

## Optimal Mass Transport in Thin Domains

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### Abstract

We find the behavior of the solution of the optimal transport problem for the Euclidean distance (and its approximation by  $p$ -Laplacian problems) when the involved measures are supported in a domain that is contracted in one direction.

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*Key words.* Optimal mass transport,  $p$ -Laplacian equations, thin domains.

## 1 Introduction.

In this paper we study the behaviour of the solutions (Kantorovich potentials and mass transport plans) for the Monge-Kantorovich mass transport problem when the involved masses (that we assume to be absolutely continuous with respect to the usual Lebesgue measure) are contained in a domain that is contracted (and therefore thin) in one direction.

Thin domains occur in applications as they can be found in problems in mechanics. For example, in ocean dynamics, one is dealing with fluid regions which are thin compared to the horizontal length scales. Other examples include lubrication, meteorology, blood circulation, etc.; they are a part of a broader study of the behaviour of various PDEs on thin  $n$ -dimensional domains, where  $n \geq 2$  (for a review see [24]).

In order to formulate precise statements as well as to put this work in context, we first need to introduce some notations, concepts and results from the Monge-Kantorovich Mass Transport Theory (we refer to [1], [13], [25] and [26] for details) that will be used in the rest of the paper.

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### 1.1 Monge-Kantorovich Mass Transport Theory

We denote by  $\mathcal{M}(\Omega)$  the set of Radon measures on  $\Omega$  and by  $\mathcal{M}^+(\Omega)$  the non-negative elements of  $\mathcal{M}(\Omega)$ . Given  $\mu, \nu \in \mathcal{M}^+(\Omega)$  satisfying the mass balance condition  $\mu(\Omega) = \nu(\Omega)$  we denote by  $\mathcal{A}(\mu, \nu)$  the set of transport maps pushing  $\mu$  to  $\nu$ , that is, the set of Borel maps  $T : \Omega \rightarrow \Omega$  such that  $T\#\mu = \nu$ , that is,  $\mu(T^{-1}(E)) = \nu(E)$  for all  $E \subset \Omega$  Borel.

**The Monge problem.** *The Monge problem, associated with the measures  $\mu$  and  $\nu$ , is to find a map  $T^* \in \mathcal{A}(\mu, \nu)$  which minimizes the cost functional*

$$\tilde{\mathcal{F}}(T) := \int_{\Omega} |x - T(x)| d\mu(x) \tag{1.1}$$

in the set  $\mathcal{A}(\mu, \nu)$ . When  $\mu$  and  $\nu$  are absolutely continuous with respect to the Lebesgue measure,  $\mu = f\mathcal{L}^N \llcorner \Omega$  and  $\nu = g\mathcal{L}^N \llcorner \Omega$ , there exists such an optimal map  $T$ . A map  $T^* \in \mathcal{A}(\mu, \nu)$  satisfying  $\tilde{\mathcal{F}}(T^*) = \min\{\tilde{\mathcal{F}}(T) : T \in \mathcal{A}(\mu, \nu)\}$ , is called an optimal transport map of  $\mu$  to  $\nu$ .

In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942, L. V. Kantorovich in [17] proposed a relaxed version of the problem and introduced a dual variational principle. Let  $\pi_t(x, y) := (1 - t)x + ty$ . Given a Radon measure  $\gamma$  in  $\Omega \times \Omega$ , its marginals are defined by  $proj_x(\gamma) := \pi_0\#\gamma$ ,  $proj_y(\gamma) := \pi_1\#\gamma$ .

**The Monge-Kantorovich problem.** *The Monge-Kantorovich problem, [17], is the minimization problem*

$$\min \left\{ \int_{\Omega \times \Omega} |x - y| d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\},$$

where  $\Pi(\mu, \nu) := \{\text{Radon measures } \gamma \text{ in } \Omega \times \Omega : \pi_0\#\gamma = \mu, \pi_1\#\gamma = \nu\}$ . The elements  $\gamma \in \Pi(\mu, \nu)$  are called transport plans between  $\mu$  and  $\nu$ , and a minimizer  $\gamma^*$  an optimal transport plan. A minimizer always exists.

The Monge-Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [25, Theorem 1.14]).

**Kantorovich-Rubinstein Theorem.** *The following duality result holds:*

$$\min \left\{ \int_{\Omega \times \Omega} |x - y| d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\} = \max \left\{ \int_{\Omega} u d(\mu - \nu) : u \in K_1(\overline{\Omega}) \right\}, \tag{1.2}$$

where  $K_1(\overline{\Omega}) := \{u : \overline{\Omega} \rightarrow \mathbb{R} : |u(x) - u(y)| \leq |x - y| \ \forall x, y \in \overline{\Omega}\}$  is the set of 1-Lipschitz functions in  $\overline{\Omega}$ . The maximizers  $u^*$  of the right hand side of (1.2) are called Kantorovich potentials.

Kantorovich potentials can be obtained taking the limit as  $p \rightarrow \infty$  in a  $p$ -Lapla-cian problem. Assume that  $\mu = f\mathcal{L}^N \llcorner \Omega$  and  $\nu = g\mathcal{L}^N \llcorner \Omega$  and consider

$$\begin{cases} -\Delta_p u_p = f - g & \text{in } \Omega, \\ |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega, \\ u_p(0) = 0. \end{cases} \tag{1.3}$$

The condition  $u_p(0) = 0$  is just a normalization (we assume here that  $0 \in \Omega$ ). We have the following result, see [14] and Section 5 in this paper.

**Evans-Gangbo Theorem.** *The solutions to (1.3) converge, along subsequen-ces, uniformly in  $\overline{\Omega}$ ,*

$$\lim_{p \rightarrow \infty} u_p = u^*,$$

where  $u^*$  is a Kantorovich potential, that is, a maximizer for the right hand side of (1.2). In fact, this limit procedure gives much more since it allows to construct an optimal transport map.

For later reference, we will call  $TC(f, g)_\Omega$  the total cost of the transport of  $f\mathcal{L}^N \llcorner \Omega$  to  $g\mathcal{L}^N \llcorner \Omega$ , that is given by the minimum or the maximum in (1.2).

### 1.2 The Monge-Kantorovich problem in a thin domain.

We consider a product domain  $\Omega_1 \times \Omega_2 = \Omega \subset \mathbb{R}^n$ , with  $\Omega_1 \subset \mathbb{R}^k$ ,  $\Omega_2 \subset \mathbb{R}^l$  and, for simplicity, we assume that  $|\Omega_1| = |\Omega_2| = 1$  and that  $(0, 0) \in \Omega_1 \times \Omega_2$ . We are given two nonnegative  $L^1$  functions  $f_+(x, y)$  and  $f_-(x, y)$ , with  $x \in \mathbb{R}^k$ ,  $y \in \mathbb{R}^l$ , supported in  $\Omega$ , with the same total mass,

$$\int_\Omega f_+(x, y) dx dy = \int_\Omega f_-(x, y) dx dy := M. \tag{1.4}$$

Now we take  $\varepsilon > 0$  small and contract the second variable,  $y$ , that is, we consider

$$\Omega_\varepsilon = \Omega_1 \times \varepsilon\Omega_2 = \{(x, \varepsilon y) : x \in \Omega_1, y \in \Omega_2\}.$$

In this set  $\Omega_\varepsilon$  we define

$$f_+^\varepsilon(\bar{x}, \bar{y}) = f_+\left(\bar{x}, \frac{\bar{y}}{\varepsilon}\right) \frac{1}{\varepsilon^l}, \quad \text{and} \quad f_-^\varepsilon(\bar{x}, \bar{y}) = f_-\left(\bar{x}, \frac{\bar{y}}{\varepsilon}\right) \frac{1}{\varepsilon^l}, \quad \text{for } (\bar{x}, \bar{y}) \in \Omega_\varepsilon.$$

These functions still satisfy the mass balance condition in  $\Omega_\varepsilon$ , indeed, it holds that,

$$\int_{\Omega_\varepsilon} f_+^\varepsilon(\bar{x}, \bar{y}) d\bar{x}d\bar{y} = \int_{\Omega_\varepsilon} f_-^\varepsilon(\bar{x}, \bar{y}) d\bar{x}d\bar{y} = M.$$

We will keep the notation  $(x, y)$  for the variables in the reference domain,  $\Omega_1 \times \Omega_2$ , and  $(\bar{x}, \bar{y})$  for the variables in the contracted domain,  $\Omega_1 \times \varepsilon\Omega_2$ , along the whole paper.

