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Symbolic Calculus and Boundedness of Multi-parameter and Multi-linear Pseudo-differential Operators *

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Abstract

Since the work of Hörmander on linear pseudo-differential operators, the applications of pseudo-differential operators have played an important role in partial differential equations, harmonic analysis, theory of several complex variables and other branches of modern analysis (e.g., they are used to construct parametrices and establish the regularity of solutions to PDEs such as the $\overline{\partial}$ problem, etc.). The work of Coifman and Meyer on multi-linear Fourier multipliers and pseudo-differential operators has stimulated further such applications. In [2], the authors developed a fairly satisfactory theory of symbolic calculus for multi-linear pseudo-differential operators. Motivated by this work [2] and L^p estimates of [34, 35] on multi-parameter and multi-linear Fourier multipliers and of [12] on multi-parameter and multi-linear pseudo-differential operators, we study and carry out the theory of symbolic calculus for multi-parameter and multi-linear pseudo-differential operators. Our results include the symbol estimates of the adjoints, asymptotic behavior, kernel estimates and boundedness properties and extend those in [2] to the multi-parameter and multi-linear setting. The estimates of the distributional kernel of associated multi-parameter and multi-linear pseudo-differential operators can be found useful in establishing the boundedness of such multi-parameter and multi-linear pseudo-differential operators.

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1 Introduction

It is well-known that linear pseudo-differential operators play an important role in partial differential equations, harmonic analysis, several complex variables, etc. Pseudo-differential operators are also employed to construct parametrices and establish regularity properties of solutions to partial differential equations, see, e. g., [15], [20], [21], [22], [25], [38], [24], [40], [41], [42], etc.

The Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$ of linear pseudo-differential operators are defined to consist of operators in the form

$$T_p(f) = \int\int_{\mathbb{R}^n\times\mathbb{R}^n} p(x,\xi)\cdot\widehat{f}(\xi)\cdot e^{ix\xi}d\xi \tag{1.1}$$

where $x,\xi,\eta\in\mathbb{R}^n$ and the symbol p satisfies

$$|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for all multi-indices α,β and some positive constants $C_{\alpha,\beta}$ depending on α,β. The function f is taken initially from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ (see [20, 21]).

The continuity properties of such classes of pseudo-differential operators have been extensively studied in the literature since the work of Hörmander [20, 21] in which he proved the operators with symbols in $S^0_{\rho,\delta}$ are L^2 bounded when $0\le\delta<\rho\le 1$. In a celebrated paper, Calderón and Vaillancourt [6] established the L^2 boundedness when $0\le\delta=\rho<1$. C. Fefferman further extended in [14] to the L^p boundedness ($1<p<\infty$) for operators with symbols in $S^{-m}_{\rho,\delta}$ with $0\le\delta<\rho\le 1$ and $m\ge n|\frac{1}{p}-\frac{1}{2}|(1-\rho)$. The result of C. Fefferman is sharp in the sense that for $m<n|\frac{1}{p}-\frac{1}{2}|(1-\rho)$, then the L^p boundedness fails. The critical case of $\delta=\rho$ was later established by Palvarinth and E. Somersalo in [39] by establishing h^p to h^p boundedness for all $0<p<\infty$, where h^p is the local Hardy space of Goldberg [17] (and the classical Hardy space H^p has the relation $H^p\subset h^p\subset L^p$ for all $0<p<\infty$ and $h^p=L^p$ for $1<p<\infty$). The result of [39] strengthens the H^1 to L^1 boundedness of Coifman and Meyer [9] when $m=\frac{n}{2}$. We also refer to the more extensive treatment of pseudo-differential operators and their applications in PDEs to [1], [15], [38], [24], [25], [40], [41], [42], etc.

The bilinear analogue of such pseudo-differential operators are defined to be the class $BS^m_{\rho,\delta}(\mathbb{R}^{2n})$ consisting of operators of the form. Let $f_1,f_2\in\mathcal{S}(\mathbb{R}^n)$. Then we can define

$$T_p(f_1,f_2) = \int\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(x,\xi,\eta)\cdot\widehat{f_1}(\xi)\cdot\widehat{f_2}(\eta)\cdot e^{ix(\xi+\eta)}d\xi d\eta$$

where $x,\xi,\eta\in\mathbb{R}^n$ and p satisfies

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma p(x,\xi,\eta)| \leqslant C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{1.2}$$

for all multi-indices α,β,γ and some positive constants $C_{\alpha,\beta,\gamma}$ depending on α,β,γ.

The first work of bilinear singular integrals and pseudo-differential operators is due to Coifman and Meyer [10, 11] which originated from specific problems about Calderón's commutators. Many people became interested in the study of general boundedness properties of multilinear Fourier multipliers and pseudo-differential operators, see Christ and Journé [8], Lacey and Thiele [27, 28], Lacey [26], Kenig and Stein [23], Grafakos and Torres [19], Gilbert and Nahmod [16], Grafakos and Li [18, 29], Benyi and Torres [3], Benyi, Maldonado, Naibo and Torres [2], Bernicot [5] (see also [32], [33], [30], [31]), etc.

The development of the symbolic calculus for bilinear pseudo-differential operators started in the work of Bényi and Torres [3] and was continued in Bényi et al. [2]. As in the linear case, many of the results obtained were motivated by the Calderón-Zygmund theory and its bilinear counterpart as developed in [19], [8], [23], [32]. See also e.g. Bényi et al. [2] for further references.

In [2], the authors Bényi et al. developed a fairly satisfactory theory of symbolic calculus for multi-linear pseudo-differential operators. Motivated by this work and L^p estimates of Muscalu, Pipher, Tao and Thiele [34, 35, 36, 37] on multi-parameter and multi-linear Fourier multipliers (see also an alternative proof of these results by Chen and Lu [7] for multipliers with minimal smoothness) and of Dai and Lu [12] on multi-parameter and multi-linear pseudo-differential operators, we study and carry out the theory of symbolic calculus for multi-parameter and multi-linear pseudo-differential operators. In this article, we want to understand the properties of all the multi-parameter and multilinear analogs of the linear Hörmander classes of pseudo-differential operators. Our results and methods are motivated by the work in the linear and bilinear settings [25] and [2]. In particular, our techniques are inspired by the work in bilinear setting [2] and our results extend those in [2] to the multi-paramter and multi-linear setting.

For the simplicity of presentations, in this paper we will only consider the case of bi-parameter and bilinear pseudo-differential operators. It is not hard to generalize them to any multi-parameter and multi-linear setting. Our goal is to carry out the symbolic calculus for the bi-parameter and bi-linear operators with symbols $BBS^m_{\rho,\delta}$ of all the possible values of ρ and δ. Though some of our computations are reminiscent of those for the bilinear pseudo-differential operators or Fourier integral operators, the symbolic calculus for bi-parameter and bilinear operators become more complicated.

In this article, we will first need to formulate the conditions that a symbol of multi-parameter and multilinear pseudo-differential operators should satisfy.
Fix $m = (m^{(1)}, m^{(2)})$, $\rho = (\rho^{(1)}, \rho^{(2)})$, $\delta = (\delta^{(1)}, \delta^{(2)}) \in \mathbb{R} \times \mathbb{R}$, we will study the following type of bi-parameter and bilinear pseudo-differential operators defined for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2n})$:

$$T_p(f_1, f_2) = \int\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(x,\xi,\eta)\cdot \widehat{f_1}(\xi)\cdot \widehat{f_2}(\eta)\cdot e^{ix(\xi+\eta)} d\xi d\eta$$

where $x = (x_1, x_2), \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and p satisfies

$$\begin{aligned} |\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\gamma_1}_{\eta_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_2}_{\eta_2} p(x,\xi,\eta)| \leqslant \quad & C_{\alpha,\beta,\gamma}(1 + |\xi_1| + |\eta_1|)^{m^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \\ & \cdot(1 + |\xi_2| + |\eta_2|)^{m^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)} \end{aligned} \tag{1.3}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2)$, and some positive constants $C_{\alpha,\beta,\gamma}$ depending on α, β, γ.

We denote the class of such symbols by $BBS^m_{\rho,\delta}(\mathbb{R}^{4n})$. This is motivated by the terms "bi-parameter" and "bilinear". We often denote it by $BBS^m_{\rho,\delta}$ when the underlying space $\mathbb{R}^{2n}$ is evident.

It is clear that the estimates in (1.3) that the bi-parameter and bilinear symbol $p(x, \xi, \eta)$ satisfies are weaker than those in (1.2) satisfied by the bilinear symbol. It is these estimates which make the substantial difference between the bilinear pseudo-differential operators and the bi-parameter and bi-linear pseudo-differential operators. Such different symbols also make the study of the bi-parameter and bilinear pseudo-differential operators more difficult and considerably more complicated. In particular, the estimates for the kernels corresponding to the multi-parameter and multilinear pseudo-differential operators are fairly involved and a very careful analysis is needed (Theorem 1.4). Such kernel estimates will find useful in future applications of proving boundedness of multi-parameter and multilinear pseudo-differential operators.

Given the above bi-parameter and bilinear operator $T = T_p$, we can define its adjoints T^{*1} and T^{*2} as follows:

$$\langle T(f_1, f_2), f_3\rangle = \langle T^{*1}(f_3, f_2), f_1\rangle = \langle T^{*2}(f_1, f_3), f_2\rangle \quad \text{for all } f_1, f_2 \in \mathcal{S}(\mathbb{R}^n).$$

We now collect several lemmas that will be needed in the proofs of our main theorems. The proofs of these lemmas are fairly straightforward and we shall omit them.

We first consider the first transpose T^{*1} of T. Then we have the following result.

Lemma 1.1. *We can rewrite T^{*1} as a compound operator*

$$T^{*1}(f_3, f_2)(x) = \int_y \int_\eta \int_\xi C(y,\xi,\eta)\cdot f_3(y)\cdot \widehat{f_2}(\eta)\cdot e^{-i(y-x)\xi}\cdot e^{ix\cdot\eta} d\xi d\eta dy$$

where $C(y,\xi,\eta) = p(y,-\xi-\eta,\eta)$ and we have used the notation $\int_y$ to express the integral with respect to y.

By straightforward calculations, we can show that

Lemma 1.2. *The function C in Lemma 1.1 satisfies the same differential inequalities as p does, i.e*

$$\left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2}C(x,\xi,\eta)\right| \leqslant C_{\alpha,\beta,\gamma}(1+|\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot(1+|\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}.$$

By appropriately changing variables of integration, we can show

Lemma 1.3. *The symbol of T^{*1} is given in terms of C by the following expression:*

$$a(x,\xi,\eta) = \int\int C(x+y, z+\xi,\eta)e^{-iz\cdot y}dydz.$$

We now state the first main theorem of this paper.

Theorem 1.1. *Assume that $0 \leqslant \delta^{(i)} \leqslant \rho^{(i)} \leqslant 1, \delta^{(i)} < 1, i = 1,2$ and $p \in BBS^m_{\rho,\delta}$. Then for $j = 1,2$, $T_p^{*j} = T_{p^{*j}}$, where $p^{*j} \in BBS^m_{\rho,\delta}$.*

Before we state the next main theorem, we need to introduce the following notion of asymptotic expansion of a bi-parameter and bilinear symbol.

Definition 1.1. *We say $p \sim \sum_{j=0}^{\infty} p_j$, if there is a non-increasing sequence $m_N^{(1)}, m_N^{(2)} \searrow -\infty$ such that $p - \sum_{j=0}^{N} p_j \in BBS^{m_N}_{\rho,\delta}, \forall N > 0, m_N = (m_N^{(1)}.m_N^{(2)})$.*

Then our next two main theorems are the following asymptotic expansions of symbols of the bi-parameter and bilinear pseudo-differential operators.

Theorem 1.2. *Assume that $a_j \in BBS^{m_j}_{\rho,\delta}, j \geq 0$, and $m_j^{(1)}, m_j^{(2)} \searrow -\infty$ as $j \to \infty$. Then there exist $a \in BBS^{m_0}_{\rho,\delta}$ such that $a \sim \sum_{j=0}^{\infty} a_j$. Moreover, if $b \in BBS^{\infty}_{\rho,\delta} = \bigcup_m BBS^m_{\rho,\delta}$ and $b \sim \sum_{j=0}^{\infty} a_j$, then $a - b \in BBS^{-\infty}_{\rho,\delta} = \bigcap_m BBS^m_{\rho,\delta}$.*

Theorem 1.3. *Assume that $a_j \in BBS^{m_j}_{\rho,\delta}, j \geq 0$ and $m_j^{(1)}, m_j^{(2)} \searrow -\infty$ as $j \to \infty$. Let $a \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ be such that*

$$\left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_1}^{\gamma_1}\partial_{\eta_2}^{\gamma_2}a(x,\xi,\eta)\right| \lesssim C_{\alpha,\beta,\gamma}\cdot(1+|\xi_1|+|\eta_1|)^{\mu^{(1)}}\cdot(1+|\xi_2|+|\eta_2|)^{\mu^{(2)}},$$

for some positive constants $C_{\alpha,\beta,\gamma}$ *and* $\mu = \mu(\alpha,\beta,\gamma) = (\mu^{(1)}, \mu^{(2)})$. *If there exist* $\mu_N^{(1)}, \mu_N^{(2)} \to \infty$ *such that*

$$\left|a(x,\xi,\eta) - \sum_{j=0}^{N-1} a_j(x,\xi,\eta)\right| \leqslant C_N \cdot (1 + |\xi_1| + |\eta_1|)^{-\mu_N^{(1)}} \cdot (1 + |\xi_2| + |\eta_2|)^{-\mu_N^{(2)}}.$$

Then $a \in BBS^{m_0}_{\rho,\delta}$, *and* $a \sim \sum_{j=0}^{\infty} a_j$.

The next theorem is about the estimate of the distributional kernel of associated bilinear and bi-parameter pseudo-differential operators.

