Advanced Nonlinear Studies **14** (2014), 199–237

Regularity for Fully Nonlinear P-Laplacian Parabolic Systems: the Degenerate Case

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Received in revised form 02 March 2013
Communicated by Shair Ahmad

Abstract

This paper introduces new *nonlinear heat approximation* and L^∞ preserving homotopy techniques to investigate regularity properties of bounded weak solutions of strongly coupled p-Laplacian parabolic systems which consist of more than one equation defined on a domain of any dimension. The main results imply everywhere Hölder continuity of bounded weak solutions and the global existence of strong solutions to nonlinear p-Laplacian systems.

2010 Mathematics Subject Classification. 49J20, 35J92.
Key words. Optimal mass transport, $p-$Laplacian equations, thin domains.

1 Introduction

We study the Hölder regularity of bounded weak solutions of nonlinear p-Laplacian parabolic systems of the form

$$u_t = \operatorname{div}(A(u, Du)) + F(u, Du), \tag{1.1}$$

in a domain $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, with Ω being an open subset of $\mathbb{R}^n$, $n \geq 1$. The *vector valued* functions u, F take values in $\mathbb{R}^m$, $m \geq 1$. Du denotes the spatial derivative of u. Here, $A(u, \zeta)$ is a nonlinear map from $\mathbb{R}^m \times \mathbb{R}^{nm}$ into $\mathbb{R}^{nm}$.

A weak solution u to (1.1) is a function $u \in W_2^{1,0}(Q, \mathbb{R}^m)$ such that

$$\iint_Q [\langle -u, \phi_t\rangle + \langle A(u, Du), D\phi\rangle]\, dz = \iint_Q \langle F(u, Du), \phi\rangle\, dz$$

for all $\phi \in C^1_0(Q, \mathbb{R}^m)$. Here, we write $dz = dxdt$.

The time dependent p-Laplacian scalar equation has been one of most widely studied nonlinear degenerate parabolic equations. The particular feature of (1.1) is its gradient-dependent diffusivity. Such systems, and their stationary counterparts, appear in different models in non-Newtonian fluids, turbulent flows in porous media, certain diffusion or heat transfer processes, and recently in image processing.

A large body of literature on p-Laplacian systems has been devoted to the following system

$$u_t = \mathrm{div}(|Du|^{p-2}Du) + F(u, Du)$$

which is a special case of (1.1) where $A(u, Du) = |Du|^{p-2}Du$ does not depend explicitly on u. In this case, the regularity theory of bounded weak solutions was then almost settled and masterfully presented in the text book [3] (see also [6] for the stationary counterpart). The techniques and results also hold for systems where A depends smoothly on x, t. In fact, under suitable assumptions on F, we now know that bounded weak solutions to the above systems have Hölder continuous spatial derivatives. The theory was then based on a far-reaching combination of generalized DiGiorgi and Moser's methods for scalar equations.

However, this method breaks down in dealing with systems (1.1) allowing more general structural conditions and with the diffusivity A depending explicitly on the unkown u. First of all, the dispersion of the eigenvalues of the derivative of A with respect to the second variable Du will prevent the Moser type iteration techniques in [3, Chapter IX] from being applicable in order to show that $|Du|$ is locally bounded, a starting and crucial point in defining the scaled cylinders in the next steps. Secondly, the presence of u in A will create extra terms when one differentiates the system in order to obtain a new system satisfied by Du. These extra terms may not be well defined if u is not yet known to be Hölder continuous.

In this work, we choose a different approach. We will establish the Hölder continuity of weak solutions to (1.1) by using a homotopy argument. We assume that the system (1.1) can be imbedded in a family of systems and at least one of which has the property that its bounded weak solutions are Hölder continuous and satisfy a *scaling decay estimate*. Under suitable assumptions, we show that this property will be carried onto bounded weak solutions of the considered system. This type of decay estimate with scaling was also used in [3] using the local supremum norm of $|Du|$. In our case, $|Du|$ is not yet known to be locally bounded and the best we can say is that $|Du|^q$ is locally integrable for some $q > p$. The scaled cylinders in this work must then be defined differently. We will use the average mean of $|Du|^p$ instead of its unavailable supremum norm.

Thus, we will consider a family of parabolic systems parameterized by $\nu \in [0, 1]$

$$u_t = \mathrm{div}(A(\nu, u, Du)) + F(\nu, u, Du), \text{ in } Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}. \tag{1.2}$$

We assume that (1.1) is the above system when $\nu = 1$ and Hölder continuity results are known for the system when $\nu = 0$. Inspired by [3, Proposition 3.1], which gives Hölder continuity for weak solutions to *scalar* degenerate equations, we introduce here the so called *scaling decay property* D). We then consider a subset $\mathcal{I}$ of parameters in $[0, 1]$ where bounded weak solutions of the above system satisfy this property. The main goal is then to prove that $\mathcal{I}$ is both open and closed in $[0, 1]$ so that $\mathcal{I} = [0, 1]$ and the desired Hölder continuity for solutions to (1.1) is obtained. Our first two main results concerning the open and closed properties of $\mathcal{I}$ will be presented under two sets of conditions as they will be established by using different tools, and they may be independently of interest in other applications.

The main vehicle in the proof of $\mathcal{I}$ being open is Proposition 4.3, which is the p-Laplacian version of the *nonlinear* heat approximation result in our recent work [14]. Basically, it asserts that if a vector valued function u *almost* and weakly solves a system like (1.2), with $v \in \mathcal{I}$, then it can be approximated in certain controllable way by a solution v of the system. By this, property D) of v can be carried over to u. The proof of this p-Laplacian approximation version is not a simple extension of the result in [14] as our systems are degenerate (or singular) and many more technical tools are needed. Among them is a measure theoretic result Lemma 3.8 in Section 3 establishing uniform continuity of the integrals of the derivatives of approximated solutions. As a consequence of this, in Lemma 3.11, we also present a result on higher integrability of the derivatives of "almost" weak solutions to a p-Laplacian system. Similar results for weak solutions to p-Laplacian systems were first reported in [10]. We should mention that the *linear* approximation technique was used in [2] (Lemma 4.1) to obtain *partial* regularity of weak solutions. Partial regularity up to the boundary was also studied in [1, 2]. In this paper, we restrict ourself to the interior regularity property as the boundary case can be studied as well but require additional technicalities.

On the other hand, the above argument is local by nature and cannot be used to prove that $\mathcal{I}$ is closed as it lacks certain uniform estimates in order to show that limits of a sequence of regular solutions are also regular. To this end, we will use a different approach deriving uniform and global estimates for the integrals of spatial derivatives of regular solutions with uniform bounded norms.

In this paper, our main results only concern the degenerate case, i.e. $p > 2$. The singular case, $p < 2$, can be dealt with in a similar way but much more subtle and will be reported in a forthcoming work. However, most of our main tools work for both cases and we report them here in Section 3 and Section 4 for future references. We will specifically state the range of p for which our results hold.

The paper is organized as follows. In Section 2, we will introduce notations and discuss in details our hypotheses and main theorems. Section 3 collects technical lemmas. Section 4 presents our main vehicles - the p-Laplacian nonlinear approximation results. The proof that $\mathcal{I}$ is open will be given in Section 5. Finally, Section 6 details the proof of $\mathcal{I}$ being closed and concludes our paper.

Acknowledgement: The author would like to express his gratitude to the anonymous referee's careful reading of the original manuscript and constructive comments which greatly impact the presentation of this paper.

2 Main results

Throughout this paper, for some $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $R, \rho > 0$, $Q_{R,\rho}(z_0)$ denotes the parabolic cylinder centered at z_0 with radius R, ρ. That is, $Q_{R,\rho}(z_0) = B_R(x_0) \times [t_0 - \rho, t_0]$. We also abbreviate by $S_{R,\rho}(z_0) = B_R(x_0) \times \{t_0 - \rho\} \cup \partial B_R(x_0) \times [t_0 - \rho, t_0]$, the parabolic boundary of $Q_R(z_0)$. If the center z_0 is understood, we simply write $Q_{R,\rho}, S_{R,\rho}$ for $Q_{R,\rho}(z_0)$ and $S_{R,\rho}(z_0)$, respectively.

For a given cylinder $Q = B \times [a, b]$ and $p > 1$, we consider the space $V_p(Q) = V_p(Q, \mathbb{R}^m)$ of vector valued functions $u : Q \to \mathbb{R}^m$ with norm $\|\cdot\|_{V_p(Q)}$ defined by

$$\|u\|_{V_p(Q)} = \sup_{t \in [a,b]} \|u(\cdot, t)\|_{L^2(B)} + \|Du\|_{L^p(Q)}.$$

By $V_p^0(Q)$ we denote the closure of $C_0^1(Q)$ in $V_p(Q)$ with respect to the above norm.

Let $z_0 = (x_0, t_0)$ and $Q_{R,\rho}(z_0)$ be any parabolic cylinder in R^{n+1}, the following scaled norm will also be used

$$\|u\|_{V(Q_{R,\rho})} = \sup_{t \in [t_0 - \rho, t_0]} \left(R^{-n} \int_{B_R(x_0)} |u(x,t)|^2 \, dx \right)^{\frac{1}{2}} + \left(R^{p-n} \rho^{-1} \iint_{Q_{R,\rho}} |Du|^p \, dz \right)^{\frac{1}{p}}.$$

Obviously, this norm is invariant by dilations.

For any integrable function $u : Q \to \mathbb{R}^m$ and any measurable subset A of Q, we write

$$u_A = \frac{1}{|A|}\int_A u(z)dz = \fint\!\!\fint_A u(z)\,dz.$$

If A is a cylinder $Q_{R,\rho} = Q_{R,\rho}(z_0)$ and there is no possibility of ambiguity, we simply write $u_{R,\rho} = u_{Q_{R,\rho}}$. Furthermore, if ρ is defined in term of R and the relation between R, ρ is clear we also abbreviate $u_{R,\rho}$ by u_R for the sake of simplicity.

As our results are local in nature, without loss of generality, we will simply consider Q being the unit parabolic cylinder $B_1(0) \times [-1, 0]$ throughout this paper. We then consider a family of systems

$$u_t = \operatorname{div}(A(v,u,Du)) + F(v,u,Du), \text{ in } Q = B_1 \times (-1,0) \subset \mathbb{R}^{n+1} \text{ and } v \in [0,1]. \tag{2.3}$$

By a bounded weak solution u to this system we mean a bounded vector valued function u satisfying

$$\iint_Q [-u\phi_t + \langle A(v,u,Du), D\phi\rangle]\,dz - \iint_Q \langle F(v,u,Du), \phi\rangle\,dz = 0, \quad \forall \phi \in C_0^1(Q, \mathbb{R}^m).$$

For simplicity, we will mainly consider the case $F \equiv 0$ in our discussion. The presence of F can be treated with minor modifications and we will briefly discuss this case at the end of this section.

We will always consider matrices $A(v,u,\zeta)$ satisfying the following ellipticity condition

E) There are positive constants λ, Λ such that for all $\zeta, \eta \in \mathbb{R}^{nm}$, $u \in \mathbb{R}^m$ and $v \in [0,1]$

$$\langle A(v,u,\zeta), \zeta\rangle \ge \lambda|\zeta|^p, \quad |\langle A(v,u,\zeta), \eta\rangle| \le \Lambda|\zeta|^{p-1}|\eta|. \tag{2.4}$$

In the study of the Hölder regularity of a weak solution u, it is now well known that (see [5]) one needs to establish a *mean oscillation decay estimate*: For some $\tau, \alpha \in (0,1)$ and any $Q_R = Q_{R,\rho}(x,t) \subset Q$, there are positive constant $\tau_0, C(\tau_0)$ such that

$$\fint\!\!\fint_{Q_{\tau R,\tau\rho}} |u - u_{\tau R,\tau\rho}|^2\,dz \le C(\tau_0)\tau^\alpha \fint\!\!\fint_{Q_{R,\rho}} |u - u_{R,\rho}|^2\,dz \quad \forall \tau \in (0,\tau_0). \tag{2.5}$$

However, due to the degeneracy/singularity of the diffusion matrix A, such decay estimates do not hold in general for cylinders whose space-time configuration are not uniform for all solutions and depend only on the parameters defining the systems. Roughly speaking, we can only obtain here (2.5) when R, ρ are linked via an intrinsic scaling determined by the solution u itself. Yet this decay property still gives the desired Hölder continuity. The idea of using scaled cylinders was known in literature (see [3]) where scalar equations were studied so that Harnack type inequalities could be established and gave the Hölder continuity. Here, such techniques are no longer available and we have to scale the cylinders by using the average oscillations of weak solutions. Being inspired by [3, Proposition 3.1], we introduce the following decay property. Note that the mean oscillation of vector valued solutions is used here in place of the essential oscillation for scalar solutions in [3].

We say that a bounded vector valued function $v : Q \to \mathbb{R}^m$ satisfies a *scaling decay property* if

D) *Let $M = \sup_Q |v|$. For any $R_0 > 0, \eta \in (0,1)$, and $(x_0, t_0) \in Q$ there are positive numbers $A, K, L, \alpha_0, \omega_0$ depending on M, η (with K, A sufficiently large) such that there exist the following sequences*

$$R_k = \frac{R_0}{K^k}, \quad \omega_{k+1} = \max\{\eta\omega_k, LR_k^{\alpha_0}\}, \quad S_k = \frac{\omega_k}{A}, Q_k = B_{R_k}(x_0) \times [t_0 - S_k^{2-p}R_k^p, t_0], \tag{2.6}$$

such that if

$$\omega_0^p \geq \fint\!\!\fint_{Q_0} |v - (v)_0|^p \, dz, \quad (v)_0 = \fint\!\!\fint_{Q_0} v \, dz$$

then for any integer $k = 1, \ldots$

$$\omega_k^p \geq \fint\!\!\fint_{Q_k} |v - (v)_k|^p \, dz, \quad (v)_k = \fint\!\!\fint_{Q_k} v \, dz. \tag{2.7}$$

Our first main result shows that if (2.3), for some parameter $\nu \in [0,1]$, is a "nice" system in the sense that its bounded weak solutions satisfy the decay property D), then "near-by" systems are also nice. To be more precise, let us describe this result in details here. We first suppose that the family of systems (2.3) contains at least a "nice" one (Examples may include diagonal systems or those with no explicit dependence of u in the matrix A, see Chapters III and IX in [3]).

