P(x_*) = Q^{-1} x_\theta(\cdot, x_*) = x_*.$$

By uniqueness of solutions, we have

$$x(t + \theta, x_*) = Qx(t, x_*) \text{ for all } t \geq 0.$$

This is the end of the proof.

**Remark 2.1** Assume that  $Q = \tilde{I} = \begin{pmatrix} 0 & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & 0 \end{pmatrix}_{n \times n}$  or

$$Q = \hat{I} = \begin{pmatrix} O_{m \times m} & I_{m \times m} \\ -I_{m \times m} & O_{m \times m} \end{pmatrix}.$$

Since  $\tilde{I}^{-2k} = I_{n \times n}$ ,  $\tilde{I}^{-2k+1} = \tilde{I}_{n \times n}$  for all  $k \in \mathbb{Z}_+^1$ , and  $\hat{I}^{-4k} = I_{2m \times 2m}$ ,  $\hat{I}^{-4k+1} = \hat{I}$ ,  $\hat{I}^{-4k+2} = -I_{2m \times 2m}$ ,  $\hat{I}^{-4k+3} = -\hat{I}$  for all  $k \in \mathbb{Z}_+^1$ , where  $I$  is the identity matrix, Hypothesis 2.1 holds for all  $Q^{-2k}$ . In both cases, we consider  $(Q^2, 2\theta)$ -affine-periodic solutions.

### 3 Functional differential systems

In this section, we give an analogous result for functional differential systems. Consider the functional differential system

$$x' = F(t, x_t), \tag{3.1}$$

where  $F : \mathbb{R}^1 \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz in the second variable. Assume that  $F(t, \varphi)$  is  $(Q, \theta)$ -affine-periodic, i.e.

$$F(t + \theta, \varphi) = QF(t, Q^{-1}\varphi) \text{ for all } (t, \varphi),$$

where  $Q \in GL(n)$ ,  $\theta > 0$  is a constant, and takes any bounded sets in  $\mathbb{R}^1 \times C([-r, 0], \mathbb{R}^n)$  into bounded sets in  $\mathbb{R}^n$ . Denote by  $x(t, s, \varphi)$  the solution of (3.1) with initial value  $x_s = \varphi$ , by  $|\cdot|$  the usual supremum norm. For simplicity, let  $x(t, \varphi) = x(t, 0, \varphi)$ ,  $x_t(\tau) = x(t + \tau)$ ,  $\tau \in [-r, 0]$  with  $r > 0$ ,  $x_t(\cdot, \varphi) = x(\cdot + t, 0, \varphi)$  and  $C = C([-r, 0], \mathbb{R}^n)$ . Define the projections  $J_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^l$  as in section 2. We introduce the following definition of ADR system.

**Definition 3.1** The system (3.1) is said to be ADR if there exist  $B, b > 0$  such that for any  $a \geq d, b', T' > 0$ , there are  $T = T(a, b') > 0, M = M(a, b', T')$  such that the following hold all solutions  $x(t, \varphi)$  with initial value  $x_0 = \varphi = (\psi_1, \psi_2)$  satisfying:

- i)  $|x(t, \varphi)| \leq M$ , whenever  $t \in [0, T'], |\varphi| \leq a + b + b'$ ,
- ii)  $|J_1 Q^{-k} x(t + k\theta, \varphi)| \leq B$ , whenever  $t + k\theta \geq T, |\psi_1| \leq a$  and  $|\psi_2| \leq b$ ,
- iii)  $|J_2 Q^{-k} x(t + k\theta, \varphi)| > 0$ , whenever  $0 \leq t + k\theta \leq T, |\psi_1| \leq a$  and  $b \leq |\psi_2| \leq b + b'$ ;  $|J_2 Q^{-k} x(t + k\theta, \varphi)| > b$ , whenever  $t + k\theta \geq T, |\psi_1| \leq a$  and  $b \leq |\psi_2| \leq b + b'$ ,

where  $k \in \mathbb{Z}_+^1$  and  $t \geq 0$ .