Theorem 1.4. *Let* $p \in BBS^m_{\rho,\delta}, 0 < \rho^{(i)} \leqslant 1, 0 \leqslant \delta^{(i)} < 1,\ i = 1,2,\ m = (m^{(1)}, m^{(2)}) \in \mathbb{R} \times \mathbb{R}$ *and let* $K(x,y,z)$ *denote the distributional kernel of associated bilinear and biparameter pseudo-differential operators* T_p. *Let* $\mathbb{Z}_+$ *denote the set of non-negative integers and for* $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n$, *set*

$$R(x,y,z) = |x-y| + |x-z| + |y-z| \approx |x-y| + |y-z|.$$

(1) Given $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+$, *there exist* $N_0, N_0' \in \mathbb{Z}_+$ *such that for each* $N_1 \geq N_0, N_2 \geq N_0'$

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0 \\ R(x_2,y_2,z_2)>0}} \left|D_x^\alpha D_y^\beta D_z^\gamma K(x,y,z)\right| \leqslant C_{N_1,N_2,p} \cdot R(x_1,y_1,z_1)^{-N_1} \cdot R(x_2,y_2,z_2)^{-N_2};$$

(2) Suppose that p has compact support in (ξ,η) *uniform in x. Then K is smooth, and given* $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+$, *and* $N_1, N_2 \in \mathbb{Z}_+$, *we have*

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0 \\ R(x_2,y_2,z_2)>0}} \left|D_x^\alpha D_y^\beta D_z^\gamma K(x,y,z)\right| \leqslant C_{N_1,N_2,p} \cdot (1 + R(x_1,y_1,z_1))^{-N_1} \cdot (1 + R(x_2,y_2,z_2))^{-N_2};$$

(3) when $m^{(1)} + 2n < 0$, *we have that*

$$|K(x,y,z)| \lesssim \begin{cases} R(x_2,y_2,z_2)^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)} + 2n > 0 \\ \left|log|R(x_2,y_2,z_2)|\right|, & if\ m^{(2)} + 2n = 0; \end{cases}$$

and K is a bounded continuous function when $m^{(2)} + 2n < 0$
(4) when $m^{(1)} + 2n = 0$, *we have that*

$$|K(x,y,z)| \lesssim \begin{cases} \left|log|R(x_1,y_1,z_1)|\right| \cdot |R(x_2,y_2,z_2)|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)} + 2n > 0 \\ \left|log|R(x_1,y_1,z_1)|\right| \cdot \left|log|R(x_2,y_2,z_2)|\right|, & if\ m^{(2)} + 2n = 0; \end{cases}$$

(5) when $m^{(1)} + 2n > 0, m^{(2)} + 2n > 0$, *we have that*

$$|K(x,y,z)| \lesssim |X_1 - Y_1|^{-\frac{m^{(1)}+2n}{\rho^{(1)}}} \cdot |X_2 - Y_2|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}} = |R(x_1,y_1,z_1)|^{-\frac{m^{(1)}+2n}{\rho^{(1)}}} \cdot |R(x_2,y_2,z_2)|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}.$$

Now we give a definition of the bi-paramter Sobolev spaces on $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 1.2. *Let* s_1, s_2 *be two real numbers and let* $1 < p < \infty$. *The inhomogeneous bi-parameter Sobolev space* $L^p_{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^n)$ *is defined as the space of all tempered distributions u in* $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ *with the property that* $((1+|\xi_1|^2)^{\frac{s_1}{2}}(1+|\xi_2|^2)^{\frac{s_2}{2}}\widehat{u})^\wedge$ *is an element of* $L^p(\mathbb{R}^n \times \mathbb{R}^n)$. *For such distributions u we define*

$$\left\|u\right\|_{L^p_{s_1,s_2}} = \left\|((1 + |\xi_1|^2)^{\frac{s_1}{2}}(1 + |\xi_2|^2)^{\frac{s_2}{2}}\widehat{u})^\wedge\right\|_{L^p}$$

where $\widehat{u}(\xi_1,\xi_2) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x_1,x_2) e^{-2\pi i x_1\cdot\xi_1} e^{-2\pi i x_2\cdot\xi_2} dx_1 dx_2.$

Note that the function $(1+|\xi_1|^2)^{\frac{s_1}{2}}(1+|\xi_2|^2)^{\frac{s_2}{2}}$ has at most polynomial growth at infinity.

Definition 1.3. *We say that* $f_1 \in C_c^{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^n)$, *if* $f_1(\xi_1,\cdot) \in C_c^{s_1}(\mathbb{R}^n)$ *and* $f_1(\cdot,\xi_2) \in C_c^{s_2}(\mathbb{R}^n)$.

Definition 1.4. $\dot{L}^p_{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^n)$ *is defined to be the closure of* $C_c^{\infty,\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ *in* $L^p_{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^n)$.

By freezing f_2, $T_p(\cdot, f_2)$ can be regarded as a linear pseudo-differential operator (with a symbol depending on f_2), that is,

$$\begin{aligned} T_p(f_1,f_2)(x) &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\widehat{f_2}(\eta)e^{ix\cdot\eta}d\eta\right)\widehat{f_1}(\xi)e^{ix\cdot\xi}d\xi \\ &\doteq \int_{\mathbb{R}^{2n}} p_2(x,\xi)\widehat{f_1}(\xi)e^{ix\cdot\xi}d\xi \end{aligned}$$

where $p_2(x,\xi) = \int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\widehat{f_2}(\eta)e^{ix\cdot\eta}d\eta$.

Moreover, the well-known L^2 boundedness of a linear pseudo-differential operator states that if $p \in S^0_{\rho,\delta}, 0 \leqslant \delta \leqslant \rho \leqslant 1, \delta < 1$, then there exist constants C_0 and $k \in \mathbb{N}$ (independent of p) such that

$$\left\|T_p(u)\right\|_{L^2} \leqslant C_0\left|p\right|_k\left\|u\right\|_{L^2}, u \in \mathcal{S}(\mathbb{R}^n),$$

where $\left|p\right|_k = \max_{|\alpha|,|\beta|\leqslant k}\sup_{x,\xi}\left|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\right|(1+|\xi|)^{-\delta|\alpha|+\rho|\beta|}$.
In fact, k can be taken equal to $[n/2]+1$, see [11].

Next result is the boundedness of the bi-parameter and bilinear pseudo-differential operators. For simplicity of presentation, we just consider the case when $\delta^{(1)} = \delta^{(2)} = \delta$ and $\rho^{(1)} = \rho^{(2)} = \rho$.

Theorem 1.5. *[Boundedness of pseudo-differential operators] Let* $p \in BBS^0_{\rho,\delta}, 0 \leqslant \delta \leqslant \rho \leqslant 1, \delta < 1$. *Then*

$$T_p : L^2 \times \dot{L}^\infty_{s_1,s_2} \to L^2$$

where s_1, s_2 *are any integers satisfying* $s_1 > \frac{[n/2]+1}{1-\delta} + n$, $s_2 > \frac{[n/2]+1}{1-\delta} + n$. *Moreover, if* $f_2 \in C_c^{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^n)$, *and define* p_2 *as before, then* $p_2 \in S^0_{\rho,\delta}$ *and*

$$\left|p_2\right|_{[n/2]+1} \leqslant \left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}} := \sup_{|\gamma_1|\leqslant s_1,|\gamma_2|\leqslant s_2}\left\|D^{\gamma_1}D^{\gamma_2}f_2\right\|_{L^\infty}$$

with a constant depending only on the $BBS^0_{\rho,\delta}$ *norm of* p *up to order* $n+2$.

Theorem 1.6. *Let* $m \geq 0$ *and* $p \in BBS^m_{\rho,\delta}, 0 \leqslant \delta \leqslant \rho \leqslant 1, \delta < 1$. *Then if* s_1, s_2 *are any integers satisfying* $s_1 > \frac{[n/2]+1}{1-\delta} + n$, $s_2 > \frac{[n/2]+1}{1-\delta} + n$, *the following fractional Leibniz rule type inequality holds true*

$$\left\|T_p(f_1,f_2)\right\|_{L^2} \lesssim \left(\left\|f_1\right\|_{L^2_{2m,0}}\left\|f_2\right\|_{\dot{L}^\infty_{s_1+2m,s_2}} + \left\|f_1\right\|_{\dot{L}^\infty_{s_1+2m,s_2}}\left\|f_2\right\|_{L^2_{2m,0}}\right).$$

We end up this section by mentioning some recent result of Lu and L. Zhang [30] on a Calderón-Vaillancourt type theorem for multi-parameter and multi-linear pseudo-differential operators with minimal smoothness which extends the work of Miyachi and Tomita [33]. We state it in the bi-parameter and bilinear setting.

Theorem 1.7. *Let $m \in \mathbb{R}$, $1 \leqslant p, q, r \leqslant \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

(a) All the operators of class $Op(BBS^{m}_{0,0})$ are bounded in $L^p \times L^q \to L^r$ if

$$m < m(p,q) = -n\left\{\max(\frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r})\right\}.$$

(b) If the operators of class $Op(BBS^{m}_{0,0})$ are bounded in $L^p \times L^q \to L^r$, then we must have

$$m \leqslant m(p,q) = -n\left\{\max(\frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r})\right\}.$$

The organization of the paper is as follows: Section 2 will give the proofs of Theorems 1.1, 1.2, 1.3, namely, the estimates for the symbols of the adjoint operators and asymptotic estimates of the symbols. Section 3 will offer the proof of Theorem 1.4, i.e., the kernel estimates for the bi-parameter and bilinear pseudo-differential operators. In Section 4, we will carry out the proof of the boundedness of the multi-parameter and multilinear pseudo-differential operators, namely Theorems 1.5 and 1.6.

2 Proofs of Theorem 1.1, Theorem 1.2, Theorem 1.3

We start with the proof of Theorem 1.1. As in the cases dealing with the linear or bilinear pseudo-differential operators, the main techniques are using appropriate integration by parts (see e.g., [25], [2]). Nevertheless, the bi-parameter nature makes these more involved and complicated.

Proof. From Lemma1.3, it suffices to show

$$a(x,\xi,\eta) = \int\int C(x+y, z+\xi, \eta)e^{-iz\cdot y}dydz \in BBS^{m}_{\rho,\delta},$$

$$i.e. \quad \left|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\gamma_1}_{\eta_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_2}_{\eta_2}a(x,\xi,\eta)\right|$$
$$\leqslant C_{\alpha,\beta,\gamma}(1+|\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2)$.

First we show the above inequality is true for $\alpha = \beta = \gamma = 0$. Fix $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and set

$$E \doteq 1 + |\xi_1| + |\eta_1|, F \doteq 1 + |\xi_2| + |\eta_2|.$$

We will prove that $|a(x,\xi,\eta)| \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}}$ with a constant independent of the support of p.

Let $l_1 \in \mathbb{N}, l_2 \in \mathbb{N}, 2l_1 > n, 2l_2 > n$. We have

$$\begin{aligned}
a(x,\xi,\eta) &= \int\int C(x+y, z+\xi, \eta)e^{-iz\cdot y}dydz \\
&= \int\int C(x+y, z+\xi, \eta)(1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}(e^{-iz_1\cdot y_1}) \\
&\cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \cdot (1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2}(e^{-iz_2\cdot y_2})dydz \\
&= \int\int \frac{(1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1} \cdot (1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2} \cdot C(x+y, z+\xi, \eta)}{(1+E^{2\delta^{(1)}}|y_1|^2)^{l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{l_2}}e^{-iz\cdot y}dydz
\end{aligned}$$

$$\doteq \int\int q(x,y,z,\xi,\eta)e^{-iz\cdot y}dydz,$$

where $$q(x,y,z,\xi,\eta) = \frac{(1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}\cdot(1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2}\cdot C(x+y,z+\xi,\eta)}{(1+E^{2\delta^{(1)}}|y_1|^2)^{l_1}\cdot(1+F^{2\delta^{(2)}}|y_2|^2)^{l_2}}.$$

Now we want to estimate $(-\triangle_{y_1})^{L_1}\cdot(-\triangle_{y_2})^{L_2}\cdot q$ for $L_1, L_2 \in \mathbb{N}$. Let $\alpha = (\alpha^1,\alpha^2,...,\alpha^n), \beta = (\beta^1,\beta^2,...,\beta^n)$. We have

$$\begin{aligned}
&(-\triangle_{y_1})^{L_1}\cdot(-\triangle_{y_2})^{L_2}\cdot q\\
&=\sum_{\substack{|\alpha|=2L_1\\ \alpha^i even}}\sum_{\substack{|\beta|=2L_2\\ \beta^i even}} C_{\alpha,\beta}\partial_{y_1}^{\alpha}\partial_{y_2}^{\beta}\left[\frac{(1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}\cdot(1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2}\cdot C(x+y,z+\xi,\eta)}{(1+E^{2\delta^{(1)}}|y_1|^2)^{l_1}\cdot(1+F^{2\delta^{(2)}}|y_2|^2)^{l_2}}\right]\\
&=\sum_{\substack{|\alpha|=2L_1\\ \alpha^i even}}\sum_{\substack{|\beta|=2L_2\\ \beta^i even}}\sum_{\beta_1\leqslant\beta} C_{\alpha,\beta}C_{\alpha_1,\alpha}C_{\beta_1,\beta}\partial_{y_1}^{\alpha_1}(1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1}\cdot\partial_{y_2}^{\beta_1}(1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2}\\
&\quad\cdot\partial_{y_1}^{\alpha-\alpha_1}\partial_{y_2}^{\beta-\beta_1}\left[(1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}\cdot(1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2}\cdot C(x+y,z+\xi,\eta)\right]
\end{aligned}$$

Set $P_{l_k} = \{\gamma = (\gamma^1,\gamma^2,...,\gamma^n) : \gamma^i$ even and $|\gamma| = 2j, j = 0,1,2,...,l_k\}, k = 1,2$, then we have

$$\begin{aligned}
&((1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}\cdot(1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2})C(x+y,z+\xi,\eta)\\
&=\sum_{\gamma_1\in P_{l_1}}\sum_{\gamma_2\in P_{l_2}} C\gamma_1,\gamma_2 E^{\delta^{(1)}\cdot|\gamma_1|}\cdot F^{\delta^{(2)}\cdot|\gamma_2|}\cdot\partial_{\xi_1}^{\gamma_1}\cdot\partial_{\xi_2}^{\gamma_2}C(x+y,z+\xi,\eta).
\end{aligned}$$

Using the estimate

$$\left|\partial_{y_1}^{\alpha_1}(1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1}\right| \leqslant C_{\alpha_1,l_1}E^{\delta^{(1)}\cdot|\alpha_1|}\Big(1+E^{2\delta^{(1)}}|y_1|^2\Big)^{-l_1}$$

and

$$\left|\partial_{y_2}^{\beta_1}(1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2}\right| \leqslant C_{\beta_1,l_2}F^{\delta^{(2)}\cdot|\beta_1|}\Big(1+F^{2\delta^{(2)}}|y_2|^2\Big)^{-l_2},$$

we can get

$$\begin{aligned}
&\left|\partial_{y_1}^{\alpha-\alpha_1}\partial_{y_2}^{\beta-\beta_1}\left[(1+E^{2\delta^{(1)}}(-\triangle_{z_1}))^{l_1}\cdot(1+F^{2\delta^{(2)}}(-\triangle_{z_2}))^{l_2}\cdot C(x+y,z+\xi,\eta)\right]\right|\\
&=\sum_{\gamma_1\in P_{l_1}}\sum_{\gamma_2\in P_{l_2}} C\gamma_1,\gamma_2E^{\delta^{(1)}\cdot|\gamma_1|}\cdot F^{\delta^{(2)}\cdot|\gamma_2|}\cdot\left|\partial_{y_1}^{\alpha-\alpha_1}\partial_{y_2}^{\beta-\beta_1}\partial_{\xi_1}^{\gamma_1}\cdot\partial_{\xi_2}^{\gamma_2}C(x+y,z+\xi,\eta)\right|\\
&\leqslant\sum_{\gamma_1\in P_{l_1}}\sum_{\gamma_2\in P_{l_2}} C\gamma_1,\gamma_2E^{\delta^{(1)}\cdot|\gamma_1|}\cdot F^{\delta^{(2)}\cdot|\gamma_2|}\cdot(1+|z_1+\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}(|\alpha|-|\alpha_1|)-\rho^{(1)}(|\gamma_1|)}\\
&\quad\cdot(1+|z_2+\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}(|\beta|-|\beta_1|)-\rho^{(2)}(|\gamma_2|)}.
\end{aligned}$$