Now we consider the Monge-Kantorovich problem for the measures  $f_+^\varepsilon$  and  $f_-^\varepsilon$  in the thin domain  $\Omega_\varepsilon$ .

From previous results (see [1], [13], [25] and [26]) we know that there exist  $\bar{\mu}^\varepsilon$  an optimal transport plan and  $\bar{u}^\varepsilon$  a Kantorovich potential for this problem defined in  $\Omega_\varepsilon$ . In addition if we consider the  $p$ -Laplacian approximation given by (1.3) with  $f = f_+^\varepsilon$  and  $g = f_-^\varepsilon$  in the thin domain  $\Omega_\varepsilon$  we know that the solutions  $\bar{u}_p^\varepsilon$  to the  $p$ -Laplacian type problems (1.3) in  $\Omega_\varepsilon$  provide an approximation to a Kantorovich potential.

**Main goal.** Our main concern in this paper is to study the behaviour as  $\varepsilon \rightarrow 0$  of all the relevant variables for this problem; the total costs  $TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon}$ , the optimal transport plans,  $\bar{\mu}^\varepsilon$ , the Kantorovich potentials,  $\bar{u}^\varepsilon$ , and the  $p$ -Laplacian approximations,  $\bar{u}_p^\varepsilon$ .

We find that when  $\varepsilon \rightarrow 0$  the limit problem that appears is the mass transport problem in  $\Omega_1$  where the involved masses are given by the projections of  $f_+$  and  $f_-$  in the  $x$  variable, that is,

$$g_+(x) = \int_{\Omega_2} f_+(x, y) dy \quad \text{and} \quad g_-(x) = \int_{\Omega_2} f_-(x, y) dy. \tag{1.5}$$

Associated with the mass transport problem for the projections we have optimal transport plans (denoted by  $\eta$  in the sequel) and Kantorovich potentials (denoted by  $u$ ) and approximating sequences of solutions to  $p$ -Laplacians (denoted by  $u_p$ ).

Our main results can be summarized as follows:

**Theorem 1.1** *With the above notations we have the following commutative diagram (for all the involved functions rescaled to the fixed reference domain  $\Omega$ )*

$$\begin{array}{ccc}
 u_p^\varepsilon & \longrightarrow & u^\varepsilon \\
 \downarrow & \circlearrowleft & \downarrow \quad (\varepsilon \rightarrow 0). \\
 u_p & \longrightarrow & u
 \end{array}$$

$(p \rightarrow \infty)$

*This means that Kantorovich potentials (and their  $p$ -Laplacian approximations) for the problem in the thin domain converge to a Kantorovich potential (and to the  $p$ -Laplacian approximation) for the problem for the projections of the involved measures.*

*Concerning optimal plans, it holds that the optimal plans in the thin domain  $\bar{\mu}^\varepsilon$  rescaled back to  $\Omega \times \Omega$  converge weakly- $*$  in the sense of measures to a measure,  $\nu$ , that allows us to construct an optimal plan for the projections,  $\eta$ .*

*In addition, we find that the error is of order  $\varepsilon$ , in the sense that the difference of the total cost of transporting  $f_+^\varepsilon$  to  $f_-^\varepsilon$  and the total cost of transporting the projections  $g_+$  to  $g_-$  is less or equal to  $2M\text{diam}(\Omega_2)\varepsilon$ .*

**Remark 1.1** With the same methods and ideas we can handle the case of  $\Omega$  being a general domain in  $\mathbb{R}^{k+l}$  (not necessarily a product domain). In this case we just consider

$$\Omega_\varepsilon = \{(x, \varepsilon y) : (x, y) \in \Omega\},$$

$f_\pm^\varepsilon$  are defined as above and the projections are given by  $g_\pm(x) = \int_{\mathbb{R}^l} f_\pm(x, y) dy$ . All our results (and their proofs) can be obtained for this more general case. The only place at which there is a difference is when we take the limit as  $\varepsilon \rightarrow 0$  of the approximations sequence  $\bar{u}_p^\varepsilon$  (with fixed  $p$ ). In this case there appears a weight in the limit PDE (that is the constant  $|\Omega_2|$  for a product domain, but that depends on  $x$  in the general case). We include a remark on this point when appropriate (in Section 5). We prefer to present our results for a product domain to clarify the arguments involved.

**Remark 1.2** The same ideas can be used to handle the situation in which the measures are contained in a domain that lies between two parallel hyperplanes that are close to each other. We don't include the details for simplicity. Also, the methods used here could be extended with domains that concentrate along a surface, that is, domains of the form  $\Omega_\varepsilon = S + B(0, \varepsilon)$  where  $S$  is a  $k$ -dimensional surface in  $\mathbb{R}^l$ .

**Remark 1.3** In general, the transport problem for the projections is simpler than the original one (since it involves measures in a smaller dimension). This fact together with the bound for the error allows us to build approximate transport maps when the projections are one-dimensional, that is,  $\Omega_1 = (a, b) \subset \mathbb{R}$ . We provide examples in Section 6.

To conclude the introduction we briefly comment on the previous bibliography and the methods and ideas involved in the proofs. Optimal transport problems is by now a classical subject that still deserves attention. We refer to [2], [3], [4], [6], [21], [22], [23] and the surveys and books [1], [13], [25] and [26]. It has many applications, for example in economics (matching problems), [5], [7], [8], [9], [10], [11], [20]. Closely related to this article is the case in which the involved measures are concentrated in a small strip around the boundary of a fixed domain. This has been considered in [15] (see also [16] for singular measures supported on the boundary). In [19] the role of boundary conditions (Dirichlet and/or Neumann) in the  $p$ -Laplacian approximation was clarified (note that

in our case we use Neumann boundary conditions since no mass is to be taken/bringed to/from outside of the domain). The first paper that uses the approximation by  $p$ -Laplacian type problems is [14] where the authors use Dirichlet boundary conditions in a sufficiently large ball, we can not use Dirichlet boundary conditions here since, as we want to contract the domain in one direction, is it likely that some mass will be taken to/from the boundary of the domain if we impose Dirichlet boundary conditions (we will elaborate more on this issue in Section 7).

Concerning the methods used in the proofs we have: to pass to the limit in the Kantorovich potentials, we first rescale back to  $\Omega$  and then, using that Kantorovich potentials are Lipschitz functions to gain compactness and that they are solutions to a variational formulation we find that any possible uniform limit is a solution to a maximization limit problem. Then we find that the limit function is independent of the  $y$  variable and just observe that integration in  $y$  gives the projections of  $f_{\pm}$ . The proof of the convergence of the optimal transport plans is similar but we have to work in the space of Borel measures. To obtain convergence of the  $p$ -Laplacian approximations we use mainly the variational characterization of the solutions to the  $p$ -Laplacian as minimizers of an adequate functional in the Sobolev space  $W^{1,p}$ . We include here the details of the approximation of a Kantorovich potential with solutions to the  $p$ -Laplacian problems as  $p \rightarrow \infty$  for completeness.

The paper is organized as follows: In Section 2 we prove the existence of Kantorovich potentials  $\bar{u}_\varepsilon$  and study their limit as  $\varepsilon \rightarrow 0$ ; in Section 3 we study the behaviour of the optimal transport plans; in Section 4 we show estimates for the difference of the total costs of the  $\varepsilon$ -problem and the limit problem; in Section 5 we deal with the  $p$ -Laplacian approximations and their behaviour as  $\varepsilon \rightarrow 0$ ; in Section 6 we collect some examples that show that we can construct approximate transport maps when the limit problem is one-dimensional; finally in Section 7 we comment on the possibility of considering other boundary conditions than homogeneous Neumann ones in the  $p$ -Laplacian approximations.