I) There is a nonempty set $\mathcal{I} \subset [0,1]$ such that for any ν in $\mathcal{I}$ the decay property D) holds for any bounded weak solution to (2.3). The same assumption applies to the systems with u being replaced by any constant vector C, i.e.

$$\iint_Q [-u\phi_t + \langle A(\nu, C, Du), D\phi \rangle] \, dz - \iint_Q F(\nu, C, Du)\phi \, dz = 0, \quad \forall \phi \in C_0^1(Q, \mathbb{R}^m)$$

We then consider the following structural assumptions on the matrices $A(\nu, u, Du)$.

O.1) (Uniform ellipticity) For any $\nu \in [0,1]$, $A(\nu, u, \zeta)$ satisfies the ellipticity condition E) for some positive constants λ, Λ.

O.2) (Monotonicity) For any $w \in \mathbb{R}^m$ and $U, V \in \mathbb{R}^{nm}$, there holds

$$\langle A(w, U) - A(w, V), U - V \rangle \geq \lambda_0(U, V)|U - V|^2, \tag{2.8}$$

where

$$\lambda_0(U, V) = c_0 \begin{cases} \min\{|U|^{p-2}, |V|^{p-2}\} & U \neq 0 \text{ or } V \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for some positive constant c_0.

O.3) (Continuity) $A(\nu, u, \zeta)$ is Hölder in $\nu \in [0,1]$ and Lipschitz in u. That is,

$$|A(\nu, u, \zeta) - A(\mu, u, \zeta)| \leq C|\nu - \mu|^\theta |\zeta|^{p-1}, \tag{2.9}$$

$$|A(\nu, u, \zeta) - A(\nu, v, \zeta)| \leq C|u - v||\zeta|^{p-1} \tag{2.10}$$

for some $C > 0$, $\theta \in (0,1)$ and any $\nu, \mu \in [0,1]$, $u, v \in \mathbb{R}^m$, $\zeta \in \mathbb{R}^{nm}$.

O.4) (Existence) For $\nu \in \mathcal{I}$ and any cylinder $Q' \subset Q$, and any vector valued function $g \in C^1(Q')$, the system

$$\begin{cases} \iint_{Q_R} [-u\phi_t + \langle A(\nu, u, Du), D\phi \rangle] \, dz = 0, \quad \forall \phi \in C_0^1(Q', \mathbb{R}^m), \\ u = g \text{ on } S_{Q'} \end{cases} \tag{2.11}$$

has a bounded weak solution u (and thus satisfies the property D)).

O.5) (Uniform maximum principle) For any $\nu \in [0, 1]$ and $Q' \subset Q$, if $g \in C^1(Q')$ then there is a constant $C(\|g\|_{L^\infty(Q')})$ such that any weak solution u to the system (2.11) is bounded and satisfies the estimate $\|u\|_{L^\infty(Q')} \le C(\|g\|_{L^\infty(Q')})$.

Remark 2.1 We should remark that if the argument u in $A(\nu, u, Du)$ is replaced by a constant vector and the data g is bounded then the existence of a weak solution in O.4) is granted by classical approximation methods as in [11] or [18]. Otherwise, the existence condition O.4) can be satisfied by using Galërkin's method if g is sufficiently regular and the solution u is a-priori known to be Hölder continuous. Since $\mathcal{I}$ is the set of parameters for which the systems has bounded weak solutions being Hölder continuous, O.4) is justified.

Our first main result then asserts that the set $\mathcal{I}$, where the property D) holds, is open.

Theorem 2.2 *Suppose that I) and O.1)-O.5) hold and $p > 2$. Then $\mathcal{I}$ is open in the usual topology of* $[0, 1]$. *Moreover, bounded weak solutions to the system (2.3) with $\nu \in \mathcal{I}$ are Hölder continuous in the interior of Q.*

The above theorem relies on a nontrivial generalization of the so called nonlinear heat approximation lemma which was introduced in our earlier work [14] concerning nondegenerate systems.

Next, we will give conditions for the set $\mathcal{I}$ to be closed in $[0, 1]$. To this end, we take a sequence $\{\nu_k\}$ in $\mathcal{I}$ such that $\nu_k \to \mu$ and we will show that $\mu \in \mathcal{I}$. We first require that any bounded weak solution u to (2.3), with $\nu = \mu$, can be weakly approximated by "nice" solutions.

II) For each $\nu \in \mathcal{I}$, the system (2.3) satisfies the existence condition O.4) and maximum principle O.6). Moreover, if $\nu_k \in \mathcal{I}$ and $\nu_k \to \mu$ then for any bounded weak solution u to (2.3) with $\nu = \mu$ there is a sequence $\{v_k\}$ of Hölder continuous solutions to (2.3), with $\nu = \nu_k$, such that Dv_k converges weakly to Du in $L^p(Q)$ and the L^∞ norms of v_k are bounded uniformly in terms of that of u.

Although it will be shown in Section 4 that the above assumption holds under O.1)-O.5) via nonlinear heat approximation, we state II) here for our next result, which can be of interest in itself, so that it is independent of Theorem 2.2. Of course, II) could also be verified by other means via weaker assumptions than O.1)-O.5).

We then consider $\nu \in \mathcal{I}$ and a C^1 solution v to

$$\begin{cases} v_t = \mathrm{div}(A(\nu, v, Dv)) & \text{in } Q, \\ v = g & \text{on } S. \end{cases} \tag{2.12}$$

The boundary condition g is assumed to be smooth. By II), the above system satisfies the maximum principle and we can define

$$M_{\nu,v} = \sup_{Q_{\frac{3}{4}}} |v|.$$

For any bounded weak solution v to (2.12), we then impose the following assumptions on the structure of the system.

M.1) The matrix $(A^{ij}_{kl}) = \dfrac{\partial A}{\partial \zeta}(\nu, v, \zeta)$ is elliptic with the ellipticity constants $\lambda_{\nu,v}, \Lambda_{\nu,v}$. That is

$$\sum_{i,j=1}^{m} \sum_{k,l=1}^{n} A^{ij}_{kl} \eta^i_k \eta^j_l \ge \lambda_{\nu,v} |\eta|^2, \quad \sum_{i,k} (\sum_{j,l} A^{ij}_{kl} \eta^j_l)^2 \le \Lambda^2_{\nu,v} |\eta|^2 \quad \forall \eta \in \mathbb{R}^{nm}. \tag{2.13}$$

Moreover, for some positive constants λ_ν, Λ_ν we have

$$\lambda_{\nu,v} \geq \lambda_\nu |\zeta|^{p-2}, \quad \Lambda_{\nu,v} \leq \Lambda_\nu |\zeta|^{p-2}. \tag{2.14}$$

If $n > 2$, we also assume that

$$\sup\{\frac{\Lambda_{\nu,v}}{\lambda_{\nu,v}} \ : \ v \text{ is a bounded weak solution to (2.12)}\} < \frac{n}{n-2}. \tag{2.15}$$

M.2) For every ν in $\mathcal{I}$ and any bounded weak solution v to (2.12), there exists a positive constant $a_{\nu,v}$ such that

$$|\frac{\partial A}{\partial v}(\nu, v, \xi)| \leq a_{\nu,v}|\xi|^{p-1} \text{ with } 2a_{\nu,v}M_{\nu,v}(p+n-1) < \sigma_0 \widehat{\lambda}_\nu, \tag{2.16}$$

where σ_0 is a fixed number in (0, 1) and

$$\widehat{\lambda}_\nu = (1-\delta^2)\lambda_\nu \text{ and } \delta = \frac{n-2}{n} \sup\{\frac{\Lambda_{\nu,v}}{\lambda_{\nu,v}} \ : \ v \text{ is a bounded weak solution to (2.12)}\}.$$

Note that $\widehat{\lambda}_{\nu,v} > 0$ due to (2.15). Meanwhile, (2.15) (when $n > 2$) requires that the principal eigenvalues $\Lambda_{\nu,v}, \lambda_{\nu,v}$ of $\frac{\partial A}{\partial \zeta}$ are not too far apart and this condition also imposes some restriction on the range of p. We are indebted for the latter remark to one of the referees. We should also remark that the constants λ_ν, Λ_ν could be allowed to depend on v as long as there are fixed positive numbers c_1, c_2 such that quotient $c_1 \leq \lambda_{\nu,v}/\Lambda_{\nu,v} \leq c_2$. We assume however that they are constants for the sake of simplicity.

Our next main result reads

Theorem 2.3 *Assume the conditions II) and M.1)- M.2) and that $p > 2$. The set $\mathcal{I}$ is closed in [0,1].*

Combining with the results of the previous theorem, as we remarked that O.1)-O.5) are sufficient for II), we then have

Theorem 2.4 *Assuming I), O.1)-O.5) and M.1)- M.2) and $p > 2$, then $I = [0, 1]$. Thus, bounded weak solutions of (2.3) are locally Hölder continuous.*

Finally, we remark that our proof continues to hold for systems like

$$u_t = \operatorname{div}(A(\nu, u, Du)) + F(\nu, u, Du), \tag{2.17}$$

if the nonlinearity F satisfies

$$|\frac{\partial F}{\partial v}(\nu, v, \xi)| \leq C + C|v|^l + \varepsilon_0|\xi|^p, \quad |\frac{\partial F}{\partial \xi}(\nu, v, \xi)| \leq C + C|v|^l + \varepsilon_0|\xi|^{p-1}$$

for some $l > 0$ and sufficiently small $\varepsilon_0 > 0$. The presence of F would cause extra terms in our arguments but they could be easily treated by invoking the Hölder and Young inequalities.

3 Technical lemmas

In this section, we present various estimates on a vector valued function u weakly satisfying certain differential inequalities. Although our main results in this work concern the degenerate situation when $p > 2$ and the singular case ($p < 2$) will be treated in future works, several results in this section hold for $p < 1$ and we will specify the range of p in each statement for future reference.

Throughout this paper, the constants $C, C_1, \ldots$ can change line by line but they are all universal constants as they depend only on the initially fixed parameters (such as n, m). For any two quantities A, B, we write $A \sim B$ if there are universal positive constants C_1, C_2 such that $C_1 A \le B \le C_2 A$.

First of all, we recall the following Sobolev inequalities in a ball of $\mathbb{R}^n$ and the dependence of the constants on the size of the ball. Let ϕ be a function in suitable Sobolev spaces on B_1 with zero trace on the boundary. By scaling, with $x = \frac{1}{R}\bar{x}$ and $\phi(x) = \bar{\phi}(\bar{x})$, we have the followings facts on the kth spatial derivatives $D^{(k)}$

$$\|D_x^{(k)}\phi\|^p_{L^p(B_1)} \sim CR^{pk-n}\|D_{\bar{x}}^{(k)}\bar{\phi}\|^p_{L^p(B_R)} \text{ and } \|D_x\phi\|^p_{L^q(B_1)} \sim CR^{(1-\frac{n}{q})p}\|D_{\bar{x}}\bar{\phi}\|^p_{L^q(B_R)}.$$

In particular, for $q = p_* = pn/(n+p)$, we easily see that $\|D_x\phi\|^p_{L^q(B_1)} \sim CR^{-n}\|D_{\bar{x}}\bar{\phi}\|^p_{L^q(B_R)}$. Using this in the Sobolev-Poincaré inequality on B_1, we get

$$\|\phi\|^p_{L^p(B_1)} \le C(n)\|D_x\phi\|^p_{L^q(B_1)} \Rightarrow \|\bar{\phi}\|^p_{L^p(B_R)} \le C(n)R^{n+(1-\frac{n}{q})p}\|D_{\bar{x}}\bar{\phi}\|^p_{L^q(B_R)}$$

for all p such that $q \le p \le q^* = qn(n-q)$.

On the other hand, with $\|\psi\|_{W_0^{k,p}(B_1)} = \int_{B_1} |D^{(k)}\psi|^p\, dx$, we have

$$\|\phi\|_{W^{-k,p'}(B_1)} = \sup_{\psi} \frac{1}{\|\psi\|_{W_0^{k,p}(B_1)}}\int_{B_1}\phi\psi dx \le \sup_{\psi}\frac{C}{R^{\frac{pk-n}{p}}\|\bar{\psi}\|_{W_0^{k,p}(B_R)}}\frac{1}{R^n}\int_{B_R}\bar{\phi}\bar{\psi}d\bar{x}.$$

This also implies $\|\phi\|^p_{W^{-k,p'}(B_1)} \sim CR^{-pk+n-pn}\|\bar{\phi}\|^p_{W^{-k,p'}(B_R)}$.

We begin with the following lemma.