With the above estimates, we obtain

$$\begin{aligned}
&\left|(-\triangle_{y_1})^{L_1}\cdot(-\triangle_{y_2})^{L_2}\cdot q\right|\\
&\leqslant\sum_{\substack{|\alpha|=2L_1\\ \alpha^i even}}\sum_{\alpha_1\leqslant\alpha}\sum_{\substack{|\beta|=2L_2\\ \beta^i even}}\sum_{\beta_1\leqslant\beta} C_{\alpha_1,l_1}E^{\delta^{(1)}\cdot|\alpha_1|}(1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1}\cdot C_{\beta_1,l_2}F^{\delta^{(2)}\cdot|\beta_1|}\\
&\quad(1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2}C_{\alpha,\beta}C_{\alpha_1,\alpha}C_{\beta_1,\beta}\sum_{\gamma_1\in P_{l_1}}\sum_{\gamma_2\in P_{l_2}} C\gamma_1,\gamma_2E^{\delta^{(1)}\cdot|\gamma_1|}\cdot F^{\delta^{(2)}\cdot|\gamma_2|}
\end{aligned}$$

$$
\begin{aligned}
&\cdot (1+|z_1+\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}(|\alpha|-|\alpha_1|)-\rho^{(1)}(|\gamma_1|)} \cdot (1+|z_2+\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}(|\beta|-|\beta_1|)-\rho^{(2)}(|\gamma_2|)} \\
&\lesssim (1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\substack{|\alpha|=2L_1 \\ \alpha^i even}} \sum_{\alpha_1 \leqslant \alpha} \sum_{\substack{|\beta|=2L_2 \\ \beta^i even}} \sum_{\beta_1 \leqslant \beta} E^{\delta^{(1)}|\alpha_1|} F^{\delta^{(2)}\cdot|\beta_1|} \\
&\sum_{\gamma_1 \in P_{l_1}} \sum_{\gamma_2 \in P_{l_2}} E^{\delta^{(1)}\cdot|\gamma_1|} \cdot F^{\delta^{(2)}\cdot|\gamma_2|} \cdot (1+|z_1+\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}(|\alpha|-|\alpha_1|)-\rho^{(1)}(|\gamma_1|)} \\
&\cdot (1+|z_2+\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}(|\beta|-|\beta_1|)-\rho^{(2)}(|\gamma_2|)}. \qquad (2.4)
\end{aligned}
$$

Recall
$$a(x,\xi,\eta) = \int\int q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz.$$

Define the sets

$$
\begin{aligned}
\Gamma_1 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}, |z_2| \leqslant \frac{F^{\delta^{(2)}}}{2} \right\}, \\
\Gamma_2 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}, \frac{F^{\delta^{(2)}}}{2} \leqslant |z_2| \leqslant F \right\}, \\
\Gamma_3 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}, |z_2| \leqslant F \right\}, \\
\Gamma_4 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{E^{\delta^{(1)}}}{2} \leqslant |z_1| \leqslant E, |z_2| \leqslant \frac{F^{\delta^{(2)}}}{2} \right\}, \\
\Gamma_5 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{E^{\delta^{(1)}}}{2} \leqslant |z_1| \leqslant E, \frac{F^{\delta^{(2)}}}{2} \leqslant |z_2| \leqslant F \right\}, \\
\Gamma_6 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{E^{\delta^{(1)}}}{2} \leqslant |z_1| \leqslant E, |z_2| \geq F \right\}, \\
\Gamma_7 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \geq E, |z_2| \leqslant \frac{F^{\delta^{(2)}}}{2} \right\}, \\
\Gamma_8 &= \left\{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \geq E, \frac{F^{\delta^{(2)}}}{2} \leqslant |z_2| \leqslant F \right\}, \\
\Gamma_9 &= \{ z=(z_1,z_2) \in \mathbb{R}^n \times \mathbb{R}^n : |z_1| \geq E, |z_2| \geq F \}.
\end{aligned}
$$

Then we have
$$
\begin{aligned}
a(x,\xi,\eta) &= \int\int q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz \\
&= \left(\int_{\Gamma_1}\int_y + \int_{\Gamma_2}\int_y + \ldots + \int_{\Gamma_9}\int_y \right) q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz \\
&=: I_1 + I_2 + \ldots + I_9.
\end{aligned}
$$

We note

In $\Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \cup \Gamma_5$: $\frac{E}{2} \leqslant 1+|z_1+\xi_1|+|\eta_1| \leqslant \frac{3E}{2}, \frac{F}{2} \leqslant 1+|z_2+\xi_2|+|\eta_2| \leqslant \frac{3F}{2}$,

In $\Gamma_3 \cup \Gamma_6$: $\frac{E}{2} \leqslant 1+|z_1+\xi_1|+|\eta_1| \leqslant \frac{3E}{2}, 1+|z_2+\xi_2|+|\eta_2| \leqslant F+|z_2| \leqslant 3|z_2|$,

In $\Gamma_7 \cup \Gamma_8$: $1+|z_1+\xi_1|+|\eta_1| \leqslant E+|z_1| \leqslant 3|z_1|, \frac{F}{2} \leqslant 1+|z_2+\xi_2|+|\eta_2| \leqslant \frac{3F}{2}$, In $\Gamma_9 : 1 + |z_1 + \xi_1| + |\eta_1| \leqslant E + |z_1| \leqslant 3|z_1|, 1 + |z_2 + \xi_2| + |\eta_2| \leqslant F + |z_2| \leqslant 3|z_2|$.

We start the estimation for I_1. Using the estimation (2.4) with $L_1 = L_2 = 0$, $\delta - \rho \leqslant 0$ and $z \in \Gamma_1$, we have

$$\begin{aligned}|q| &\lesssim (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\gamma_1 \in P_{l_1}} \sum_{\gamma_2 \in P_{l_2}} E^{\delta^{(1)}\cdot|\gamma_1|} \cdot F^{\delta^{(2)}\cdot|\gamma_2|} \\ &\quad \cdot (1 + |z_1 + \xi_1| + |\eta_1|)^{m^{(1)} - \rho^{(1)}(|\gamma_1|)} \cdot (1 + |z_2 + \xi_2| + |\eta_2|)^{m^{(2)} - \rho^{(2)}(|\gamma_2|)} \\ &\lesssim (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\gamma_1 \in P_{l_1}} \sum_{\gamma_2 \in P_{l_2}} E^{m^{(1)} + (\delta^{(1)} - \rho^{(1)})|\gamma_1|} \cdot F^{m^{(2)} + (\delta^{(2)} - \rho^{(2)})|\gamma_2|} \\ &\lesssim (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \cdot E^{m^{(1)}} \cdot F^{m^{(2)}}.\end{aligned}$$

So

$$\begin{aligned}|I_1| &= \left| \int_{\Gamma_1} \int_y q(x, y, z, \xi, \eta) e^{-iz\cdot y} dydz \right| \leqslant E^{m^{(1)}} \cdot F^{m^{(2)}} \int_{\Gamma_1} \int_y |q(x, y, z, \xi, \eta)| dydz \\ &\lesssim E^{m^{(1)}} \cdot F^{m^{(2)}} \int_{\Gamma_1} \int_y \frac{1}{(1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2}} dydz \\ &\lesssim E^{m^{(1)}} \cdot F^{m^{(2)}} \int_{|z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}} \int_{|z_2| \leqslant \frac{F^{\delta^{(2)}}}{2}} E^{-\delta^{(1)} n} \cdot F^{-\delta^{(2)} n} dz_1 dz_2 \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}};\end{aligned}$$

To estimate I_2, using integration by parts gives

$$\begin{aligned}\int_y q(x, y, z, \xi, \eta) e^{-iz\cdot y} dy &= \frac{1}{|z_2|^{2l_2}} \int_y q(x, y, z, \xi, \eta)(- \triangle_{y_2})^{l_2} e^{-iz_2\cdot y_2} e^{-iz_1\cdot y_1} dy \\ &= \frac{1}{|z_2|^{2l_2}} \int_y (- \triangle_{y_2})^{l_2} q(x, y, z, \xi, \eta) e^{-iz\cdot y} dy.\end{aligned}$$

Using the estimate (2.4) with $L_1 = 0, L_2 = l_2, \delta^{(i)} - \rho^{(i)} \leqslant 0, i = 1, 2$ and $z \in \Gamma_2$, we have

$$\begin{aligned}&\left|(-\triangle_{y_2})^{l_2} \cdot q\right| \lesssim (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\substack{|\beta| = 2l_2 \\ \beta^i\, even}} \sum_{\beta_1 \leqslant \beta} F^{\delta^{(2)}\cdot|\beta_1|} \sum_{\gamma_1 \in P_{l_1}} \sum_{\gamma_2 \in P_{l_2}} \\ &E^{\delta^{(1)}\cdot|\gamma_1|} \cdot F^{\delta^{(2)}\cdot|\gamma_2|} \cdot (1 + |z_1 + \xi_1| + |\eta_1|)^{m^{(1)} - \rho^{(1)}(|\gamma_1|)} \cdot (1 + |z_2 + \xi_2| + |\eta_2|)^{m^{(2)} + \delta^{(2)}(|\beta| - |\beta_1|) - \rho^{(2)}(|\gamma_2|)} \\ &\lesssim (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\gamma_1 \in P_{l_1}} E^{m^{(1)} + (\delta^{(1)} - \rho^{(1)})|\gamma_1|} \sum_{\substack{|\beta| = 2l_2 \\ \beta^i\, even}} \sum_{\beta_1 \leqslant \beta} \sum_{\gamma_2 \in P_{l_2}} \\ &\quad F^{\delta^{(2)}\cdot|\beta_1|} \cdot F^{\delta^{(2)}\cdot|\gamma_2|} \cdot F^{m^{(2)} + \delta^{(2)}(|\beta| - |\beta_1|) - \rho^{(2)}(|\gamma_2|)} \\ &= (1 + E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\gamma_1 \in P_{l_1}} E^{m^{(1)} + (\delta^{(1)} - \rho^{(1)})|\gamma_1|} \\ &\quad \sum_{\substack{|\beta| = 2l_2 \\ \beta^i\, even}} \sum_{\beta_1 \leqslant \beta} \sum_{\gamma_2 \in P_{l_2}} F^{m^{(2)} + \delta^{(2)}|\beta| + (\delta^{(2)} - \rho^{(2)})|\gamma_2|} \lesssim \frac{E^{m^{(1)}} \cdot F^{m^{(2)} + 2l_2\delta^{(2)}}}{(1 + E^{2\delta^{(1)}}|y_1|^2)^{l_1} \cdot (1 + F^{2\delta^{(2)}}|y_2|^2)^{l_2}}.\end{aligned}$$

Recalling that $2l_1 > n, 2l_2 > n$, we get

$$|I_2| = \left| \int_{\Gamma_2} \int_y q(x, y, z, \xi, \eta) e^{-iz\cdot y} dydz \right|$$

$$
\begin{aligned}
&= \Big| \int_{\Gamma_2} \frac{1}{|z_2|^{2l_2}} \int_y (-\triangle_{y_2})^{l_2} q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz \Big| \\
&\lesssim \int_{\Gamma_2} \frac{1}{|z_2|^{2l_2}} \int_y \frac{E^{m^{(1)}} \cdot F^{m^{(2)}+2l_2\delta^{(2)}}}{(1+E^{2\delta^{(1)}}|y_1|^2)^{l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{l_2}} dydz| \\
&\lesssim E^{m^{(1)}} \cdot E^{-\delta^{(1)}n} \int_{|z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}} dz_1 \cdot F^{m^{(2)}+2l_2\cdot\delta^{(2)}-\delta^{(2)}n} \int_{|z_2| \geq \frac{F^{\delta^{(2)}}}{2}} |z_2|^{-l_2} dz_2 \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}};
\end{aligned}
$$

We now give the estimation for I_3.
Let $l \in \mathbb{N}$ be chosen later, using again the integration by parts gives

$$
\int_y q(x,y,z,\xi,\eta) e^{-iz\cdot y} dy \quad = \quad \frac{1}{|z_2|^{2l}} \int_y (-\triangle_{y_2})^l q(x,y,z,\xi,\eta) e^{-iz\cdot y} dy.
$$

Using the estimation (2.4), defining $m_+ = max(0,m)$, then for $z \in \Gamma_3$, we get

$$
\begin{aligned}
&\Big|(-\triangle_{y_2})^l \cdot q\Big| \lesssim (1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\substack{|\beta|=2l \\ \beta^i even}} \sum_{\beta_1 \leqslant \beta} F^{\delta^{(2)}\cdot|\beta_1|} \sum_{\gamma_1 \in P_{l_1}} \sum_{\gamma_2 \in P_{l_2}} \\
&E^{\delta^{(1)}\cdot|\gamma_1|} \cdot F^{\delta^{(2)}\cdot|\gamma_2|} \cdot (1+|z_1+\xi_1|+|\eta_1|)^{m^{(1)}-\rho^{(1)}(|\gamma_1|)} \cdot (1+|z_2+\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}(|\beta|-|\beta_1|)-\rho^{(2)}(|\gamma_2|)} \\
&\lesssim (1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2} \sum_{\gamma_1 \in P_{l_1}} E^{m^{(1)}+(\delta^{(1)}-\rho^{(1)})|\gamma_1|} \sum_{\substack{|\beta|=2l \\ \beta^i even}} \sum_{\beta_1 \leqslant \beta} \sum_{\gamma_2 \in P_{l_2}} \\
&\quad \cdot |z_2|^{\delta^{(2)}\cdot|\beta_1|+|\gamma_2|} \cdot |z_2|^{m_+^{(2)}+\delta^{(2)}\cdot(|\beta|-|\beta_1|)} \\
&\lesssim (1+E^{2\delta^{(1)}}|y_1|^2)^{-l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{-l_2} E^{m^{(1)}} \cdot |z_2|^{m_+^{(2)}+\delta^{(2)}\cdot(2l+2l_2)};
\end{aligned}
$$

Recalling that $2l_1 > n, 2l_2 > n$, we can derive

$$
\begin{aligned}
|I_3| &= \Big| \int_{\Gamma_3} \int_y q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz \Big| = \Big| \int_{\Gamma_3} \frac{1}{|z_2|^{2l}} \int_y (-\triangle_{y_2})^l q(x,y,z,\xi,\eta) e^{-iz\cdot y} dydz \Big| \\
&\lesssim \Big| \int_{\Gamma_3} \frac{1}{|z_2|^{2l_2}} \int_y \frac{E^{m^{(1)}} \cdot |z_2|^{m_+^{(2)}+\delta^{(2)}\cdot(2l+2l_2)}}{(1+E^{2\delta^{(1)}}|y_1|^2)^{l_1} \cdot (1+F^{2\delta^{(2)}}|y_2|^2)^{l_2}} dydz \Big| \\
&\lesssim E^{m^{(1)}} \cdot E^{-\delta^{(1)}n} \cdot F^{-\delta^{(2)}n} \int_{|z_1| \leqslant \frac{E^{\delta^{(1)}}}{2}} dz_1 \cdot \int_{|z_2| \geq \frac{F}{2}} |z_2|^{m_+^{(2)}+2l_2\delta^{(2)}+2l(\delta^{(2)}-1)} dz_2 \\
&\lesssim E^{m^{(1)}} \cdot F^{-\delta^{(2)}n} \int_{|z_2| \geq \frac{F}{2}} |z_2|^{m_+^{(2)}+2l_2\delta^{(2)}+2l(\delta^{(2)}-1)} dz_2;
\end{aligned}
$$

For $0 \leqslant \delta < 1$, we can choose $l \in \mathbb{N}$ so that $m_+^{(2)} + 2l_2\delta^{(2)} + 2l(\delta^{(2)} - 1) < -n$ and $-\delta^{(2)}n + m_+^{(2)} + 2l_2\delta^{(2)} + 2l(\delta^{(2)} - 1) + n < m^{(2)}$.