## 2 Behavior of the Kantorovich potentials.

**Lemma 2.1** *Given  $f_+$ ,  $f_-$  and  $\Omega$ , for each  $\varepsilon$  there exists a Kantorovich potential,  $\bar{u}^\varepsilon$ , that is, a solution to*

$$\begin{aligned} \max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0, 0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x}d\bar{y}. \end{aligned} \tag{2.1}$$

*Proof.* Let  $K = \{ \bar{v} : \bar{\Omega}_\varepsilon \rightarrow \mathbb{R} : |\nabla \bar{v}| \leq 1, v(0, 0) = 0 \}$ , and, for  $\bar{v} \in K$ , consider

$$L(\bar{v}) = \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y}))d\bar{x}d\bar{y}.$$

If we take  $(\bar{x}, \bar{y}), (\bar{z}, \bar{w}) \in \bar{\Omega}_\varepsilon$  we have,

$$|\bar{v}(\bar{x}, \bar{y}) - \bar{v}(\bar{z}, \bar{w})| \leq |\nabla \bar{v}(\bar{\xi})| |(\bar{x}, \bar{y}) - (\bar{z}, \bar{w})| \leq |(\bar{x}, \bar{y}) - (\bar{z}, \bar{w})| \leq \text{diam}(\Omega_\varepsilon), \tag{2.2}$$

where  $\bar{\xi}$  lies on the segment between  $(\bar{x}, \bar{y})$  and  $(\bar{z}, \bar{w})$ . Now, (1.4) implies

$$\begin{aligned} L(\bar{v}) &= \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})f_+^\varepsilon(\bar{x}, \bar{y})d\bar{x}d\bar{y} - \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})f_-^\varepsilon(\bar{x}, \bar{y})d\bar{x}d\bar{y} \\ &\leq 2\text{diam}(\Omega_\varepsilon) \int_{\Omega_\varepsilon} f_+^\varepsilon(\bar{x}, \bar{y})d\bar{x}d\bar{y} = 2M\text{diam}(\Omega_\varepsilon), \end{aligned}$$

for all  $\bar{v} \in K$ . Hence  $L$  is bounded above in  $K$ . Let  $(\bar{v}_j)_{j \in \mathbb{N}}$  be a sequence in  $K$  such that

$$L(\bar{v}_j) \nearrow \sup_{\bar{v} \in K} L(\bar{v}).$$

This sequence is equicontinuous and equibounded by (2.2), using the condition  $\bar{v}(0, 0) = 0$ . So we can extract a subsequence  $(\bar{v}_{j_k})_{k \in \mathbb{N}}$  such that  $\bar{v}_{j_k} \rightrightarrows \bar{u}^\varepsilon$  in  $\bar{\Omega}_\varepsilon$ , uniformly. We have,

$$\lim_{k \rightarrow \infty} \int_{\Omega_\varepsilon} \bar{v}_{j_k}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x}d\bar{y} = L(\bar{u}^\varepsilon) = \sup_{\bar{v} \in K} L(\bar{v}).$$

To conclude we need to check that  $\bar{u}^\varepsilon \in K$ . This follows from the fact that  $\bar{v}_{j_k}(0, 0) = 0$  and that, from (2.2) we get,  $|\bar{v}_{j_k}(\bar{x}, \bar{y}) - \bar{v}_{j_k}(\bar{z}, \bar{w})| \leq |(\bar{x}, \bar{y}) - (\bar{z}, \bar{w})|$ . When we take the limit as  $k \rightarrow \infty$ , we obtain,  $\bar{u}^\varepsilon(0, 0) = 0$  and  $|\bar{u}^\varepsilon(\bar{x}, \bar{y}) - \bar{u}^\varepsilon(\bar{z}, \bar{w})| \leq |(\bar{x}, \bar{y}) - (\bar{z}, \bar{w})|$ . So  $\bar{u}^\varepsilon \in K$  and then it is the desired maximizer.

Now we can state the following theorem concerning the behaviour as  $\varepsilon \rightarrow 0$  of the Kantorovich potentials.

**Theorem 2.1** *Let  $\bar{u}^\varepsilon$  be a maximizer of (2.1) defined in  $\Omega_\varepsilon$  and rescale it to  $\Omega$  as*

$$u^\varepsilon(x, y) = \bar{u}^\varepsilon(x, \varepsilon y).$$

Then

$$u^\varepsilon(x, y) \rightrightarrows u(x), \quad \text{when } \varepsilon \rightarrow 0, \tag{2.3}$$

uniformly in  $\bar{\Omega}$  along subsequences. The limit  $u$  only depends on  $x$  and is a Kantorovich potential for the projections of  $f_+$  and  $f_-$ , that is,  $u$  is a maximizer for

$$\max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0) = 0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x)) dx, \tag{2.4}$$

with  $g_+$  and  $g_-$  given by (1.5).

*Proof.* We have that  $\bar{u}^\varepsilon$  is defined in  $\Omega_\varepsilon$  and we want to rescale it to  $\Omega$ . We let  $\bar{x} = x, \bar{y} = \varepsilon y$ , and we obtain, using that  $\bar{u}^\varepsilon$  is a Kantorovich potential that

$$\begin{aligned} & \int_{\Omega_\varepsilon} \bar{u}^\varepsilon(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x}d\bar{y} \\ &= \varepsilon^l \int_{\Omega} \bar{u}^\varepsilon(x, \varepsilon y)(f_+^\varepsilon(x, \varepsilon y) - f_-^\varepsilon(x, \varepsilon y)) dx dy \\ &\geq \varepsilon^l \int_{\Omega} v(x)(f_+^\varepsilon(x, \varepsilon y) - f_-^\varepsilon(x, \varepsilon y)) dx dy, \end{aligned} \tag{2.5}$$

for any  $v$  such that  $|\nabla_x v(x)| \leq 1$  and  $v(0) = 0$ . The function  $u^\varepsilon$  verifies  $u^\varepsilon(0, 0) = \bar{u}^\varepsilon(0, 0) = 0$  and

$$\begin{aligned} |\nabla_x u^\varepsilon(x, y)| &= |\nabla_x \bar{u}^\varepsilon(x, \varepsilon y)| \Rightarrow |\nabla_x u^\varepsilon(x, y)| \leq 1, \\ |\nabla_y u^\varepsilon(x, y)| &= |\nabla_y \bar{u}^\varepsilon(x, \varepsilon y)| \varepsilon \Rightarrow |\nabla_y u^\varepsilon(x, y)| \leq \varepsilon. \end{aligned}$$

Hence  $u^\varepsilon$  is an equicontinuous and equibounded family and therefore we can extract a uniformly convergent subsequence, that is, there is  $(\varepsilon_j)_{j \in \mathbb{N}}$ , with  $\varepsilon_j \rightarrow 0$  such as  $u^{\varepsilon_j} \rightrightarrows u$ , uniformly in  $\bar{\Omega}$ . Now we check that  $u$  only depends on  $x$ . First we have,

$$|u^\varepsilon(x, y_1) - u^\varepsilon(x, y_2)| \leq |\nabla_y u^\varepsilon(x, \xi)| |y_1 - y_2| \leq \varepsilon \text{diam}(\Omega_2)$$

where  $\xi$  lies on the segment between  $y_1$  and  $y_2$ . Now if  $\varepsilon_j \rightarrow 0$  we conclude

$$|u(x, y_1) - u(x, y_2)| \leq 0.$$

Hence,  $u(x, y)$  only depends on  $x$ . So we write  $u(x)$  and next we show that  $u$  is a Kantorovich potential for the projections of  $f_+$  and  $f_-$ . We need to check that  $u(x)$  satisfy  $|\nabla_x u(x)| \leq 1$ . We have

$$|u^\varepsilon(x_1, y) - u^\varepsilon(x_2, y)| \leq |\nabla_x u^\varepsilon(\xi, y)| |x_1 - x_2| \leq |x_1 - x_2|$$

where  $\xi$  lies on the segment between  $x_1$  and  $x_2$ . Now taking  $\varepsilon_j \rightarrow 0$  we conclude that

$$|u(x_1) - u(x_2)| \leq |x_1 - x_2|.$$

So  $|\nabla_x u(x)| \leq 1$  and, therefore the limit  $u$  is 1-Lipschitz. To see that  $u$  is a Kantorovich potential for the projections of  $f_+$  and  $f_-$  we argue as follows:

$$\begin{aligned} & \varepsilon^l \int_{\Omega} \bar{u}^{\varepsilon_j}(x, \varepsilon y) (f_+^\varepsilon(x, \varepsilon y) - f_-^\varepsilon(x, \varepsilon y)) \, dx dy \\ &= \int_{\Omega} u^{\varepsilon_j}(x, y) (\varepsilon^l f_+^\varepsilon(x, \varepsilon y) - \varepsilon^l f_-^\varepsilon(x, \varepsilon y)) \, dx dy. \end{aligned}$$