Lemma 3.1 *Let $Q_{R,\rho} = B_R \times (-\rho, 0)$ and $u : Q_{R,\rho} \to \mathbb{R}^m$ be in $V_p(Q_{R,\rho})$ for some $p > 1$. Assume that there is a function $G \in L^1(Q_{R,\rho})$ such that*

$$\left|\iint_{Q_{R,\rho}} u\phi_t\, dz\right| \le C\iint_{Q_{R,\rho}} |G||D\phi|\, dz \quad \forall \phi \in C_0^1(Q_{R,\rho}). \tag{3.1}$$

Let q be such that $p_* = np/(n+p) \le q < p$. Then, for any $\varepsilon > 0$ there exist positive constants $C, C(\varepsilon)$ such that

$$\begin{aligned}\fint\!\!\fint_{Q_{R,\rho}} |u-u_R|^p\, dz \quad &\le \quad \varepsilon R^p \fint\!\!\fint_{Q_{R,\rho}} |Du|^p\, dz + (C+\varepsilon)R^p\left(\fint\!\!\fint_{Q_{R,\rho}} |Du|^q\, dz\right)^{\frac{p}{q}} \\ &\quad + C(\varepsilon)(\tfrac{\rho}{R})^p\left(\fint\!\!\fint_{Q_{R,\rho}} |G|\, dz\right)^p.\end{aligned} \tag{3.2}$$

Proof. For $s, r \in (-\rho, 0)$ and $\varepsilon > 0$, take $\phi = \psi(x)\eta(t)$ where $\psi \in C_0^1(B_R)$ and $\eta \equiv 1$ in (s, r), η is linear in $(s-\varepsilon, s)$ and $(r, r+\varepsilon)$, η is zero elsewhere. By the Young inequality, the estimate (3.1) becomes

$$\left|\int_{B_R}\left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} udt - \frac{1}{\varepsilon}\int_r^{r+\varepsilon} udt\right)\psi dx\right| \le C\left(\iint_{Q_{R,\rho}} |G\eta|\, dz\right)\|\psi\|_{C_0^1(B_R)}.$$

Let $l > (n+p)/p$ and $m = l - 1 > n/p$. By scaling, with $x = \frac{1}{R}\bar{x}$ and $\phi(x) = \bar{\phi}(\bar{x})$, we have for $\phi \in W_0^{m,p}(B_1)$ that

$$\|\phi\|_{L^\infty(B_1)} \leq C(n)\|D_x^{(m)}\phi\|_{L^p(B_1)} \Rightarrow \|\bar{\phi}\|_{L^\infty(B_R)} \leq CR^{\frac{pm-n}{p}}\|D_{\bar{x}}^{(m)}\bar{\phi}\|_{L^p(B_R)}.$$

Using this for $\bar{\phi} = D\psi$, we obtain $\|\psi\|_{C_0^1(B_R)} \leq CR^{\frac{pl-p-n}{p}}\|\psi\|_{W_0^{l,p}(B_R)}$. Hence, letting $\varepsilon \to 0$, we obtain the following estimate for $s, r \in (-\rho, 0)$

$$\left|\int_{B_R}(u(\cdot,s) - u(\cdot,r))\psi(x)dx\right| \leq CR^{\frac{pl-p-n}{p}}\left(\iint_{Q_{R,\rho}}|G|\,dz\right)\|\psi\|_{W_0^{l,p}(B_R)} \quad \forall\psi \in W_0^{l,p}(B_R).$$

Setting $H(t,s,\cdot) = u(\cdot,s) - u(\cdot,t)$, we just proved that

$$\|H(t,s,\cdot)\|_{W^{-l,p'}(B_R)} \leq CR^{\frac{pl-p-n}{p}}\left(\iint_{Q_{R,\rho}}|G|\,dz\right) \quad \forall s,t \in (-\rho,0). \tag{3.3}$$

Let q be such that $p_* = np/(n+p) \leq q < p$. We have $W^{1,q}(B_1) \subset L^p(B_1) \subset W^{-l,p'}(B_1)$. Because $W^{1,q}(B_1)$ is compactly imbedded in $L^p(B_1)$, a simple argument by contradiction gives the following interpolation inequality

$$\|\phi\|_{L^p(B_1)} \leq \varepsilon\|\phi\|_{W^{1,q}(B_1)} + C(\varepsilon)\|\phi\|_{W^{-l,p'}(B_1)}, \quad \forall\phi \in W^{1,q}(B_1).$$

Using the norm $\|\phi\|_{W^{1,q}(B_1)} = \|D\phi\|_{L^q(B_1)} + \|\phi\|_{L^p(B_1)}$ and choosing ε small, we have

$$\|\phi\|_{L^p(B_1)} \leq \varepsilon\|D\phi\|_{L^q(B_1)} + C(\varepsilon)\|\phi\|_{W^{-l,p'}(B_1)}, \quad \forall\phi \in W^{1,q}(B_1).$$

By a simple scaling argument, we derive from the above that

$$R^{-n}\|\phi\|_{L^p(B_R)}^p \leq \varepsilon R^{(1-\frac{n}{q})p}\|D\phi\|_{L^q(B_R)}^p + C(\varepsilon)R^{-lp+n-pn}\|\phi\|_{W^{-l,p'}(B_R)}^p, \quad \forall\phi \in W^{1,q}(B_R).$$

Applying this to $H(t,s,\cdot)$ and using (3.3), we have

$$R^{-n}\|H(t,s,\cdot)\|_{L^p(B_R)}^p \leq \varepsilon R^{(1-\frac{n}{q})p}\|DH(t,s,\cdot)\|_{L^q(B_R)}^p + C(\varepsilon)R^{-p-pn}\left(\iint_{Q_{R,\rho}}|G|\,dz\right)^p. \tag{3.4}$$

We now choose s such that

$$\int_{B_R}|Du(x,s)|^q\,dx \leq \frac{1}{\rho}\int_{-\rho}^0\int_{B_R}|Du(x,t)|^q\,dxdt.$$

Then

$$\|Du(\cdot,s)\|_{L^q(B_R)}^p \leq \left(\frac{1}{\rho}\int_{-\rho}^0\int_{B_R}|Du(x,t)|^q\,dxdt\right)^{\frac{p}{q}} = \rho^{-p/q}\left(\iint_{Q_{R,\rho}}|Du|^q\,dz\right)^{\frac{p}{q}}. \tag{3.5}$$

On the other hand, by Sobolev-Poincaré's inequality, we also have

$$R^{-n}\|u(\cdot,s) - u(\cdot,s)_R\|_{L^p(B_R)}^p \leq CR^{(1-\frac{n}{q})p}\|Du(\cdot,s)\|_{L^q(B_R)}^p \leq CR^p\left(⨏\!\!\!\!\!\!\iint_{Q_{R,\rho}}|Du|^q\,dz\right)^{\frac{p}{q}}.$$

Obviously,

$$\|u(\cdot,t) - u(\cdot,t)_R\|_{L^p(B_R)}^p \leq C\|u(\cdot,t) - u(\cdot,s)\|_{L^p(B_R)}^p + \|u(\cdot,s) - u(\cdot,s)_R\|_{L^p(B_R)}^p$$

and $|DH(t,s,\cdot)| \le |Du(\cdot,t)| + |Du(\cdot,s)|$. A simple use of Hölder's inequality gives

$$\|Du(\cdot,t)\|^p_{L^q(B_R)} \le CR^{n(\frac{p}{q}-1)}\|Du(\cdot,t)\|^p_{L^p(B_R)}.$$

Therefore, when such s is fixed, the above yields

$$\begin{aligned}\tfrac{1}{R^n}\|u(\cdot,t)-u(\cdot,t)_R\|^p_{L^p(B_R)} \quad &\le \quad \varepsilon R^{p-n}\|Du(\cdot,t)\|^p_{L^p(B_R)} + (C+\varepsilon)CR^p\left(\fint\!\!\!\fint_{Q_{R,\rho}} |Du|^q\,dz\right)^{\frac{p}{q}} \\ &\quad + C(\varepsilon)R^{-p-pn}\left(\iint_{Q_{R,\rho}} |G|\,dz\right)^p.\end{aligned}$$

Integrating the above over $t \in [-\rho, 0]$ and dividing by ρ, we get (3.2) and the proof of the lemma is then complete. ∎

The following Poincaré type inequality is an immediate consequence of the above lemma.

Lemma 3.2 *Under the assumption of Lemma 3.1, if* $\rho = S^{2-p}R^2$ *and* $|G| \le |Du|^{p-1}$ *then for any* $\varepsilon > 0$ *and* $np/(n+p) \le q < p$ *there exist positive constants* $C, C(\varepsilon)$ *such that*

$$\begin{aligned}\fint\!\!\!\fint_{Q_{R,\rho}} |u-u_R|^p\,dz \quad &\le \quad \varepsilon R^p \fint\!\!\!\fint_{Q_{R,\rho}} |Du|^p\,dz + (C+\varepsilon)R^p\left(\fint\!\!\!\fint_{Q_{R,\rho}} |Du|^q\,dz\right)^{\frac{p}{q}} \\ &\quad + C(\varepsilon)(S^{2-p}R)^p\left(\fint\!\!\!\fint_{Q_{R,\rho}} |Du|^{p-1}\,dz\right)^p.\end{aligned} \tag{3.6}$$

We now consider a weak solution $u \in V_p(Q_{1,1})$ to

$$u_t = \operatorname{div}(A(u, Du) \quad \text{in } Q_{1,1}. \tag{3.7}$$

The matrix A is assumed to satisfy the ellipticity conditions E) for some positive constants λ, Λ and $p > 1$.

Using Steklov's average, we can formally test (3.7) with $|u-c|\phi^2$, with ϕ being a cutoff function for $Q_{\frac{1}{2}R,\frac{1}{2}\rho}$, $Q_{R,\rho}$, to get the following Caccioppoli type inequality

Lemma 3.3 *Let u satisfy (3.7). For any constant vector* $c \in \mathbb{R}^m$ *and any* $Q_{R,\rho} \subset Q_{1,1}$

$$\fint\!\!\!\fint_{Q_{\frac{1}{2}R,\frac{1}{2}\rho}} |Du|^p\,dz \le C\frac{1}{R^p}\fint\!\!\!\fint_{Q_{R,\rho}} |u-c|^p\,dz + C\frac{1}{\rho}\fint\!\!\!\fint_{Q_{R,\rho}} |u-c|^2\,dz. \tag{3.8}$$

A consequence of Lemma 3.1 and the above is the following reverse Hölder inequality.

Lemma 3.4 *Assume that $p > 2$ and u satisfies (3.7). If $\rho = S^{2-p}R^2$ for some $S^p \sim \iint_{Q_{R,\rho}} |Du|^p\, dz$ then we have for any given positive ε and $np/(n+p) \le q < p$*

$$\begin{aligned}\iint_{Q_{\frac{1}{2}R,\frac{1}{2}\rho}} |Du|^p\, dz &\le \varepsilon \iint_{Q_{R,\rho}} |Du|^p\, dz + (C+\varepsilon)\left(\iint_{Q_{R,\rho}} |Du|^q\, dz\right)^{\frac{p}{q}} \\ &\quad + C(\varepsilon)\left(\iint_{Q_{R,\rho}} |Du|^{p-1}\, dz\right)^{\frac{p}{p-1}}.\end{aligned} \tag{3.9}$$

Proof. For $\rho = S^{2-p}R^2$ we have (3.6). By Hölder inequality we estimate the last term in (3.6) as follows:

$$(S^{2-p}R)^p \left(\iint_{Q_{R,\rho}} |Du|^{p-1}\, dz\right)^p \le R^p \left(\frac{\iint_{Q_{R,\rho}} |Du|^p\, dz}{S^p}\right)^{(p-2)} \left(\iint_{Q_{R,\rho}} |Du|^{p-1}\, dz\right)^{\frac{p}{p-1}}.$$

If $p > 2$ then the Young inequality $(1-(p-2)/p = 2/p)$ can apply to the last term in (3.8) to yield

$$\frac{1}{\rho}\iint_{Q_{R,\rho}} |u-u_R|^2\, dz = S^{p-2}R^{-2}\iint_{Q_{R,\rho}} |u-u_R|^2\, dz \le \varepsilon S^p + C(\varepsilon)\frac{1}{R^p}\iint_{Q_{R,\rho}} |u-u_R|^p\, dz.$$

Thus, if $S^p \sim \iint_{Q_R} |Du|^p\, dz$, we can combine the above estimates with (3.8) and (3.6) to obtain (3.9) if $p > 2$. ∎

The above result also holds for $2n/(n+2) < p < 2$ but we have to treat the last term in (3.8) differently.

Lemma 3.5 *Assume that $2n/(n+2) < p < 2$ and u satisfies (3.7). If $\rho = S^{2-p}R^2$ for some S satisfying*

$$S^p \ge c(n)\left(\iint_{Q_{2R,2\rho}} |Du|^p\, dz + \frac{1}{\rho}\iint_{Q_{2R,2\rho}} |u-u_{2R}|^2\, dz\right). \tag{3.10}$$

Then we have, for $q = 2n/(n+2)$,

$$\iint_{Q_{\frac{1}{2}R,\frac{1}{2}\rho}} |Du|^p\, dz \le C\left(\iint_{Q_{2R,2\rho}} |Du|^q\, dz\right)^{\frac{p}{q}} + CS^p. \tag{3.11}$$

Proof. Let $\chi(x)$ be a nonnegative smooth function in x with compact support in $B_{\frac{3}{2}R}(x_0)$ such that $\chi \equiv 1$ in B_R, $\chi(x) \in [0,1]$ and $|D\chi(x)| \le c/R$ for all x. We define

$$u^{\chi}_{x_0,R}(t) = \frac{1}{\chi_R}\int_{B_{\frac{3}{2}R}(x_0)} u(x,t)dx \text{ with } \chi_R = \int_{B_{\frac{3}{2}R}(x_0)} \chi(x)dx.$$

For any s, t such that $-\rho < s < t < 0$, we test (3.7) with $(u^{\chi}_{x_0,R}(t) - u^{\chi}_{x_0,R}(s))\chi(x)\eta(\tau)$ (η is defined as in Lemma 3.1), to get the following.

$$|u^{\chi}_{x_0,R}(t) - u^{\chi}_{x_0,R}(s)|^2 \le C\frac{t-s}{R^{n+2}}\int_{Q_{2R,\rho}} |A(u,Du)|^2 dz \le C\frac{t-s}{R^{n+2}}\int_{Q_{2R,\rho}} |Du|^{2p-2} dz.$$

Thus, if $|t-s|\le\rho=S^{2-p}R^2$ then

$$\frac{1}{\rho}|u^{\chi}_{x_0,R}(t)-u^{\chi}_{x_0,R}(s)|^2\le CS^{2-p}\fint\!\!\!\fint_{Q_{2R,\rho}}|Du|^{2p-2}\,dz. \tag{3.12}$$

Obviously, for any $s\in(-\rho,0)$

$$(*)\qquad \begin{aligned}\frac{1}{\rho}\fint\!\!\!\fint_{Q_{R,\rho}}|u-u_R|^2\,dz &\le \frac{1}{\rho}\fint\!\!\!\fint_{Q_R}|u-u^{\chi}_{x_0,R}(s)|^2\,dz\\ &\le \tfrac{1}{\rho}\fint\!\!\!\fint_{Q_R}|u-u^{\chi}_{x_0,R}(t)|^2\,dz+\frac{1}{\rho}\sup_{t\in(-\rho,0)}|u^{\chi}_{x_0,R}(t)-u^{\chi}_{x_0,R}(s)|^2.\end{aligned}$$