Finally, we can conclude that $|I_3| \lesssim E^{m^{(1)}} \cdot F^{-\delta^{(2)}n+m_+^{(2)}+2l_2\delta^{(2)}+2l(\delta^{(2)}-1)+n} \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}}$.

Similarly, we can establish the same estimate for $I_4, I_5, ..., I_9$, i.e. we have $|I_j| \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}}, \forall j = 1,2,...,9$. Therefore,

$$
|a(x,\xi,\eta)| \lesssim E^{m^{(1)}} \cdot F^{m^{(2)}} = (1+|\xi_1|+|\eta_1|)^{m^{(1)}} \cdot (1+|\xi_2|+|\eta_2|)^{m^{(2)}}.
$$

Combining all the above estimates, we obtain that when

$$
|f_2(x,\xi,\eta)| \lesssim (1+|\xi_1|+|\eta_1|)^{M_1} \cdot (1+|\xi_2|+|\eta_2|)^{M_2}, \forall M_1, M_2 \in \mathbb{R},
$$

we have

$$|G(x,\xi,\eta)| = \left| \int\int f_2(x+y, z+\xi, \eta) e^{-iz\cdot y} dydz \right| \lesssim (1+|\xi_1|+|\eta_1|)^{M_1} \cdot (1+|\xi_2|+|\eta_2|)^{M_2}.$$

Note

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} a(x,\xi,\eta) = \int\int \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} C(x+y, z+\xi, \eta) e^{-iz\cdot y} dydz.$$

Let

$$f_2(x,\xi,\eta) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} C(x+y, z+\xi, \eta);$$

$$G(x,\xi,\eta) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} a(x,\xi,\eta);$$

$$M_1 = m^{(1)} + \delta^{(1)}|\alpha_1| - \rho^{(1)}(|\beta_1|+|\gamma_1|), M_2 = m^{(2)} + \delta^{(2)}|\alpha_2| - \rho^{(2)}(|\beta_2|+|\gamma_2|).$$

From the estimate

$$\begin{aligned} f_2(x,\xi,\eta) &= |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} C(x+y, z+\xi, \eta)| \\ &\lesssim (1+|\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)} \\ &\lesssim (1+|\xi_1|+|\eta_1|)^{M_1} \cdot (1+|\xi_2|+|\eta_2|)^{M_2}, \end{aligned}$$

we have

$$\begin{aligned} &\left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} a(x,\xi,\eta)\right| = |G(x,\xi,\eta)| \lesssim (1+|\xi_1|+|\eta_1|)^{M_1} \cdot (1+|\xi_2|+|\eta_2|)^{M_2} \\ &\lesssim (1+|\xi_1|+|\eta_1|)^{m^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}. \end{aligned}$$

This shows $a(x,\xi,\eta) \in BBS^m_{\rho,\delta}$. Similarly, we can show $T_p^{*2} = T_{p^{*2}}$ with $p^{*2} \in BBS^m_{\rho,\delta}$. □

We now prove Theorem 1.2.

Proof. First we will construct the symbol $a(x,\xi,\eta)$. Let $(\xi,\eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \xi = (\xi_1,\xi_2), \eta = (\eta_1,\eta_2)$. Set

$$\begin{aligned} \Gamma_1 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \leqslant 1, |\xi_2|+|\eta_2| \leqslant 1\}, \\ \Gamma_2 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \leqslant 1, 1 \leqslant |\xi_2|+|\eta_2| \leqslant 2\}, \\ \Gamma_3 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \leqslant 1, |\xi_2|+|\eta_2| \geq 2\}, \\ \Gamma_4 &= \{(\xi,\eta) : 1 \leqslant |\xi_1|+|\eta_1| \leqslant 2, |\xi_2|+|\eta_2| \leqslant 1\}, \\ \Gamma_5 &= \{(\xi,\eta) : 1 \leqslant |\xi_1|+|\eta_1| \leqslant 2, 1 \leqslant |\xi_2|+|\eta_2| \leqslant 2\}, \\ \Gamma_6 &= \{(\xi,\eta) : 1 \leqslant |\xi_1|+|\eta_1| \leqslant 2, |\xi_2|+|\eta_2| \geq 2\}, \\ \Gamma_7 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \geq 2, |\xi_2|+|\eta_2| \leqslant 1\}, \\ \Gamma_8 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \geq 2, 1 \leqslant |\xi_2|+|\eta_2| \leqslant 2\}, \\ \Gamma_9 &= \{(\xi,\eta) : |\xi_1|+|\eta_1| \geq 2, |\xi_2|+|\eta_2| \geq 2\}. \end{aligned}$$

Let $\psi \in C_c^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ be such that $0 \leqslant \psi \leqslant 1, \psi(\xi,\eta) = 0$ on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, and $\psi(\xi,\eta) = 1$ on $\Gamma_6 \cup \Gamma_7 \cup \Gamma_8 \cup \Gamma_9$.

For

$$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

$$= \{(\xi,\eta) : |\xi_1| + |\eta_1| \leqslant 1\} \cup (\{(\xi,\eta) : |\xi_2| + |\eta_2| \leqslant 1\} \setminus \{(\xi,\eta) : |\xi_1| + |\eta_1| \geq 2\})$$

and

$$\Gamma_6 \cup \Gamma_7 \cup \Gamma_8 \cup \Gamma_9$$
$$= \{(\xi,\eta) : |\xi_1| + |\eta_1| \geq 2\} \cup (\{(\xi,\eta) : |\xi_2| + |\eta_2| \geq 2\} \setminus \{(\xi,\eta) : |\xi_1| + |\eta_1| \leqslant 1\}),$$

we define

$$a(x,\xi,\eta) = \sum_{j=0}^{\infty} \psi(\epsilon_j\xi, \epsilon_j\eta) a_j(x,\xi,\eta), \tag{2.5}$$

where $\epsilon_j = (\epsilon_j^{(1)}, \epsilon_j^{(2)}) \in (0,1)\times(0,1), \epsilon_j^{(1)} \searrow 0, \epsilon_j^{(2)} \searrow 0$ as $j \to \infty$ which we will choose later, and $\epsilon_j\xi = (\epsilon_j^{(1)}\xi_1, \epsilon_j^{(2)}\xi_2), \epsilon_j\eta = (\epsilon_j^{(1)}\eta_1, \epsilon_j^{(2)}\eta_2), \forall j = 0,1,2,...,$.

Fix $\epsilon = (\epsilon^{(1)}, \epsilon^{(2)})$, we have

$$\psi(\epsilon\xi,\epsilon\eta) = 0 \text{ on } E_1 := \left\{(\xi,\eta) : |\xi_1| + |\eta_1| \leqslant \frac{1}{\epsilon^{(1)}}\right\} \bigcup \left(\left\{(\xi,\eta) : |\xi_2| + |\eta_2| \leqslant \frac{1}{\epsilon^{(2)}}\right\} \setminus \left\{(\xi,\eta) : |\xi_1| + |\eta_1| \geq \frac{2}{\epsilon^{(1)}}\right\}\right);$$

$$\psi(\epsilon\xi,\epsilon\eta) = 1 \text{ on } E_2 := \left\{(\xi,\eta) : |\xi_1| + |\eta_1| \geq \frac{2}{\epsilon^{(1)}}\right\} \bigcup \left(\left\{(\xi,\eta) : |\xi_2| + |\eta_2| \geq \frac{2}{\epsilon^{(2)}}\right\} \setminus \left\{(\xi,\eta) : |\xi_1| + |\eta_1| \leqslant \frac{1}{\epsilon^{(1)}}\right\}\right).$$

For all β, γ such that $|\beta| + |\gamma| \geq 1$ and for $(\xi,\eta) \in E_1$ or $(\xi,\eta) \in E_2$, we have

$$\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta) = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_1}^{\gamma_1}\partial_{\eta_2}^{\gamma_2}\psi(\epsilon^{(1)}\xi_1, \epsilon^{(2)}\xi_2, \epsilon^{(1)}\eta_1, \epsilon^{(2)}\eta_2) = 0.$$

Furthermore, for any (ξ,η) we have

$$|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta)| \lesssim (\epsilon^{(1)})^{|\beta_1|+|\gamma_1|}(\epsilon^{(2)})^{|\beta_2|+|\gamma_2|}.$$

Thus, when $(\xi,\eta) \in E_1 \cup E_2$ and $(\beta,\gamma) \neq (0,0)$, we have

$$|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta)| = 0 \leqslant (1 + |\xi_1| + |\eta_1|)^{-(|\beta_1|+|\gamma_1|)} \cdot (1 + |\xi_2| + |\eta_2|)^{-(|\beta_2|+|\gamma_2|)},$$

and for $(\beta,\gamma) = (0,0)$, we have

$$|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta)| \lesssim 1 \lesssim (1 + |\xi_1| + |\eta_1|)^{-(|\beta_1|+|\gamma_1|)} \cdot (1 + |\xi_2| + |\eta_2|)^{-(|\beta_2|+|\gamma_2|)}.$$

Moreover, $\forall(\xi,\eta) \in (E_1 \cup E_2)^c \subseteq \left\{(\xi,\eta) : \frac{1}{\epsilon^{(1)}} \leqslant |\xi_1| + |\eta_1| \leqslant \frac{2}{\epsilon^{(1)}}, \frac{1}{\epsilon^{(2)}} \leqslant |\xi_2| + |\eta_2| \leqslant \frac{2}{\epsilon^{(2)}}\right\}$, we have

$$|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta)| \lesssim (\epsilon^{(1)})^{|\beta_1|+|\gamma_1|}(\epsilon^{(2)})^{|\beta_2|+|\gamma_2|} \lesssim (1 + |\xi_1| + |\eta_1|)^{-(|\beta_1|+|\gamma_1|)} \cdot (1 + |\xi_2| + |\eta_2|)^{-(|\beta_2|+|\gamma_2|)}.$$

Combining the above estimates, we have that for $\forall\beta,\gamma,\xi,\eta$,

$$|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi,\epsilon\eta)| \lesssim (1 + |\xi_1| + |\eta_1|)^{-(|\beta_1|+|\gamma_1|)} \cdot (1 + |\xi_2| + |\eta_2|)^{-(|\beta_2|+|\gamma_2|)}. \tag{2.6}$$

So the family $\{\psi(\epsilon\xi,\epsilon\eta)\}_{0<\epsilon<1}$ represents a bounded set in $BBS^0_{1,0}$.

Based on this estimate and $a_j \in BBS^{m_j}_{\rho,\delta}$, we have

$$\begin{aligned}&|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}a_j(x,\xi,\eta)|\\ &\lesssim (1+|\xi_1|+|\eta_1|)^{m^{(1)}_j+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{m^{(2)}_j+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}.\end{aligned}$$

We will control each of the terms in the sum that defines $a(x,\xi,\eta)$.

By using Leibniz' rule, we immediately obtain

$$\begin{aligned}&\left|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}\psi(\epsilon^{(1)}\xi_1,\epsilon^{(2)}\xi_2,\epsilon^{(1)}\eta_1,\epsilon^{(2)}\eta_2)\cdot a_j(x,\xi,\eta)\right|\\ &=\Big|\sum_{\beta'\leqslant\beta,\gamma'\leqslant\gamma} C_{\beta',\beta,\gamma',\gamma}\partial^{\beta'_1}_{\xi_1}\partial^{\beta'_2}_{\xi_2}\partial^{\gamma'_1}_{\eta_1}\partial^{\gamma'_2}_{\eta_2}\psi(\epsilon^{(1)}\xi_1,\epsilon^{(2)}\xi_2,\epsilon^{(1)}\eta_1,\epsilon^{(2)}\eta_2)\\ &\qquad\cdot\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1-\beta'_1}_{\xi_1}\partial^{\beta_2-\beta'_2}_{\xi_2}\partial^{\gamma_1-\gamma'_1}_{\eta_1}\partial^{\gamma_2-\gamma'_2}_{\eta_2}a_j(x,\xi,\eta)\Big|\\ &\lesssim C_{j,\alpha,\beta,\gamma}\cdot(1+|\xi_1|+|\eta_1|)^{-(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{-(|\beta_2|+|\gamma_2|)}\cdot\\ &\quad(1+|\xi_1|+|\eta_1|)^{m^{(1)}_j+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{m^{(2)}_j+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}.\end{aligned}$$

Let us now select $\epsilon^{(1)}_j,\epsilon^{(2)}_j$ such that $C_{j,\alpha,\beta,\gamma}\cdot\epsilon^{(1)}_j\cdot\epsilon^{(2)}_j\leqslant 2^{-j}$ for all $|\alpha+\beta+\gamma|\leqslant j$, then

$$C_{j,\alpha,\beta,\gamma}\leqslant 2^{-j}\cdot(\epsilon^{(1)}_j)^{-1}\cdot(\epsilon^{(2)}_j)^{-1}\leqslant 2^{-j}\cdot(1+|\xi_1|+|\eta_1|)\cdot(1+|\xi_2|+|\eta_2|).$$

Thus $$\begin{aligned}&\left|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}\psi(\epsilon^{(1)}\xi_1,\epsilon^{(2)}\xi_2,\epsilon^{(1)}\eta_1,\epsilon^{(2)}\eta_2)\cdot a_j(x,\xi,\eta)\right|\\ &\lesssim 2^{-j}\cdot(1+|\xi_1|+|\eta_1|)^{m^{(1)}_j+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{m^{(2)}_j+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}\\ &for\ all\ |\alpha+\beta+\gamma|\leqslant j,\end{aligned}\tag{2.7}$$

Fix x,ξ,η, if the sum defining $a(x,\xi,\eta)$ is infinite, then there exist $\{\epsilon_{j_k}\}\subseteq\{\epsilon_j\}$ such that $\forall k,\psi(\epsilon_{j_k}\xi,\epsilon_{j_k}\eta)\cdot a_{j_k}(x,\xi,\eta)\neq 0$. When

$$(\xi,\eta)\in(E_1)^c\subseteq\left\{(\xi,\eta):|\xi_1|+|\eta_1|\geqslant\frac{1}{\epsilon^1_{j_k}}\right\}\bigcap\left\{(\xi,\eta):|\xi_2|+|\eta_2|\geqslant\frac{1}{\epsilon^2_{j_k}}\right\}$$

we have $|\xi_1|+|\eta_1|\geqslant\frac{1}{\epsilon^1_{j_k}}\longrightarrow\infty$, $|\xi_2|+|\eta_2|\geqslant\frac{1}{\epsilon^2_{j_k}}\longrightarrow\infty$, as $k\longrightarrow\infty$, this is a contradiction to that x,ξ,η were fixed. Thus the sum defining $a(x,\xi,\eta)$ is finite. In particular, we also have $a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$. Fix a triple of multi-indices (α,β,γ) and let $J\in\mathbb{N}$ be such that $|\alpha+\beta+\gamma|\leqslant J$. We split

$$a=\sum_{j=0}^{\infty}\psi(\epsilon_j\xi,\epsilon_j\eta)a_j(x,\xi,\eta):=S_1(a)+S_2(a),$$

where $S_1(a)=\sum\limits_{j=0}^{J-1}\psi(\epsilon_j\xi,\epsilon_j\eta)a_j(x,\xi,\eta)$, $S_2(a)=\sum\limits_{j=J}^{\infty}\psi(\epsilon_j\xi,\epsilon_j\eta)a_j(x,\xi,\eta)$.