We have the following result on the existence of affine-periodic solutions of functional differential system (3.1).

**Theorem 3.1** Assume that  $F : \mathbb{R}^1 \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is continuous and locally Lipschitz continuous with respect to the second variable and takes any bounded sets in  $\mathbb{R}^1 \times C([-r, 0], \mathbb{R}^n)$  into bounded sets in  $\mathbb{R}^n$ . Under Hypothesis 2.1, if system (3.1) is  $(Q, \theta)$ -affine-periodic and ADR, then it admits a  $(Q, \theta)$ -affine-periodic solution.

*Proof.* Put  $a = B + 1$ ,  $b' = 1$ ,  $T' = N\theta + 2\theta$ ,  $M_0 = M(B + 1, 1, N\theta + 2\theta)$ , where  $N$  is a prime number such that  $N\theta > T(a, b') + \theta$ . Since  $F$  maps any bounded sets in  $\mathbb{R}^1 \times C$  into bounded sets in  $\mathbb{R}^n$ , there is a constant  $L > 0$  such that for any  $t \in [0, N\theta + 2\theta]$  and  $|\varphi| \leq 2M_0$ ,

$$|F(t, \varphi)| \leq L - 1.$$

Let  $D = S_1(a) \times S_2(b)$ , where

$$S_1(a) = \{J_1\varphi(\cdot) : \varphi \in C, |J_1\varphi(\cdot)| < a\},$$

$$S_2(b) = \{J_2\varphi(\cdot) : \varphi \in C, |J_2\varphi(\cdot)| < b\}.$$

Define

$$X = \{\varphi \in C : |\varphi| \leq 2M_0, |\varphi(s_1) - \varphi(s_2)| \leq L|s_1 - s_2|, s_1, s_2 \in [-r, 0]\}.$$

Then  $X$  is a compact and convex set in  $C$ . Hence there is a continuous retract  $\alpha : C \rightarrow X$ . Obviously,  $\alpha$  is compact.

Define a Poincaré map  $P : C \rightarrow C$  by

$$P(\varphi) = Q^{-1}x_{\theta}(\cdot, \varphi).$$

Thus,

$$P^k(\varphi) = Q^{-k}x_{k\theta}(\cdot, \varphi).$$

In particular, by i)-ii) and the structure of  $D$ , we have that in  $\bar{D} \cap X$ ,

$$P^k(\alpha(\varphi)) = P^k(\varphi) \quad \forall k \leq N + 2.$$

This leads to considering the following compact homotopy

$$\begin{aligned} & H(\varphi, \mu) \\ &= (\mu J_1 Q^{-N} x_{N\theta}(\cdot, (\psi_{*1} - \mu\psi_{*1} + \mu J_1(\alpha \circ \varphi), J_2(\alpha \circ \varphi))), \\ & \quad (1 - \mu)J_2(\alpha \circ \varphi) + J_2 Q^{-N} x_{N\theta}(\cdot, (\psi_{*1} - \mu\psi_{*1} + \mu J_1(\alpha \circ \varphi), J_2(\alpha \circ \varphi))) \end{aligned}$$

for all  $(\varphi, \mu) \in C \times [0, 1]$ , where  $\psi_* = (\psi_{*1}, \psi_{*2}) \in D \cap X$  is fixed. It follows from i)-iii) that

$$0 \notin (\text{id} - H)(\partial D \times [0, 1]).$$

Consequently,

$$\begin{aligned} \text{deg}(\text{id} - P^N, D, 0) &= \text{deg}(\text{id} - P^N(\alpha \circ \cdot), D, 0) \\ &= \text{deg}(\text{id} - H(\cdot, 1), D, 0) \\ &= \text{deg}(\text{id} - H(\cdot, 0), D, 0) \\ &= \text{deg}(\text{id} - J_2 H(\cdot, 0), S_2(b), 0). \end{aligned}$$