Using (2.5) we obtain

$$\begin{aligned} & \varepsilon^l \int_{\Omega} \bar{u}^{\varepsilon_j}(x, \varepsilon y) (f_+^\varepsilon(x, \varepsilon y) - f_-^\varepsilon(x, \varepsilon y)) \, dx dy \\ &= \int_{\Omega} u^{\varepsilon_j}(x, y) (f_+(x, y) - f_-(x, y)) \, dx dy \\ &\geq \varepsilon^l \int_{\Omega} v(x) (f_+^\varepsilon(x, \varepsilon y) - f_-^\varepsilon(x, \varepsilon y)) \, dx dy \\ &= \int_{\Omega} v(x) (f_+(x, y) - f_-(x, y)) \, dx dy \\ &= \int_{\Omega_1} v(x) \int_{\Omega_2} (f_+(x, y) - f_-(x, y)) \, dy dx, \end{aligned}$$

for all  $v$  such that  $|\nabla_x v(x)| \leq 1$  and  $v(0) = 0$ . Now we take limits as  $\varepsilon_j \rightarrow 0$ , using that  $u^{\varepsilon_j} \rightrightarrows u$ , and (1.5), we get,

$$\int_{\Omega_1} u(x) (g_+(x) - g_-(x)) \, dx dy \geq \int_{\Omega_1} v(x) (g_+(x) - g_-(x)) \, dx,$$

for all  $v$  such that  $|\nabla_x v(x)| \leq 1$  and  $v(0) = 0$ .

Also from the previous proof we obtain the following result:

**Corollary 2.1** *Under the same hypothesis of Theorem 2.1 we have,*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \bar{u}^\varepsilon(\bar{x}, \bar{y}) (f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) \, d\bar{x} d\bar{y} = \int_{\Omega_1} u(x) (g_+(x) - g_-(x)) \, dx.$$

That is, we have that

$$\lim_{\varepsilon \rightarrow 0} TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} = TC(g_+, g_-)_{\Omega_1}.$$

### 3 Behaviour of the transport plans.

We consider measures  $\bar{\mu}^\varepsilon$  in  $\Omega_\varepsilon \times \Omega_\varepsilon$  that are solutions to the minimization problem

$$\begin{aligned} \min & \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} |(\bar{x}, \bar{y}) - (\bar{\theta}, \bar{\xi})| d\bar{\mu}((\bar{x}, \bar{y}), (\bar{\theta}, \bar{\xi})). \\ \text{proj}_{(\bar{x}, \bar{y})}(\bar{\mu}) = f_+^\varepsilon & \\ \text{proj}_{(\bar{\theta}, \bar{\xi})}(\bar{\mu}) = f_-^\varepsilon & \end{aligned} \tag{3.1}$$

Now, for  $F \subset \Omega$  we let  $S_\varepsilon(F) = \{(\theta, \varepsilon\xi) : (\theta, \xi) \in F\}$  and we define the rescaled measure as

$$\mu^\varepsilon(E \times F) = \bar{\mu}^\varepsilon(S_\varepsilon(E) \times S_\varepsilon(F)). \tag{3.2}$$

Concerning the limit as  $\varepsilon \rightarrow 0$  of optimal transport plans we have the following result:

**Theorem 3.1** *Let  $\mu^\varepsilon$  be the measure in  $\Omega \times \Omega$  given by (3.2) where  $\bar{\mu}^\varepsilon$  is a minimizer of (3.1). Then*

$$\mu^\varepsilon \rightarrow \nu$$

weakly-\* as  $\varepsilon \rightarrow 0$  along a subsequence. If we let

$$\eta(x, \theta) = \int_{\Omega_2} \int_{\Omega_2} d\nu((x, y), (\theta, \xi)), \tag{3.3}$$

it holds that  $\eta$  depends only on the first coordinates  $(x, \theta)$  and is an optimal transport plan for the projections of  $f_+$  and  $f_-$ , that is,  $\eta$  is a minimizer of

$$\begin{aligned} \min & \int_{\Omega_1} \int_{\Omega_1} |x - \theta| d\eta(x, \theta). \\ \text{proj}_x(\eta) = g_+ & \\ \text{proj}_\theta(\eta) = g_- & \end{aligned}$$

*Proof.* We have

$$\mu^\varepsilon(\Omega \times F) = \bar{\mu}^\varepsilon(\Omega_1 \times \varepsilon\Omega_2 \times S_\varepsilon(F)) = \int_{S_\varepsilon(F)} f_-^\varepsilon(\bar{\theta}, \bar{\xi}) d\bar{\theta}d\bar{\xi} = \int_F f_-(\theta, \xi) d\theta d\xi.$$

Therefore, we have that  $\text{proj}_{\theta, \xi}(\mu^\varepsilon) = f_-$ . Analogously, we obtain  $\text{proj}_{x, y}(\mu^\varepsilon) = f_+$ . Hence,  $\mu^\varepsilon$  are nonnegative measures with bounded total mass,

$$\mu^\varepsilon(\Omega \times \Omega) = \int_\Omega f_+ = M,$$

and therefore there exists a sequence  $\varepsilon_j \rightarrow 0$  such that

$$\mu^{\varepsilon_j} \rightharpoonup \nu$$

weakly-\* in the sense of measures. It follows that  $\text{proj}_{\theta, \xi}(\nu) = f_-$ , and  $\text{proj}_{x, y}(\nu) = f_+$ . Now we observe that, taking into account (3.2),

$$\begin{aligned} & \int_{\Omega_{\varepsilon_j}} \int_{\Omega_{\varepsilon_j}} |(\bar{x}, \bar{y}) - (\bar{\theta}, \bar{\xi})| d\bar{\mu}^{\varepsilon_j}((\bar{x}, \bar{y}), (\bar{\theta}, \bar{\xi})) \\ &= \int_\Omega \int_\Omega |(x, \varepsilon_j y) - (\theta, \varepsilon_j \xi)| d\mu^{\varepsilon_j}((x, y), (\theta, \xi)). \end{aligned}$$

Hence, the limit as  $\varepsilon_j \rightarrow 0$  is given by

$$\int_{\Omega} \int_{\Omega} |x - \theta| d\nu((x, y), (\theta, \xi)).$$

Finally, we easily obtain that the measure  $\eta$  given by (3.3) is a minimizer for

$$\begin{aligned} \min \quad & \int_{\Omega_1} \int_{\Omega_1} |x - \theta| d\eta(x, \theta). \\ \text{proj}_x(\eta) = g_+ \quad & \\ \text{proj}_\theta(\eta) = g_- \quad & \end{aligned}$$

### 4 A bound for the error.

In this section our main goal is to estimate the error committed in the total cost when we replace the optimal transport problem in  $\Omega_\varepsilon$  with the transport problem of the projections, that is, we want to obtain a bound for

$$\left| TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} - TC(g_+, g_-)_{\Omega_1} \right| = \left| \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) - \int_{\Omega_1} u(g_+ - g_-) \right|$$

in terms of  $\varepsilon$ . Our main result in this direction is the following:

**Theorem 4.1** *There exists a constant  $C := 2M \text{diam}(\Omega_2)$  independent of  $\varepsilon$  such that*

$$\left| \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) - \int_{\Omega_1} u(g_+ - g_-) \right| \leq C\varepsilon.$$

*Proof.* Changing variables as before  $\bar{x} = x, \bar{y} = \varepsilon y$  and  $\bar{u}^\varepsilon(\bar{x}, \bar{y}) = u^\varepsilon(x, y)$  we get

$$\int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon)(\bar{x}, \bar{y}) d\bar{x}d\bar{y} = \int_{\Omega} u^\varepsilon (f_+ - f_-)(x, y) dx dy$$

with  $u^\varepsilon$  verifying  $|\nabla_x u^\varepsilon|^2 + \varepsilon^{-2} |\nabla_y u^\varepsilon|^2 \leq 1$ . As  $u$  depends only on  $x$  and verifies  $|\nabla_x u| \leq 1$  it competes with  $u^\varepsilon$  in the maximization problem, hence we have

$$\int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \geq \int_{\Omega_1} u(g_+ - g_-).$$

Let

$$h^\varepsilon(x) = \int_{\Omega_2} u^\varepsilon(x, y) dy.$$

Now, we observe that, from the fact that  $|\nabla_x u^\varepsilon| \leq 1$  we get that this function  $h^\varepsilon$  competes with  $u$  in its maximization problem, then,

$$\int_{\Omega_1} h^\varepsilon(g_+ - g_-) \leq \int_{\Omega_1} u(g_+ - g_-).$$

In addition, we have

$$|u^\varepsilon(x, y) - h^\varepsilon(x)| \leq \text{diam}(\Omega_2)\varepsilon.$$

It follows that (recall that we assumed  $|\Omega_2| = 1$ )

$$\begin{aligned} & \left| \int_{\Omega_1} u(g_+ - g_-) - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon(f_+^\varepsilon - f_-^\varepsilon) \right| \\ & \leq \int_{\Omega_1} \int_{\Omega_2} |h^\varepsilon - u^\varepsilon|(f_+ + f_-) \leq 2M \text{diam}(\Omega_2)\varepsilon. \end{aligned}$$

This ends the proof.

**Remark 4.1** The bound depends in a sharp way of the relevant quantities as it can be seen taken two masses concentrated near points  $(x_1, y_1)$  and  $(x_1, y_2)$  with  $|y_1 - y_2| \sim \text{diam}(\Omega_2)$ . Note that since both concentration points have the same first coordinate, we have  $TC(g_+, g_-)_{\Omega_1} \sim 0$  and for the total cost  $TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} \sim \varepsilon|y_1 - y_2|M \sim \text{diam}(\Omega_2)M\varepsilon$ .

We can also characterize when we have equality of the total cost for the original functions and the projections.

**Theorem 4.2** *There is a Kantorovich potential for the transport of  $f_+^\varepsilon$  to  $f_-^\varepsilon$  that depends only in the  $x$  variable, that is, of the form  $\bar{u}(\bar{x}, \bar{y}) = \bar{u}(\bar{x})$ , if and only if the total cost of sending  $f_+^\varepsilon$  to  $f_-^\varepsilon$  is the same as the total cost for the projections  $g_+$  to  $g_-$ .*

*Proof.* Using that  $\bar{u}(\bar{x})$  is a Kantorovich potential for the transport of  $f_+^\varepsilon$  to  $f_-^\varepsilon$  and the previous proof we obtain that

$$\begin{aligned} & \max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x} d\bar{y} \\ & = \int_{\Omega_\varepsilon} \bar{u}(\bar{x})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x} d\bar{y} = \int_{\Omega_1} \bar{u}(x)(g_+(x) - g_-(x)) dx \\ & \leq \max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0) = 0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x)) dx \\ & \leq \max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x} d\bar{y}, \end{aligned}$$

and hence we conclude that the total costs for  $f_+^\varepsilon$  to  $f_-^\varepsilon$  and for  $g_+$  to  $g_-$  coincide.

Conversely, if the costs coincide, then take  $\bar{u}(x)$  a Kantorovich potential for the projections and observe that

$$\begin{aligned} & \int_{\Omega_1} \bar{u}(x)(g_+(x) - g_-(x)) dx = \max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0) = 0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x)) dx \\ & = \max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y})) d\bar{x} d\bar{y}, \end{aligned}$$

and we conclude that  $\bar{u}$  is a Kantorovich potential for  $f_+^\varepsilon$  to  $f_-^\varepsilon$  that depends only on  $x$ .

## 5 A $p$ -Laplacian approximation and its behaviour as $\varepsilon \rightarrow 0$ .

We consider

$$\min_{\substack{\bar{v} \in W^{1,p}(\Omega_\varepsilon) \\ \bar{v}(0) = 0}} \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon). \tag{5.1}$$

Note that we have normalized the gradient term in the functional with  $\frac{1}{\varepsilon^l}$ . This is the right scale to compensate the fact that  $|\Omega_\varepsilon| \sim \varepsilon^l$ . This scaling factor is not needed in the second term since we have normalized  $f_\pm^\varepsilon$  in such a way that they have constant total mass  $M$ .

**Lemma 5.1** *There exists a unique minimizer of (5.1), that we will call  $\bar{u}_p^\varepsilon$ .*

*Proof.* We just observe that the functional

$$L_p(\bar{v}) = \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon)$$

is bounded below in  $W^{1,p}(\Omega_\varepsilon)$ . Indeed, for  $\bar{v} \in W^{1,p}(\Omega_\varepsilon)$  with  $\bar{v}(0,0) = 0$ , calling  $f^\varepsilon = f_+^\varepsilon - f_-^\varepsilon$  we have,

$$\int_{\Omega_\varepsilon} (\bar{v} f^\varepsilon) \leq \|\bar{v}\|_{L^p(\Omega_\varepsilon)} \|f^\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} \leq C_1 \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)},$$

where  $C_1$  is a constant that depends on  $f^\varepsilon$ . So

$$L_p(\bar{v}) = \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon) \geq \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - C_1 \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)}. \tag{5.2}$$

Using Young's inequality  $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$  with  $a = \varepsilon^{l/p} C_1$ ,  $b = \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)}$ , we get

$$L_p(\bar{v}) \geq \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \frac{\varepsilon^{l/(p-1)}(C_1)^{p'}}{p'} - \frac{(\|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)})^p}{\varepsilon^l p} = -\frac{\varepsilon^{l/(p-1)}(C_1)^{p'}}{p'}.$$

So  $L_p(\bar{v}) \geq C$  for all  $\bar{v} \in W^{1,p}(\Omega_\varepsilon)$  with  $\bar{v}(0,0) = 0$ . Take  $\bar{v}_n$  a minimizing sequence. From (5.2) and the fact that  $\bar{v}_n(0,0) = 0$  we get that  $\bar{v}_n$  is bounded in  $W^{1,p}(\Omega_\varepsilon)$  and extracting a subsequence if necessary we can assume that  $\bar{v}_n \rightarrow \bar{u}_p^\varepsilon$  weakly in  $W^{1,p}(\Omega_\varepsilon)$ . From the lower semicontinuity of  $L_p$  we conclude that  $\bar{u}_p^\varepsilon$  is a minimizer of  $L_p$ .

Uniqueness follows from the strict convexity of  $L_p$ .

From the fact that  $\bar{u}_p^\varepsilon$  is a minimizer of (5.1) we have that  $\bar{u}_p^\varepsilon$  is a weak solution to the following PDE problem

$$\begin{cases} -\frac{1}{\varepsilon^l} \Delta_p \bar{u}_p^\varepsilon = f_+^\varepsilon - f_-^\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{1}{\varepsilon^l} |\nabla \bar{u}_p^\varepsilon|^{p-2} \frac{\partial \bar{u}_p^\varepsilon}{\partial \eta} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \bar{u}_p^\varepsilon(0) = 0. \end{cases} \tag{5.3}$$

**Theorem 5.1** *Let  $\bar{u}_p^\varepsilon$  be a minimizer of (5.1). Then, extracting a subsequence if necessary,*

$$\bar{u}_p^\varepsilon \rightarrow \bar{u}^\varepsilon$$

as  $p \rightarrow \infty$  uniformly in  $\Omega_\varepsilon$  where  $\bar{u}^\varepsilon$  is a Kantorovich potential for the transport problem of the mass  $f_+^\varepsilon$  to the mass  $f_-^\varepsilon$ .