Since $\forall j\in 0,1,2,...,J-1,\psi(\epsilon_j\xi,\epsilon_j\eta)\cdot a_j(x,\xi,\eta)\in BBS^{m_j}_{\rho,\delta}\subseteq BBS^{m_0}_{\rho,\delta}$, it is clear to have that $S_1(a)\in BBS^{m_0}_{\rho,\delta}$.

To estimate $S_2(a)$, for all $j\geq J, m_j+1\leqslant m_J+1\leqslant m_0$, using (2.7) we have

$$\begin{aligned}&\left|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}S_2(a)\right|=\Big|\sum_{j=J}^{\infty}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta_1}_{\xi_1}\partial^{\beta_2}_{\xi_2}\partial^{\gamma_1}_{\eta_1}\partial^{\gamma_2}_{\eta_2}(\psi(\epsilon_j\xi,\epsilon_j\eta)a_j(x,\xi,\eta))\Big|\\ &\leqslant\sum_{j=J}^{\infty}2^{-j}\cdot(1+|\xi_1|+|\eta_1|)^{m^{(1)}_j+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)}\cdot(1+|\xi_2|+|\eta_2|)^{m^{(2)}_j+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}\end{aligned}$$

$$\leqslant \Big(\sum_{j=J}^{\infty} 2^{-j}\Big) \cdot (1+|\xi_1|+|\eta_1|)^{m_0^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m_0^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}$$
$$\leqslant (1+|\xi_1|+|\eta_1|)^{m_0^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m_0^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}.$$

This implies $S_2(a) \in BBS^{m_0}_{\rho,\delta}$. Thus, we conclude that $a \in BBS^{m_0}_{\rho,\delta}$.

Next, we consider the asymptotic expansion of a. We first have

$$a - \sum_{j=0}^{N-1} a_j = \sum_{j=0}^{N-1} (\psi(\epsilon_j\xi, \epsilon_j\eta) - 1) \cdot a_j + \sum_{j=N}^{\infty} \psi(\epsilon_j\xi, \epsilon_j\eta) \cdot a_j$$

For the first term, we just need to care about when

$$(\xi, \eta) \in (E_2)^c \subseteq \Big\{(\xi, \eta) : |\xi_1| + |\eta_1| \leqslant \frac{2}{\epsilon^1_{N-1}}, |\xi_2| + |\eta_2| \leqslant \frac{2}{\epsilon^2_{N-1}}\Big\}.$$

From (2.7), we get the desired estimate of this term.

When we consider the second term, as discussed about $S_2(a)$, we can show that

$$\sum_{j=N}^{\infty} \psi(\epsilon_j\xi, \epsilon_j\eta) \cdot a_j \in BBS^{m_N}_{\rho,\delta}.$$

Therefore, $a - \sum_{j=0}^{N-1} a_j \in BBS^{m_N}_{\rho,\delta}$, i.e. $a \sim \sum_{j=0}^{\infty} a_j$.

Finally, when $b \in BBS^{\infty}_{\rho,\delta} = \cup_m BBS^m_{\rho,\delta}$ and $b \sim \sum_{j=0}^{\infty} a_j$, we have that for all $N > 0, a - b = \Big(a - \sum_{j=0}^{N-1} a_j\Big) - \Big(b - \sum_{j=0}^{N-1} a_j\Big) \in BBS^{m_N}_{\rho,\delta}$. Recall that $m_N \searrow -\infty$ as $N \to \infty$, we then have $a - b \in BBS^{-\infty}_{\rho,\delta} = \cap_m BBS^m_{\rho,\delta}$. □

We will now prove Theorem 1.3.

Proof. From Theorem 1.2, $a_j \in BBS^{m_j}_{\rho,\delta}$, and $m_j \searrow -\infty$ as $j \to \infty$, we have that $\exists\, b \in BBS^{m_0}_{\rho,\delta}$ such that $b \sim \sum_{j=0}^{\infty} a_j$, so we just need to show that $a - b \in BBS^{-\infty}_{\rho,\delta}$.

From $\mu_N \searrow \infty$, and $m_N \searrow -\infty$ as $N \to \infty$, we get

$$\Big|a(x,\xi,\eta) - b(x,\xi,\eta)\Big| \leqslant \Big|a(x,\xi,\eta) - \sum_{j=0}^{N-1} a_j(x,\xi,\eta)\Big| + \Big|b(x,\xi,\eta) - \sum_{j=0}^{N-1} a_j(x,\xi,\eta)\Big|$$
$$\leqslant C_N \cdot (1+|\xi_1|+|\eta_1|)^{-\mu^{(1)}_{N-1}} \cdot (1+|\xi_2|+|\eta_2|)^{-\mu^{(2)}_{N-1}}$$
$$+ C'_N \cdot (1+|\xi_1|+|\eta_1|)^{m^{(1)}_N} \cdot (1+|\xi_2|+|\eta_2|)^{m^{(2)}_N}$$
$$\leqslant C_N \cdot (1+|\xi_1|+|\eta_1|)^{-N} \cdot (1+|\xi_2|+|\eta_2|)^{-N}$$

Now we employ the interpolation result which can be found in many standard books, e.g., in [41] to estimate the derivatives of $a - b$.

Choosing compact sets K_1, K_2 such that $K_1 \subseteq K_2^o \subseteq K_2$ and $u \in C_c^2(\mathbb{R}^n)$, then

$$\sum_{|\alpha|=1} \sup_{z\in K_1} \left|D^\alpha u(z)\right| \lesssim \sup_{z\in K_2} \left|u(z)\right| \sum_{|\alpha|\leqslant 2} \sup_{z\in K_2} \left|D^\alpha u(z)\right|.$$

In particular,

$$\left\|\partial u/\partial x_j\right\|_{L^\infty}^2 \lesssim \left\|u\right\|_{L^\infty} \cdot \left\|\partial^2 u/\partial x_j^2\right\|_{L^\infty}, u \in C_c^2(R^n)$$

Let K be a compact set such that $x \in K$, set $K_1 = K \times (0,0) \times (0,0)$ and let K_2 be a compact neighborhood of K_1. Fix $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n, \eta = (\eta_1, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^n, \zeta = (\zeta_1, \zeta_2), \zeta' = (\zeta_1', \zeta_2')$, and define

$$F_{\xi,\eta}(x, \zeta, \zeta') = a(x, \xi + \zeta, \eta + \zeta') - b(x, \xi + \zeta, \eta + \zeta').$$

Then we have

$$\begin{aligned}
&\sup_{x\in K} \left|\nabla_{x,\xi,\eta}(a-b)(x,\xi,\eta)\right|^2 = \sup_{(x,\zeta,\zeta')\in K_1} \left|\nabla_{(x,\zeta,\zeta')} F_{\xi,\eta}(x,\zeta,\zeta')\right|^2 \\
&\lesssim \sup_{(x,\zeta,\zeta')\in K_2} \left|F_{\xi,\eta}(x,\zeta,\zeta')\right| \cdot \sum_{|\alpha|\leqslant 2} \sup_{(x,\zeta,\zeta')\in K_2} \left|D^\alpha_{x,\zeta,\zeta'}(a-b)(x,\xi+\zeta,\eta+\zeta')\right| \\
&\leqslant C_N \sup_{(x,\zeta,\zeta')} (1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{-N} \cdot (1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{-N} \\
&\left\{ \sum_{|\alpha|\leqslant 2} \sup_{(x,\zeta,\zeta')\in K_2} \left|D^\alpha_{x,\zeta,\zeta'} a(x,\xi+\zeta,\eta+\zeta')\right| + \sum_{|\alpha|\leqslant 2} \sup_{(x,\zeta,\zeta')\in K_2} \left|D^\alpha_{x,\zeta,\zeta'} b(x,\xi+\zeta,\eta+\zeta')\right| \right\} \\
&\leqslant C_N \sup_{(x,\zeta,\zeta')} (1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{-N} \cdot (1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{-N} \\
&\Big\{(1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{\mu^{(1)}} \cdot (1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{\mu^{(2)}} \\
&+(1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{m_0^{(1)}+2(\delta^{(1)}-\rho^{(1)})} \cdot (1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{m_0^{(2)}+2(\delta^{(2)}-\rho^{(2)})}\Big\} \\
&\leqslant C_N \sup_{(x,\zeta,\zeta')} (1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{-N} \cdot (1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{-N} \\
&\qquad \cdot(1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'|)^{max(\mu^{(1)}, m_0^{(1)}+2(\delta^{(1)}-\rho^{(1)}))} \\
&\qquad \cdot(1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'|)^{max(\mu^{(2)}, m_0^{(2)}+2(\delta^{(2)}-\rho^{(2)}))};
\end{aligned}$$

Choosing K_2 such that $|\zeta| \leqslant 1/3, |\zeta'| \leqslant 1/3$, then by the triangle inequality we have

$$\begin{aligned}
1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'| &\geq 1 + |\xi_1| - |\zeta_1| + |\eta_1| - |\zeta_1'| \\
&\geq 1 - \frac{2}{3} + |\xi_1| + |\eta_1| \geq \frac{1}{3}(1 + |\xi_1| + |\eta_1|); \\
1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'| &\leqslant 1 + |\xi_1| + |\zeta_1| + |\eta_1| + |\zeta_1'| \\
&\leqslant 1 + \frac{2}{3} + |\xi_1| + |\eta_1| \leqslant 2(1 + |\xi_1| + |\eta_1|); \\
\text{i.e.} \qquad \frac{1}{3}(1 + |\xi_1| + |\eta_1|) &\leqslant 1 + |\xi_1 + \zeta_1| + |\eta_1 + \zeta_1'| \leqslant 2(1 + |\xi_1| + |\eta_1|);
\end{aligned}$$

Similarly, $\frac{1}{3}(1 + |\xi_2| + |\eta_2|) \leqslant 1 + |\xi_2 + \zeta_2| + |\eta_2 + \zeta_2'| \leqslant 2(1 + |\xi_2| + |\eta_2|)$. Thus

$$\sup_{x\in K} |\nabla_{x,\xi,\eta}(a-b)(x,\xi,\eta)|^2$$

$$\leqslant \cdot (1+|\xi_1|+|\eta_1|)^{max(\mu^{(1)},m_0^{(1)}+2(\delta^{(1)}-\rho^{(1)}))} \cdot (1+|\xi_2|+|\eta_2|)^{max(\mu^{(2)},m_0^{(2)}+2(\delta^{(2)}-\rho^{(2)}))}$$
$$\leqslant C_N \sup_{(x,\zeta,\zeta')} 3^N \cdot 3^N \cdot 2^{2max(|\mu|,|m_0|+2(|\delta|+|\rho|))} \cdot (1+|\xi_1|+|\eta_1|)^{-N} \cdot (1+|\xi_2|+|\eta_2|)^{-N}.$$

So, for all $N > 0$,

$$\left|\partial_x \partial_\xi \partial_\eta (a-b)(x,\xi,\eta)\right| \leqslant C_N \sup_{(x,\zeta,\zeta')} 3^N \cdot 3^N \cdot 2^{2max(|\mu|,|m_0|+2(|\delta|+|\rho|))}$$
$$\cdot (1+|\xi_1|+|\eta_1|)^{-N} \cdot (1+|\xi_2|+|\eta_2|)^{-N}.$$

Similarly, we can get the following estimate for all the derivatives of $a-b$

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (a-b)(x,\xi,\eta)\right| \leqslant C_N(\alpha,\beta,\gamma) \cdot (1+|\xi_1|+|\eta_1|)^{-N} \cdot (1+|\xi_2|+|\eta_2|)^{-N}$$

where $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2), \gamma = (\gamma_1,\gamma_2), \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdot \partial_{x_2}^{\alpha_2}, \partial_\xi^\beta = \partial_{\xi_1}^{\beta_1} \cdot \partial_{\xi_2}^{\beta_2}, \partial_\eta^\gamma = \partial_{\eta_1}^{\gamma_1} \cdot \partial_{\eta_2}^{\gamma_2}$.
So $a-b \in BBS_{\rho,\delta}^{-\infty}$.

Next, we will show that $a \in BBS_{\rho,\delta}^{m_0}$, and $a \sim \sum_{j=0}^{\infty} a_j$.

From $b \in BBS_{\rho,\delta}^{m_0}$, we have

$$\left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} b(x,\xi,\eta)\right|$$
$$\leqslant C(1+|\xi_1|+|\eta_1|)^{m_0^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m_0^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}$$

So

$$\left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} a(x,\xi,\eta)\right|$$
$$\leqslant \left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2}(a-b)(x,\xi,\eta)\right| + \left|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\eta_1}^{\gamma_1}\partial_{\xi_2}^{\beta_2}\partial_{\eta_2}^{\gamma_2} b(x,\xi,\eta)\right|$$
$$\leqslant C_N(\alpha,\beta,\gamma)[(1+|\xi_1|+|\eta_1|)^{-N} \cdot (1+|\xi_2|+|\eta_2|)^{-N}$$
$$+(1+|\xi_1|+|\eta_1|)^{m_0^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)} \cdot (1+|\xi_2|+|\eta_2|)^{m_0^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}$$
$$\leqslant C_N(\alpha,\beta,\gamma) \cdot (1+|\xi_1|+|\eta_1|)^{m_0^{(1)}+\delta^{(1)}|\alpha_1|-\rho^{(1)}(|\beta_1|+|\gamma_1|)}$$
$$\cdot (1+|\xi_2|+|\eta_2|)^{m_0^{(2)}+\delta^{(2)}|\alpha_2|-\rho^{(2)}(|\beta_2|+|\gamma_2|)}.$$

Combining with $b \sim \sum_{j=0}^{\infty} a_j$, we obtain $a \in BBS_{\rho,\delta}^{m_0}$, and $a \sim \sum_{j=0}^{\infty} a_j$. □

3 Proof of Theorem 1.4: The kernel estimates

Now we will prove Theorem 1.4. It is more difficult to carry out the kernel estimates in our bi-parameter and bilinear setting than that in the one-parameter and bilinear setting. In the one-parameter and bilinear case, it can be reduced to the linear case and then use the known estimates in the linear case as outlined in [2]. However, in our bi-parameter case, such reduction is not available. Therefore, we will carry out a different argument with a careful analysis here.