To calculate the degree, consider another compact homotopy

$$\begin{aligned} H_1(\varphi, \mu) &= J_2(\alpha \circ \varphi) + \\ & \quad J_2 Q^{-N} x_{\mu N\theta}(\cdot, (\psi_{*1}, J_2(\alpha \circ \varphi))) \quad \forall (\varphi, \mu) \in C \times [0, 1]. \end{aligned}$$

By iii), we derive that

$$0 \notin (\text{id} - H_1)(\partial S_2(b) \times [0, 1]).$$

Thereby,

$$\begin{aligned} \deg(\text{id} - J_2 H(\cdot, 0), S_2(b), 0) &= \deg(\text{id} - H_1(\cdot, 1), S_2(b), 0) \\ &= \deg(\text{id} - H_1(\cdot, 0), S_2(b), 0) \\ &= \deg(-J_2 Q^{-N}(\psi_{*1}, J_2(\alpha \circ \varphi))^T, S_2(b), 0) \\ &= \pm 1. \end{aligned}$$

Applying the modular degree theorem, ii) and iii) yields

$$\deg(\text{id} - P, D, 0) = \deg(\text{id} - P^N, D, 0) \pmod{N},$$

which together with  $\deg(\text{id} - P^N, D, 0) = \pm 1$  implies

$$\deg(\text{id} - P, D, 0) \neq 0.$$

Hence, there is a  $\varphi_0 \in D$  such that

$$P(\varphi_0) = Q^{-1} x_\theta(\cdot, \varphi_0) = \varphi_0.$$

By uniqueness of solutions, we have

$$x(t + \theta, \varphi_0) = Qx(t, \varphi_0) \text{ for all } t \geq 0$$

as desired. This is the end of the proof.

## 4 Comments and Examples

First we give some examples to illustrate our results, and for the sake of simplicity, we prefer to make them not complicated.

**Example 4.1** Consider the system

$$\begin{aligned} x' + 2x &= e^{-t}, \\ y' - 3y &= -e^{2t}. \end{aligned}$$

The general solutions of this system are obviously:

$$\begin{aligned} x &= e^{-2t} c_1 + e^{-t}, \\ y &= e^{3t} c_2 + e^{2t}. \end{aligned}$$

Note that for each  $\theta > 0$ ,

$$\begin{aligned} e^{-(t+\theta)} &= e^{-\theta} e^{-t}, \\ e^{2(t+\theta)} &= e^{2\theta} e^{2t}. \end{aligned}$$

Put  $Q = \begin{pmatrix} e^{-\theta} & 0 \\ 0 & e^{2\theta} \end{pmatrix}$ . Then the system is  $(Q, \theta)$ -affine-periodic, and it is easy to verify conditions i) and ii) in Definition 2.1. Therefore the system admits  $(Q, \theta)$ -affine-periodic solutions  $X(t) = (x(t), y(t))^T$  such that

$$X(t + \theta) = QX(t).$$

Actually, the system has a unique  $(Q, \theta)$ -affine-periodic solution

$$x = e^{-t}, \quad y = e^{2t}.$$

**Remark 4.1** The above simple example shows that an affine-periodic solution might be non-periodic even unbounded.

**Example 4.2** Consider the system

$$\begin{aligned} x' + ax &= e^{\lambda t}, \\ y' - by &= e^{\mu t}, \end{aligned}$$

where  $a, b > 0$ ,  $\lambda = \alpha_1 + i\beta_1 \neq 0$ ,  $\mu = \alpha_2 + i\beta_2 \neq 0$ . On  $\mathbb{C}^2$ , the general solution is

$$\begin{aligned} x &= \begin{cases} e^{-at}(c_1 + \frac{1}{a+\lambda}e^{(a+\lambda)t}), & a + \lambda \neq 0, \\ e^{-at}(c_1 + t), & a + \lambda = 0; \end{cases} \\ y &= \begin{cases} e^{bt}(c_2 + \frac{1}{-b+\mu}e^{(-b+\mu)t}), & -b + \mu \neq 0, \\ e^{bt}(c_2 + t), & -b + \mu = 0. \end{cases} \end{aligned}$$

