*Advanced Nonlinear Studies* **15** (2015), 135–156

# On the Brezis-Nirenberg Problem with a Kirchhoff Type Perturbation

**Daisuke Naimen**

*Department of Mathematics, Graduate School of Science, Osaka City University*
*3-3-138 Sugimoto Sumiyoshi-ku, Osaka-shi, Osaka 558-8585 JAPAN*
*e-mail: d12sax0J51@ex.media.osaka-cu.ac.jp*

Received 28 March 2014
*Communicated by Zhi-Qiang Wang*

**Abstract**

In this paper, we investigate a Kirchhoff type elliptic problem,

$$\begin{cases} -\left(1 + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = \lambda u + u^5, \ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^3$ is an open ball, $\lambda \in \mathbb{R}$ and $b \geq 0$. We give an extension of the result by Brezis-Nirenberg in 1983 to the case $b > 0$. In particular, we can observe several effects of the nonlocal coefficient on the well known results related to the existence, nonexistence and uniqueness.

*2010 Mathematics Subject Classification.* 35J60, 35J61, 35J62.
*Key words.* Kirchhoff; nonlocal; elliptic; critical; variational method

## 1 Introduction

We investigate the Kirchhoff type elliptic problem,

$$\begin{cases} -(1 + b \int_\Omega |\nabla u|^2 dx) \Delta u = \lambda u + u^5 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

here $\Omega$ is a 3 dimensional open ball. We regard $\lambda \in \mathbb{R}$ as a given constant and $b \geq 0$ as a parameter. In this paper we prove the existence of solutions of (P).

The characteristic principal term of the equation of (P) has its origin in the theory of nonlinear vibrations. In [27], we can find the following equation which describes the free vibration of stretched strings,

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(1 + \frac{Ea}{2T_0 L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{$P_0$}$$

where $c^2 := T_0/m$ and $u : [0, L] \to \mathbb{R}$ denotes the amplitude of the vibrating string. Moreover $m, T_0, E, a, L$ are several physical quantities which respectively denote the mass per unit length, base tension, Young modulus, cross section and base length of the string. The nonlocal term appears as a consequence of taking account the change of the tension on the vibrating string which is induced by the change of its length. The equations of this type were first considered by Kirchhoff [16] and several researchers on physics.

In addition, we can also find such Kirchhoff type wave equations on mathematics. In [7][8][11] [15][21][30], we can observe the equations such as,

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, t, u), \tag{$P_1$}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is some continuous function. By their works the solvability of the equation ($P_1$) under several boundary conditions, initial data and hypotheses on $M$ is investigated. For the details see the survey [6]. Our problem (P) can be regarded as a stationary problem of ($P_1$).

After the work by Alves, Corrêa and Ma [4], several Kirchhoff type elliptic problems have been investigated extensively. We refer to [3]-[5][12][13][18]-[20][23]-[25][28][33][34][37][38] and so on. They apply the variational or topological methods to their problems and successfully prove the existence of their solutions. In this paper we also investigate the solvability of such a Kirchhoff type problem (P).

Here we note that the right hand side of the equation of (P) has the critical term $u^5$. Thus (P) has the typical difficulty in proving the existence of solutions. In our problem this difficulty is caused by the lack of the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. In view of this, the critical problems of Kirchhoff type equations for three dimensional case have been already investigated in [3][5][13][24][33][37][38]. See [25] for the four dimensional case and [12], [20] for the general dimensional one. Here note that if we do not have the lower order perturbation term in the equation of (P), that is, if $\lambda = 0$ and further $\Omega$ is strictly star-shaped, (P) has no solution (as proved in Section 2). Thus it is natural for us to try to get the solvability of (P) with $\lambda > 0$ following the idea by Brezis-Nirenberg [9]. In this point of view the problem (P) with $\lambda u^q$ and $1 < q < 5$ instead of $\lambda u$ has already been investigated by [12], [24] and [38]. By their works a certain extension from the result in [9] to the case $b > 0$ is accomplished. In [12], Figueiredo considers the case $1 < q < 5$ (we remark that he rather deals with more general dimension and the nonlocal coefficient). He gets the existence if $\lambda > 0$ is sufficiently large. In [38], Xie, Wu and Tang treat the case $3 < q < 5$ and conclude the existence for all $\lambda > 0$. A similar problem is investigated in [24] with additional existence and nonexistence results. But there is no work for the delicate case $q = 1$. In this paper, we deal with such a linear perturbation case. We emphasize that in this case we can find several effects of the nonlocal coefficient on the well-known existence, nonexistence [9] and uniqueness [1][2][17][26][31][39] results for the case $b = 0$.

## 1.1 Main result

Here we note a result by Brezis-Nirenberg. In [9], (P) with $b = 0$ is treated. Their result for the three dimensional case can be read as, "If and only if $\lambda_1/4 < \lambda < \lambda_1$ there exists a solution of (P) if $b = 0$", here $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ on the open ball. In this paper, we extend this result above to the case $b > 0$. To make the comparison between our result and the one by [9] clear, we fit our situation closely to the one in [9]. That is, we assume $\Omega$ is an open ball and further, consider the coefficient of $-\Delta u$ in (P) as $(1+b\int_\Omega |\nabla u|^2)$ which is usually treated as $(a+b\int_\Omega |\nabla u|^2)$ for arbitrary $a > 0$. Moreover we regard $\lambda \in \mathbb{R}$ as a given constant and $b \geq 0$ as a parameter in contrast to the treatments in [12], [24] and [38]. Then the existence for the case $b = 0$ is completely given by [9]. Our interest is to find several effects of a Kirchhoff type perturbation on the result above. Our main theorem is the following.

**Theorem 1.1.** *Let $\Omega$ be an open ball and $\lambda \in \mathbb{R}$ be a given constant. Then the following assertions hold.*

*(i) If $\lambda \leq \lambda_1/4$, (P) has no solution for all $b \geq 0$.*

*(ii) If $\lambda_1/4 < \lambda < \lambda_1$, there exists a constant $B_2 = B_2(\lambda) > 0$ such that (P) has a solution for all $0 \leq b < B_2$.*

*(iii) If $\lambda = \lambda_1$, there exists a constant $B_3 = B_3(\lambda_1) > 0$ such that (P) has a solution for all $0 < b < B_3$ and (P) has no solution for $b = 0$.*

*(iv) If $\lambda_1 < \lambda$, there exists a constant $B_4 = B_4(\lambda) > 0$ such that (P) has a solution for all $b > B_4$.*

**Remark 1.1.** The conditions for $b > 0$ to be small in (ii) and (iii) are essential. In fact we can prove that (P) has no solution if $0 < \lambda \leq \lambda_1$ and $b > 0$ is large. See Section 3.2 below.

**Remark 1.2.** We can expect the multiplicity of solutions for (P) for the case $\lambda > \lambda_1$. We give a remark on the multiplicity in Section 4.2.

We make a comment on Theorem 1.1 (iii) and (iv). As we stated before, in the case $\lambda \geq \lambda_1$, (P) has no solution if $b = 0$. But our theorem says that even if $\lambda \geq \lambda_1$, (P) does have a solution, thanks to $b > 0$ in the appropriate region. In many works on Kirchhoff type elliptic problems, the authors often seem to consider that the Kirchhoff type perturbation disturbs the existence of solutions of their problem or prevent them from proving the existence. Consequently some of them put some strong conditions on their nonlinear terms (for example, the 3-superlinear condition is assumed to get the mountain pass solution which is observed in [34] etc.) and the others elaborate some improvement for the proof to overcome such difficulties. But our result implies that the Kirchhoff type perturbation can help the solvability of the problem. Furthermore as remarked above, the Kirchhoff type perturbation permits the problem to have multiple solutions for the case $\lambda > \lambda_1$. This fact suggests that the nonlocal perturbation may break the uniqueness of solutions observed by [1][2][17][26][31][39] for the case $b = 0$. A breaking uniqueness effect of the nonlocal coefficient on the inhomogeneous problem has also found in an earlier work, see [10]. These existence and breaking uniqueness phenomena induced by the nonlocal perturbation are new knowledges among recent researches in Kirchhoff type nonlinear elliptic problems.