*Proof.* Throughout this proof  $\varepsilon$  is fixed and  $C$  denotes a constant that is independent of  $p$  but may depend on  $\varepsilon$  and change from one line to another. Let  $\bar{u}^\varepsilon$  be a Kantorovich potential for the transport of  $f_+^\varepsilon$  to  $f_-^\varepsilon$  (its existence is guaranteed by Lemma 2.1). We have  $|\nabla \bar{u}^\varepsilon| \leq 1$  and  $\bar{u}^\varepsilon(0) = 0$  and hence  $\bar{u}^\varepsilon$  is bounded in  $\Omega_\varepsilon$  and  $\bar{u}^\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ . Using that  $\bar{u}_p^\varepsilon$  is a minimizer of  $L_p$  we get

$$\begin{aligned} & \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{u}_p^\varepsilon|^p - \int_{\Omega_\varepsilon} \bar{u}_p^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \leq \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{u}^\varepsilon|^p - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \\ & \leq \frac{1}{\varepsilon^l} \frac{|\Omega_\varepsilon|}{p} - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \leq C. \end{aligned} \tag{5.4}$$

It follows that

$$\begin{aligned} & \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{u}_p^\varepsilon|^p \leq C + \int_{\Omega_\varepsilon} \bar{u}_p^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \\ & \leq C + C \|\bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq C + CS_p \|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)}, \end{aligned}$$

here  $S_p$  is the best Sobolev constant that can be bounded by  $Cp$  (see [12]). Therefore, we get

$$\|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq (Cp)^{1/p}.$$

Now, fix  $q$  with  $n < q < p$  and observe that

$$\|\nabla \bar{u}_p^\varepsilon\|_{L^q(\Omega_\varepsilon)} \leq |\Omega_\varepsilon|^{\frac{p-q}{pq}} \|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq |\Omega_\varepsilon|^{\frac{p-q}{pq}} (Cp)^{1/p}.$$

Hence, we have that  $(\bar{u}_p^\varepsilon)_{p>q}$  is bounded in  $W^{1,q}(\Omega_\varepsilon)$ . Therefore, by a diagonal procedure, we can extract a subsequence (that we call  $\bar{u}_{p_n}^\varepsilon$ ) such that

$$\bar{u}_{p_n}^\varepsilon \rightarrow \bar{v} \quad \text{as } p_n \rightarrow \infty$$

weakly in every  $W^{1,q}(\Omega_\varepsilon)$  and, therefore, uniformly in  $\bar{\Omega}_\varepsilon$  (we are using here the compact embedding  $W^{1,q}(\Omega_\varepsilon) \hookrightarrow C^\alpha(\Omega_\varepsilon)$  when  $q > n$ ). Since  $\bar{u}_{p_n}^\varepsilon(0) = 0$  we get  $\bar{v}(0) = 0$ . From the semicontinuity of the norm we get  $\|\nabla \bar{v}\|_{L^q(\Omega_\varepsilon)} \leq |\Omega_\varepsilon|^{1/q}$ , and hence, taking  $q \rightarrow \infty$ , we obtain

$$\|\nabla \bar{v}\|_{L^\infty(\Omega_\varepsilon)} \leq 1.$$

From (5.4) we have

$$-\int_{\Omega_\varepsilon} \bar{u}_{p_n}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \leq \frac{1}{\varepsilon^l} \frac{|\Omega_\varepsilon|}{p_n} - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon).$$

Now, taking  $p \rightarrow \infty$  we obtain

$$-\int_{\Omega_\varepsilon} \bar{v} (f_+^\varepsilon - f_-^\varepsilon) \leq -\int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon),$$

from where we conclude that  $\bar{v}$ , the limit of  $\bar{u}_{p_n}^\varepsilon$ , as  $p_n \rightarrow \infty$  is a Kantorovich potential.

**Remark 5.1** With the arguments used in the previous proof we can obtain an alternative proof of the existence of a Kantorovich potential for the transport of  $f_+^\varepsilon$  to  $f_-^\varepsilon$ .

Now we study the limit as  $\varepsilon \rightarrow 0$  of  $\bar{u}_p^\varepsilon$ .

**Theorem 5.2** *Let*

$$u_p^\varepsilon(x, y) = \bar{u}_p^\varepsilon(\bar{x}, \bar{y}), \quad x = \bar{x}, \varepsilon y = \bar{y},$$

where  $\bar{u}_p^\varepsilon$  is a minimizer of (5.1). Then

$$u_p^\varepsilon \rightharpoonup u_p$$

as  $\varepsilon \rightarrow 0$  uniformly in  $\bar{\Omega}$  and weakly in  $W^{1,p}(\Omega)$  where  $u_p$  depends only on  $x$  and is a solution to the minimization problem

$$\min_{\substack{v \in W^{1,p}(\Omega_1) \\ v(0) = 0}} \frac{1}{p} \int_{\Omega_1} |\nabla_x v|^p - \int_{\Omega_1} v(g_+ - g_-) \tag{5.5}$$

*Proof.* We have  $\nabla_{\bar{x}} \bar{u}_p^\varepsilon(\bar{x}, \bar{y}) = \nabla_x u_p^\varepsilon(x, y)$  and  $\varepsilon \nabla_{\bar{y}} \bar{u}_p^\varepsilon(\bar{x}, \bar{y}) = \nabla_y u_p^\varepsilon(x, y)$ . Hence,  $u_p^\varepsilon$  is a minimizer of

$$\frac{1}{p} \int_{\Omega_1} \int_{\Omega_2} \left( \sqrt{|\nabla_x v|^2 + \varepsilon^{-2} |\nabla_y v|^2} \right)^p dx dy - \int_{\Omega_1} \int_{\Omega_2} v(f_+ - f_-) dx dy$$

in  $W^{1,p}(\Omega)$  with  $v(0) = 0$ .

By the same arguments used in Lemma 5.1 we obtain the existence of a unique minimizer of (5.5) that we call  $u_p$ . As  $u_p \in W^{1,p}(\Omega_1)$  we can consider it as a function of  $W^{1,p}(\Omega_1 \times \Omega_2)$  and then it competes with  $u_p^\varepsilon$ . We get

$$\begin{aligned} & \frac{1}{p} \int_{\Omega_1} \int_{\Omega_2} \left( \sqrt{|\nabla_x u_p^\varepsilon|^2 + \varepsilon^{-2} |\nabla_y u_p^\varepsilon|^2} \right)^p dx dy - \int_{\Omega_1} \int_{\Omega_2} u_p^\varepsilon (f_+ - f_-) dx dy \\ & \leq \frac{1}{p} \int_{\Omega_1} |\nabla_x u_p|^p dx - \int_{\Omega_1} u_p (g_+ - g_-) dx \end{aligned} \tag{5.6}$$

(we recall that, for simplicity, we have assumed that  $|\Omega_2| = 1$ ). Therefore there exists a constant  $C$  independent of  $\varepsilon$  such that

$$\frac{1}{p} \int_{\Omega_1} \int_{\Omega_2} \left( \sqrt{|\nabla_x u_p^\varepsilon|^2 + \varepsilon^{-2} |\nabla_y u_p^\varepsilon|^2} \right)^p dx dy \leq C + \int_{\Omega_1} \int_{\Omega_2} u_p^\varepsilon (f_+ - f_-) dx dy. \tag{5.7}$$

Taking  $\varepsilon < 1$  and arguing as in the proof of Theorem 5.1 we get that  $u_p^\varepsilon$  is bounded in  $W^{1,p}(\Omega)$  uniformly in  $\varepsilon$ . Therefore, we can extract a subsequence such that

$$u_p^{\varepsilon_n} \rightharpoonup v, \quad \text{as } \varepsilon_n \rightarrow 0,$$

weakly in  $W^{1,p}(\Omega)$  and (using that  $p > n$ ) uniformly in  $\Omega$ . In addition we have

$$\nabla_x u_p^{\varepsilon_n} \rightharpoonup \nabla_x v \text{ and } \nabla_y u_p^{\varepsilon_n} \rightharpoonup \nabla_y v \quad \text{weakly in } L^p(\Omega).$$

Now we observe that from (5.7) we obtain that there exists a constant  $C$  independent of  $\varepsilon$  such that

$$\left( \int_{\Omega_1} \int_{\Omega_2} |\nabla_y u_p^{\varepsilon_n}|^p dx dy \right)^{1/p} \leq C \varepsilon_n.$$

Therefore,

$$\nabla_y u_p^{\varepsilon_n} \rightarrow 0 \quad \text{strongly in } L^p(\Omega)$$

and we obtain that the limit  $v$  is independent of  $y$ .