Proof. We first note

$$T_p(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(x,\xi,\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

$$= \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(x,\xi,\eta)\left\{\int_{\mathbb{R}^{2n}} f_1(y)e^{-iy\cdot\xi}dy\right\}\left\{\int_{\mathbb{R}^{2n}} f_2(z)e^{-iz\cdot\xi}dz\right\}e^{ix\cdot(\xi+\eta)}d\xi d\eta$$
$$= \int_{\mathbb{R}^{4n}}\left\{\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot e^{i(x_1-y_1)\cdot\xi_1}\cdot e^{i(x_2-y_2)\cdot\xi_2}\cdot e^{i(x_1-z_1)\cdot\eta_1}\cdot e^{i(x_2-z_2)\cdot\eta_2}d\xi d\eta\right\}f_1(y)f_2(z)dydz.$$

Thus the kernel of the operator T_p is

$$K(x,y,z) = \int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot e^{i(x_1-y_1)\cdot\xi_1}\cdot e^{i(x_2-y_2)\cdot\xi_2}\cdot e^{i(x_1-z_1)\cdot\eta_1}\cdot e^{i(x_2-z_2)\cdot\eta_2}d\xi d\eta.$$

Set

$$X_1 = (x_1,x_1), X_2 = (x_2,x_2), Y_1 = (y_1,z_1), Y_2 = (y_2,z_2), \zeta_1 = (\xi_1,\eta_1), \zeta_2 = (\xi_2,\eta_2).$$

Then

$$K(x,y,z) = \int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot e^{i(X_1-Y_1)\cdot\zeta_1}\cdot e^{i(X_2-Y_2)\cdot\zeta_2}d\zeta_1 d\zeta_2.$$

First, we prove (1) of Theorem 1.4. Note

$$K(x,y,z) = |X_1-Y_1|^{-N_1}\cdot|X_2-Y_2|^{-N_2}\cdot\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot\sum_{\substack{|\alpha|=N_1\\|\beta|=N_2}}\partial^\alpha_{\zeta_1}(e^{i(X_1-Y_1)\cdot\zeta_1})\cdot\partial^\beta_{\zeta_2}(e^{i(X_2-Y_2)\cdot\zeta_2})d\zeta_1 d\zeta_2.$$

Thus

$$\begin{aligned}
&\left||X_1-Y_1|^{N_1}\cdot|X_2-Y_2|^{N_2}K(x,y,z)\right|\\
= &\left|\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot\sum_{\substack{|\alpha|=N_1\\|\beta|=N_2}}\partial^\alpha_{\zeta_1}(e^{i(X_1-Y_1)\cdot\zeta_1})\cdot\partial^\beta_{\zeta_2}(e^{i(X_2-Y_2)\cdot\zeta_2})d\zeta_1 d\zeta_2\right|\\
= &\left|\int_{\mathbb{R}^{4n}}\sum_{\substack{|\alpha|=N_1\\|\beta|=N_2}}\partial^\alpha_{\zeta_1}\partial^\beta_{\zeta_2}p(x,\xi,\eta)\cdot e^{i(X_1-Y_1)\cdot\zeta_1}\cdot e^{i(X_2-Y_2)\cdot\zeta_2}d\zeta_1 d\zeta_2\right|\\
\leqslant &\int_{\mathbb{R}^{4n}}\left|\sum_{\substack{|\alpha|=N_1\\|\beta|=N_2}}\partial^\alpha_{\zeta_1}\partial^\beta_{\zeta_2}p(x,\xi,\eta)\right|d\zeta_1 d\zeta_2\\
\leqslant &\int_{\mathbb{R}^{4n}}\left|\sum_{\substack{|\alpha|=N_1\\|\beta|=N_2}}(1+|\xi_1|+|\eta_1|)^{m^{(1)}-\rho^{(1)}(|\alpha|)}(1+|\xi_2|+|\eta_2|)^{m^{(2)}-\rho^{(2)}(|\beta|)}\right|d\xi_1 d\xi_2 d\eta_1 d\eta_2 \lesssim 1,
\end{aligned}$$

when N_1, N_2 are sufficiently large.

Since $|X_1-Y_1| = |(x_1,x_1)-(y_1,z_1)| = |x_1-y_1|+|x_1-z_1| = R(x_1,y_1,z_1)$ and $|X_2-Y_2| = R(x_2,y_2,z_2)$, we have

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0\\R(x_2,y_2,z_2)>0}} R(x_1,y_1,z_1)^{N_1}\cdot R(x_2,y_2,z_2)^{N_2}\cdot|K(x,y,z)| < \infty,$$

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0\\R(x_2,y_2,z_2)>0}} |K(x,y,z)| \lesssim R(x_1,y_1,z_1)^{-N_1}\cdot R(x_2,y_2,z_2)^{-N_2}.$$

Similarly, we can get

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0\\R(x_2,y_2,z_2)>0}} |D^\alpha_x D^\beta_y D^\gamma_z K(x,y,z)| \lesssim R(x_1,y_1,z_1)^{-N_1}\cdot R(x_2,y_2,z_2)^{-N_2}.$$

Next, we prove (2) of Theorem 1.4. Note that

$$(1+|X_1-Y_1|)\cdot e^{i\frac{\zeta_1}{N}\cdot(X_1-Y_1)} = (I+N\partial_{\zeta_1})(e^{i\frac{\zeta_1}{N}\cdot(X_1-Y_1)}).$$

The we can rewrite

$$K(x,y,z) = (1+|X_1-Y_1|)^{-N_1}\cdot(1+|X_2-Y_2|)^{-N_2}\cdot\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)$$
$$\cdot\Big[(I+N\partial_{\zeta_1})(e^{i\frac{\zeta_1}{N_1}\cdot(X_1-Y_1)})\Big]^{N_1}\cdot\Big[(I+N\partial_{\zeta_2})(e^{i\frac{\zeta_2}{N_2}\cdot(X_2-Y_2)})\Big]^{N_2} d\zeta_1 d\zeta_2.$$

Thus,

$$\begin{aligned}
&\Big|(1+|X_1-Y_1|)^{N_1}\cdot(1+|X_2-Y_2|)^{N_2}\cdot K(x,y,z)\Big| \\
= &\Big|\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot\Big[(I+N\partial_{\zeta_1})(e^{i\frac{\zeta_1}{N_1}\cdot(X_1-Y_1)})\Big]^{N_1}\cdot\Big[(I+N\partial_{\zeta_2})(e^{i\frac{\zeta_2}{N_2}\cdot(X_2-Y_2)})\Big]^{N_2} d\zeta_1 d\zeta_2\Big| \\
= &\Big|\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot\Big[(I+N\partial_{\zeta_1})(e^{i\frac{\zeta_1}{N_1}\cdot(X_1-Y_1)})\Big]^{N_1}\cdot\Big[(I+N\partial_{\zeta_2})(e^{i\frac{\zeta_2}{N_2}\cdot(X_2-Y_2)})\Big]^{N_2} d\zeta_1 d\zeta_2\Big| \\
= &\Big|\int_{\mathbb{R}^{4n}} p(x,\xi,\eta)\cdot\sum_{\substack{|\alpha|\leqslant N_1\\|\beta|\leqslant N_2}} C_{|\alpha|,|\beta|,|N_1,N_2}\partial^\alpha_{\zeta_1}(e^{i\zeta_1\cdot(X_1-Y_1)})\partial^\beta_{\zeta_2}(e^{i\zeta_2\cdot(X_2-Y_2)})d\zeta_1 d\zeta_2\Big| \\
= &\Big|\int_{\mathbb{R}^{4n}} \sum_{\substack{|\alpha|\leqslant N_1\\|\beta|\leqslant N_2}} C_{|\alpha|,|\beta|,|N_1,N_2}\partial^\alpha_{\zeta_1}\partial^\beta_{\zeta_2} p(x,\xi,\eta)\cdot e^{i\zeta_1\cdot(X_1-Y_1)}e^{i\zeta_2\cdot(X_2-Y_2)}d\zeta_1 d\zeta_2\Big| \\
\leqslant & C_{N_1,N_2}\int_{supp(p)}\sum_{\substack{|\alpha|\leqslant N_1\\|\beta|\leqslant N_2}} |\partial^\alpha_{\zeta_1}\partial^\beta_{\zeta_2} p(x,\xi,\eta)|d\zeta_1 d\zeta_2 \\
\leqslant & C_{N_1,N_2}\int_{supp(p)}\sum_{\substack{|\alpha|\leqslant N_1\\|\beta|\leqslant N_2}} (1+|\xi_1|+|\eta_1|)^{m^{(1)}-\rho^{(1)}(|\alpha|)}(1+|\xi_2|+|\eta_2|)^{m^{(2)}-\rho^{(2)}(|\beta|)}d\zeta_1 d\zeta_2 \\
\leqslant & C_{N_1,N_2}\cdot|supp(p)|\leqslant C_{N_1,N_2,p}.
\end{aligned}$$

Therefore,

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0\\R(x_2,y_2,z_2)>0}} |K(x,y,z)| \lesssim (1+R(x_1,y_1,z_1))^{-N_1}\cdot(1+R(x_2,y_2,z_2))^{-N_2}.$$

Similarly, we can get

$$\sup_{\substack{(x,y,z):R(x_1,y_1,z_1)>0\\R(x_2,y_2,z_2)>0}} |D^\alpha_x D^\beta_y D^\gamma_z K(x,y,z)| \lesssim (1+R(x_1,y_1,z_1))^{-N_1}\cdot(1+R(x_2,y_2,z_2))^{-N_2}.$$

Now we prove the estimates (3), (4) and (5) of Theorem 1.4. Let $\varphi\in C^\infty_c, \varphi\geq 0,\ supp\ \varphi\in[1/2,1], \int\varphi(t)dt=1$, and set

$$\varphi(\zeta_1,\zeta_2,t,s) = \varphi\Big(\frac{t}{2}-|\zeta_1|^{1-\rho^{(1)}}\Big)\cdot\varphi\Big(\frac{s}{2}-|\zeta_2|^{1-\rho^{(2)}}\Big),$$

for $\zeta_1=(\xi_1,\eta_1), \zeta_2=(\xi_2,\eta_2)\in\mathbb{R}^n\times\mathbb{R}^n$.

According to (1) and (2), it is enough to estimate $K(x,y,z)$ for $|X_1-Y_1|<1$, $|X_2-Y_2|<1$. Assume that $p(x,\xi,\eta)$ vanishes identically for $|\zeta_1|\leqslant 1, |\zeta_2|\leqslant 1$. Note

$$\int_1^\infty\int_1^\infty \varphi(\zeta_1,\zeta_2,t,s)dtds$$

$$= \int_1^\infty \int_1^\infty \varphi\Big(\frac{t}{2} - |\zeta_1|^{1-\rho^{(1)}}\Big) \cdot \varphi\Big(\frac{s}{2} - |\zeta_2|^{1-\rho^{(2)}}\Big) dt ds$$
$$= \Big\{ \int_1^\infty \varphi\Big(\frac{t}{2} - |\zeta_1|^{1-\rho^{(1)}}\Big) dt \Big\} \cdot \Big\{ \int_1^\infty \cdot \varphi\Big(\frac{s}{2} - |\zeta_2|^{1-\rho^{(2)}}\Big) ds \Big\}$$
$$= 4 \cdot \Big\{ \int_{\frac{1}{2}-|\zeta_1|^{1-\rho^{(1)}}}^\infty \varphi(u) du \Big\} \cdot \Big\{ \int_{\frac{1}{2}-|\zeta_2|^{1-\rho^{(2)}}}^\infty \varphi(v) dv \Big\} \overset{supp(\varphi)}{=\!=\!=} 4\Big(\int \varphi(t) dt \Big)^2 = 4,$$

and

$$K(x, y, z) = \int_{\mathbb{R}^{4n}} p(x, \xi, \eta) \cdot e^{i(X_1-Y_1)\cdot\zeta_1} \cdot e^{i(X_2-Y_2)\cdot\zeta_2} d\zeta_1 d\zeta_2.$$

Thus

$$K(x, y, z) = \int_1^\infty \int_1^\infty \int_{\mathbb{R}^{4n}} p(x, \xi, \eta) \cdot e^{i(X_1-Y_1)\cdot\zeta_1} \cdot e^{i(X_2-Y_2)\cdot\zeta_2} \cdot \varphi(\zeta_1, \zeta_2, t, s) d\zeta_1 d\zeta_2 dt ds.$$

Set

$$K(x, y, z, t, s) = 4 \cdot \int_{\mathbb{R}^{4n}} p(x, \xi, \eta) \cdot e^{i(X_1-Y_1)\cdot\zeta_1} \cdot e^{i(X_2-Y_2)\cdot\zeta_2} \cdot \varphi(\zeta_1, \zeta_2, t, s) d\zeta_1 d\zeta_2.$$

Thus $\forall \beta_1, \beta_2 \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} & |X_1 - Y_1|^{-\beta_1} \cdot |X_2 - Y_2|^{-\beta_2} K(x, y, z, t, s) \\ = \quad & \sum_{\alpha_1 \leqslant \beta_1} \sum_{\alpha_2 \leqslant \beta_2} C_{\alpha,\beta} \int_{\mathbb{R}^{4n}} e^{i(X_1-Y_1)\cdot\zeta_1} \cdot e^{i(X_2-Y_2)\cdot\zeta_2} \cdot \partial_{\zeta_1}^{\alpha_1} \partial_{\zeta_2}^{\alpha_2} p(x, \xi, \eta) \cdot \partial_{\zeta_1}^{\beta_1-\alpha_1} \partial_{\zeta_2}^{\beta_2-\alpha_2} \varphi(\zeta_1, \zeta_2, t, s) d\zeta_1 d\zeta_2. \end{aligned}$$

We note that in the support of φ, we have $\zeta_1 \approx t^{\frac{1}{1-\rho}}, \zeta_2 \approx s^{\frac{1}{1-\rho}}$. Thus

$$\begin{aligned} & |\partial_{\zeta_1}^{\beta_1-\alpha_1} \partial_{\zeta_2}^{\beta_2-\alpha_2} \varphi(\zeta_1, \zeta_2, t, s)| \\ \leqslant \quad & C|\zeta_1|^{-\rho^{(1)}(|\beta_1|-|\alpha_1|)} \cdot |\zeta_2|^{-\rho^{(2)}(|\beta_2|-|\alpha_2|)} \\ \leqslant \quad & Ct^{-\frac{\rho^{(1)}}{1-\rho^{(1)}}(|\beta_1|-|\alpha_1|)} \cdot s^{-\frac{\rho^{(2)}}{1-\rho^{(2)}}(|\beta_2|-|\alpha_2|)}. \end{aligned}$$

For $p \in BBS^m_{\rho,\delta}$, we have

$$\begin{aligned} & |\partial_{\zeta_1}^{\alpha_1} \partial_{\zeta_2}^{\alpha_2} p(x, \xi, \eta)| = |\partial_{\xi_1}^{\alpha_{11}} \partial_{\xi_2}^{\alpha_{21}} \partial_{\eta_1}^{\alpha_{12}} \partial_{\eta_2}^{\alpha_{22}} p(x, \xi, \eta)| \\ & \lesssim (1 + |\zeta_1|)^{m^{(1)}-\rho^{(1)}|\alpha_1|} (1 + |\zeta_2|)^{m^{(2)}-\rho^{(2)}|\alpha_2|} \lesssim t^{\frac{m^{(1)}-\rho^{(1)}|\alpha_1|}{1-\rho^{(1)}}} \cdot s^{\frac{m^{(2)}-\rho^{(2)}|\alpha_2|}{1-\rho^{(2)}}}. \end{aligned}$$