Finally we note that on Theorem 1.1 we still have a question, the existence or nonexistence for the case $\lambda > \lambda_1$ with small $b > 0$. At least we can say that, from the variational point of view, we clearly expect the condition $b > 0$ to be large in (iv) is essential.

## 1.2 Organization of this paper

This paper is organized as follows. In Section 2, we prove Theorem 1.1 (i). In Section 3, we demonstrate Theorem 1.1 (ii), (iii) and the nonexistence for large $b > 0$. In Section 4, we conclude Theorem 1.1 (iv) and make a remark on the multiplicity of solutions for the case $\lambda > \lambda_1$. In the proof, we use a same character $C$ to denote several positive constants. Note also that we denote

$B(x, r)$ as a 3 dimensional open ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$ or an open ball in $H_0^1(\Omega)$ topology centered at $x \in H_0^1(\Omega)$ with radius $r$. In the followings we denote the usual $H_0^1(\Omega)$ norm as $\|\cdot\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$.

### 1.3 Weak solutions of (P)

Here we give the definition of the weak solutions of (P). We say $u \in H_0^1(\Omega)$ is a weak solution of (P), if and only if $u$ satisfies

$$\left(1 + b\|u\|^2\right) \int_\Omega \nabla u \cdot \nabla h dx - \lambda \int_\Omega u_+ h dx - \int_\Omega u_+^5 h dx = 0, \tag{1.1}$$

for all $h \in H_0^1(\Omega)$, where $u_+ := \max\{0, u\}$. Applying the usual elliptic regularity argument and the strong maximum principle, we can conclude that every weak solution of (P) is a classical solution of (P) even if $b > 0$.

### 1.4 A priori estimate

We can get a priori estimate for the solution $u$ of (P) if $b > 0$ as follows. Let $\lambda_1/4 < \lambda$ and $b > 0$ be in the appropriate region. We have if $\lambda < \lambda_1$,

$$0 < \|u\| < \left\{\frac{4}{b}\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)\right\}^{\frac{1}{2}}, \tag{1.2}$$

and if $\lambda \geq \lambda_1$,

$$\left\{\frac{1}{b}\left(\frac{\lambda}{\lambda_1} - 1\right)\right\}^{\frac{1}{2}} < \|u\| < \left\{\frac{4}{b}\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)\right\}^{\frac{1}{2}}. \tag{1.3}$$

In fact, put $C := (1 + b\|u\|^2)^{-1/4}$. Then $v := Cu$ satisfies

$$\begin{cases} -\Delta v = \frac{\lambda}{1+b\|u\|^2} v + v^5, \ v > 0 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases}$$

From [9], it follows that

$$\frac{\lambda_1}{4} < \frac{\lambda}{1 + b\|u\|^2} < \lambda.$$

This proves (1.2) and (1.3).

## 2 The case $\lambda \leq \lambda_1/4$

In this section, we prove Theorem 1.1 (i). The argument is strictly based on that in [9]. Firstly we consider the case $\lambda \leq 0$. In this case, we use the following Pohozaev type identity [29] (see also [32]) for the solutions of (P).

**Lemma 2.1.** *Let $u$ be a solution of (P) and put $g(t) := \lambda t + t^5$. Then the following identity holds.*

$$\frac{(1 + b\|u\|^2)}{2} \int_\Omega |\nabla u|^2 dx - 3 \int_\Omega G(u) dx + \frac{(1 + b\|u\|^2)}{2} \int_{\partial\Omega} (x \cdot \nu) \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma = 0 \tag{2.4}$$

*where $G(t) := \int_0^u g(s) ds$, $\nu$ and $\partial/\partial\nu$ denote the outer normal vector and the outer normal derivative on $\partial\Omega$ respectively, and further $d\sigma$ is the 2-dimensional surface measure on $\partial\Omega$.*

*Proof.* For every solution of (P), applying the well known procedure, the proof is straightforward.

We give the following theorem.

**Theorem 2.1.** *Let $\Omega$ be strictly star-shaped and $\lambda \leq 0$. Then (P) has no solution for all $b \geq 0$.*

*Proof.* Utilizing (2.4), we can clearly conclude the theorem by the usual argument.

Next we consider the case $0 < \lambda \leq \lambda_1/4$.

**Theorem 2.2.** *If $\Omega$ is an open ball and $0 < \lambda \leq \lambda_1/4$, there exists no solution of (P) for all $b \geq 0$.*

*Proof.* For simplicity we assume $\Omega = B(0,1)$. Then applying the similar argument to that in [14], we can confirm that every solution of (P) is radially symmetric even if $b > 0$. Consequently every solution of (P) satisfies

$$\begin{cases} -A(\|u\|^2)\left(u'' + \frac{2}{r}u'\right) = \lambda u + u^5 \text{ in } (0,1), \\ u > 0 \text{ in } [0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{$P_r$}$$

where we put $A(\|u\|^2) := 1 + b\|u\|^2$ for simplicity. Now let $0 < \lambda \leq \lambda_1/4$ and $u$ be a solution of $(P_r)$. We take a smooth function $\psi$ such that $\psi(0) = 0$. By a similar procedure to that on Lemma 1.4 in [9], we have a variant of the Pohozaev type identity,

$$\int_0^1 u^2\left(\frac{A}{4}\psi''' + \lambda\psi'\right)r^2 dr = \frac{2}{3}\int_0^1 u^6\left(r\psi - r^2\psi'\right)dr + \frac{A}{2}|u'(1)|^2\psi(1). \tag{2.5}$$

For reader's convenience, we show the proof of (2.5). Multiplying the equation in $(P_r)$ by $r^2\psi u'$ and integrating by parts, we have

$$\begin{aligned} &A\int_0^1 |u'|^2\left(r\psi - \frac{r^2\psi'}{2}\right)dr + \frac{A}{2}|u'(1)|^2\psi(1) \\ &\qquad = \lambda\int_0^1 u^2\left(r\psi + \frac{r^2\psi'}{2}\right)dr + \frac{1}{6}\int_0^1 u^6\left(2r\psi + r^2\psi'\right)dr. \end{aligned} \tag{2.6}$$

On the other hand, multiplying the equation by $\left(r^2\psi'/2 - r\psi\right)u$, similarly we get

$$\begin{aligned} &A\int_0^1 |u'|^2\left(\frac{1}{2}r^2\psi' - r\psi\right)dr - \frac{A}{4}\int_0^1 u^2 r^2\psi''' dr \\ &\qquad = \lambda\int_0^1 u^2\left(\frac{1}{2}r^2\psi' - r\psi\right)dr + \int_0^1 u^6\left(\frac{1}{2}r^2\psi' - r\psi\right)dr. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) we obtain (2.5). Now we take

$$\psi(r) = \sin\left((4\lambda/A)^{1/2}\, r\right).$$

Observe that $\psi(0) = 0$ and $\psi(1) \geq 0$ since $\lambda_1 = \pi^2$ (with the first eigenfunction $|x|^{-1}\sin(\pi|x|)$) and $A \geq 1$. Then we have a similar contradiction to that in [9] by (2.5) and the facts that

$$\frac{A}{4}\psi''' + \lambda\psi' = 0 \text{ and } r\psi - r^2\psi' > 0 \text{ in } (0,1].$$

This completes the proof.

*Proof of Theorem 1.1 (i).* Assume $\Omega$ is an open ball. Then the proof is obvious by Theorem 2.1 and Theorem 2.2.

# 3 The case $\lambda_1/4 < \lambda \le \lambda_1$

In this section, we deal with the case $\lambda_1/4 < \lambda \le \lambda_1$. We show the existence for small $b > 0$ and nonexistence for large $b > 0$.

## 3.1 Existence for small $b > 0$

As the main argument, we prove Theorem 1.1 (ii) and (iii). We suppose $\Omega = B(0,1)$ for simplicity and fix $\lambda_1/4 < \lambda \le \lambda_1$. If $b = 0$, the conclusions are in [9]. Hence here, we give the proof for the case $b > 0$. But if $b > 0$, the minimizing argument in [9] does not seem to work. Thus we apply the mountain pass theorem here. We define the energy functional associated to (P) so that

$$I_b(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega u_+^2 dx - \frac{1}{6}\int_\Omega u_+^6 dx.$$

Clearly $I_b$ is well-defined and continuously Fréchet differentiable on $H_0^1(\Omega)$. We shall ensure the existence of a nontrivial critical point of $I_b$. The one of the main arguments lies in ensuring the local PS condition for $I_b$. To this aim, we give the following lemma.