We consider the first term on the right and estimate it by

$$\iint_{Q_{R,\rho}}|u-u^{\chi}_{x_0,R}(t)|^2\,dz\le\left(\sup_t\int_{B_R}|u-u^{\chi}_{x_0,R}(t)|^2dx\right)^{1-\frac{q}{2}}\int_{-\rho}^{0}\left(\int_{B_R}|u-u^{\chi}_{x_0,R}(t)|^2dx\right)^{\frac{q}{2}}dt,$$

where $q=2n/(n+2)$. The last factor can be bounded via Poincaré-Sobolev's inequality (in the x variable) by

$$\int_{-\rho}^{0}\left(\int_{B_R}|u-u^{\chi}_{x_0,R}(t)|^2dx\right)^{\frac{q}{2}}dt\le\iint_{Q_{2R,\rho}}|Du|^q\,dz.$$

In addition, because $\chi\equiv1$ in B_R and the integral $\int_{B_{\frac{3}{2}R}}|u-c|^2\chi dx$ is minimized at $c=u^{\chi}_{x_0,R}(t)$ and $\chi(x)\in[0,1]$, we have for any $t\in(-\rho,0)$

$$\int_{B_R}|u-u^{\chi}_{x_0,R}(t)|^2dx\le\int_{B_{\frac{3}{2}R}}|u-u^{\chi}_{x_0,R}(t)|^2\chi dx\le\int_{B_{\frac{3}{2}R}}|u-u_{2R}|^2\chi dx\le\int_{B_{\frac{3}{2}R}}|u-u_{2R}|^2dx.$$

We estimate the last integral by testing the equation (3.7) for u with $(u-u_{2R})\phi^p$, where ϕ is a cut-off function for $Q_{\frac{3}{2}R,\frac{3}{2}R\rho}$ and $Q_{2R,2\rho}$. We easily obtain

$$\sup_{t\in(-\rho,0)}\int_{B_{\frac{3}{2}R}}|u-u_{2R}|^2dx\le\frac{1}{\rho}\iint_{Q_{2R,2\rho}}|u-u_{2R}|^2\,dz+\frac{1}{R^p}\iint_{Q_{2R,2\rho}}|u-u_{2R}|^p\,dz.$$

We deduce from the above two estimates the following

$$\sup_{t\in(-\rho,0)}\int_{B_R}|u-u^{\chi}_{x_0,R}(t)|^2dx\le\frac{1}{\rho}\iint_{Q_{2R,2\rho}}|u-u_{2R}|^2\,dz+\frac{1}{R^p}\iint_{Q_{2R,2\rho}}|u-u_{2R}|^p\,dz. \tag{3.13}$$

Applying Hölder inequality to the right hand side of the Poincaré inequality (3.6), we have

$$\frac{1}{R^p}\fint\!\!\!\fint_{Q_{2R,2\rho}}|u-u_{2R}|^p\,dz\le C\fint\!\!\!\fint_{Q_{2R,2\rho}}|Du|^p\,dz+CS^{(2-p)p}\left(\fint\!\!\!\fint_{Q_{2R,2\rho}}|Du|^p\,dz)\right)^{p-1}.$$

By (3.10), the right hand side is bounded by CS^p. Thus,

$$\frac{1}{R^p}\iint_{Q_{2R,2\rho}}|u-u_{2R}|^p\,dz=\frac{S^{2-p}R^{n+2}}{R^p}\fint\!\!\!\fint_{Q_{2R,2\rho}}|u-u_{2R}|^p\,dz\le CR^{n+2}S^2.$$

(3.10) also gives

$$\frac{1}{\rho}\iint_{Q_{2R,2\rho}} |u-u_{2R}|^2\,dz = \frac{|Q_{2R,2\rho}|}{\rho}⨏⨏_{Q_{2R,2\rho}} |u-u_{2R}|^2\,dz \le C|Q_{R,\rho}|S^p = CR^{n+2}S^2.$$

The above estimates and (3.13) yield

$$\left(\sup_{t\in(-\rho,0)}\int_{B_R} |u-u^\chi_{x_0,R}(t)|^2 dx\right)^{1-q/2} \le C(R^{n+2}S^2)^{\frac{2}{n+2}} = CR^2S^{\frac{4}{n+2}}.$$

Hence, the first term on the right of (*) is bounded by

$$⨏⨏_{Q_{R,\rho}} |u-u^\chi_{x_0,R}(t)|^2\,dz \le CS^{p-2+\frac{4}{n+2}}⨏⨏_{Q_{R,\rho}} |Du|^q\,dz = CS^{p-q}⨏⨏_{Q_{R,\rho}} |Du|^q\,dz.$$

Next, the last term on right of (*) is estimated by (3.12) and Young's inequality

$$\frac{1}{\rho}|u^\chi_{x_0,R}(t)-u^\chi_{x_0,R}(s)|^2 \le CS^p + C\left(⨏⨏_{Q_{2R,\rho}} |Du|^{2p-2}\,dz\right)^{\frac{p}{2p-2}}.$$

Putting these in (*) and using the Young and Hölder inequalities ($2p-2<q$), we obtain

$$\frac{1}{\rho}⨏⨏_{Q_{R,\rho}} |u-u_R|^2\,dz \le C\left(⨏⨏_{Q_{R,\rho}} |Du|^q\,dz\right)^{p/q} + CS^p.$$

From the assumption (3.10) on S and the Caccioppoli inequality (3.8) we derive the desired reverse Hölder inequality (3.11). ∎

Finally, in order to obtain certain uniform continuity for the integrals of gradients of weak solutions, we will need the following measure theoretic result which could be of interest in its own right. (Recall the convention: for any two quantities A, B, we write $A \sim B$ if there are universal positive constants C_1, C_2 such that $C_1A \le B \le C_2A$.)

Lemma 3.6 *Let F, G_k ($k = 1,\dots,M$) be nonnegative integrable functions defined on $Q_{1,1}$ and α,β,m_k be real numbers with $\alpha+1,\beta>0$ and $m_k\in(0,1)$. Assume that for any scaled cylinders $Q_{R,\rho}\subset Q_{2R,2\rho}\subset Q_{1,1}$, with $\rho=S^\alpha R^\beta$ and $S\sim ⨏⨏_{Q_{R,\rho}} F\,dz$, the following holds.*

$$⨏⨏_{Q_{R,\rho}} F\,dz \le \varepsilon⨏⨏_{Q_{2R,2\rho}} F\,dz + \sum_{k=1}^M\left(⨏⨏_{Q_{2R,2\rho}} G_k\,dz\right)^{1/m_k}. \tag{3.14}$$

If $\varepsilon>0$ is sufficiently small (depending on $\|F\|_{L^1(Q_{1,1})},\alpha,\beta,m_k$) then for any subset A of $Q_{\frac12,\frac12}$, $m\in(0,1)$ and

$$t^{\alpha+1} \ge ⨏⨏_{Q_{1,1}} F\,dz, \tag{3.15}$$

there is a positive constant $C=C(n)$ such that

$$\iint_{\Phi_t} F\,dz \le Ct^{1-m}\iint_{\Phi_t} F^m\,dz + C\sum_{k=1}^M t^{1-m_k}\iint_{\Gamma_t^k} G_k\,dz,$$

where $\Phi_t=\{z : z\in A \text{ and } F(z)>t\}$ and $\Gamma_t^k=\{z : z\in A \text{ and } G_k(z)>t^{m_k}\}$.

Proof. For simplicity, we will consider the case when $M = 1$ since it is easy to extend the argument to the case $M > 1$.

Let $P = Q_{\frac{1}{2},\frac{1}{2}}$ and $t = \lambda_0 S$, with $\lambda_0 = \lambda_0(n)$ being a constant to be determined later. We have from (3.15)

$$\lambda_0^{\alpha+1} S^{\alpha+1} \geq \fint\!\!\fint_{Q_{1,1}} F\,dz. \tag{3.16}$$

Define

$$J(r) = \fint\!\!\fint_{Q_{r,S^\alpha r^\beta}} F\,dz, \quad r > 0. \tag{3.17}$$

Obviously, for $r \in [\frac{1}{10}, \frac{1}{2}]$ and $\lambda_0 = \lambda_0(n)$ sufficiently small we have

$$J(r) \leq S^{-\alpha} C(n) \fint\!\!\fint_{Q_{1,1}} F\,dz = S C(n) \frac{1}{S^{\alpha+1}} \fint\!\!\fint_{Q_{1,1}} F\,dz \leq \frac{1}{4} S \quad \forall r \in [\frac{1}{10}, \frac{1}{2}]. \tag{3.18}$$

Consider a point $z \in P$ such that $F(z) > S$. Lebesgue's theorem yields $\lim_{r\to 0} J(r) > S$. Thus, by the continuity of the integral and the above inequality, we can find $r(z) \in (0, \frac{1}{10})$ such that $J(r(z)) = \frac{1}{2}S$ and $J(\rho) \leq \frac{1}{2}S$ for any $\rho \in [r(z), \frac{1}{10}]$. This and (3.18) imply that $J(2r(z))$ and $J(5r(z))$ are bounded by $\frac{1}{2}S$. Moreover, there is a constant c_0 depending on n such that $J(r(z)) \leq c_0 J(2r(z))$ and therefore $J(2r(z)) \geq \frac{1}{2}c_0^{-1}S$. Hence, for $\rho(z) = S^\alpha r^\beta(z)$

$$S \sim \fint\!\!\fint_{Q_{2r(z),S^\alpha(2r)^\beta(z)}} F\,dz \text{ and } \fint\!\!\fint_{Q_{5r(z),S^\alpha(5r)^\beta(z)}} F\,dz \leq \frac{1}{2}S. \tag{3.19}$$

We apply the Calderon-Zygmund lemma to P to obtain a countable family of disjoint subcubes $\{Q_i\} = \{Q_{2r(z_i),S^\alpha(2r)^\beta(z_i)}\}$ such that $\{z : z \in P \text{ and } F(z) > S\} \subset \cup_i \hat{Q}_i$. Here, $\hat{Q}_i = Q_{5r(z_i),S^\alpha(5r)^\beta(z_i)}$.

Let A be a subset of $Q_{\frac{1}{2},\frac{1}{2}}$ and $\Phi_t = \{z : z \in A \text{ and } F(z) > t\}$ and $\Gamma_t = \{z : z \in A \text{ and } G(z) > t^{m_1}\}$. We will only consider subcubes Q_i's such that $Q_i \cap A \neq \emptyset$. For such a subcube, $Q_{2r(z_i),S^\alpha(2r)^\beta(z_i)} \subset Q_{4r(z_i),S^\alpha(4r)^\beta(z_i)} \subset Q_{1,1}$ and by (3.19) we see that (3.14) holds and we have two cases (with $\bar{Q}_{r,\rho} = Q_{2r,2\rho}$)

$$\fint\!\!\fint_{Q_i} F\,dz \leq 2\varepsilon \fint\!\!\fint_{\bar{Q}_i} F\,dz \text{ or } \fint\!\!\fint_{Q_i} F\,dz \leq 2\left(\fint\!\!\fint_{\bar{Q}_i} G\,dz\right)^{1/m_1}. \tag{3.20}$$

If the second case of (3.20) holds then because $S \sim \fint\!\!\fint_{Q_i} F\,dz$ we have the following

$$c_1(n)^{-m_1} S^{m_1} < \left(\fint\!\!\fint_{Q_i} F\,dz\right)^{m_1} \leq 2^{m_1} \fint\!\!\fint_{\bar{Q}_i} G\,dz \leq c_2(n) 2^{m_1} \fint\!\!\fint_{\hat{Q}_i} G\,dz.$$

Hence, by splitting the integral on $\hat{Q}_i$ into those on $\hat{Q}_i \cap \Gamma_t$ and $\hat{Q}_i \setminus \Gamma_t$, we have for some $c = c(n)$

$$S^{m_1} |\hat{Q}_i| < (2c)^{m_1} \iint_{\hat{Q}_i \cap \Gamma_t} G\,dz + (2c)^{m_1} \iint_{\hat{Q}_i \setminus \Gamma_t} G\,dz.$$

This gives

$$S^{m_1} |\hat{Q}_i| < (2c)^{m_1} \iint_{\hat{Q}_i \cap \Gamma_t} G\,dz + (2c)^{m_1} t^{m_1} |\hat{Q}_i|$$

and furthermore

$$S^{m_1} |\hat{Q}_i| \leq c_3(n) \iint_{\hat{Q}_i \cap \Gamma_t} G\,dz,$$

if $\lambda_0 = t/S$ is sufficiently small (such that $(2c)^{m_1} t^{m_1} \le \frac{1}{2} S^{m_1}$ and (3.18) still holds). We then fix such λ_0.