Now, from (5.6) we get

$$\begin{aligned} & \frac{1}{p} \int_{\Omega_1} \int_{\Omega_2} |\nabla_x u_p^\varepsilon|^p dx dy - \int_{\Omega_1} \int_{\Omega_2} u_p^\varepsilon (f_+ - f_-) dx dy \\ & \leq \frac{1}{p} \int_{\Omega_1} |\nabla_x u_p|^p dx - \int_{\Omega_1} u_p (g_+ - g_-) dx. \end{aligned}$$

Taking  $\varepsilon_n \rightarrow 0$  and using that  $v$  is independent of  $y$  we conclude that

$$\frac{1}{p} \int_{\Omega_1} |\nabla_x v|^p dy dx - \int_{\Omega_1} v (g_+ - g_-) dx \leq \frac{1}{p} \int_{\Omega_1} |\nabla_x u_p|^p dx - \int_{\Omega_1} u_p (g_+ - g_-) dx.$$

Hence the limit  $v$  is a minimizer. By uniqueness we must have  $v = u_p$  and then it holds that  $\lim_{\varepsilon \rightarrow 0} u_p^\varepsilon = u_p$ .

**Corollary 5.1** *Under the same assumptions of Theorem 5.2 we have that*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \min_{\substack{\bar{v} \in W^{1,p}(\Omega_\varepsilon) \\ \bar{v}(0) = 0}} \frac{1}{\varepsilon^l} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v} (f_+^\varepsilon - f_-^\varepsilon) \right\} \\ = \min_{\substack{v \in W^{1,p}(\Omega_1) \\ v(0) = 0}} \frac{1}{p} \int_{\Omega_1} |\nabla_x v|^p - \int_{\Omega_1} v (g_+ - g_-). \end{aligned}$$

**Remark 5.2** The unique minimizer  $u_p$  of (5.5) is a weak solution to

$$\begin{cases} -\Delta_p u_p = g_+ - g_- & \text{in } \Omega_1, \\ |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega_1, \\ u_p(0) = 0. \end{cases}$$

**Remark 5.3** When we deal with a general domain  $\Omega$  (instead of a product domain) and we take the limit as  $\varepsilon \rightarrow 0$  the limit problem that appears involve the weight

$$\omega(x) = |\{y : (x, y) \in \Omega\}|.$$

In fact, with the same arguments used before, we get that the uniform limit of  $u_p^\varepsilon$  as  $\varepsilon \rightarrow 0$  is a weak solution to

$$\begin{cases} -\operatorname{div}(\omega |\nabla u_p|^{p-2} \nabla u_p) = g_+ - g_- & \text{in } \Omega_1, \\ \omega |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega_1, \\ u_p(0) = 0. \end{cases}$$

**Theorem 5.3** *Let  $u_p$  be the unique minimizer of (5.5). Then*

$$u_p \rightarrow u$$

*uniformly in  $\overline{\Omega}_1$  where  $u$  is Kantorovich potential for the transport of the projections,  $g_+$  to  $g_-$ .*

*Proof.* The proof is analogous to the one of Theorem 5.1 and hence we omit the details.

## 6 Examples.

In this section we look for a method to define, using an optimal transport map from the projections, an approximation for the original problem. The construction of such a transport map is known in the literature as the Knothe map, [18].

To simplify, let us suppose that we are in  $\mathbb{R}^2$ , and we have  $\Omega_1 = (a, b)$  and  $\Omega_2 = (c, d)$ . Hence the projections are defined as  $g_+ : \Omega_1 = (a, b) \rightarrow \mathbb{R}$  and  $g_- : \Omega_1 = (a, b) \rightarrow \mathbb{R}$ . Let us assume that the support of the projections are also intervals, that is,  $supp(g_+(x)) = [\alpha, \beta]$  and  $supp(g_-(y)) = [\gamma, \delta]$ .

Now in one dimension we are going to see two ways to define an optimal transport map for the projections  $T : [\alpha, \beta] \rightarrow [\gamma, \delta]$ . This optimal transport map must satisfy for all  $E \in (\gamma, \delta)$ ,

$$\int_{T^{-1}(E)} g_+(x)dx = \int_E g_-(y)dy.$$

Therefore, assuming that  $T$  is differentiable, we get

$$\int_{T^{-1}(E)} g_+(x)dx = \int_E g_-(y)dy = \int_{T^{-1}(E)} g_-(T(x))|T'(x)|dx.$$

Now we have two options, to consider  $T'(x) \geq 0$  or  $T'(x) \leq 0$ . We will call these two possible applications as  $T_D$  and  $T_I$ . First we will take  $T'(x) \geq 0$  and look for  $T_D$  a solution to the ODE problem,

$$\begin{cases} g_+(x) = g_-(T_D(x))T'_D(x), \\ T_D(\alpha) = \gamma. \end{cases}$$

Observe that we move the mass "directly", it means  $T_D$  preserves orientation. An alternative way to define  $T_D$  for all  $x \in [\alpha, \beta]$  is the following:

$$T(x) = \inf \left\{ y \in [\gamma, \delta] : \int_{\alpha}^x g_+ = \int_{\gamma}^y g_- \right\}.$$

The other choice to define  $T$  is to consider  $T'(x) \leq 0$ . We call it  $T_I$  and have the ODE,

$$\begin{cases} g_+(x) = -g_-(T_I(x))T'_I(x), \\ T_I(\alpha) = \delta. \end{cases}$$

Observe that this time we move the mass reversing the orientation of the interval. An alternative way to define  $T_I$  for all  $x \in [\alpha, \beta]$  is given by,

$$T(x) = \sup \left\{ y \in [\gamma, \delta] : \int_{\alpha}^x g_+ = \int_y^{\delta} g_- \right\}.$$

The two options are optimal.

Now we go back to the original problem and show how we can use these optimal maps in  $\mathbb{R}^2$  to obtain a transport map  $S : supp(f_+) \rightarrow supp(f_-)$ . Let us suppose further that exist  $g_{11}, g_{12}, g_{21}$  and  $g_{22}$  functions which allow us to write:  $supp(f_+) = \{(x, y) \in \mathbb{R}^2 : g_{11}(x) \leq y \leq g_{12}(x)\}$  and  $supp(f_-) = \{(x, y) \in \mathbb{R}^2 : g_{21}(x) \leq y \leq g_{22}(x)\}$ . We will propose  $S$  to be of the form  $S(x, y) = (T_1(x), T_2(x, y))$  (with  $T_1$  equal to  $T_D$  or  $T_I$ ). Hence we want for all  $E \subset \Omega_1 \times \Omega_2$ ,

$$\int_E f_+(x, y)dxdy = \int_{S^{-1}(E)} f_-(x, y)dxdy = \int_E f_-(S(x, y))|det(DS(x, y))|dxdy.$$

Since  $S(x, y) = (T_1(x), T_2(x, y))$  with  $T_1$  independent of  $y$ , we have,

$$DS = \begin{pmatrix} T_1'(x) & \frac{\partial T_2}{\partial x}(x, y) \\ 0 & \frac{\partial T_2}{\partial y}(x, y) \end{pmatrix}.$$

Therefore,

$$|\det(DS(x, y))| = \left| T_1'(x) \frac{\partial T_2}{\partial y}(x, y) \right|.$$

And we obtain,

$$f_+(x, y) = f_-(T_1(x), T_2(x, y)) \left| T_1'(x) \frac{\partial T_2}{\partial y}(x, y) \right|. \tag{6.1}$$

This equation can be seen as an ODE for  $T_2$  as a function of  $y$  (here  $x$  plays the role of a parameter). Now, again, we have two options for  $T_2$  given by consider  $T_2$  increasing or decreasing as a function of  $y$ . In each case we choose as initial conditions to complement (6.1),

$$\begin{cases} T_2(x, g_{11}(x)) = g_{21}(x), & \text{if } \frac{\partial T_2}{\partial y} \geq 0, \\ T_2(x, g_{12}(x)) = g_{22}(x), & \text{if } \frac{\partial T_2}{\partial y} \leq 0. \end{cases}$$

In this way we can construct a transport map  $S$  (that is in general not optimal) moving  $f_+$  to  $f_-$ .