**Lemma 3.1.** *Let $b > 0$, $\lambda < \lambda_1$ and $\{u_j\} \subset H_0^1(\Omega)$ be a (PS)$_c$ sequence with*

$$c < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

*where $S > 0$ is the usual Sobolev constant defined by*

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega u^6 dx\right)^{\frac{1}{3}}}$$

*and $C_K = C_K(b) := \left(bS^3 + \sqrt{(bS^3)^2 + 4S^3}\right)/2$. Then there exists a function $u \in H_0^1(\Omega)$ such that $(u_j)_+ \to u_+$ in $L^6(\Omega)$ up to subsequences.*

*Proof.* We first claim that $\{u_j\}$ is bounded in $H_0^1(\Omega)$. In fact, since $I_b(u_j) \to c$ and $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$, the Poincarè inequality and our assumption $\lambda \le \lambda_1$ imply

$$\begin{aligned} c + 1 &\ge I_b(u_j) - \frac{1}{6}\langle I_b'(u_j), u_j\rangle + \frac{1}{6}\langle I_b'(u_j), u_j\rangle \\ &\ge \frac{1}{3}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_j\|^2 + \frac{b}{12}\|u_j\|^4 - \|u_j\| \\ &\ge \frac{b}{12}\|u_j\|^4 - \|u_j\|, \end{aligned}$$

for large $j \in \mathbb{N}$. As $b > 0$, this proves the claim. Hence by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embedding, we have

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_j \to u \text{ in } L^2(\Omega), \\ &u_j \to u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences but still denoted $\{u_j\}$. Furthermore by the second concentration compactness lemma [22], we can assume that there exist an at most countable set $J$, points $\{x_k\}_{k\in J} \subset \overline{\Omega}$, and values $\{\mu_k\}_{k\in J}, \{\nu_k\}_{k\in J} \subset \mathbb{R}^+$ with

$$S\nu_k^{\frac{1}{3}} \le \mu_k \ (k \in J) \tag{3.8}$$

such that,

$$\begin{aligned} |\nabla u_j|^2 &\rightharpoonup d\mu \ge |\nabla u|^2 + \sum_{k\in J} \mu_k \delta_{x_k}, \\ (u_j)_+^6 &\rightharpoonup d\nu = u_+^6 + \sum_{k\in J} \nu_k \delta_{x_k}, \end{aligned}$$

in the measure sense, here $\delta_x$ denotes the Dirac delta measure concentrated at $x \in \mathbb{R}^3$ with mass 1. Now we claim $J = \emptyset$. To show this, we assume $J \neq \emptyset$ on the contrary. Then fix $k \in J$. For $\varepsilon > 0$, define a smooth test function $\phi_\varepsilon$ in $\mathbb{R}^3$ such that $\phi_\varepsilon = 1$ on $B(x_k, \varepsilon)$, $\phi_\varepsilon = 0$ on $B(x_k, 2\varepsilon)^c$ and $0 \le \phi_\varepsilon \le 1$ otherwise. We also assume $|\nabla\phi_\varepsilon| \le 2/\varepsilon$. Again since $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$ and $\{u_j\}$ is bounded, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle I_b'(u_j), u_j\phi_\varepsilon\rangle \\ &= \lim_{j\to\infty} \left\{ \left(1 + b\int_\Omega |\nabla u_j|^2 dx\right) \int_\Omega \nabla u_j \cdot \nabla(u_j\phi_\varepsilon) dx \right. \\ &\qquad \left. - \int_\Omega (u_j)_+^2 \phi_\varepsilon dx - \int_\Omega (u_j)_+^6 \phi_\varepsilon dx \right\}. \end{aligned} \tag{3.9}$$

Here note that

$$\lim_{j\to\infty} \left| \int_\Omega (\nabla u_j \cdot \nabla\phi_\varepsilon) u_j dx \right| = o(1) \tag{3.10}$$

and

$$\lim_{j\to\infty} \left| \int_\Omega (u_j)_+^2 \phi_\varepsilon dx \right| = o(1) \tag{3.11}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. (3.11) is obvious by the $L^2(\Omega)$ convergence of $\{u_j\}$ and the assumption on $\phi_\varepsilon$. We show (3.10). Using the boundedness and $L^2(\Omega)$ convergence of $\{u_j\}$ and utilizing the well-known inequalities and the assumption on $\phi_\varepsilon$, we calculate as

$$\begin{aligned} &\lim_{j\to\infty} \left| \left(1 + b\int_\Omega |\nabla u_j|^2 dx\right) \int_\Omega (\nabla u_j \cdot \nabla\phi_\varepsilon) u_j dx \right| \\ &\le C \lim_{j\to\infty} \left\{ \left( \int_\Omega |\nabla u_j|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega\cap B(x_k,2\varepsilon)} |u_j \nabla\phi_\varepsilon|^2 dx \right)^{\frac{1}{2}} \right\} \\ &\le C \left( \int_{\Omega\cap B(x_k,2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \left( \int_{\Omega\cap B(x_k,2\varepsilon)} |\nabla\phi_\varepsilon|^3 dx \right)^{\frac{1}{3}} \\ &\le C \left( \int_{\Omega\cap B(x_k,2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \to 0 \text{ as } \varepsilon \to 0, \end{aligned}$$

where $C > 0$ is some constant. This shows (3.11). By (3.9)-(3.11) and the measure convergence above, we get

$$0=\lim_{j\to\infty}\left\{\left(1+b\int_\Omega|\nabla u_j|^2dx\right)\int_\Omega|\nabla u_j|^2\phi_\varepsilon dx-\int_\Omega(u_j)_+^6\phi_\varepsilon dx\right\}+o(1)$$
$$=\left(1+b\int_{\overline{\Omega}}d\mu\right)\int_{\overline{\Omega}}\phi_\varepsilon d\mu-\int_{\overline{\Omega}}\phi_\varepsilon d\nu+o(1)$$

Taking $\varepsilon\to 0$, we conclude

$$0\geq(1+b\mu_k)\mu_k-\nu_k.$$

Finally using (3.8) we obtain,

$$\mu_k\geq C_K \tag{3.12}$$

and

$$\nu_k\geq\frac{C_K^3}{S^3}. \tag{3.13}$$

Then since $I_b(u_j)\to c$ and $I_b'(u_j)\to 0$ in $H^{-1}(\Omega)$, we have

$$c=\lim_{j\to\infty}\left\{I_b(u_j)-\frac{1}{4}\langle I_b'(u_j),u_j\rangle\right\}$$
$$\geq\lim_{j\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{4}\right)\|u_j\|^2-\lambda\left(\frac{1}{2}-\frac{1}{4}\right)\int_\Omega(u_j)_+^2dx+\left(\frac{1}{4}-\frac{1}{6}\right)\int_\Omega(u_j)_+^6dx\right\}$$
$$\geq\left(\frac{1}{2}-\frac{1}{4}\right)\mu_k+b\left(\frac{1}{4}-\frac{1}{4}\right)\mu_k^2+\left(\frac{1}{4}-\frac{1}{6}\right)\nu_k$$
$$\geq\frac{1}{2}C_K+\frac{b}{4}C_K^2-\frac{1}{6S^3}C_K^3$$

where for the second inequality we use the Poincarè inequality and the assumption $\lambda\leq\lambda_1$ and for the last inequality we use (3.12), (3.13) and the fact that $C_K+bC_K^2-C_K^3/S^3=0$. This contradicts our hypothesis on $c$. Thus $J=\emptyset$. It follows that

$$\int_\Omega(u_j)_+^6dx\to\int_\Omega u_+^6dx\text{ as }j\to\infty.$$

This leads us to the proof.

**Remark 3.1.** We can easily confirm that $u$ is nonnegative. In fact, since $\{u_j\}$ is bounded, we can assume that $\|u_j\|^2\to B$ for some value $B\geq 0$. If $B=0$, we finish the proof. If not, noting that we can suppose $u_j\rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $(u_j)_+\to u_+$ in $L^p(\Omega)$ for all $1\leq p<6$, we have

$$(1+bB)\int_\Omega\nabla u\cdot\nabla hdx=\lambda\int_\Omega u_+hdx+\int_\Omega u_+^5hdx,$$

for all $h\in H_0^1(\Omega)$. Taking $h=-\min\{u,0\}=:u_-$, we ensure the claim.