Arguing similarly, with G, m_1 now being εF and 1, we see that if the first case of (3.20) holds and ε is small then

$$S|\hat{Q}_i| \le 4\varepsilon \iint_{\hat{Q}_i \cap \Phi_S} F\, dz.$$

Combining the above estimates for $|\hat{Q}_i|$ in both cases, we have

$$S|\hat{Q}_i| \le 4\varepsilon \iint_{\hat{Q}_i \cap \Phi_S} F\, dz + c_3(n) S^{1-m_1} \iint_{\hat{Q}_i \cap \Gamma_t} G\, dz.$$

Since $\Phi_S \subset \cup_i \hat{Q}_i$, we now have, by (3.19) and the fact that Q_i's are disjoint,

$$\iint_{\Phi_S} F\, dz \le \sum_{\hat{Q}_i \cap \Phi_S \neq \emptyset} \iint_{\hat{Q}_i} F\, dz \le \frac{1}{2} S \sum_{\hat{Q}_i \cap \Phi_S \neq \emptyset} |\hat{Q}_i| = C(n) S \sum_i |Q_i| = C(n) S\, |\cup Q_i|\,.$$

By Vitali's covering lemma, we can find a subsequence of disjoint subcubes $\{\Pi_i\}$ of $\{\hat{Q}_i\}$ such that $\cup Q_i \subset \cup \hat{\Pi}_i$ and therefore

$$|\cup Q_i| \le \left|\cup \hat{\Pi}_i\right| \le \sum |\hat{\Pi}_i| \le C_1(n) \sum |\Pi_i|.$$

Thus, as Π's are disjoint, we have from the above estimates for $\Pi_i = \hat{Q}_i$ that

$$\begin{aligned} \iint_{\Phi_S} F\, dz &\le C_2(n) S \sum |\Pi_i| \le C_3(n) \sum \left(4\varepsilon \iint_{\Pi_i \cap \Phi_S} F\, dz + S^{1-m_1} \iint_{\Pi_i \cap \Gamma_t} G\, dz\right) \\ &\le C_3(n) 4\varepsilon \iint_{\Phi_S} F\, dz + C_3(n) S^{1-m_1} \iint_{\Gamma_t} G\, dz. \end{aligned}$$

On the other hand, as λ_0 is small, $t < S$ and therefore $\Phi_S \subset \Phi_t$. We then have

$$\iint_{\Phi_t \setminus \Phi_S} F\, dz \le S^{1-m} \iint_{\Phi_t} F^m\, dz \le 2C_4(n) t^{1-m} \iint_{\Phi_t} F^m\, dz.$$

Hence, by choosing ε sufficiently small, we get

$$\iint_{\Phi_t} F\, dz \le C t^{1-m} \iint_{\Phi_t} F^m\, dz + C t^{1-m_1} \iint_{\Gamma_t} G\, dz.$$

This completes the proof. ∎

Remark 3.7 Note that the number S in the proof is fixed and needs only satisfy (3.16).

We now have the following result on the uniform continuity of integrals.

Lemma 3.8 *Let $\{F^{(i)}\}$ and $\{G_k^{(i)}\}$, $k = 1, \ldots, M$, be bounded sequences of functions in $L^1(Q_{1,1})$ satisfying (3.14) of Lemma 3.6. Assume that the sequences $\{G_k^{(i)}\}$ are weakly convergent in $L^1(Q_{1,1})$. If ε is sufficiently small then there is a subsequence $\{F^{(i_k)}\}$ such that the integrals of $F^{(i_k)}$ are uniformly continuous in the following sense: for any given $\delta > 0$ there is $\mu(\delta) > 0$ such that*

$$\iint_A F^{(i_k)}\, dz < \delta \text{ for all } i_k \text{ if } A \subset Q_{\frac{1}{2},\frac{1}{2}} \text{ and } |A| \le \mu(\delta).$$

Proof. We can again consider the case $M = 1$. Fix a t satisfying (3.15) of Lemma 3.6. We have shown that if ε is small enough then we have for $F = F^{(i)}$ and $G = G^{(i)}$.

$$\begin{aligned}\iint_A F\,dz &= \iint_{A\setminus\Phi_t} F\,dz + \iint_{\Phi_t} F\,dz \le t|A| + Ct^{1-m}\iint_{\Phi_t} F^m\,dz + Ct^{1-m_1}\iint_{\Gamma_t} G\,dz\\ &\le t|A| + Ct^{1-m}\iint_A F^m\,dz + Ct^{1-m_1}\iint_A G\,dz.\end{aligned}$$

Since $m < 1$, $|F^{(i)}|^m$ is bounded in L^q for some $q > 1$ so that there is a subsequence $\{F^{(i_k)}\}$ converges weakly in L^1. By assumption, $G^{(i_k)}$ is also weakly convergent in $L^1(Q_{1,1})$. We can apply [4, Corollary IV.11] on the uniform continuity of integrals to see that the last two integrals in the above estimate are uniformly small if $|A|$ is small. The assertion then follows easily. ∎

Another consequence of Lemma 3.6 is the following higher integrability result.

Lemma 3.9 *Let $\{F^{(i)}\}$ and $\{G_k^{(i)}\}$, $k = 1,\dots,M$, be bounded sequences of functions in $L^1(Q_{1,1})$ satisfying (3.14) of Lemma 3.6 (with $\rho = S^\alpha R^\beta$). If ε is sufficiently small then there is some $r > 1$ such that if $G_i \in L^{r-m_i+1}(Q_{1,1})$ then the following estimate holds:*

$$\iint_{Q_{\frac12,\frac12}} F^r\,dz \le C\left(\iint_{Q_{1,1}} F\,dz\right)^{\frac{r-1}{1+\alpha}+1} + C\Sigma\iint_{Q_{1,1}} G_i^{r-m_i+1}\,dz. \tag{3.21}$$

Proof. For $m \in (0,1)$ and $r > 1$, we define

$$\phi(t) = \iint_{\Phi_t} F^m\,dz, \quad \omega_i(t) = \iint_{\Gamma_t^i} G_i\,dz,$$

and

$$I_r(t) = \iint_{\Phi_t} F^r\,dz.$$

For $A = Q_{\frac12,\frac12}$, the assertion in Lemma 3.6 can be written as

$$-\int_t^\infty \tau^{1-m}d\phi(\tau) \le C[t^{1-m}\phi(t) + \Sigma_i t^{1-m_i}\omega_i(t)] \quad \forall t \ge a := \left(⨌_{Q_{1,1}} F\,dz\right)^{\frac{1}{\alpha+1}}.$$

A simple modification of the Gehring lemma in [6, Lemma 6.3, p.200] provides some $r > 1$ such that

$$-\int_a^\infty u^{r-m}d\phi(u) \le -Ca^{r-1}\int_a^\infty u^{1-m}d\phi(u) - C\Sigma\int_a^\infty u^{r-m_i}d\omega_i(u).$$

This gives (see [6])

$$\iint_{Q_{\frac12,\frac12}} F^r\,dz \le Ca^{r-1}\iint_{Q_{1,1}} F\,dz + C\Sigma\iint_{Q_{1,1}} G_i^{r-m_i+1}\,dz.$$

The definition of a then gives the desired (3.21). ∎

We now consider a sequence of vector functions u_k which almost solve (3.7) in the following sense:

$$\left|\iint_{Q_{R,\rho}} -u\phi_t + \langle A_k(u,Du), D\phi\rangle\,dz\right| \le \delta\|Du\|_{L^p(Q_{R,\rho})}^{p-1}\|D\phi\|_{L^p(Q_{R,\rho})} \tag{3.22}$$

for all $\phi \in V_p^0(Q_{R,\rho})$, $Q_{R,\rho} \subset Q_{1,1}$.

Lemma 3.10 *Assume that $p > 2n/(n+2)$ and $p \neq 2$. Let $\{u_k\}$ be a sequence of vector functions satisfying (3.22) and and assume that the norms $\|Du_k\|_{L^p(Q_{1,1})}$ are uniformly bounded. Furthermore, the sequence of matrices A_k is assumed to satisfy the ellipticity condition E) of Section 2. If δ is sufficiently small then the integrals of $|Du_{k_m}|^{p+\varepsilon}$ are uniformly continuous for some subsequence $\{k_m\}$ and $\varepsilon > 0$.*

Proof. We will apply Lemma 3.8 here by taking $F^{(i)} = |Du_i|^p$ and $G_k^{(i)}$ $(k = 1, 2)$ to be either $|Du_i|^q$ or $|Du_i|^{p-1}$ with $q = np/(n+p)$. Consider first the case $p > 2$. It is easy to see that u_i satisfies the Poincaré and Caccioppoli type inequalities of Lemma 3.1 and Lemma 3.3. Hence the reverse Hölder inequality (3.9) of Lemma 3.4 holds for Du_i with $\varepsilon = 2\delta$ and

$$\rho = S^{2-p}R^2 \text{ and } S^p \sim \fint\!\!\fint_{Q_{R,\rho}} |Du|^p\, dz.$$

Hence, the assumption (3.14) of Lemma 3.6 is verified with α, S there being $(2-p)/p, S^p$ respectively, $\rho = S^{\alpha}R^{\beta}$ and $\beta = 2$.

As the norms $\|Du_k\|_{L^p(Q_{1,1})}$ are uniformly bounded, G_i's are uniformly bounded in L^r for some $r > 1$ and they are weakly convergent in L^1. Lemma 3.8 then applies here to give our lemma if ε, or δ, is sufficiently small.

If $p < 2$, as in the proof of Lemma 3.6, we fix a number S such that (see (3.16))

$$\lambda_0^{\frac{2}{p}} S^2 \geq \left(\fint\!\!\fint_{Q_{1,1}} |Du_k|^p\, dz + \frac{1}{S^{2-p}} \fint\!\!\fint_{Q_{1,1}} |u_k - (u_k)_{Q_{1,1}}|^2\, dz \right).$$

Define (see (3.17))

$$J(r) = \fint\!\!\fint_{Q_{r,S^{2-p}r^2}} |Du_k|^p\, dz + \frac{1}{S^{2-p}r^2} \fint\!\!\fint_{Q_{r,S^{2-p}r^2}} |u_k - (u_k)_{Q_{1,1}}|^2\, dz.$$

We will be interested in the set where $|Du_k|^p > S^p$. At each point z of this set, the argument leading to (3.19) in the proof allows us to find a cylinder $Q_{R,S^{2-p}R^2}(z)$ and a positive constant $c_1(n)$ such that

$$c_1(n) \left(\fint\!\!\fint_{Q_{R,S^{2-p}R^2}} |Du|^p\, dz + \frac{1}{S^{2-p}R^2} \fint\!\!\fint_{Q_{R,S^{2-p}R^2}} |u - u_R|^2\, dz \right) \leq S^p.$$

Therefore, the condition (3.10) on S is verified and a reverse Hölder inequality (3.14) for $F^{(k)} = |Du_k|^p$ is available again. Noting that S is fixed so that the functions $G_k^{(i)}$ are now constant, we see that the proof can continue as before. ∎

We should remark that the above result also holds for $p = 2$ and this case is just a simple consequence of higher integrability of gradients (see [5]).

We also have the following L^q estimates for Du as a result of Lemma 3.9.

Lemma 3.11 *Assume that $p > 2n/(n+2)$. Consider a vector functions u satisfying (3.22) where A_k is assumed to satisfy the ellipticity condition E) of Section 2. If δ is sufficiently small then there is $\varepsilon > 0$ such that*

$$R^{p+\varepsilon} \fint\!\!\fint_{Q_{\frac{1}{2}R, \frac{1}{2}R^p}} |Du|^{p+\varepsilon}\, dz \leq C \tag{3.23}$$

for some constant C depending on $R^p \fint\!\!\fint_{Q_{R,R^p}} |Du|^p\, dz$. If $2n/(n+2) < p < 2$, the above constant C also depends on $\frac{1}{R^p} \fint\!\!\fint_{Q_{R,R^p}} |u - u_R|^2\, dz$.

Proof. We make the scaling $x \to x/R$, $t \to t/R^p$ in the equation for u and need only show (3.23) when $R = 1$. We now set $F = |Du|^p$ and let G_i be either $|Du|^{p-1}$ or $|Du|^q$ with $q = np/(n+p)$. Accordingly, m_i will be either $(p-1)/p$ or $q/p = n/(n+p)$ respectively. Thus, $G_i = F^{m_i}$ and belongs to L^{r-m_i+1} if $(r - m_i + 1)m_i \le 1$. We can find such $r > 1$ if $(2 - m_i)m_i < 1$ but this requirement is just one of the followings

$$(2 - \frac{p-1}{p})\frac{p-1}{p} < 1 \Leftrightarrow p^2 - 1 < p^2 \text{ and } (2 - \frac{n}{n+p})\frac{n}{n+p} < 1 \Leftrightarrow n(n+2p) < (n+p)^2.$$

Thus, Lemma 3.9 applies here with $\alpha = (2-p)/p$ and $r = 1 + \varepsilon/p$ to give (3.23) when $R = 1$. The dependence of the constant C on $\frac{1}{R^p}\iint_{Q_{R,R^p}} |u - u_R|^2\, dz$ comes from the choice of S in the proof of Lemma 3.10. ∎

Remark 3.12 When $p > 2$, from the estimate (3.21) and the above proof it is easy to see that the quantity

$$\left(R^{p+\varepsilon} \iint_{Q_{\frac{1}{2}R,\frac{1}{2}R^p}} |Du|^{p+\varepsilon}\, dz \right)^{\frac{p}{p+\varepsilon}}$$

can be bounded by

$$C_0 \max\{\left(R^p \iint_{Q_{R,R^p}} |Du|^p\, dz \right)^{\frac{p(2+\varepsilon)}{2(p+\varepsilon)}}, R^p \iint_{Q_{R,R^p}} |Du|^p\, dz\}$$

for some constant C_0 independent of Du. We also remark that the exponent $\frac{p(2+\varepsilon)}{2(p+\varepsilon)} > 1$ when $p > 2$.

4 The approximation lemmas

In this section, we establish one of the main tools of our work - the nonlinear approximation lemmas for p-Laplacian systems. To begin, let us fix a cylinder $Q_R = B_R \times [-R^p, 0]$ and consider two collections $\mathcal{A}$ of matrix-valued functions A and $\mathcal{B}$ of vector valued functions B satisfying the following:

a.1) There are positive constants λ, Λ such that the ellipticity condition (2.4) in E) holds for each $A \in \mathcal{A}$.

a.2) For any $B \in \mathcal{B}$, $B(u, \zeta)$ is Lipschitz in u and there is a positive constant $C_{\mathcal{B}}$ such that

$$|B(u,\zeta)| \le C_{\mathcal{B}} + C_{\mathcal{B}}|\zeta|^{p-1}. \tag{4.1}$$

a.3) For each $A \in \mathcal{A}$, $B \in \mathcal{B}$ and any given function $g \in C^1(Q_\rho)$, where $Q_\rho \subset Q_R$, the system

$$\iint_{Q_\rho} -u\phi_t + \langle A(u,Du), D\phi\rangle - \langle B(u,Du), \phi\rangle\, dz = 0,\ \forall \phi \in C_0^1(Q_\rho), \quad u = g \text{ on } S_\rho$$

has a bounded weak solution u, with $\|u\|_{L^\infty(Q_\rho)} \le C(\|g\|_{L^\infty(Q_\rho)})$.

a.4) (Monotonicity) There is a positive constant λ_0 such that

$$\iint_{Q_\rho} \langle A(u, Du) - A(v, Dv), Du - Dv\rangle \, dz \geq \lambda_0 \iint_{Q_\rho} |Du - Dv|^p \, dz$$

for any $u, v \in V_p(Q_\rho)$.