**Example 1.** To start with, let us consider the simplest situation. In  $\mathbb{R}^2$  consider  $f_+$  and  $f_-$  two measures supported on two points with mass 1/2, that is

$$f_+ = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,0)} \quad \text{and} \quad f_- = \frac{1}{2}\delta_{(1,1)} + \frac{1}{2}\delta_{(2,1)}.$$

So, for the projections we have the optimal transport maps  $T_D = x + 1$  and  $T_I = 2 - x$ , and then all possible transport maps  $S$  are given by all possible assignments of  $\{(0, 0), (1, 0)\} \rightarrow \{(1, 1), (2, 1)\}$ . We obtain,

$$S_1(x, y) = (x + 1, y + 1), \quad \text{and} \quad S_2(x, y) = (2 - x, y + 1).$$

Let us compute the total costs corresponding to these maps. We have,

$$\tilde{\mathcal{F}}(S_1) = \sqrt{2} = 1,4142 < \tilde{\mathcal{F}}(S_2) = \frac{1}{2}(1 + \sqrt{5}) = 1,6180.$$

In the contracted domain  $\Omega_1 \times \varepsilon\Omega_2$  we get

$$S_1(x, y) = (x + 1, \varepsilon(y + 1)), \quad \text{and} \quad S_2(x, y) = (2 - x, \varepsilon(y + 1)),$$

with approximate costs (up to the first nontrivial order in  $\varepsilon$ ),

$$\tilde{\mathcal{F}}(S_1) \sim 1 + \frac{\varepsilon^2}{2} + o(\varepsilon^2) < \tilde{\mathcal{F}}(S_2) \sim 1 + \frac{\varepsilon}{2} + o(\varepsilon).$$

**Example 2.** As a second example we consider as  $f_{\pm}$  the characteristic functions of the triangles,  $C_1 = \text{conv}\{(0, 0), (1, 0), (1, 1)\}$  and  $C_2 = \text{conv}\{(3, 0), (3, 1), (2, 1)\}$ . So, for the projections we have the optimal transport maps,

$$T_D = \sqrt{-(x^2 - 2x)} + 2, \quad \text{and} \quad T_I = 3 - x.$$

Then we can obtain four different  $S(x, y)$  transport maps given by the construction that we explained before, these are given by,

$$S_1(x, y) = (\sqrt{-(x^2 - 2x)} + 2, y \frac{\sqrt{-(x^2 - 2x)}}{-x+1} + 3 - x),$$

$$S_2(x, y) = (\sqrt{-(x^2 - 2x)} + 2, y \frac{\sqrt{-(x^2 - 2x)}}{-x+1} + 1),$$

$$S_3(x, y) = (3 - x, y + x),$$

$$S_4(x, y) = (3 - x, 1 - y).$$

Now, we approximate the total cost in the thin triangles

$$E_1 = \text{conv}\{(0, 0), (1, 0), (1, \varepsilon)\}$$

and

$$E_2 = \text{conv}\{(3, 0), (3, \varepsilon), (2, \varepsilon)\}$$

with the transport maps

$$R_1(x, y) = (3 - x, \varepsilon(1 - \frac{y}{\varepsilon})) = (3 - x, \varepsilon - y),$$

$$R_2(x, y) = (3 - x, y + \varepsilon x).$$

We estimate the cost as follows:

$$\begin{aligned} \tilde{\mathcal{F}}(R_1) &= \int_0^1 \int_0^{\varepsilon x} \|(x, y) - (3 - x, \varepsilon - y)\| f_+^\varepsilon(x, y) dy dx, \\ &= \int_0^1 \int_0^{\varepsilon x} \|(2x - 3, 2y - \varepsilon)\| \frac{1}{\varepsilon} dy dx, \\ &= \int_0^1 \int_0^{\varepsilon x} \sqrt{(2x - 3)^2 + (2y - \varepsilon)^2} \frac{1}{\varepsilon} dy dx. \end{aligned}$$

We take  $z = \frac{y}{\varepsilon}$  and we obtain

$$\tilde{\mathcal{F}}(R_1) = \int_0^1 \int_0^x \sqrt{(2x - 3)^2 + \varepsilon^2(2y - 1)^2} \frac{1}{\varepsilon} \varepsilon dy dx = A(\varepsilon^2),$$

and hence

$$\tilde{\mathcal{F}}(R_1) = A(0) + A'(0) \varepsilon^2 + O(\varepsilon^4) = \frac{5}{6} + \frac{1}{78} (27 \ln(3) - 26) \varepsilon^2 + O(\varepsilon^4).$$

We perform the same computations for  $R_2(x, y) = (3 - x, y + \varepsilon x)$  and we obtain,

$$\tilde{\mathcal{F}}(R_2) = \frac{5}{6} + \frac{27}{32} (\ln(3) - 1) \varepsilon^2 + O(\varepsilon^4).$$

Since  $\frac{1}{78} (27 \ln(3) - 26) < \frac{27}{32} (\ln(3) - 1)$ , we see that  $\tilde{\mathcal{F}}(R_1) < \tilde{\mathcal{F}}(R_2)$  for  $\varepsilon$  small.

We just note that in this example we obtain that the two possible transport maps, constructed as explained before, considering  $T_1$  increasing or decreasing, may have different costs.

## 7 Boundary conditions.

In this last section we comment briefly on the possibility of using Dirichlet boundary conditions instead of Neumann. Throughout this paper we have used Neumann boundary conditions for the  $p$ -Laplacian approximations. This choice is due to the fact that we want to transport the whole mass of  $f_+^\varepsilon$  to cover the whole mass of  $f_-^\varepsilon$  inside  $\Omega_\varepsilon$ . If we impose Dirichlet boundary conditions, we allow for some mass to be imported (created) at some point on the boundary or exported (eliminated) at other points on the boundary, paying in this case an extra import/export tax per unit of mass given by the value of the Dirichlet datum in addition to the usual transport cost given by the Euclidean distance. This problem was analyzed in detail in [19]. Here we contract the domain in one direction. Therefore, if we impose Dirichlet boundary conditions on the boundary  $\Omega_1 \times \partial\varepsilon\Omega_2$ , it will be more convenient to import/export some part of the mass through the boundary than to transport it inside  $\Omega_\varepsilon$  (since the distance of our masses to that part of the boundary is of order  $\varepsilon$  and hence negligible as  $\varepsilon \rightarrow 0$  while the distance between masses remains of order one as  $\varepsilon \rightarrow 0$ ). Hence, the choice of homogeneous Neumann boundary conditions on  $\Omega_1 \times \partial\varepsilon\Omega_2$  seems natural. However, we can impose Dirichlet boundary conditions on  $\partial\Omega_1 \times \varepsilon\Omega_2$ , but to pass to the limit as  $\varepsilon \rightarrow 0$  we need to take a constant as Dirichlet datum. If we do this we arrive at a limit problem that corresponds to an optimal mass transport problem between the projections in  $\Omega_1$  with import/export taxes at the boundary of  $\Omega_1$  equal to the constant Dirichlet datum.

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