Here as in [9], we shall prove a lemma which shows a mountain pass level of $I_b$ is below the desired value. To the first, let us recall the argument by [9]. They choose the Talenti function [36] to estimate their mountain pass level. Note that by an appropriate rescaling, the function can be considered as a positive solution of the critical problem in whole space,

$$\begin{cases}-\Delta U=U^5\text{ in }\mathbb{R}^3,\\ U(x)\to 0\text{ as }|x|\to\infty,\end{cases}$$

which is given by

$$U_\varepsilon(x) = \frac{C_0 \varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + |x - x_0|^2)^{\frac{1}{2}}}$$

for some constant $C_0 > 0$, every value $\varepsilon > 0$ and point $x_0 \in \mathbb{R}^3$. Actually if we rescale $U_\varepsilon$ by an appropriate constant,

$$V_\varepsilon := \left(\frac{C_K}{S^{\frac{3}{2}}}\right)^{\frac{1}{2}} U_\varepsilon,$$

we can easily check that $V_\varepsilon$ satisfies the Kirchhoff type critical problem in whole space,

$$\begin{cases} -\left(1 + b\int_{\mathbb{R}^3} |\nabla V|^2 dx\right)\Delta V = V^5 \text{ in } \mathbb{R}^3, \\ V(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

Furthermore the energy of $V_\varepsilon$ is obtained by

$$\frac{1}{2}\|V_\varepsilon\|_{1,2}^2 + \frac{b}{4}\|V_\varepsilon\|_{1,2}^4 - \frac{1}{6}\int_{\mathbb{R}^3} V_\varepsilon^6 dx = \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

where $\|\cdot\|_{1,2} := (\int_{\mathbb{R}^3} |\nabla \cdot|^2 dx)^{1/2}$. Thus it is natural to use the Talenti function to estimate our mountain pass level too.

Now recall our assumption, $\Omega$=B(0,1). For all $\varepsilon > 0$ we introduce the cut off Talenti function,

$$u_\varepsilon(x) := \frac{\varepsilon^{\frac{1}{2}} \tau(|x|)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}$$

where $\tau(r) := \cos(\pi r/2)$ is a cut off function which satisfies $\tau(0) = 1$ and $\tau'(0) = \tau(1) = 0$. We normalize $u_\varepsilon$ as $v_\varepsilon := u_\varepsilon/(\int_\Omega u_\varepsilon^6 dx)^{1/6}$. Then we estimate

$$\begin{cases} \int_\Omega |\nabla v_\varepsilon|^2 dx = S + \frac{\alpha\lambda_1}{4}\varepsilon + O(\varepsilon^2), \\ \int_\Omega v_\varepsilon^6 dx = 1, \\ \int_\Omega v_\varepsilon^2 dx = \alpha\varepsilon + O(\varepsilon^2), \end{cases} \tag{3.14}$$

where $\alpha > 0$ is some constant. We prove the next lemma.

**Lemma 3.2.** *For every $\lambda_1/4 < \lambda$, there exists a constant $B = B(\lambda) > 0$ such that for all $0 < b < B$,*

$$\sup_{t \geq 0} I_b(tv_\varepsilon) < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

*if $\varepsilon > 0$ is small enough.*

*Proof.* Let $v_\varepsilon$ be defined as above. Take $t > 0$. Then we have after some calculation that

$$\begin{aligned} I_b(tv_\varepsilon) &= \frac{t^2}{2}\left(\|v_\varepsilon\|^2 - \lambda\int_\Omega v_\varepsilon^2 dx\right) + \frac{bt^4}{4}\|v_\varepsilon\|^4 - \frac{t^6}{6}\int_\Omega v_\varepsilon^6 dx \\ &\leq \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3 - C\left(\lambda - \frac{\lambda_1}{4} - \frac{b\lambda_1 C_K(b)}{4}\right)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{3.15}$$

for some constant $C > 0$. The calculation to obtain the last inequality is straightforward but a little bit complicated. Thus we show the proof in Appendix A for reader's convenience. Notice that since $\lambda > \lambda_1/4$, there exists a constant $B = B(\lambda) > 0$ such that

$$\lambda - \frac{\lambda_1}{4} - \frac{b\lambda_1 C_K(b)}{4} > 0$$

for all $0 < b < B$ (it is enough if we take $B(\lambda) = 2\,(\lambda/\lambda_1 - 1/4)\,(\lambda_1/\lambda)^{\frac{1}{2}}\,S^{-\frac{3}{2}}$). Fix such a $b$. Then we can conclude that

$$\sup_{t\geq 0} I_b(tv_\varepsilon) < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

for small $\varepsilon > 0$. This completes the proof.

*Proof of Theorem 1.1 (ii) and (iii).* Fix $\lambda_1/4 < \lambda \leq \lambda_1$. The conclusions for the case $b = 0$ are in [9]. Let us consider the case $b > 0$. Take $B = B(\lambda) > 0$ from Lemma 3.2. For all $0 < b < B$, we shall ensure the followings.

(a) There exist constants $\alpha, \rho > 0$ such that $I_b(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$.

(b) There exists a function $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| > \rho$ and $I_b(v_0) \leq 0$.

We confirm (a). Take $\rho > 0$ and $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. Then the Poincarè inequality and the Sobolev embedding imply

$$\begin{aligned} I_b(u) &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^6 \\ &\geq \frac{b}{4}\rho^4 - C\rho^6 \end{aligned}$$

for some constant $C > 0$. Thanks to $b > 0$, we obtain the conclusion. Next we show (b). Take any nontrivial function $v \in H_0^1(\Omega)$ with $v \geq 0$. Then we have for all $t \geq 0$,

$$I_b(tv_0) \leq \frac{t^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \frac{t^6}{6}\int_\Omega v^6 dx.$$

Then obviously we get $I_b(tv) \to -\infty$ as $t \to \infty$. Thus we have a constant $t_0 > 0$ such that $\|t_0 v\| > \alpha$ and $I_b(t_0 v) \leq 0$. Put $v_0 := t_0 v$. This proves (b). Now recalling Lemma 3.2, we choose $\varepsilon > 0$ small enough. With large $t_0 > 0$, we take the function in (b) as $v_0 := t_0 v_\varepsilon$. Then we define

$$\Gamma_b := \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0\ ,\ \gamma(1) = v_0\}$$

and

$$c_b := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_b(\gamma(t)).$$

By our choice of $b > 0$, the definition of $c_b$ and Lemma 3.2, we get

$$c_b < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3.$$

Finally let us confirm the $(PS)_{c_b}$ condition for $I_b$. Let $\{u_j\} \subset H_0^1(\Omega)$ be a $(PS)_{c_b}$ sequence. Then by Lemma 3.1 and Remark 3.1 , we can assume that there exists a nonnegative function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_j \to u \text{ in } L^2(\Omega), \\ &(u_j)_+ \to u \text{ in } L^6(\Omega). \end{aligned}$$

Furthermore since $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$, we have

$$\langle I_b'(u_j), u_j - u \rangle = o(1)$$

where $o(1) \to 0$ as $j \to \infty$. Then the $L^2(\Omega)$ convergence of $\{u_j\}$ and the $L^6(\Omega)$ convergence of $\{(u_j)_+\}$ show

$$(1 + b\|u_j\|^2) \int_\Omega \nabla u_j \cdot \nabla(u_j - u)dx = o(1) \text{ as } j \to \infty.$$

By the weak convergence, we obtain $u_j \to u$ in $H_0^1(\Omega)$. This concludes the proof.

## 3.2 Nonexistence for large $b > 0$

To show the nonexistence result for the case $0 < \lambda \leq \lambda_1$, we use a priori estimate (1.2). We state the following.