The monotonicity condition a.4) has been frequently assumed in literature concerning the uniqueness of weak solutions to the systems described in a.3). We will state the first version of our nonlinear approximation results under this condition in order to streamline our presentation and ideas. Later, we will replace a.4) by more practical assumptions and the proof will be similar modulo some technical modifications.

We will first prove

Proposition 4.1 *Assume a.1)- a.4) and $p > 2n/(n+2)$. For any given $M, \varepsilon > 0$ there exists $\delta \in (0, 1]$ that depends only on $\lambda, \Lambda, M, \varepsilon$ and $C_{\mathcal{B}}$ such that if $A \in \mathcal{A}, B \in \mathcal{B}$ and $u \in V_p(Q_R)$ satisfying*

$$⨌_{Q_R} |u|^2 \, dz + R^p ⨌_{Q_R} |Du|^p \, dz \leq M, \tag{4.2}$$

and

$$\left| \iint_{Q_R} - u\phi_t + \langle A(u, Du), D\phi\rangle - \langle B(u, Du), \phi\rangle \, dz \right| \leq \delta \|Du\|^{\frac{p}{q}}_{L^p(Q_R)} \|D\phi\|_{L^p(Q_R)} \tag{4.3}$$

for $q = p'$ and for all $\phi \in V_p^0(Q_R)$, then either

$$R^p ⨌_{Q_R} |Du|^p \, dz < \varepsilon, \tag{4.4}$$

or there exists $v \in V_p(Q_{R/2})$ such that

$$⨌_{Q_{R/2}} |v|^2 \, dz + R^p ⨌_{Q_{R/2}} |Dv|^p \, dz \leq C(\lambda, \Lambda, \lambda_0, M)$$

and

$$\iint_{Q_{R/2}} - v\phi_t + \langle A(v, Dv), D\phi\rangle - \langle B(v, Dv), \phi\rangle \, dz = 0 \tag{4.5}$$

for all $\phi \in C_0^1(Q_{R/2})$, and

$$⨌_{Q_{R/2}} |v - u|^2 \, dz + ⨌_{Q_{R/2}} |v - u|^p \, dz \leq \varepsilon R^p ⨌_{Q_R} |Du|^p \, dz. \tag{4.6}$$

Moreover, there is $\beta > 0$ that depends only on λ, Λ, M and $C_{\mathcal{B}}$ such that

$$⨌_{Q_{R/2}} |Dv|^p \, dz \leq \beta ⨌_{Q_{R/2}} |Du|^p \, dz.$$

Proof. For simplicity we will present the proof when B is identically zero. It is not difficult to see that the presence of B, satisfying our assumptions, will introduce some extra terms which can be easily treated by the same argument and a simple use of Young's inequality.

The proof is by contradiction. We then assume that there exist $\varepsilon_0 > 0$ and sequences $\{A_k\}, \{u_k\}, \{\varepsilon_k\}$ and cylinders $Q_{R_k}(x_k, t_k)$ such that for $Q_{R_k} = Q_{R_k, R_k^p}(x_k, t_k)$ we have

$$|\iint_{Q_{R_k}} -u_k\phi_t + \langle A_k(u_k, Du_k), D\phi\rangle\, dz| \le \varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_{R_k})}\|D\phi\|_{L^p(Q_{R_k})} \tag{4.7}$$

for all $\phi \in V^0_p(Q_{R_k})$ but

$$\limsup_{k\to\infty} R_k^p \fint\!\!\fint_{Q_{R_k}} |Du_k|^p\, dz > 0, \tag{4.8}$$

and

$$\fint\!\!\fint_{Q_{R_k/2}} |v-u_k|^2\, dz + \fint\!\!\fint_{Q_{R_k/2}} |v-u_k|^p\, dz > \varepsilon_0 R_k^p \fint\!\!\fint_{Q_{R_k}} |Du_k|^p\, dz \tag{4.9}$$

for all v satisfying

$$\iint_{Q_{R_k}} -v\phi_t + \langle A_k(v, Dv), D\phi\rangle\, dz = 0 \quad \text{for all } \phi \in V^0_p(Q_{R_k/2}).$$

We then make a change of variables

$$\widetilde{u}_k(x,t) = u_k(x_k + R_k x, t_k + R_k^p t), \quad (x,t) \in Q_1.$$

By the boundedness assumption (4.2), the norms $\|u_k\|_{V_p(Q_{R_k})}$ are uniformly bounded by M and so are $\|\widetilde{u}_k\|_{V_p(Q_1)}$. Thus, by scaling and translation, we can assume $R = 1$ in (4.7)-(4.9) and note that (4.2) and (4.3) continue to hold. This also proves that δ is independent of R.

For any nonzero real number h and any vector valued function f in $L^1(Q_1)$, we denote by $f_h = J_h * f$ the standard mollifier of f. That is, for some smooth nonnegative function J with compact support in the unit ball Q_1 of $\mathbb{R}^{n+1}$ and $\|J\|_{L^1(\mathbb{R}^{n+1})} = 1$, we write

$$f_h(Z) = J_h * f(Z) = \frac{1}{h^{n+1}} \iint_{Q_1} J\left(\frac{|Z-z|}{h}\right) f(z)\, dz, \quad Z \in \mathbb{R}^{n+1}.$$

Let $\{h_k\}$ be some sequence of positive reals converging to 0 and $g_k = (u_k)_{h_k}$, which is in $C^1(Q_1)$. We then define U_k to be the solutions of

$$\begin{cases} \iint_{Q_{\frac{2}{3}}} -U_k\phi_t + \langle A_k(U_k, DU_k), D\phi\rangle\, dz = 0 & \forall \phi \in C^1_0(Q_{\frac{2}{3}}) \\ U_k = g_k & \text{on } S_{\frac{2}{3}}. \end{cases} \tag{4.10}$$

Note that, by a.3), $\|U_k\|_\infty \le C(\|g_k\|_\infty)$.

The following claims provide a contradiction to (4.8) and (4.9) and prove our proposition.

Claim I: There is a constant C such that $\|DU_k\|_{L^p(Q_{\frac{2}{3}})} \le C\|Du_k\|_{L^p(Q_{\frac{2}{3}})}$.

Claim II: $u_k - U_k \to 0$ in $L^2(Q_{\frac{2}{3}})$ and $L^p(Q_{\frac{2}{3}})$.

Proof of Claim I: Let $(\alpha)_k = A_k(u_k, Du_k)_{h_k}$ and replace ϕ in the inequality for u_k by ϕ_{-h_k}, whose support is in $Q_{5/6}$ if $|h_k|$ is sufficiently small. From (4.7), the following holds

$$-\varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_1)}\|D\phi\|_{L^p(Q_1)} \le \iint_{Q_1} [-\langle g_k, \phi_t\rangle + \langle(\alpha)_k, D\phi\rangle]\, dz \le \varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_1)}\|D\phi\|_{L^p(Q_1)},$$

for all $\phi \in C_0^1(Q_1)$. For any $\tau \in (-(\frac{2}{3})^p, 0)$ and sufficiently small positive h, let $\eta(s)$ be a C^1 function such that $\eta(s) = 1$ if $s < \tau$, $\eta(s) = 0$ if $s > \tau + h$ and η is linear in $(\tau, \tau + h)$.

Subtracting each side of the equation of U_k from the corresponding side of the above inequalities and testing the result with $(U_k(x,s) - g_k(x,s))\eta^2(s)$, which vanishes on the boundary of $Q_{\frac{2}{3}}$, we easily obtain the following for $\phi = U_k - g_k$, when we send h to 0:

$$\sup_{s\in[-\frac{4}{9},\tau]} \int_{B_{\frac{2}{3}}} |\phi|^2\, dx + \iint_{Q^\tau_{\frac{2}{3}}} \langle A_k(U_k, DU_k) - (\alpha)_k, D\phi\rangle\, dz \le \varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_1)} \|D\phi\|_{L^p(Q_1)}.$$

Here, for any cylinder Q, we denote $Q^t = Q \cap \{(x,s) : s \le t\}$. The above then implies

$$\sup_{s\in[-\frac{4}{9},\tau]} \int_{B_{\frac{2}{3}}} |\phi|^2\, dx + \iint_{Q^\tau_{\frac{2}{3}}} \langle A_k(U_k, DU_k), DU_k\rangle\, dz \le$$
$$\iint_{Q^\tau_{\frac{2}{3}}} \langle A_k(U_k, DU_k), Dg_k\rangle + \langle (\alpha)_k, (DU_k - Dg_k)\rangle\, dz + \varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_1)} \|D\phi_{h_k}\|_{L^p(Q_1)}.$$

Since $\|D(U_k)_{h_k}\|_{L^p(Q_{\frac{2}{3}})} \le C\|DU_k\|_{L^p(Q_{\frac{2}{3}})}$, a simple use of the Young inequality with ε_k sufficiently small and the ellipticity of A_k in a.1) give

$$\begin{aligned}\sup_{s\in[-\frac{4}{9},\tau]} \int_{B_{\frac{2}{3}}} |U_k - g_k|^2\, dx + \iint_{Q^\tau_{\frac{2}{3}}} |DU_k|^p\, dz \le \\ C\iint_{Q^\tau_{\frac{2}{3}}} [|Dg_k|^p + |(\alpha)_k|^q + |Du_k|^p]\, dz + \varepsilon_k \|Du_k\|^{\frac{p}{q}}_{L^p(Q_1)} \|D\phi_{h_k}\|_{L^p(Q_1)}.\end{aligned} \tag{4.11}$$

Because $\|Dg_k\|_{L^p(Q^\tau_{\frac{2}{3}})} \le C\|Du_k\|_{L^p(Q^\tau_{\frac{2}{3}})}$ and $\|(\alpha)_k\|^q_{L^q(Q^\tau_{\frac{2}{3}})} = \|A_k(u_k, Du_k)_{h_k}\|^q_{L^q(Q^\tau_{\frac{2}{3}})}$, which can be bounded by $C\|Du_k\|^p_{L^p(Q^\tau_{\frac{2}{3}})}$ for some constant C, we obtain from the above and the Young inequality the following estimate

$$\|DU_k\|^p_{L^p(Q^\tau_{\frac{2}{3}})} \le C\|Du_k\|^p_{L^p(Q^\tau_{\frac{2}{3}})} + \varepsilon_k \|DU_k\|^p_{L^p(Q_{\frac{2}{3}})} \quad \forall \tau \in [-(\frac{2}{3})^p, 0]. \tag{4.12}$$

This established our first claim if we take $\tau = 0$.

Proof of Claim II: Now, for any $\rho, r \in (0, \frac{2}{3})$, we write $Q_1' = Q_{\frac{2}{3}} \cap \{(x,s) : s \le -(\frac{2}{3})^p + \rho^2$, $Q_2' = Q_{\frac{2}{3}} \cap \{(x,t) : |x| > \frac{2}{3} - r\}$ and $Q_3' = Q_1' \cap Q_2'$. These sets are the thin layers at the base and the lateral sides of the cylinder $Q_{\frac{2}{3}}$.

Let $\phi(x,t) = \psi(x)\eta(t)$, where ψ, η are respectively cut-off functions in x, t. That is, ψ is a cut-off function for $B_{\frac{2}{3}-r}$ and $B_{\frac{2}{3}}$ and η is a cut-off function for $[-(\frac{2}{3})^p + \rho^2, 0]$ and $[-(\frac{2}{3})^p, 0]$. We can assume that $|D\psi| \le 1/r$ and $|\eta_t| \le 1/\rho^2$.