Fix  $\theta > 0$  and put  $Q = \begin{pmatrix} e^{\lambda\theta} & 0 \\ 0 & e^{\mu\theta} \end{pmatrix}$ . Then the system is  $(Q, \theta)$ -affine-periodic in  $t$ . If

$$a + \lambda \neq 0, \quad -b + \mu \neq 0, \quad \alpha_1 \leq 0, \quad \alpha_2 \geq 0, \quad a + \alpha_1 \geq 0, \quad b - \alpha_2 \geq 0,$$

then

$$\begin{aligned} |e^{-k\lambda\theta}x(t+k\theta, x_0)| &= |e^{\lambda t}| \cdot |e^{-(a+\lambda)(t+k\theta)}c_1 + \frac{1}{a+\lambda}|, \\ |e^{-k\mu\theta}y(t+k\theta, y_0)| &= |e^{\mu t}| \cdot |e^{(b-\mu)(t+k\theta)}c_2 + \frac{1}{\mu-b}|. \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and the existence of  $(Q, \theta)$ -affine-periodic solutions follows. Actually, the system admits a unique  $(Q, \theta)$ -affine-periodic solution

$$\begin{aligned} x &= \frac{1}{a+\lambda}e^{\lambda t} = \frac{1}{a+\lambda}e^{\alpha_1 t}(\cos \beta_1 t + i \sin \beta_1 t), \\ y &= \frac{1}{-b+\mu}e^{\mu t} = \frac{1}{-b+\mu}e^{\alpha_2 t}(\cos \beta_2 t + i \sin \beta_2 t). \end{aligned}$$

**Remark 4.2** In case of  $\alpha_1 = \alpha_2 = 0$  and  $\beta_1\beta_2 \neq 0$ , the  $(Q, \theta)$ -affine-periodic solution is

1.  $\theta$ -periodic if  $\frac{\beta_1}{2\pi}\theta, \frac{\beta_2}{2\pi}\theta \in \mathbb{N}^1$ ;
2.  $k\theta$ -subharmonic for some  $k \in \mathbb{N}^1$  if  $\frac{\beta_1}{2\pi}\theta, \frac{\beta_2}{2\pi}\theta \in \mathbb{Q}^1$  and one of them is at least a reduced fraction;
3. quasi-periodic if  $\beta_1/\beta_2 \notin \mathbb{Q}^1$ .

When  $\alpha_1\alpha_2 \neq 0$ ,  $x$  and  $y$  have their own vibration frequency  $\frac{\beta_1}{2\pi}$  and  $\frac{\beta_2}{2\pi}$ , however, amplitudes of  $x$

and  $y$  will vary according to the change of time  $t$ . Particularly, in case of  $\alpha_1 < 0$  and  $\alpha_2 > 0$ , as  $t \rightarrow \infty$ ,

$$\begin{aligned} |x(t)| &= \frac{1}{a + \lambda} e^{\alpha_1 t} \rightarrow 0, \\ |y(t)| &= \frac{1}{-b + \mu} e^{\alpha_2 t} \rightarrow \infty. \end{aligned}$$

Generally, consider the system

$$\begin{aligned} x' &= -|z|^{2\alpha} x + e^{i2\pi\Phi t} \mathbf{1}_x = f_1(t, x, y), \\ y' &= |z|^{2\beta} y + e^{i2\pi\Psi t} \mathbf{1}_y = f_2(t, x, y), \end{aligned} \tag{4.1}$$

where  $\alpha, \beta \geq 0$ ,

$$\begin{aligned} z &= (x, y)^T, x \in \mathbb{C}^m, y \in \mathbb{C}^l, \\ \Phi &= \text{diag}(\theta_1, \theta_2, \dots, \theta_m), \Psi = \text{diag}(\theta_{m+1}, \theta_{m+2}, \dots, \theta_n), \\ \mathbf{1}_x &= (1, 1, \dots, 1)^T \in \mathbb{R}^m, \mathbf{1}_y = (1, 1, \dots, 1)^T \in \mathbb{R}^l. \end{aligned}$$