**Theorem 3.1.** *Let* $0 < \lambda \leq \lambda_1$. *Then there exists no solution of (P) if*

$$b \geq \frac{4}{\sqrt{3}} \left( \frac{\lambda}{\lambda_1} - \frac{1}{4} \right) \left( \frac{\lambda_1}{\lambda} \right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

*Proof.* Let $u$ be a solution for (P). Multiplying the equation of (P) by $u$, integrating by parts, we have

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda \int_\Omega u^2 dx - \int_\Omega u^6 dx \\ &> \left( 1 - \frac{\lambda}{\lambda_1} \right) \|u\|^2 + b\|u\|^4 - \frac{1}{S^3} \|u\|^6, \end{aligned}$$

where we use the Poincarè and the Sobolev inequalities for the last inequality. Solving this inequality for $\|u\| \neq 0$, we get

$$\|u\|^2 > \frac{1}{2} \left( bS^3 + \sqrt{(bS^3)^2 + 4\left(1 - \frac{\lambda}{\lambda_1}\right) S^3} \right).$$

Combining this estimate with (1.2) we obtain

$$\frac{1}{2} \left( bS^3 + \sqrt{(bS^3)^2 + 4\left(1 - \frac{\lambda}{\lambda_1}\right) S^3} \right) < \frac{4}{b} \left( \frac{\lambda}{\lambda_1} - \frac{1}{4} \right).$$

Thus we must have

$$b < \frac{4}{\sqrt{3}} \left( \frac{\lambda}{\lambda_1} - \frac{1}{4} \right) \left( \frac{\lambda_1}{\lambda} \right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

This completes the proof.

# 4 The case $\lambda > \lambda_1$

In this section we consider the case $\lambda > \lambda_1$. We can get the possibly multiple solutions with the aid of the Kirchhoff type nonlocal perturbation.

## 4.1 Existence for large $b > 0$

First we prove Theorem 1.1 (iv). We remark that for the conclusion, it is enough if we assume $\Omega$ is a bounded domain with smooth boundary. Fix $\lambda > \lambda_1$. As in the previous section, we define the energy functional,

$$I_b(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega u_+^2 dx - \frac{1}{6}\int_\Omega u_+^6 dx.$$

We shall prove the existence of a nontrivial critical point, in particular a local minimum, of $I_b$. To this aim, inspired by [33], we apply the method of the Nehari manifold. We also refer to the original work [36]. To the beginning, we define

$$\Lambda := \{u \in H_0^1(\Omega) \mid \langle I_b'(u), u\rangle = 0\},$$

and split $\Lambda$ into 3 parts,

$$\Lambda^+ := \{u \in \Lambda \mid b\|u\|^4 > 2\int_\Omega u_+^6 dx\}, \tag{4.16}$$

$$\Lambda^0 := \{u \in \Lambda \mid b\|u\|^4 = 2\int_\Omega u_+^6 dx\}, \tag{4.17}$$

$$\Lambda^- := \{u \in \Lambda \mid b\|u\|^4 < 2\int_\Omega u_+^6 dx\}. \tag{4.18}$$

Now we choose

$$b > 2\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}}, \tag{4.19}$$

where $S$ is the Sobolev constant as defined in the previous section. Since we fix $b > 0$ as above, we denote $I_b$ as $I$ for simplicity. Let us begin with the following lemmas.

**Lemma 4.1.** $\Lambda^+ \neq \emptyset$ *and* $\Lambda^0 = \{0\}$.

*Proof.* We consider the principal eigenfunction $\phi_1 > 0$ of $-\Delta$ on $\Omega$ with $\|\phi_1\| = 1$. Then for $t > 0$, we put a function

$$f(t) := -\left(\frac{\lambda}{\lambda_1} - 1\right) + bt^2 - \left(\int_\Omega \phi_1^6 dx\right)t^4.$$

Noting our assumption $\lambda > \lambda_1$ and (4.19), we can find a constant $\tau_0 > 0$ such that $f(\tau_0) = 0$ and $f'(\tau_0) > 0$. This implies $\tau_0\phi_1 \in \Lambda^+$. Next we assume that there exists a nontrivial function $u \in \Lambda^0$ on the contrary. Firstly suppose $\|u\|^2 \geq \lambda\int_\Omega u_+^2 dx$. Then as $u \in \Lambda^0$, considering (4.17), we obtain

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda\int_\Omega u_+^2 dx - \int_\Omega u_+^6 dx \\ &\geq \frac{b}{2}\|u\|^4. \end{aligned}$$

This is impossible. Secondary we assume $\|u\|^2 < \lambda\int_\Omega u_+^2 dx$. Then since $u \in \Lambda \setminus \{0\}$, we have

$$\begin{aligned} b &= \frac{\lambda\int_\Omega u_+^2 dx - \|u\|^2}{\|u\|^4} + \frac{\int_\Omega u_+^6 dx}{\|u\|^4} \\ &= \frac{2\int_\Omega u_+^6 dx\left(\lambda\int_\Omega u_+^2 dx - \|u\|^2\right)}{b\|u\|^8} + \frac{\int_\Omega u_+^6 dx}{\|u\|^4} \\ &\leq \frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1\right)S^{-3} + \frac{b}{2} \end{aligned}$$

where for the second equality, we use (4.17) and for the last inequality, we apply the Poincarè and the Sobolev inequalities and again (4.17). Solving this inequality with respect to $b > 0$, we get

$$b \leq 2\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

This contradicts (4.19). The proof is done.

**Lemma 4.2.** *There holds*

$$\|u\| < \left\{\frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1\right)\right\}^{\frac{1}{2}}$$

*for all* $u \in \Lambda^+$.

*Proof.* For any $u \in \Lambda^+$, we have by the Poincarè inequality and (4.16),

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda \int_\Omega u_+^2 dx - \int_\Omega u_+^6 dx \\ &> \frac{b}{2}\|u\|^2 \left\{\|u\|^2 - \frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1\right)\right\}. \end{aligned}$$

Noting our assumption $\lambda > \lambda_1$, we get the conclusion.

**Lemma 4.3.** *For every* $u \in \Lambda^+$*, there exists a constant* $\varepsilon > 0$ *and a* $C^1$ *functional* $t$ *on* $B(0, \varepsilon) \subset H_0^1(\Omega)$ *such that* $t(0) = 1$*,* $t(w) > 0$*,* $t(w)(u - w) \in \Lambda$ *for all* $w \in B(0, \varepsilon)$*, and further,*

$$\langle t'(0), h\rangle = \frac{(1 + 2b\|u\|^2) \int_\Omega \nabla u \cdot \nabla h dx - \lambda \int_\Omega u_+ h dx - 3 \int_\Omega u_+^5 h dx}{b\|u\|^4 - 2\int_\Omega u_+^6 dx}, \tag{4.20}$$

*for all* $h \in H_0^1(\Omega)$.

*Proof.* For every $u \in \Lambda^+$, we define a $C^1$ map,

$$g(w, t) := \|u - w\|^2 + bt^2\|u - w\|^4 - \lambda \int_\Omega (u - w)_+^2 dx - t^4 \int_\Omega (u - w)_+^6 dx.$$

Then by (4.16), we can easily verify that

$$g(0, 1) = 0 \text{ and } g_t(0, 1) \neq 0,$$

where $g_t(w, t)$ is the first partial derivative of $g(w, t)$ with respect to $t$. The implicit function theorem concludes the proof.

**Lemma 4.4.** $-\infty < \inf_{u\in\Lambda^+\cup\Lambda^0} < 0$.

*Proof.* For all $u \in \Lambda^+ \cup \Lambda^0$, using the Poincarè inequality we get

$$\begin{aligned} I(u) &= I(u) - \frac{1}{6}\langle I'(u), u\rangle \\ &\geq -\frac{1}{3}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + \frac{b}{12}\|u\|^4. \end{aligned} \tag{4.21}$$

On the other hand, we fix a function $u \in \Lambda^+$. Then we have from (4.16),

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ &< -\frac{b}{12}\|u\|^4. \end{aligned} \tag{4.22}$$

(4.21) and (4.22) ensure the proof.