Note the volume of the support of $\varphi(\zeta_1, \zeta_2, t, s)$ can be controlled by $t^{\frac{2n}{1-\rho^{(1)}}-1} \cdot s^{\frac{2n}{1-\rho^{(2)}}-1}$. Therefore, we have

$$\begin{aligned} & \Big| |X_1 - Y_1|^{-\beta_1} \cdot |X_2 - Y_2|^{-\beta_2} K(x, y, z, t, s) \Big| \\ & \leqslant Ct^{\frac{m^{(1)}-\rho^{(1)}|\alpha_1|}{1-\rho^{(1)}}} \cdot t^{-\frac{\rho^{(1)}}{1-\rho^{(1)}}(|\beta_1|-|\alpha_1|)} \cdot s^{\frac{m-\rho|\alpha_2|}{1-\rho}} \cdot s^{-\frac{\rho^{(2)}}{1-\rho^{(2)}}(|\beta_2|-|\alpha_2|)} \cdot t^{\frac{2n}{1-\rho^{(1)}}-1} \cdot s^{\frac{2n}{1-\rho^{(2)}}-1} \\ & = Ct^{\frac{m^{(1)}-\rho^{(1)}|\beta_1|}{1-\rho^{(1)}} + \frac{2n}{1-\rho^{(1)}} - 1} \cdot s^{\frac{m^{(2)}-\rho^{(2)}|\beta_2|}{1-\rho^{(2)}} + \frac{2n}{1-\rho^{(2)}} - 1} = Ct^{\frac{m^{(1)}+2n-\rho^{(1)}|\beta_1|}{1-\rho^{(1)}} - 1} \cdot s^{\frac{m^{(2)}+2n-\rho^{(2)}|\beta_2|}{1-\rho^{(2)}} - 1}. \end{aligned}$$

Choosing $\beta = 0$ and $\beta^{(i)} = (N, N) = [(m^{(i)} + 2n)/\rho^{(i)}] + 1, i = 1, 2$, and using $a \leqslant \frac{1}{\frac{1}{a}+\frac{1}{b}} \leqslant b$, if $0 \leqslant a \leqslant b$, we can get

$$|K(x, y, z, s, t)| \leqslant C \cdot \frac{t^{\frac{m^{(1)}+2n}{1-\rho^{(1)}}-1}}{t^{1+\frac{\rho^{(1)}N}{1-\rho^{(1)}}} \cdot |X_1 - Y_1|^N} \cdot \frac{s^{\frac{m^{(2)}+2n}{1-\rho^{(2)}}-1}}{s^{1+\frac{\rho^{(2)}N}{1-\rho^{(2)}}} \cdot |X_2 - Y_2|^N}.$$

This leads to

$$\begin{aligned}|K(x,y,z)| &\leqslant \int_1^\infty\int_1^\infty |K(x,y,z,s,t)|dtds\\ &\lesssim \int_1^\infty\int_1^\infty \frac{t^{\frac{m^{(1)}+2n}{1-\rho^{(1)}}-1}}{t^{1+\frac{\rho^{(1)}N}{1-\rho^{(1)}}}\cdot|X_1-Y_1|^N}\cdot\frac{s^{\frac{m^{(2)}+2n}{1-\rho^{(2)}}-1}}{s^{1+\frac{\rho^{(2)}N}{1-\rho^{(2)}}}\cdot|X_2-Y_2|^N}dtds.\end{aligned}$$

Thus we have the following estimates.

(a) When $m^{(1)}+2n<0$, we have that

$$\begin{aligned}|K(x,y,z)| &\lesssim \begin{cases}|X_2-Y_2|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)}+2n>0\\ \big|log|X_2-Y_2|\big|, & if\ m^{(2)}+2n=0;\end{cases}\\ &= \begin{cases}R(x_2,y_2,z_2)^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)}+2n>0\\ \big|log|R(x_2,y_2,z_2)|\big|, & if\ m^{(2)}+2n=0;\end{cases}\end{aligned}$$

and K is a bounded continuous function when $m^{(2)}+2n<0$

(b) When $m^{(1)}+2n=0$, we have

$$\begin{aligned}|K(x,y,z)| &\lesssim \begin{cases}\big|log|X_1-Y_1|\big|\cdot|X_2-Y_2|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)}+2n>0\\ \big|log|X_1-Y_1|\big|\cdot\big|log|X_2-Y_2|\big|, & if\ m^{(2)}+2n=0;\end{cases}\\ &= \begin{cases}\big|log|R(x_1,y_1,z_1)|\big|\cdot|R(x_2,y_2,z_2)|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}, & if\ m^{(2)}+2n>0\\ \big|log|R(x_1,y_1,z_1)|\big|\cdot\big|log|R(x_2,y_2,z_2)|\big|, & if\ m^{(2)}+2n=0;\end{cases}\end{aligned}$$

(c) When $m^{(1)}+2n>0, m^{(2)}+2n>0$, we have

$$|K(x,y,z)|\lesssim|X_1-Y_1|^{-\frac{m^{(1)}+2n}{\rho^{(1)}}}\cdot|X_2-Y_2|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}=|R(x_1,y_1,z_1)|^{-\frac{m^{(1)}+2n}{\rho^{(1)}}}\cdot|R(x_2,y_2,z_2)|^{-\frac{m^{(2)}+2n}{\rho^{(2)}}}.$$

Using the same techniques, we can prove (3), (4) and (5) of Theorem 1.4 when $M>0$. □

4 Proofs of Theorem 1.5 and Theorem 1.6

The main purpose of this section is to prove Theorem 1.5 and Theorem 1.6. We first give the proof of Theorem 1.5.

Proof. Let $f_2\in C_c^{s_1,s_2}(\mathbb{R}^n\times\mathbb{R}^n)$ and $p\in BBS^0_{\rho,\delta}, 0\leqslant\delta\leqslant\rho\leqslant 1, \delta<1$. Assume p has compact support, we can prove the result with constants independent of the support of p. Thus, the result can be extended to the case that p does not have compact support.

Fix α,β satisfying $|\alpha|,|\beta|\leqslant[n/2]+1, l_1,l_2\in\mathbb{N}$, define $E=1+|\xi_1|, F=1+|\xi_2|$ and

$$P_{l_i}=\{\gamma=(\gamma^1,\gamma^2,...,\gamma^n):\gamma^i \text{ is even and } |\gamma|=2j, j=0,1,2,...,l_i\}, i=1,2.$$

Then

$$p_2(x,\xi)=\int_{\mathbb{R}^{2n}}p(x,\xi,z)\widehat{f_2}(z)e^{ix\cdot z}dz=\int_z p(x,\xi,z)\Big(\int_y f_2(x-y)e^{iz\cdot y}dy\Big)dz.$$

Note that

$$e^{iz\cdot y}=e^{iz_1\cdot y_1}\cdot e^{iz_2\cdot y_2}=(1+E^{2\delta}|y_1|^2)^{-l_1}(1+F^{2\delta}|y_2|^2)^{-l_2}\cdot(1+E^{2\delta}(-\Delta_{z_1})^2)^{l_1}(e^{iz_1\cdot y_1})(1+F^{2\delta}(-\Delta_{z_2})^2)^{l_2}(e^{iz_2\cdot y_2}).$$

We then have

$$\begin{aligned}
&\partial_x^\alpha\partial_\xi^\beta p_2(x,\xi)=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p_2(x_1,x_2,\xi_1,\xi_2)\\
&=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\left[\int_z p(x,\xi,z)\Big(\int_y f_2(x-y)e^{iz\cdot y}dy\Big)dz\right]\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1\\|\gamma_2|\leqslant\alpha_2}}C_{\gamma,\alpha}\int_z\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)\left\{\int_y\partial_{x_1}^{\alpha_1-\gamma_1}\partial_{x_2}^{\alpha_2-\gamma_2}f_2(x_1-y_1,x_2-y_2)e^{iz\cdot y}dy\right\}dz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1\\|\gamma_2|\leqslant\alpha_2}}C_{\gamma,\alpha}\int_z\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)\left\{\int_y\partial_{y_1}^{\alpha_1-\gamma_1}\partial_{y_2}^{\alpha_2-\gamma_2}f_2(x_1-y_1,x_2-y_2)e^{iz\cdot y}dy\right\}dz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1\\|\gamma_2|\leqslant\alpha_2}}C_{\gamma,\alpha}\int_z\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)\left\{\int_y f_2(x-y)\partial_{y_1}^{\alpha_1-\gamma_1}(e^{iz_1\cdot y_1})\partial_{y_2}^{\alpha_2-\gamma_2}(e^{iz_2\cdot y_2})dy\right\}dz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1\\|\gamma_2|\leqslant\alpha_2}}C_{\gamma,\alpha}\int_z\int_y\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)z_1^{\alpha_1-\gamma_1}z_2^{\alpha_2-\gamma_2}f_2(x-y)(1+E^{2\delta}|y_1|^2)^{l_1}\\
&\quad(1+F^{2\delta}|y_2|^2)^{l_2}(1+E^{2\delta}|-\Delta_{z_1}|^2)^{l_1}(e^{iz_1\cdot y_1})(1+F^{2\delta}|-\Delta_{z_2}|^2)^{l_2}(e^{iz_2\cdot y_2})dydz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1\\|\gamma_2|\leqslant\alpha_2}}C\int_z\int_y(1+E^{2\delta}|-\Delta_{z_1}|^2)^{l_1}(1+F^{2\delta}|-\Delta_{z_2}|^2)^{l_2}\\
&\quad\cdot\Big(\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)z_1^{\alpha_1-\gamma_1}z_2^{\alpha_2-\gamma_2}\Big)\frac{e^{iz\cdot y}f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1,|\gamma_2|\leqslant\alpha_2\\|\theta_1|\in P_{l_1},|\theta_2|\in P_{l_2}}}C\int_z\int_y E^{\delta|\theta_1|}F^{\delta|\theta_2|}\partial_{z_1}^{\theta_1}\partial_{z_2}^{\theta_2}\Big(\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}p(x,\xi,z)z_1^{\alpha_1-\gamma_1}z_2^{\alpha_2-\gamma_2}\Big)\\
&\qquad\cdot\frac{e^{iz\cdot y}f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz\\
&=\sum_{\substack{|\gamma_1|\leqslant\alpha_1,|\gamma_2|\leqslant\alpha_2,|\theta_1|\in P_{l_1},|\theta_2|\in P_{l_2}\\\omega_1\leqslant min\{\theta_1,\alpha_1-\gamma_1\},\omega_2\leqslant min\{\theta_2,\alpha_2-\gamma_2\}}}C\int_z\int_y E^{\delta|\theta_1|}F^{\delta|\theta_2|}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{z_1}^{\theta_1-\omega_1}\partial_{z_2}^{\theta_2-\omega_2}\\
&\qquad\cdot\Big(p(x,\xi,z)z_1^{\alpha_1-\gamma_1-\omega_1}z_2^{\alpha_2-\gamma_2-\omega_2}\Big)\frac{e^{iz\cdot y}f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz.
\end{aligned}$$

Fix $|\gamma_1|\leqslant\alpha_1$, $|\gamma_2|\leqslant\alpha_2$, $|\theta_1|\in P_{l_1}$, $|\theta_2|\in P_{l_2}$, $\omega_1\leqslant min\{\theta_1,\alpha_1-\gamma_1\}$, $\omega_2\leqslant min\{\theta_2,\alpha_2-\gamma_2\}$ and set

$$\begin{aligned}
&p(x_1,x_2,\xi_1,\xi_2)\\
=\;&\int_z\int_y E^{\delta|\theta_1|}F^{\delta|\theta_2|}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{z_1}^{\theta_1-\omega_1}\partial_{z_2}^{\theta_2-\omega_2}\Big(p(x,\xi,z)z_1^{\alpha_1-\gamma_1-\omega_1}z_2^{\alpha_2-\gamma_2-\omega_2}\Big)\frac{e^{iz\cdot y}f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz.
\end{aligned}$$

Define the sets $\Gamma_1,\cdots,\Gamma_9$ in the same way as done in Theorem 2.1. We have

$$p(x,\xi)=\Big(\int_{\Gamma_1}\int_y+\int_{\Gamma_2}\int_y+...+\int_{\Gamma_9}\int_y\Big)q(x,y,z,\xi,\eta)e^{-iz\cdot y}dydz=:I_1+I_2+...+I_9.$$

Note

$$\begin{aligned}&\text{In } \Gamma_1\cup\Gamma_2\cup\Gamma_4\cup\Gamma_5 : E \leqslant 1+|\xi_1|+|z_1| \leqslant 2E, F \leqslant 1+|\xi_2|+|z_2| \leqslant 2F,\\ &\text{In } \Gamma_3\cup\Gamma_6 : E \leqslant 1+|\xi_1|+|z_1| \leqslant 2E, |z_2| \leqslant 1+|\xi_2|+|z_2| \leqslant 2|z_2|,\\ &\text{In } \Gamma_7\cup\Gamma_8 : |z_1| \leqslant 1+|\xi_1|+|z_1| \leqslant 2|z_1|, F \leqslant 1+|\xi_2|+|z_2| \leqslant 2F,\\ &\text{In } \Gamma_9 : |z_1| \leqslant 1+|\xi_1|+|z_1| \leqslant 2|z_1|, |z_2| \leqslant 1+|\xi_2|+|z_2| \leqslant 2|z_2|.\end{aligned}$$

We now estimate the integrals.

Estimate for I_1:

Choose l_1, l_2 such that $2l_1 > n, 2l_2 > n$. Then

$$\begin{aligned}|I_1| &= \Big|\int_{\Gamma_1}\int_y E^{\delta|\theta_1|}F^{\delta|\theta_2|}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{z_1}^{\theta_1-\omega_1}\partial_{z_2}^{\theta_2-\omega_2}p(x,\xi,z)z_1^{\alpha_1-\gamma_1-\omega_1}\\ &\qquad z_2^{\alpha_2-\gamma_2-\omega_2}\frac{e^{iz\cdot y}f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz\Big|\\ &\lesssim E^{\delta|\theta_1|}F^{\delta|\theta_2|}\|f_2\|_{L^\infty}\int_{\Gamma_1}\int_y(1+|\xi_1|+|z_1|)^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1-\omega_1|)}\\ &\qquad (1+|\xi_2|+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}\frac{|z_1|^{|\alpha_1|-|\gamma_1|-|\omega_1|}|z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|}}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz\\ &\lesssim E^{\delta|\theta_1|}F^{\delta|\theta_2|}\|f_2\|_{L^\infty}E^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1|-|\omega_1|)}F^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2|-|\omega_2|)}\\ &\qquad E^{\delta(|\alpha_1|-|\gamma_1|-|\omega_1|)}F^{\delta(|\alpha_2|-|\gamma_2|-|\omega_2|)}\int_{\Gamma_1}\int_y\frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}}dydz\\ &\lesssim E^{\delta|\theta_1|}F^{\delta|\theta_2|}\|f_2\|_{L^\infty}E^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1|-|\omega_1|)}F^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2|-|\omega_2|)}\\ &\qquad\cdot E^{\delta(|\alpha_1|-|\gamma_1|-|\omega_1|)}F^{\delta(|\alpha_2|-|\gamma_2|-|\omega_2|)}\\ &= E^{(\delta-\rho)\cdot(|\theta_1|-|\omega_1|)}E^{\delta|\alpha_1|-\rho|\beta_1|}F^{(\delta-\rho)\cdot(|\theta_2|-|\omega_2|)}F^{\delta|\alpha_2|-\rho|\beta_2|}\|f_2\|_{L^\infty}\\ &\leqslant E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\|f_2\|_{L^\infty}.\end{aligned}$$

The last inequality holds for $\delta-\rho<0, |\theta_1|-|\omega_1|>0, |\theta_2|-|\omega_2|>0$. So we obtain

$$|I_1| \lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\|f_2\|_{\dot{L}^\infty_{s_1,s_2}}.$$

Estimate for I_2:

Let $l\in\mathbb{N}$ be chosen later. Then

$$\begin{aligned}&\Big|(-\Delta_{y_2})^l\Big(\frac{f_2(x-y)}{(1+F^{2\delta}|y_2|^2)^{l_2}}\Big)\Big|\\ &=\Big|\sum_{\substack{|\mu|=2l\\ \mu^i\,even,\mu_1\leqslant\mu}}C_{\mu,\mu_1}\partial_{y_\mu 2}^{\mu_1}(1+F^{2\delta}|y_2|^2)^{-l_2}\partial_{y_2}^{\mu-\mu_1}f_2(x-y)\Big|\\ &\leqslant\sum_{\substack{|\mu|=2l\\ \mu^i\,even,\mu_1\leqslant\mu}}C_{\mu,\mu_1,l_2}\left\|D_{y_2}^{\mu-\mu_1}f_2\right\|_{L^\infty}F^{\delta|\mu_1|}(1+F^{2\delta}|y_2|^2)^{-l_2}.\end{aligned}$$

Then we have

$$|I_2| = \Big|\int_{\Gamma_2}\int_y E^{\delta|\theta_1|}F^{\delta|\theta_2|}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{z_1}^{\theta_1-\omega_1}\partial_{z_2}^{\theta_2-\omega_2}p(x,\xi,z)z_1^{\alpha_1-\gamma_1-\omega_1}z_2^{\alpha_2-\gamma_2-\omega_2}$$

$$\cdot \frac{e^{iz\cdot y} f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}} dydz\Big|$$

$$= \Big| \int_{\Gamma_2} \int_y A^{\delta|\theta_1|} F^{\delta|\theta_2|} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \partial_{z_1}^{\theta_1-\omega_1} \partial_{z_2}^{\theta_2-\omega_2} p(x,\xi,z) z_1^{\alpha_1-\gamma_1-\omega_1} z_2^{\alpha_2-\gamma_2-\omega_2}$$

$$\cdot \frac{e^{iz_1\cdot y_1} f_2(x-y)}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}} \cdot \frac{(-\triangle_{y_2})^l (e^{iz_2 \dot{y}_2})}{|z_2|^{2l}} dydz\Big|$$

$$= \Big| \int_{\Gamma_2} E^{\delta|\theta_1|} F^{\delta|\theta_2|} z_1^{\alpha_1-\gamma_1-\omega_1} \frac{z_2^{\alpha_2-\gamma_2-\omega_2}}{|z_2|^{2l}} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \partial_{z_1}^{\theta_1-\omega_1} \partial_{z_2}^{\theta_2-\omega_2} p(x,\xi,z)$$

$$\cdot \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}} (-\triangle_{y_2})^l \Big(\frac{f_2(x-y)}{(1+F^{2\delta}|y_2|^2)^{l_2}} \Big) e^{iz\cdot y} dydz\Big|$$

$$= \int_{\Gamma_2} E^{\delta|\theta_1|} F^{\delta|\theta_2|} \big| z_1^{\alpha_1-\gamma_1-\omega_1} \big| \cdot \frac{\big| z_2^{\alpha_2-\gamma_2-\omega_2} \big|}{|z_2|^{2l}} \cdot \big| \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \partial_{z_1}^{\theta_1-\omega_1} \partial_{z_2}^{\theta_2-\omega_2} p(x,\xi,z) \big|$$

$$\cdot \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}} \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} C_{\mu,\mu_1,l_2} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} F^{\delta|\mu_1|} \frac{1}{(1+F^{2\delta}|y_2|^2)^{l_2}} dydz$$

$$\leqslant \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} C_{\mu,\mu_1,l_2} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} \int_{\Gamma_2} E^{\delta|\theta_1|} F^{\delta|\theta_2|} E^{\delta(\alpha_1-\gamma_1-\omega_1)} \cdot \frac{|z_2^{\alpha_2-\gamma_2-\omega_2}|}{|z_2|^{2l}}$$

$$\cdot (1+|\xi_1|+|z_1|)^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1-\omega_1|)} \cdot (1+|\xi_2|+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}$$

$$\cdot F^{\delta|\mu_1|} \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}} dydz$$

$$= \int_{\Gamma_2} E^{\delta|\theta_1|} F^{\delta|\theta_2|} \big| z_1^{\alpha_1-\gamma_1-\omega_1} \big| \cdot \frac{\big| z_2^{\alpha_2-\gamma_2-\omega_2} \big|}{|z_2|^{2l}} \cdot \big| \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \partial_{z_1}^{\theta_1-\omega_1} \partial_{z_2}^{\theta_2-\omega_2} p(x,\xi,z) \big|$$

$$\cdot \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}} \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} F^{\delta|\mu_1|} \frac{1}{(1+F^{2\delta}|y_2|^2)^{l_2}} dydz$$

$$\leqslant \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} C_{\mu,\mu_1,l_2} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} \int_{\Gamma_2} E^{\delta|\theta_1|} F^{\delta|\theta_2|} E^{\delta(\alpha_1-\gamma_1-\omega_1)} \cdot \frac{\big|z_2^{\alpha_2-\gamma_2-\omega_2}\big|}{|z_2|^{2l}}$$

$$\cdot E^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1-\omega_1|)} (F+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)} F^{\delta|\mu_1|}$$

$$\cdot \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}} dydz$$

$$\leqslant \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} C_{\mu,\mu_1,l_2} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} E^{\delta|\theta_1|} F^{\delta|\theta_2|} E^{\delta(\alpha_1-\gamma_1-\omega_1)} E^{\delta\cdot|\gamma_1|-\rho(|\beta_1|+|\theta_1-\omega_1|)}$$

$$\cdot F^{\delta|\mu_1|} \int_{\Gamma_2} |z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|-2l} (F+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)} dz$$

$$\cdot \int_y \frac{1}{(1+E^{2\delta}|y_1|^2)^{l_1}(1+F^{2\delta}|y_2|^2)^{l_2}} dy$$

$$\leqslant \sum_{\substack{|\mu|=2l \\ \mu^i \, even, \mu_1 \leqslant \mu}} C_{\mu,\mu_1,l_2} \big\| D_{y_2}^{\mu-\mu_1} f_2 \big\|_{L^\infty} E^{(\delta-\rho)\cdot(|\theta_1|-|\omega_1|)} E^{\delta|\alpha_1|-\rho|\beta_1|} F^{\delta(|\theta_2|+|\mu_1|-n)}$$

$$
\begin{aligned}
&\cdot \int_{\frac{F^{\delta}}{2}\leqslant |z_2|\leqslant F} |z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|-2l}(F+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}dz_2\\
\leqslant & \sum_{\substack{|\mu|=2l\\ \mu^i even, \mu_1\leqslant\mu}} C_{\mu,\mu_1,l_2}\left\|D_{y_2}^{\mu-\mu_1} f_2\right\|_{L^\infty} E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta(|\theta_2|+|\mu_1|-n|)}F^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}\\
&\cdot \int_{\frac{F^{\delta}}{2}\leqslant |z_2|\leqslant F} |z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|-2l}dz_2\\
\lesssim & \left\|f_2\right\|_{\dot{L}^\infty_{0,2l}} E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta(|\theta_2|+|\mu_1|-n|)}F^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}F^{\delta(|\alpha_2|-|\gamma_2|-|\omega_2|-2l)}.
\end{aligned}
$$

If l is chosen so that $|\alpha_2|-|\gamma_2|-|\omega_2|-2l+n<n$, we have

$$
\begin{aligned}
|I_2| &\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}\left\|f_2\right\|_{\dot{L}^\infty_{0,2l}} F^{\delta|\alpha_2|-\rho|\beta_2|}F^{(\delta-\rho)(|\theta_2|-|\omega_2|)}\\
&\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{0,2l}}.
\end{aligned}
$$

Choosing $l=s_2/2$, we can get

$$
|I_2|\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{0,s_2}}\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}.
$$

Estimate for I_3:

As in the estimate of I_2, we have

$$
\begin{aligned}
|I_3|\leqslant & \sum_{\substack{|\mu|=2l\\ \mu^i even, \mu_1\leqslant\mu}} C_{\mu,\mu_1,l_2}\left\|D_{y_2}^{\mu-\mu_1} f_2\right\|_{L^\infty} E^{(\delta-\rho)\cdot(|\theta_1|-|\omega_1|)}E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta(|\theta_2|+|\mu_1|-n)}\\
&\cdot \int_{|z_2|\geq F} |z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|-2l}(F+|z_2|)^{\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}dz_2\\
\leqslant & \sum_{\substack{|\mu|=2l\\ \mu^i even, \mu_1\leqslant\mu}} C_{\mu,\mu_1,l_2}\left\|D_{y_2}^{\mu-\mu_1} f_2\right\|_{L^\infty} E^{(\delta-\rho)\cdot(|\theta_1|-|\omega_1|)}E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta(|\theta_2|+2l-n)}\\
&\cdot \int_{|z_2|\geq F} |z_2|^{|\alpha_2|-|\gamma_2|-|\omega_2|-2l+\delta\cdot|\gamma_2|-\rho(|\beta_2|+|\theta_2-\omega_2|)}dz_2.
\end{aligned}
$$

Now, let us choose $l=s_2/2$, we can get

$$
\left|I_3\right|\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}.
$$

Similarly, we can obtain the same estimation for $I_4,...,I_9$, i.e.

$$
\left|I_i\right|\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}, i=1,2,...,9.
$$

So $\left|p(x,\xi)\right|\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}$. Then

$$
\begin{aligned}
\left|p_2(x,\xi)\right| &\lesssim E^{\delta|\alpha_1|-\rho|\beta_1|}F^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}\\
&\lesssim (1+|\xi_1|)^{\delta|\alpha_1|-\rho|\beta_1|}(1+|\xi_2|)^{\delta|\alpha_2|-\rho|\beta_2|}\left\|f_2\right\|_{\dot{L}^\infty_{s_1,s_2}}.
\end{aligned}
$$

So T_p map $L^2\times\dot{L}^\infty_{s_1,s_2}$ into L^2, where $s_1>\frac{[n/2]+1}{1-\delta}+n,\ s_2>\frac{[n/2]+1}{1-\delta}+n.$ □

Finally, we give the proof of Theorem 1.6.

Proof. If $p \in BBS^m_{\rho,\delta}$ and φ is a C^∞ function on $\mathbb{R}$ such that $0 \leqslant \varphi \leqslant 1$, $supp(\varphi) \subseteq [-2, 2]$ and $\varphi(r) + \varphi(\frac{1}{r}) = 1$ on $[0, \infty)$, then we define the symbols p_1 and p_2 by

$$p_1(x,\xi,\eta) := p(x,\xi,\eta)\varphi\Big(\frac{(1+|\xi_1|)(1+|\eta_1|)}{(1+|\xi_2|)(1+|\eta_2|)}\Big)(1+|\xi_1|)^{-2m}(1+|\eta_1|)^{-2m};$$
$$p_2(x,\xi,\eta) := p(x,\xi,\eta)\varphi\Big(\frac{(1+|\xi_2|)(1+|\eta_1|)}{(1+|\xi_1|)(1+|\eta_1|)}\Big)(1+|\xi_2|)^{-2m}(1+|\eta_2|)^{-2m};$$

where $x = (x_1, x_2), \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2)$. We then get that

$$\begin{aligned} c &\leqslant |p(x,\xi,\eta)| \cdot (1+|\xi_1|)^{-2m} \cdot (1+|\eta_1|)^{-2m} \\ &\lesssim (1+|\xi_1|+|\eta_1|)^m(1+|\xi_2|+|\eta_2|)^m(1+|\xi_1|)^{-2m}(1+|\eta_1|)^{-2m} \\ &\lesssim (1+|\xi_1|)^m(1+|\eta_1|)^m(1+|\xi_2|)^m(1+|\eta_2|)^m(1+|\xi_1|)^{-2m}(1+|\eta_1|)^{-2m} \\ &\lesssim (1+|\xi_1|)^m(1+|\eta_1|)^m(1+|\xi_1|)^m(1+|\eta_1|)^m(1+|\xi_1|)^{-2m}(1+|\eta_1|)^{-2m} = 1, \end{aligned}$$

where the last inequality is true because $(1 + |\xi_1|) \cdot (1 + |\eta_1|)$ is equivalent to $(1 + |\xi_2|) \cdot (1 + |\eta_2|)$ on the support of φ. Similarly, we can get $p_2(x, \xi, \eta) \lesssim 1$ and the same estimation for any derivatives of $p_1(x, \xi, \eta), p_2(x, \xi, \eta)$. So $p_1(x, \xi, \eta), p_2(x, \xi, \eta) \in BBS^0_{\rho,\delta}$. Note

$$\begin{aligned} &T_p(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\cdot\Big[\varphi\Big(\frac{(1+|\xi_1|)(1+|\eta_1|)}{(1+|\xi_2|)(1+|\eta_2|)}\Big) \\ &\qquad +\varphi\Big(\frac{(1+|\xi_2|)(1+|\eta_1|)}{(1+|\xi_1|)(1+|\eta_1|)}\Big)\Big]\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\cdot\varphi\Big(\frac{(1+|\xi_1|)(1+|\eta_1|)}{(1+|\xi_2|)(1+|\eta_2|)}\Big)(1+|\xi_1|)^{-2m}(1+|\eta_1|)^{-2m} \\ &\qquad \cdot(1+|\xi_1|)^{2m}(1+|\eta_1|)^{2m}\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &+ \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p(x,\xi,\eta)\cdot\varphi\Big(\frac{(1+|\xi_2|)(1+|\eta_1|)}{(1+|\xi_1|)(1+|\eta_1|)}\Big)(1+|\xi_2|)^{-2m}(1+|\eta_2|)^{-2m} \\ &\qquad \cdot(1+|\xi_2|)^{2m}(1+|\eta_2|)^{2m}\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p_1(x,\xi,\eta)\cdot(1+|\xi_1|)^{2m}\widehat{f_1}(\xi_1,\xi_2)(1+|\eta_1|)^{2m}\widehat{f_2}(\eta_1,\eta_1)e^{ix\cdot(\xi+\eta)}d\xi d\eta \\ &+ \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} p_2(x,\xi,\eta)\cdot(1+|\xi_2|)^{2m}\widehat{f_1}(\xi_1,\xi_2)(1+|\eta_2|)^{2m}\widehat{f_2}(\eta_1,\eta_1)e^{ix\cdot(\xi+\eta)}d\xi d\eta. \end{aligned}$$

By Theorem 1.5, we obtain that

$$\begin{aligned} \big\|T_p(f_1, g)\big\|_{L^2} &\lesssim \big\|(1+|\xi_1|)^{2m}\widehat{f_1}\big\|_{L^2}\big\|(1+|\eta_1|)^{2m}\widehat{f_2}\big\|_{\dot{L}^\infty_{s_1,s_2}} \\ &\qquad + \big\|(1+|\xi_2|)^{2m}\widehat{f_1}\big\|_{\dot{L}^\infty_{s_1,s_2}}\big\|(1+|\eta_2|)^{2m}\widehat{f_2}\big\|_{L^2} \\ &\lesssim \big\|f_1\big\|_{L^2_{2m,0}}\big\|f_2\big\|_{\dot{L}^\infty_{s_1+2m,s_2}} + \big\|f_1\big\|_{\dot{L}^\infty_{s_1+2m,s_2}}\big\|f_2\big\|_{L^2_{2m,0}}. \end{aligned}$$

Hence, the proof is complete. □

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