**Corollary 4.1** *Let  $Q = e^{i\Theta}$ ,  $\Theta = \text{diag}(\Phi, \Psi)^T$ . Then the system (4.1) admits  $(Q, \theta)$ -affine-periodic solutions, which might be 1-periodic if  $\Theta \in \mathbb{N}^n$ ,  $k$ -subharmonic if  $\Theta \in \mathbb{Q}^n$ ,  $\Theta \neq 0$  and one of the components is a reduced fraction, quasi-periodic with frequency  $\Theta$ , otherwise.*

*Proof.* The system is obviously  $(Q, \theta)$ -affine-periodic. Set

$$V_1(x) = \frac{1}{2}|x|^2, \quad V_2(y) = \frac{1}{2}|y|^2.$$

Then along the  $x$ -component and the  $y$ -component respectively,

$$\begin{aligned} V_1'(x) &= \langle x, f_1(z) \rangle \\ &= x^T \bar{f}_1(z) \\ &= -|z|^{2\alpha}|x|^2 + x^T e^{-i2\pi\Phi t} \mathbf{1}_x \\ &\leq -|x|^{2\alpha+2} + |x| \\ &\leq -2, \quad \text{for } |x| \geq 2, \quad y \in \mathbb{C}^l, \\ V_2'(y) &= \langle y, f_2(z) \rangle \\ &= y^T \bar{f}_2(z) \\ &= |z|^{2\beta}|y|^2 + y^T e^{-i2\pi\Psi t} \mathbf{1}_y \\ &\geq |y|^{2\beta+2} - |y| \\ &\geq 2, \quad \text{for } |y| \geq 2, \quad x \in \mathbb{C}^m, \end{aligned}$$

which implies all the conditions of Theorem 2.1. It follows the desired conclusions. This is the end of the proof.

**Remark 4.3** For different choices of  $Q$ , the  $(Q, \theta)$ -affine-periodic solutions might be periodic, subharmonic or quasi-periodic. The proof of Corollary 4.1 shows that the Lyapunov function method can be used to verify the conditions of Theorem 2.1.

**Example 4.3** Consider the system

$$x' = -e^{2kt} x^{2k+1} + \sum_{j=1}^{2k-1} a_j e^{jt} x^{j+1} + a_0 e^{-t},$$

where  $k \in \mathbb{N}^1, a_j \in \mathbb{R}^1$ . Let  $Q = e^{-1}$ , there holds

$$f(t + 1, x) = Qf(t, Q^{-1}x),$$

and the system is hence  $(Q, \theta)$ -affine-periodic. Set  $V(x) = \frac{1}{2}|x|^2$ . Then as  $|x| \gg 1, t \geq 0$ ,

$$\begin{aligned} V'(x) &= x x' = -e^{2kt} x^{2k+2} + \sum_{j=1}^{2k-1} a_j e^{jt} x^{j+2} + a_0 e^{-t} x \\ &\leq -\frac{1}{2} e^{2kt} x^{2k+2} \leq -2k e^{2kt} \frac{x^2}{2} = -2k e^{2kt} V(x), \end{aligned}$$

and thereby,

$$V(x) \leq V(x_0) e^{-e^{2kt}}.$$

Note that

$$Q^{-2j} V(x(t + j)) \leq V(x_0) e^{-e^{2kt} e^{2kj} + 2j},$$

which shows that the system is  $(Q, \theta)$ -affine-dissipative. By Theorem 2.1, we obtain the existence of  $(Q, \theta)$ -affine-periodic solutions.

**Remark 4.4** This example shows that even though the coefficients of the equation are unbounded, there exist  $(Q, \theta)$ -affine-periodic solutions.

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