Now we put $c_0 := \inf_{u\in\Lambda^+\cup\Lambda^0} I(u)$. From the Ekeland variational principle, there exists a sequence $\{u_n\} \subset \Lambda^+ \cup \Lambda^0$ such that

$$I(u_n) \le \frac{1}{n} + c_0 \text{ and } I(w) > I(u_n) - \frac{1}{n}\|w - u_n\| \ (w \in \Lambda^+ \cup \Lambda^0). \tag{4.23}$$

**Lemma 4.5.** *Let $\{u_n\}$ be given as above. Then*

$$I'(u_n) \to 0 \text{ in } H^{-1}(\Omega), \tag{4.24}$$

*up to subsequences.*

*Proof.* We first claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Actually noting (4.23), we have similarly to (4.21),

$$\begin{aligned} c_0 + 1 &\ge I(u_n) - \frac{1}{6}\langle I'(u_n), u_n\rangle \\ &\ge -\frac{1}{3}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u_n\|^2 + \frac{b}{12}\|u_n\|^4. \end{aligned}$$

for large $n \in \mathbb{N}$. Since $b > 0$, this inequality implies the claim. Then by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embeddings, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_n \to u \text{ in } L^2(\Omega), \\ &u_n \to u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences but still denoted $\{u_n\}$. Since $c_0 < 0$ we can assume $u_n \ne 0$ and thus $\{u_n\} \subset \Lambda^+$. Now let us conclude (4.24). To this end, we follow the argument in the proof of Theorem 1 in [36]. We assume $\|I'(u_n)\| > 0$ for large $n \in \mathbb{N}$. For such a $n \in \mathbb{N}$ and $u_n \in \Lambda^+$, we take a constant $\varepsilon > 0$ and a $C^1$ functional $t$ on $B(0, \varepsilon) \subset H_0^1(\Omega)$ from Lemma 4.3. For all $0 < \delta < \varepsilon$, we define $t_n(\delta) := t(\delta I'(u_n)/\|I'(u_n)\|)$ and

$$w_\delta := t_n(\delta)\left(u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right).$$

Note that $w_\delta \in \Lambda^+$ for sufficiently small $\delta > 0$. Recalling (4.23), we have

$$\begin{aligned} \frac{1}{n}\|w_\delta - u_n\| &> I(u_n) - I(w_\delta) \\ &= (1 - t_n(\delta))\langle I'(u_n), u_n\rangle + \delta t_n(\delta)\langle I'(u_n), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle + o(\delta). \end{aligned}$$

Noting $\langle I'(u_n), u_n\rangle = 0$, dividing by $\delta > 0$ and taking $\delta \to 0$, we get

$$\frac{C(|t_n'(0)| + 1)}{n} \geq \|I'(u_n)\|,$$

for some constant $C > 0$, where $|t_n'(0)| := |\langle t'(0), I'(u_n)/\|I'(u_n)\|\rangle|$. Thus the proof is done once we confirm that $|t_n'(0)|$ is bounded. Let us show this. From (4.20) and the boundedness of $\{u_n\}$, there exists a constant $C > 0$ which is independent of $n \in \mathbb{N}$ such that

$$|t_n'(0)| \leq \frac{C}{b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx}.$$

We claim that we can extract a subsequence so that

$$b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx > C$$

for some constant $C > 0$. To confirm this, it is enough to show that

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) > C \tag{4.25}$$

for a value $C > 0$. Since $\{u_n\} \subset \Lambda^+$, obviously

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) \geq 0.$$

Now we suppose

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) = 0 \tag{4.26}$$

on the contrary. Since $\{u_n\}$ is bounded, we can assume that there exists a constant $B > 0$ such that

$$\|u_n\|^2 \to B. \tag{4.27}$$

Then by (4.26) and the Sobolev inequality, there holds

$$1 = \lim_{n\to\infty}\frac{2\int_\Omega (u_n)_+^6 dx}{b\|u_n\|^4} \leq \frac{2}{b}\lim_{n\to\infty}\|u_n\|^2 S^{-3} = \frac{2}{b}BS^{-3}. \tag{4.28}$$

Now as $\{u_n\} \subset \Lambda \setminus \{0\}$

$$\begin{aligned} b &= \frac{\lambda\int_\Omega (u_n)_+^2 dx - \|u_n\|^2}{\|u_n\|^4} + \frac{\int_\Omega (u_n)_+^6 dx}{\|u_n\|^4} \\ &\leq \frac{1}{\|u_n\|^2}\left(\frac{\lambda}{\lambda_1} - 1\right) + \frac{\int_\Omega (u_n)_+^6 dx}{\|u_n\|^4} \end{aligned}$$

where for the last inequality we use the Poincarè inequality. Taking $n \to \infty$ and noting (4.27), (4.28) we get

$$b \leq \frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1\right)S^{-3} + \frac{b}{2}.$$

Solving this with respect to $b > 0$, we obtain a contradiction as in the proof of Lemma 4.1. Thus (4.25) holds. This completes the proof.

Finally we ensure the existence of a nontrivial critical point of $I$.

**Lemma 4.6.** *There exists a nontrivial critical point $u \in \Lambda^+$ of $I$.*

*Proof.* Our argument is based on that in [33]. Let $\{u_n\}$ be a minimizing sequence of $c_0$ as in the paragraph above Lemma 4.5. Similarly to the argument on Lemma 4.5, we have that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then by the second concentration compactness lemma, there exist a nonnegative function $u \in H_0^1(\Omega)$, an at most countable set $J$, points $\{x_k\}_{k\in J} \subset \overline{\Omega}$, and values $\{\mu_k\}_{k\in J}, \{\nu_k\}_{k\in J} \subset \mathbb{R}^+$ with $S\nu_k^{1/3} \le \mu_k$ for all $k \in J$ such that up to subsequences,

$$|\nabla u_n|^2 \rightharpoonup d\mu \ge |\nabla u|^2 + \sum_{k\in J} \mu_k \delta_{x_k},$$
$$(u_n)_+^6 \rightharpoonup d\nu = u^6 + \sum_{k\in J} \nu_k \delta_{x_k},$$

in the measure sense. Similarly to the proof of Lemma 3.1, we shall show $J = \emptyset$. To do this, we assume $J \neq \emptyset$. Since we can assume $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ by Lemma 4.5, we estimate similarly to (3.12),

$$\mu_k \ge C_K(b) \ge bS^3.$$

But since $\{u_n\} \subset \Lambda^+$, noting Lemma 4.2 and the inequality above, we have

$$\begin{aligned} \frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1\right) &\ge \lim_{n\to\infty} \|u_n\|^2 \\ &\ge \mu_k \\ &\ge bS^3. \end{aligned}$$

This is impossible because of (4.19). It follows that

$$(u_n)_+ \to u \text{ in } L^6(\Omega).$$

Then by the same argument with the proof of Theorem 1.1, we have

$$u_n \to u \text{ in } H_0^1(\Omega).$$

As a consequence, $u$ is a critical point of $I$ and clearly $u \in \Lambda^+ \cup \Lambda^0$. Furthermore since $I(u) < 0$, $u \neq 0$. Thus Lemma 4.1 concludes $u \in \Lambda^+$. This finishes the proof.

*Proof of Theorem 1.1 (iv).* Let $\lambda > \lambda_1$. We put

$$B_4 := 2\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

Then from Lemma 4.6, (P) has a nontrivial weak solution for all $b > B_4$. This completes the proof.

## 4.2 Remark on the multiplicity

It is natural for us to expect the existence of a second solution in $\Lambda^-$ defined as (4.18). But by the similar reason to the concluding remarks in [33], it seems to be hard to prove that. Instead of that, we can get a mountain pass solution as follows.

**Theorem 4.1.** *For any $\lambda > \lambda_1$, if $b > 0$ satisfies*

$$\frac{4}{\sqrt{3}}\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}} < b < 2\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)\left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{2}} S^{-\frac{3}{2}}, \tag{4.29}$$

*which is possible if and only if*

$$\lambda_1 < \lambda < \lambda_1\left(\frac{5}{4} + \frac{\sqrt{7}}{2}\right),$$

*(P) admits a solution $u$ with $I(u) > 0$.*

**Remark 4.1.** The condition on $b > 0$ not to be too large in Theorem 4.1 is essential for the existence of the solutions with positive energies. We can check that with a similar argument to that in Section 3.2. In fact, take

$$b > \frac{4}{\sqrt{3}}\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}} \tag{4.30}$$

and let $u$ be a solution of (P) with $I(u) > 0$. Then using the Sobolev inequality, we have

$$\begin{aligned} 0 &< I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ &\leq -\frac{b}{4}\|u\|^4 + \frac{1}{3S^3}\|u\|^6. \end{aligned}$$

It follows that

$$\|u\|^2 > \frac{3}{4} b S^3. \tag{4.31}$$

In addition, since $u$ is a solution of (P), we obtain by the Poincarè and the Sobolev inequalities,

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda\int_\Omega u^2 dx - \int_\Omega u^6 dx \\ &> -\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + b\|u\|^4 - \frac{1}{S^3}\|u\|^6. \end{aligned}$$

Noting (4.30) and (4.31), we deduce

$$\|u\|^2 > \frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 - 4\left(\frac{\lambda}{\lambda_1} - 1\right)S^3}\right).$$

Combining this with (1.3), we get

$$\frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 - 4\left(\frac{\lambda}{\lambda_1} - 1\right)S^3}\right) < \frac{4}{b}\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)$$

Again we must have

$$b < \frac{4}{\sqrt{3}}\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)\left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

This concludes the remark.