Denote $H_k = u_k - U_k$ and $\Phi = H_k\phi^2$. Testing the equations for U_k with Φ and replacing ϕ in (4.7) by Φ, we get by subtracting the two results

$$\left| \iint_{Q_{\frac{2}{3}}} \left[H_k \frac{\partial \Phi}{\partial t} + \langle A_k(u_k, Du_k) - A_k(U_k, DU_k), D\Phi\rangle \right] dz \right| \le \varepsilon_k \|Du_k\|^{p/q}_{L^p(Q_{\frac{2}{3}})} \|D\Phi\|_{L^p(Q_{\frac{2}{3}})}.$$

We now write $\mathbf{A} = A_k(u_k, Du_k) - A_k(U_k, DU_k)$ and derive from the above the following

$$\begin{aligned}\sup_{s\in[-(\frac{2}{3})^p,0]} \int_{B_\rho} H_k^2\phi^2\, dx + \iint_{Q_{\frac{2}{3}}} \langle \mathbf{A}, D\Phi\rangle\, dz \le \\ \iint_Q |H_k|^2 \phi \left|\frac{\partial\phi}{\partial t}\right|\, dz + \varepsilon_k \|Du_k\|^{p/q}_{L^p(Q_{\frac{2}{3}})} \|D\Phi\|_{L^p(Q_{\frac{2}{3}})}.\end{aligned} \tag{4.13}$$

We consider the second term on the left and note that $D\Phi = \phi^2 DH_k + 2H_k\phi D\phi$. By the monotonicity assumption a.4) with $u = u_k$ and $v = U_k$, we have

$$\mathbf{A}DH_k\phi^2 \geq \lambda_0\phi^2|DH_k|^p.$$

On the other hand, because $D\phi = 0$ in $Q \setminus Q'_2$, $|D\phi| \leq 1/r$ and $|\mathbf{A}| \leq C[|Du_k|^{p-1} + |DU_k|^{p-1}]$, we also have via the Young inequality

$$\begin{aligned} \iint_{Q_{\frac{2}{3}}} \mathbf{A}H_k\phi D\phi\, dz &\leq C\iint_{Q_{\frac{2}{3}}} (|Du_k|^{p-1} + |DU_k|^{p-1})|H_k||\phi||D\phi|\, dz \\ &\leq C\iint_{Q'_2} [|Du_k|^p + |DU_k|^p]\, dz + C\frac{1}{r^p}\iint_{Q'_2} H_k^p\, dz. \end{aligned}$$

Also, because $\phi_t = 0$ in $Q \setminus Q'_1$ and $|\phi_t| \leq 1/\rho^2$, we have

$$\iint_Q |H_k|^2\phi\left|\frac{\partial\phi}{\partial t}\right|\, dz \leq C\frac{1}{\rho^2}\iint_{Q'_1} |H_k|^2\, dz.$$

In addition, as

$$\|D\Phi\|^p_{L^p(Q_{\frac{2}{3}})} \leq C\|DH_k\|^p_{L^p(Q_{\frac{2}{3}})} + C\frac{1}{r^p}\iint_{Q'_2} H_k^p\, dz,$$

a simple use of the Young inequality and the above estimates allow us to deduce from (4.13) that

$$\sup_{s\in[-(\frac{2}{3})^p,0]}\int_{B_\rho} H_k^2\phi^2\, dx + \iint_{Q_{\frac{2}{3}}} \lambda_0|DH_k|^p\phi^p\, dz \leq F_k, \tag{4.14}$$

where

$$\begin{aligned} F_k &= \varepsilon_k\iint_{Q_{\frac{2}{3}}} (|Du_k|^p + |DU_k|^p)\, dz + C\iint_{Q'_2} [|Du_k|^p + |DU_k|^p]\, dz + \\ &\quad C\tfrac{1}{r^p}\iint_{Q'_2} H_k^p\, dz + C\frac{1}{\rho^2}\iint_{Q'_1} H_k^2\, dz. \end{aligned} \tag{4.15}$$

For any given $\varepsilon > 0$ we will show that if r, ρ are sufficiently small and k is large then

$$\frac{1}{r^p}\iint_{Q'_2} |H_k|^p\, dz + \frac{1}{\rho^2}\iint_{Q'_1} |H_k|^2\, dz < \varepsilon, \tag{4.16}$$

and

$$\iint_{Q'_2} (|Du_k|^p + |DU_k|^p)\, dz < \varepsilon. \tag{4.17}$$

The above estimates yield

$$\sup_{s\in[-(\frac{2}{3})^p,0]}\int_{B_\rho} H_k^2\, dx + \iint_{Q_{\frac{2}{3}}} \lambda_0|DH_k|^p\phi^p\, dz \leq C\varepsilon_k + 2\varepsilon,$$

and thus $\|H_k\|_{L^2(Q_{\frac{2}{3}})} < \varepsilon$. By Sobolev's imbedding inequality (see [3, Proposition 3.1, p.7]), we also have $\|H_k\|_{L^p(Q_{\frac{2}{3}})} < \varepsilon$. This will establish Claim II and we obtain the desired contradiction.

Concerning (4.16), when r, ρ have been fixed, because $H_k = (u_k - g_k) + (g_k - U_k)$ and $u_k - g_k$ converge to 0 in L^p and L^2, we need only to prove that

$$\frac{1}{r^p}\iint_{Q_2'} |g_k - U_k|^p \, dz + \frac{1}{\rho^2}\iint_{Q_1'} |g_k - U_k|^2 \, dz < \varepsilon, \tag{4.18}$$

if k is large and r, ρ are sufficiently small (uniformly in k).

For the integral on Q_1', from (4.11), we find that

$$\iint_{Q_1'} |U_k - g_k|^2 \, dz \le C\rho^2 \sup_{s\in[-(\frac{2}{3})^p, -(\frac{2}{3})^p+\rho^2]} \int_{B_{\frac{2}{3}},s} |U_k - g_k|^2 \, dx \le$$
$$C\rho^2 \iint_{Q_1'} [|Dg_k|^p + |(\alpha)_{h_k}|^q] \, dz + \varepsilon_k \rho^2 \|DU_k\|^p_{L^p(Q_{\frac{2}{3}})}.$$

On the other hand, because $U_k = g_k$ on the lateral part of $S_{\frac{2}{3}}$, we can use the Poincaré inequality in x to get

$$\frac{1}{r^p}\iint_{Q_2'} |U_k - g_k|^p \, dz \le \iint_{Q_2'} |DU_k - Dg_k|^p \, dz \le \iint_{Q_2'} |Dg_k|^p + |DU_k|^p \, dz.$$

By (4.11) again,

$$\begin{aligned}\iint_{Q_2'} |DU_k|^p \, dz &\le \iint_{Q_2'\setminus Q_1'} |DU_k|^p \, dz + \iint_{Q_1'} |DU_k|^p \, dz \\ &\le \iint_{Q_2'\setminus Q_1'} |DU_k|^p \, dz + \iint_{Q_1'} [|Dg_k|^p + |(\alpha)_{h_k}|^q] \, dz.\end{aligned}$$

But

$$\iint_{Q_1'} [|Dg_k|^p + |(\alpha)_{h_k}|^q] \, dz \le C \iint_{Q_1'} |Du_k|^p \, dz.$$

Therefore, the left hand side of (4.18) can be estimated by

$$C\left[\iint_{Q_1'} |Du_k|^p \, dz + \iint_{Q_2'} |Du_k|^p \, dz + \iint_{Q_2'\setminus Q_1'} |DU_k|^p \, dz + \varepsilon_k\right]. \tag{4.19}$$

Obviously, the left hand side of (4.17) is also bounded by the above. Thus, we need only prove that the integrals in (4.19) can be arbitrarily small (uniformly in k) if r, ρ are sufficiently small. By the uniform continuity of integrals (see Lemma 3.10), the integral of $|Du_k|^p$ over Q_1' is small if the measure $|Q_1'|$, or ρ, is sufficiently small (but independent of k). Hence, the first term of (4.19) is small. Fixing such a ρ, we then repeat the argument to see that if r is small then so is the second integral in (4.19). Similarly, the integral of $|DU_k|^p$ over $Q_2' \setminus Q_1'$ is small. Therefore, the right hand side F_k of (4.14) can be arbitrarily small if r, ρ are sufficiently small and k is large ($\varepsilon_k \to 0$). As we mentioned earlier this gives the proof of the second claim and completes our proof. ∎

We now consider the following alternative of the monotonicity condition a.4).

a.4') For any $w \in \mathbb{R}^m$ and $U, V \in \mathbb{R}^{nm}$, there holds

$$\langle A(w,U) - A(w,V), U - V\rangle \ge \lambda_0 |U - V|^2 \begin{cases} \min\{|U|^{p-2}, |V|^{p-2}\} & U \ne 0 \text{ or } V \ne 0 \\ 0 & \text{otherwise.}\end{cases} \tag{4.20}$$

Moreover, $A(u,\zeta)$ is Lipschitz in u in the following sense

$$|A(u,\zeta) - A(v,\zeta)| \le C|u - v||\zeta|^{p-1}.$$

Concerning the condition (4.20), if $A(u, U)$ is differentiable in U then we note that

$$\langle A(w, U) - A(w, V), U - V\rangle = \int_0^1 \langle \frac{\partial A}{\partial \zeta}(w, sU + (1-s)V)ds(U - V), U - V\rangle.$$

Therefore, (4.20) can be verified if the matrix $\dfrac{\partial A}{\partial \zeta}$ is positive definite and

$$\langle \frac{\partial A}{\partial \zeta}(w, sU + (1-s)V)\eta, \eta\rangle \geq \lambda(sU + (1-s)V)|\eta|^2 \quad \forall s \in (0,1), \eta \in \mathbb{R}^{nm}$$

for some $\lambda(sU + (1-s)V) \geq |sU + (1-s)V|^{p-2}$.

Proposition 4.2 *The conclusion of Proposition 4.1 holds if the monotonicity condition a.4) is replaced by a.4').*

Proof. We revisit the proof of Proposition 4.1 and point out necessary modifications under a.4'). As before, for $H_k = u_k - U_k$ and $\phi(x,t) = \psi(x)\eta(t)$ with ψ, η being respectively cut-off functions in x, t for $Q_{\frac{2}{3}}$. That is, ψ is a cut-off function for $B_{\frac{2}{3}-r}$ and $B_{\frac{2}{3}}$ and η is a cut-off function for $[-(\frac{2}{3})^p + \rho^2, \tau]$ and $[-(\frac{2}{3})^p, \tau]$, where τ is any number in $[-(\frac{2}{3})^p, 0]$.

Again, the proof is by contradiction and we can see that the proof of claim I in the proof of Proposition 4.1 is still applicable here. We need only consider claim II. First of all, the assumptions (4.8) and (4.9) give that

$$\iint_{Q_{\frac{2}{3}}} |H_k|^2\, dz \geq \varepsilon_1 > 0$$

for some fixed ε_1. So that, if $r \geq 2, s > 1$ then Hölder inequality, the boundedness of H_k and the above inequality yield

$$\left(\iint_{Q_{\frac{2}{3}}} |H_k|^r\, dz\right)^{\frac{1}{s}} \leq C(M)\left(\iint_{Q_{\frac{2}{3}}} |H_k|^2\, dz\right)^{\frac{1}{s}} \leq C(M, r, s, \varepsilon_1)\iint_{Q_{\frac{2}{3}}} |H_k|^2\, dz. \tag{4.21}$$

Since a.4) was not used until we obtained (4.13), we now need only look at the integral of $\langle \mathbf{A}, DH_k\phi^2\rangle$ in (4.13), which reads

$$\sup_{s\in[-(\frac{2}{3})^p, 0]} \int_{B_{\frac{2}{3}}} H_k^2\phi^2\, dx + \iint_{Q_{\frac{2}{3}}} \langle \mathbf{A}, DH_k\phi^2\rangle\, dz \leq F_k, \tag{4.22}$$

where $\mathbf{A} = A_k(u_k, Du_k) - A_k(U_k, DU_k)$, and $F_k = F_k(h, r, \rho)$ which is defined by (4.15) and can be arbitrarily small if k is large and h, r, ρ small (uniformly in k), thanks to the argument in the proof of Proposition 4.1 without using a.4).

We now consider the following two cases.

The case $p > 2$**:** We write $Q_{\frac{2}{3}} = E_u \cup E_v$ where

$$E_u = \{z : |Du_k(z)| \leq |DU_k(z)|\},\ E_v = \{z : |DU_k(z)| < |Du_k(z)|\}.$$

We also write

$$\iint_{Q_{\frac{2}{3}}} \langle \mathbf{A}, DH_k\phi^2\rangle\, dz = \iint_{E_u} \langle \mathbf{A}, DH_k\phi^2\rangle\, dz + \iint_{E_v} \langle \mathbf{A}, DH_k\phi^2\rangle\, dz.$$

On E_u, we have

$$\langle \mathbf{A}, DH_k\rangle = \langle A_k(U_k, DU_k) - A_k(U_k, Du_k) + A_k(U_k, Du_k) - A_k(u_k, Du_k), D(U_k - u_k)\rangle.$$

By (4.20), as $|Du_k(z)| \le |DU_k(z)|$ and $p > 2$, it follows that

$$\langle A_k(U_k, DU_k) - A_k(U_k, Du_k), DU_k - Du_k\rangle \ge \lambda_0 |Du_k|^{p-2}|DU_k - Du_k|^2, \tag{4.23}$$

and this term will be kept on the left of (4.22). On the other hand, as $A_k(u,\zeta)$ is Lipschitz in u,

$$\begin{aligned}&|\langle A_k(U_k, Du_k) - A_k(u_k, Du_k), DH_k\rangle| \le C|H_k||Du_k|^{p-1}|DH_k|\\ &= C|H_k||Du_k|^{\frac{p}{2}}|Du_k|^{\frac{p-2}{2}}|DH_k| \le \tfrac{\lambda_0}{4}|Du_k|^{p-2}|DH_k|^2 + C|H_k|^2||Du_k|^p.\end{aligned}$$

The first term on the right can be absorbed into that of (4.23).

Interchange the roles of u_k, U_k and apply a similar treatment for the integral of $\langle \mathbf{A}, DH_k\phi^2\rangle$ over E_v in the above argument to see that (4.22) gives

$$\begin{aligned}\sup_{s\in[-(\frac{2}{3})^p,0]}\int_{B_{\frac{2}{3}}} H_k^2\phi^2\,dx + \iint_{E_u} |Du_k|^{p-2}|DH_k|^2\phi^2\,dz + \iint_{E_v} |DU_k|^{p-2}|DH_k|^2\phi^2\,dz\\ \le\ C\iint_{E_u}|Du_k|^p|H_k|^2\phi^2\,dz + C\iint_{E_u}|DU_k|^p|H_k|^2\phi^2\,dz + F_k.\end{aligned}$$

Since $|Du_k| \le |DU_k|$ on E_u, the above yields

$$\sup_{s\in[-(\frac{2}{3})^p,0]}\int_{B_{\frac{2}{3}}} H_k^2\phi^2\,dx \le C\iint_{Q_{\frac{2}{3}}} |DU_k|^p|H_k|^2\phi^2\,dz + F_k$$

Now, using the higher integrability of Du_k of Lemma 3.11 we also have the L^q estimate, with $q > p$, for DU_k on $Q_{\frac{2}{3}}$ by extending $U_k = u_k$ beyond the boundary of $Q_{\frac{2}{3}}$. In fact, this comes from (4.12) (for cylinders centered on the boundary of $Q_{\frac{2}{3}}$) to obtain a reverse Hölder inequality for DU_k and then use the fact that Du_k is in L^q. Thus, by Hölder's inequality and (4.21)

$$\iint_{Q_{\frac{2}{3}}} |DU_k|^p|H_k|^2\phi^2\,dz \le C(M)\left(\iint_{Q_{\frac{2}{3}}} |H_k|^r\phi^r\,dz\right)^{\frac{1}{s}} \le C(\varepsilon_1, M)\iint_{Q^\tau_{\frac{2}{3}}} |H_k|^2\,dz.$$

Again, it is easy to see that the above argument still holds if $Q_{\frac{2}{3}}$ is replace by $Q^\tau_{\frac{2}{3}} = Q_{\frac{2}{3}} \cap \{(x,t) : t \le \tau\}$ for any $\tau \le 0$. Sending ρ, r to zero to obtain

$$\sup_{s\in[-(\frac{2}{3})^p,\tau]}\int_{B_{\frac{2}{3}}} H_k^2\,dx \le C(\varepsilon_1, M)\iint_{Q^\tau_{\frac{2}{3}}} |H_k|^2\,dz + F_k. \tag{4.24}$$

Setting

$$y_k(\tau) = \sup_{s\in[-(\frac{2}{3})^p,\tau]}\int_{B_{\frac{2}{3}}} H_k^2\,dx$$

the above (4.24) becomes $y_k' \le Cy_k + F_k$ with $y(-(\frac{2}{3})^p)$ and F_k can be arbitrarily small. By the Gronwall inequality, we see that $H_k \to 0$ in $L^2(Q_{\frac{2}{3}})$ as well as in $L^p(Q_{\frac{2}{3}})$ because $p > 2$ and H_k is bounded. Our desired contradiction is obtained and the proof is complete for the case $p > 2$.