Now recall that, from the argument in the previous subsection, for every $\lambda > \lambda_1$, we can get a solution with negative energy if

$$b > 2\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

Thus combining this fact and Theorem 4.1, we conclude that (P) has two solutions if

$$\lambda_1 < \lambda < \lambda_1\left(\frac{5}{4} + \frac{\sqrt{7}}{2}\right)$$

and

$$\frac{4}{\sqrt{3}}\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}} < b < 2\left(\frac{\lambda}{\lambda_1} - \frac{1}{4}\right)\left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{2}} S^{-\frac{3}{2}}.$$

The organization of the proof for Theorem 4.1 is similar to that for Theorem 1.1 (ii) and (iii). All we should check here is the appropriate local PS condition for the case $\lambda > \lambda_1$. Define $I$ as in the previous subsection. We show the following.

**Lemma 4.7.** *Let $\lambda > \lambda_1$, take*

$$b > 2\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}},$$

*and suppose $\{u_j\}$ is a (PS)$_c$ sequence for $I$ with*

$$c < \frac{1}{2}C_K(b) + \frac{1}{4}C_K(b)^2 - \frac{1}{6S^3}C_K(b)^3.$$

*Then $\{u_n\}$ converges in $H_0^1(\Omega)$ up to subsequences.*

*Proof.* The first part of the proof is similar to that for Lemma 3.1. That is, we can show $\{u_j\}$ is bounded and hence, assume that there exist a nonnegative function $u \in H_0^1(\Omega)$, an at most countable set $J$, points $\{x_k\}_{k\in J} \subset \overline{\Omega}$, and values $\{\mu_k\}_{k\in J}, \{\nu_k\}_{k\in J} \subset \mathbb{R}^+$ with $S\nu_k^{1/3} \le \mu_k$ for all $k \in J$ such that up to subsequences,

$$|\nabla u_j|^2 \rightharpoonup d\mu \ge |\nabla u|^2 + \sum_{k\in J} \mu_k \delta_{x_k},$$
$$(u_j)_+^6 \rightharpoonup d\nu = u^6 + \sum_{k\in J} \nu_k \delta_{x_k},$$

in the measure sense. If $J \neq \emptyset$, we can estimate as before,

$$\mu_k \ge C_K(b) \text{ and } \nu_k \ge \frac{C_K(b)^3}{S^3}$$

for some $k \in J$. We obtain

$$\begin{aligned} c &= \lim_{j\to\infty}\left\{I_b(u_j) - \frac{1}{4}\langle I_b'(u_j), u_j\rangle\right\} \\ &\ge \left(\frac{1}{2} - \frac{1}{4}\right)\mu_k + b\left(\frac{1}{4} - \frac{1}{4}\right)\mu_k^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\nu_k \\ &\quad - \frac{1}{4}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + \frac{1}{12}\int_\Omega u^6 dx, \end{aligned} \tag{4.32}$$

where for the last inequality we use the measure convergence above and the Poincarè inequality. Now notice that we are considering the case $\lambda > \lambda_1$. To obtain a contradiction as in the proof of Lemma 3.1, we recall that $u \in H_0^1(\Omega)$ satisfies

$$(1 + bB) \int_\Omega \nabla u \cdot \nabla h dx = \lambda \int_\Omega u_+ h dx + \int_\Omega u_+^5 h dx, \tag{4.33}$$

for all $h \in H_0^1(\Omega)$ where

$$B := \lim_{n\to\infty} \|u_j\|^2$$

as in Remark 3.1. Notice that

$$B \geq \|u\|^2 + \mu_k \geq \|u\|^2 + bS^3.$$

From (4.33) and the estimate above, we deduce that

$$\int_\Omega u^6 dx \geq \left\{ b^2 S^3 - \left(\frac{\lambda}{\lambda_1} - 1\right)\right\} \|u\|^2 + b\|u\|^4$$

Consequently we have

$$\begin{aligned} -\frac{1}{4}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + \frac{1}{12}\int_\Omega u^6 &\geq \frac{1}{12}\left\{ b^2 S^3 - 4\left(\frac{\lambda}{\lambda_1} - 1\right)\right\} \|u\|^2 + \frac{b}{12}\|u\|^4 \\ &\geq 0, \end{aligned}$$

since $b > 0$ is large enough. Combining this inequality together with (4.32) we get

$$c \geq \frac{1}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3.$$

This is a desired contradiction. This leads us to the proof.

*Proof of Theorem 4.1.* Take any $\lambda > \lambda_1$ and fix $b$ as in (4.29). To the first we confirm the mountain pass geometry. For all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$, we have by the Poincarè and the Sobolev inequalities,

$$\begin{aligned} I(u) &\geq -\frac{1}{2}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6S^3}\|u\|^6 \\ &= -\frac{1}{2}\left(\frac{\lambda}{\lambda_1} - 1\right)\rho^2 + \frac{b}{4}\rho^4 - \frac{1}{6S^3}\rho^6. \end{aligned}$$

Considering the lower bound for $b$ in (4.29), we obtain constants $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. In addition it is clear that for any nonnegative function $u \in H_0^1(\Omega) \setminus \{0\}$, $I(tu) \to -\infty$ as $t \to \infty$. Then noting Lemma 4.7 and the argument for the proof of Theorem 1.1 (ii) and (iii), we get the conclusion.

# 5 Final remark

In contrast to previous arguments, let us regard $b \geq 0$ as a given constant and $\lambda \in \mathbb{R}$ as a parameter as the treatments in [12], [24] and [38]. Then we can trivially modify the previous arguments and get the following.

**Theorem 5.1.** *Let $\Omega$ be an open ball. Then for all $b \geq 0$, there hold the followings.*

*(i) If $(\lambda_1/4)(1 + bC_K(b)) < \lambda < \lambda_1 \left(1 + 3b^2S^3/16\right)$, (P) has a solution with positive energy.*

*(ii) If $\lambda_1 < \lambda < \lambda_1 \left(1 + b^2S^3/4\right)$, (P) has a solution with negative energy.*

**Remark 5.1.** Surely, we can show corresponding nonexistence results to those in Section 2, 3-2 and 4-2 for small $\lambda > 0$. In addition note that if $b > 0$ is too large the constant $\lambda > 0$ satisfying the condition in (i) is empty.

As long as we observe from Theorem 5.1, the interval for $\lambda > 0$ in which (P) admits a mountain pass solution seems to shift from $(\lambda_1/4, \lambda_1)$ by a mount depending on $b \geq 0$ and vanish for large $b > 0$. Instead of that, there occurs *a local minimum* of the corresponding energy functional in some interval for $\lambda > \lambda_1$ and it becomes wider and wider as $b > 0$ increases. Certainly we can conclude the existence of multiple solutions in the appropriate region for small $b > 0$ and $\lambda > \lambda_1$. This behavior of the structure of solutions set of (P) seems to come from the combined effect of the special nature of the 3-*dimensional* critical problem and the nonlocal coefficient.

# A Verification for (3.15)

In this appendix we prove (3.15) in Lemma 3.2. To the first fix $\varepsilon > 0$ and put

$$\begin{aligned} f_\varepsilon(t) &:= I_b(tv_\varepsilon) \\ &= \frac{1}{2}A_\varepsilon t^2 + \frac{b}{4}B_\varepsilon t^4 - \frac{1}{6}t^6 \end{aligned}$$

where we define $A_\varepsilon := \|v_\varepsilon\|^2 - \lambda \int_\Omega v_\varepsilon^2 dx$ and $B_\varepsilon := \|v_\varepsilon\|^4$. We compute

$$\max_{t \geq 0} f_\varepsilon(t) = \frac{1}{2}A_\varepsilon C_\varepsilon + \frac{b}{4}B_\varepsilon C_\varepsilon^2 - \frac{1}{6}C_\varepsilon^3 \tag{A.34}$$

where $C_\varepsilon := (bB_\varepsilon + \sqrt{(bB_\varepsilon)^2 + 4A_\varepsilon})/2$. Here notice that from (3.14),

$$\begin{aligned} A_\varepsilon &= S - \alpha\left(\lambda - \frac{\lambda_1}{4}\right)\varepsilon + O(\varepsilon^2), \\ B_\varepsilon &= S^2 + \frac{\alpha\lambda_1 S}{2}\varepsilon + O(\varepsilon^2), \\ C_\varepsilon &= \frac{C_K(b)}{S} + \frac{\alpha}{4}\left(\lambda_1 bS + \frac{\lambda_1 b^2 S^3 - 4\left(\lambda - \frac{\lambda_1}{4}\right)}{\sqrt{b^2S^4 + 4S}}\right)\varepsilon + O(\varepsilon^2). \end{aligned}$$

Then we estimate the right hand side of (A.34) by the order of $\varepsilon > 0$ and get

$$\begin{aligned} \max_{t \geq 0} f_\varepsilon(t) \leq\ & \frac{1}{2}C_K(b) + \frac{b}{4}C_K(b)^2 - \frac{1}{6S^3}C_K(b)^3 \\ & - \frac{\alpha C_K(b)}{2S}\left(\lambda - \frac{\lambda_1}{4} - \frac{\lambda_1 bC_K(b)}{4}\right)\varepsilon \\ & + \frac{\alpha}{8S^2}\left(\lambda_1 bS + \frac{\lambda_1 b^2S^3 - 4\left(\lambda - \frac{\lambda_1}{4}\right)}{\sqrt{b^2S^4 + 4S}}\right)\left(S^3 + bS^3C_K(b) - C_K(b)^2\right)\varepsilon \\ & + O(\varepsilon^2). \end{aligned}$$

Since $S^3 + bS^3C_K(b) - C_K(b)^2 = 0$, we obtain (3.15).

## Acknowledgement

A part of this work was done during the author's visit to Chern Institute of Mathematics, Nankai University on September in 2013. He appreciates to Prof. Zhi-Qiang Wang for his kind cooperation on his visit. He is also grateful to Prof. Yohei Sato for his suggestion on a part of the result in this paper. Finally he sincerely thanks to Prof. Futoshi Takahashi for his so many supports and helpful discussions on his research.


## References


[1] Adimurthi and S. Yadava, *An elementary proof for the uniqueness of positive radial solution of a quasilinear Dirichlet problem*, Arch. Rat. Mech. Anal. **126** (1994), 219-229.

[2] A. Aftalion and F. Pacella, *Uniqueness and nondegeneracy for some nonlinear elliptic problems in a ball*, J. Differential Equations **195** (2003), 380-397.

[3] C.O. Alves, F.J.S.A. Corrêa and G.M. Figueiredo, *On a class of nonlocal elliptic problems with critical growth*, Differ. Equ. Appl. **2** (2010), 409-417.

[4] C.O. Alves, F.J.S.A. Corrêa and T.F. Ma, *Positive Solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49** (2005), 85-93.

[5] C.O. Alves and G.M. Figueiredo, *Nonlinear perturbations of a periodic Kirchhoff equation in* $\mathbb{R}^N$, Nonlinear Anal. **75** (2012), 2750-2759.

[6] A. Arosio, Averaged Evolution Equations. The Kirchhoff String and its Treatment in Scales of Banach Spaces. Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations (Trieste, 1993), 220-254, World Sci. Publ., River Edge, NJ, 1995.

[7] A. Arosio and S. Panizzi, *On the well-posedness of Kirchhoff string*, Trans. Amer. Math. Soc. **348** (1996), 305-330.

[8] S. Bernstein, *Sur une classe d'équations fonctionnelles aux dérivées partielles*, Izk. Akad. Nauk SSSR, Ser. Mat. **4** (1940), 17-26.

[9] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. pure Appl. Math. **36** (1983), 437-477.

[10] M. Chipot, V. Valente and G. Vergara Caffarelli, *Remarks on a nonlocal problem involving the Dirichlet energy*, Rend. Sem. Mat. Univ. Padova **110** (2003), 199-220.

[11] P. D'Ancona and S. Spagnolo, *Global solvability for degenerate Kirchhoff equation with real analytic date*, Invent. Math. **108** (1992), 247-262.

[12] G.M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl. **401** (2013), 706-713.

[13] G.M. Figueiredo and J.R. Santos Jr., *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, Differential Integral Equations **25** (2012) 853-868.

[14] B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[15] M. Hosoya and Y. Yamada, *On some nonlinear wave equations I: local existence and regularity of solutions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **38** (1991), 225-238.

[16] G. Kirchhoff, Vorlesungen über Mathematische Physik: Mechanik, Teubner, Leipzig, 1876.

[17] M.K. Kwong and Y. Li, *Uniqueness of radial solutions of semilinear elliptic equations*, Trans. Amer. Math. Soc. **333** (1992), 339-363.

[18] Y. Li, F. Li and J. Shi, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations **253** (2012), 2285-2294.

[19] Z. Liang, F. Li and J. Shi, *Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2013), 155-167.

[20] S. Liang and S. Shi, *Soliton solutions to Kirchhoff type problems involving the critical growth in* $\mathbb{R}^N$, Nonlinear Anal. **81** (2013), 31-41.

[21] J.L. Lions, On Some Questions in Boundary Value Problems of Mathematical Physics, in "Contemporary Developements in Continuum Mechanics and PDE's", G.M. de la Penha and L.A. Medeiros ed.s, North-Holland, Amsterdam, 1978.

[22] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, Part 1.*, Rev, Mat. Iberoamericana **1** (1985), 145-201.

[23] T.F. Ma and J.E. Muñoz Rivera, *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett. **16** (2003), 243-248.

[24] D. Naimen, *Positive solutions of Kirchhoff type elliptic equations involving a critical Sobolev exponent*, Nonlinear Deferential Equations Appl. NoDEA. **21** (2014), 885-914.

[25] D. Naimen, *The critical problem of Kirchhoff type equations in dimension four*, J. Differential Equations 257 (2014), 1168-1193.

[26] W.M. Ni and R.D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u + f(u,r) = 0$, Comm. Pure Appl. Math. **38** (1985), 67-108.

[27] D.W. Oplinger, *Frequency response of a nonlinear stretched string*, J. Acoustic Soc. Amer. **32** (1960) 1529-1538.

[28] K. Perera and Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations **221** (2006), 246-255.

[29] S. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408-1411.

[30] S. Pohozaev, *On a class of quasilinear hyperbolic equations*, Math. Ussr Sbornik **25** (1975), 145-158.

[31] P.N. Shrikanth, *Uniqueness of solutions of nonlinear Dirichlet problems*, Diff. Int. Eq. (1993), 663-670.

[32] M. Struwe, Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 4th edition, Springer-Verlag Berlin Heidelberg, 2008.

[33] Y. Sun and X. Liu, *Existence of positive solutions for Kirchhoff type problems with critical exponent*, J. Partial Differ. Equ. **25** (2012), 187-198.

[34] J.-J. Sun and C.-L Tang, *Existence and multiplicity of solutions for Kirchhoff type equations*, Nonlinear Anal. **74** (2011), 1212-1222.

[35] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pure Appl. **110** (1976), 353-372.

[36] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **9** (1992), 281-304.

[37] J. Wang, L. Tian, J. Xu, and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differential Equations **253** (2012), 2314-2351.

[38] Q. Xie, X. Wu and C. Tang, *Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent*, Commun. Pure Appl. Anal. **12** (2013), 706-713.

[39] L. Zhang, *Uniqueness of positive solutions of* $\Delta u + u^p + u = 0$ *in a finite ball*, Comm. Part. Diff. Eq. **17** (1992), 1141-1164.