The case $2n/(n+2) < p < 2$**:** For each $z \in Q$, let us denote by $u(z), v(z)$ which are either u_k or U_k such that, with a slight abuse of notation here, $Du(z) = \max\{Du_k(z), DU_k(z)\}$ and $Dv(z) = \min\{Du_k(z), DU_k(z)\}$. We write $Q_{\frac{2}{3}} = E_u \cup E_v$ where

$$E_u = \{z : \frac{1}{2}|Du(z)| \le |Du(z) - Dv(z)|\},\ E_v = \{z : |Du(z) - Dv(z)| < \frac{1}{2}|Du(z)|\}.$$

Again, we consider the integral of $\langle \mathbf{A}, DH_k\phi^2\rangle$ in (4.22). On E_u, we write $H = u - v$, $DH = Du - Dv$ and

$$\langle \mathbf{A}, DH_k\rangle = \langle A_k(u, Du) - A_k(u, Dv) + A_k(u, Dv) - A_k(v, Dv), DH\rangle.$$

Because $p < 2$, $\min\{|Du_k|^{p-2}, |DU_k|^{p-2}\} = |Du|^{p-2}$. By (4.20) and the fact that $|Du|^{p-2} \ge 2^{p-2}|Dv - Du|^{p-2}$, we have

$$\langle A_k(u, Du) - A_k(u, Dv), Du - Dv\rangle \ge C\lambda_0|Dv - Du|^p = C\lambda_0|DH|^p, \tag{4.25}$$

and the last term will be kept on the left of (4.22). On the other hand, as $A_k(v, \zeta)$ is Lipschitz in v

$$\begin{aligned} |\langle A_k(u, Dv) - A_k(v, Dv), DH\rangle| &\le C|H||Dv|^{p-1}|DH| \le \tfrac{C\lambda_0}{4}|DH|^p + C|H|^{\frac{p}{p-1}}||Dv|^p \\ &\le \tfrac{C\lambda_0}{4}|DH|^p + C(M)|H|^2||Dv|^p. \end{aligned}$$

Here, we have used the fact that H is bounded in the last inequality. The first term on the rightmost hand side can be absorbed into that of (4.25).

On E_v we note that $|Dv - Du| < \frac{1}{2}|Du|$ and this implies $|Dv| \ge \frac{1}{2}|Du|$ (otherwise, $|Dv - Du| \ge |Du| - |Dv| > \frac{1}{2}|Du|$, contradicting the definition of E_v). We now write $H = v - u$, $DH = Dv - Du$ and

$$\langle \mathbf{A}, DH_k\rangle = \langle A_k(v, Dv) - A_k(v, Du) + A_k(v, Du) - A_k(u, Du), DH\rangle.$$

We have by (4.20)

$$\langle A_k(v, Du) - A_k(v, Dv), Dv - Du\rangle \ge C\lambda_0|Du|^{p-2}|Dv - Du|^2.$$

Again, this term will stay on the left of (4.22). Meanwhile, as $|Dv| \sim |Du|$ on E_v, we have

$$\begin{aligned} &|\langle A_k(v, Du) - A_k(u, Du), DH\rangle| \le C|H||Du|^{p-1}|DH| \le \\ &\qquad C|H||Dv|^{\frac{p}{2}}|Du|^{\frac{p-2}{2}}|DH| \le \tfrac{C\lambda_0}{4}|Du|^{p-2}|DH|^2 + C|H|^2|Dv|^p. \end{aligned}$$

Combining the above estimates and noting that $|Dv| \le |DU_k|$, we derive from (4.22)

$$\begin{aligned} \sup_{s\in[-(\frac{2}{3})^p,0]} \int_{B_{\frac{2}{3}}} H_k^2\phi^2\,dx \; &+ \iint_{E_u} |DH_k|^p\phi^2\,dz + \iint_{E_v} |Du|^{p-2}|DH_k|^2\phi^2\,dz \\ &\le C(M)\iint_{Q_{\frac{2}{3}}} |DU_k|^p|H_k|^2\phi^2\,dz + F_k. \end{aligned} \tag{4.26}$$

As in the case $p > 2$, because $2n/(n+2) < p < 2$ the higher integrability of DU_k is available, we can derive a Gronwall inequality and see that $H_k \to 0$ in $L^2(Q_{\frac{2}{3}})$. Since H_k is bounded and DU_k is $L^q(Q_{\frac{2}{3}})$ for some $q > p$, an application of Hölder's inequality shows that the right hand side of the above inequality tends to zero as $k \to \infty$. Hence,

$$\iint_{E_u} |DH_k|^p\phi^2\,dz + \iint_{E_v} |Du|^{p-2}|DH_k|^2\phi^2\,dz \to 0 \quad \text{as } k \to \infty. \tag{4.27}$$

Concerning the L^p norm of DH_k, we observe that

$$\begin{aligned}\iint_{E_v} |DH_k|^p\phi^2\,dz &= \iint_{E_v} |Du|^{\frac{(p-2)p}{2}}|DH_k|^p|Du|^{\frac{(2-p)p}{2}}\phi^2\,dz\\ &\leq \left(\iint_{E_v} |Du|^{p-2}|DH_k|^2\phi^2\,dz\right)^{\frac{p}{2}}\left(\iint_{E_v} |Du|^p\phi^2\,dz\right)^{\frac{2-p}{2}}.\end{aligned}$$

Because the integral of $|Du|^p = \max\{|Du_k|^p, |DU_k|^p\}$ over $Q_{\frac{2}{3}}$ is bounded, the above and (4.27) show that the integral of $|DH_k|^p$ over E_u and E_v, and therefore $Q_{\frac{2}{3}}$, tends to zero. By Sobolev's imbedding inequality, we see that $H_k \to 0$ in $L^p(Q_{\frac{2}{3}})$. Our proof is then complete. ∎

The first alternative (4.4) in the above propositions is not as useful as (4.6) for our later proof of the Hölder continuity for weak solutions. To this end, we will show that (4.4) allows us to approximate the considered u by solutions to nice systems whose coefficients $A(v, Dv)$ do not involve the solutions v. The proof of this fact for the singular case ($p < 2$) is much more involved and will be reported in a forthcoming work. Here, we only present the result when $p > 2$.

Proposition 4.3 *Assume that $p > 2$ and a.1)-a.3) and a.4) or a.4') hold. For any given $M, \varepsilon > 0$ exists $\delta \in (0, 1]$ that depends only on $\lambda, \Lambda, \Lambda_0, \varepsilon$ and $C_{\mathcal{B}}$ such that if $A \in \mathcal{A}, B \in \mathcal{B}$ and $u \in V(Q_R)$ satisfying*

$$⨏\!\!⨏_{Q_R} |u - u_R|^2\,dz + R^p ⨏\!\!⨏_{Q_R} |Du|^p\,dz \leq M, \tag{4.28}$$

and

$$\left|\iint_{Q_R} -u\phi_t + \langle A(u, Du), D\phi\rangle - \langle B(u, Du), \phi\rangle\,dz\right| \leq \delta\|Du\|_{L^{\frac{p}{q}}(Q_R)}\|D\phi\|_{L^p(Q_R)} \tag{4.29}$$

for any $Q_R \subset Q$, $\phi \in V_p^0(Q)$, then there exists v in $V_p(Q_{R/2})$ such that $\|v\|_{L^\infty(Q_{R/2})} \leq C(\lambda, \Lambda, \lambda_0, M)$ and

$$\begin{cases} ⨏\!\!⨏_{Q_{R/2}} |v - u|^2\,dz + ⨏\!\!⨏_{Q_{R/2}} |v - u|^p\,dz \leq \varepsilon R^p ⨏\!\!⨏_{Q_R} |Du|^p\,dz,\\ ⨏\!\!⨏_{Q_{R/2}} |Dv|^p\,dz \leq \beta ⨏\!\!⨏_{Q_{R/2}} |Du|^p\,dz.\end{cases} \tag{4.30}$$

Here, $\beta > 0$ depends only on $\lambda, \Lambda, \Lambda_0, C_{\mathcal{B}}$. Moreover, v satisfies

$$\iint_{Q_{R/2}} -v\phi_t + \langle \widetilde{A}(v, Dv), D\phi\rangle - \langle \widetilde{F}(v, Dv), \phi\rangle\,dz = 0, \quad \forall\phi \in C_0^1(Q_{R/2}), \tag{4.31}$$

where either $\widetilde{A}(v, Dv) = A(v, Dv)$ and $\widetilde{B}(v, Dv) = B(v, Dv)$ or $\widetilde{A}(v, Dv) = A(c, Dv)$ and $\widetilde{B}(v, Dv) = B(c, Dv)$ for some constant vector $c \in \mathbb{R}^m$.

Proof. Again, we will only discuss the case $B \equiv 0$ here. As in the proof of Proposition 4.2, by scaling we can assume that $R = 1$ and need only consider the case when

$$\limsup_{k\to\infty} ⨏\!\!⨏_{Q_1} |Du_k|^p\,dz = 0. \tag{4.32}$$

We now look at the solution of

$$\iint_{Q_{\frac{2}{3}}} [-v_k\phi_t + A_k((u_k)_{Q_1}, Dv_k)D\phi]\,dz = 0 \quad \forall\phi \in V_p^0(Q_{\frac{2}{3}}), \tag{4.33}$$

and $v_k = u_k$ on $S_{\frac{2}{3}}$. By testing (4.33) with $v_k - u_k$ we easily see that

$$\fint\!\!\fint_{Q_{\frac{2}{3}}} |Dv_k|^p \, dz \le C \fint\!\!\fint_{Q_1} |Du_k|^p \, dz. \tag{4.34}$$

As before, by subtracting the equations for u_k, v_k and testing the result with $(v_k - u_k)$, we have for $H_k = v_k - u_k$ the following

$$\sup_t \int_{B_{\frac{2}{3}}} H_k^2 \, dx + \iint_{Q_{\frac{2}{3}}} \langle A_k((u_k)_{Q_1}, Dv_k) - A_k(u_k, Du_k), DH_k \rangle \, dz \le 0. \tag{4.35}$$

We now write

$$\begin{aligned} A_k((u_k)_{Q_1}, Dv_k) - A_k(u_k, Du_k) \quad = \quad & A_k((u_k)_{Q_1}, Dv_k) - A_k((u_k)_{Q_1}, Du_k) \\ & + A_k((u_k)_{Q_1}, Du_k) - A_k(u_k, Du_k) \end{aligned}$$

and keep the first difference on the left of (4.35). Using the ellipticity and Lipschitz property of A_k, we obtain

$$\sup_t \int_{B_{\frac{2}{3}}} H_k^2 \, dx \le C \iint_{Q_{\frac{2}{3}}} |Du_k|^p |u_k - (u_k)_{Q_1}|^2 \, dz. \tag{4.36}$$

We now make use of the L^q estimate for Du_k, see (3.23) of Lemma 3.11 and Remark 3.12, to find some $q > p$ and $\sigma \ge 1$ such that

$$\left(\fint\!\!\fint_{Q_{\frac{2}{3}}} |Du_k|^q \, dz \right)^{\frac{p}{q}} \le C \left(\fint\!\!\fint_{Q_1} |Du_k|^p \, dz \right)^{\sigma}.$$

Hence, for $r = (p/q)'$ (4.36) gives

$$\sup_t \int_{B_{\frac{2}{3}}} |H_k|^2 \, dx \le C \left(\iint_{Q_1} |Du_k|^p \, dz \right)^{\sigma} \left(\iint_{Q_{\frac{2}{3}}} |u_k - (u_k)_{Q_1}|^{2r} \, dz \right)^{\frac{1}{r}}. \tag{4.37}$$

Since u_k is bounded,

$$\left(\iint_{Q_{\frac{2}{3}}} |u_k - (u_k)_{Q_1}|^{2r} \, dz \right)^{\frac{1}{r}} \le C(M) \left(\iint_{Q_{\frac{2}{3}}} |u_k - (u_k)_{Q_1}|^2 \, dz \right)^{\frac{1}{r}}.$$

By (4.32) and an application of the Poincaré inequality, we see that the above quantities tend to 0 as $k \to \infty$. Therefore, using the uniform boundedness of the integral of $|Du_k|^p$, we derive from (4.37) the following

$$\sup_t \int_{B_{\frac{2}{3}}} |H_k|^2 \, dx \le \varepsilon \iint_{Q_1} |Du_k|^p \, dz \tag{4.38}$$

for any given $\varepsilon > 0$ when k is sufficiently large. Integrating the above in t, we obtain

$$\iint_{Q_{\frac{2}{3}}} |u_k - v_k|^2 \, dz \le \varepsilon \iint_{Q_1} |Du_k|^p \, dz.$$

Finally, by applying the interpolation inequality in x to $(u_k - v_k)$, one has the following

$$\int_{B_{\frac{2}{3}}} |u_k - v_k|^p \, dx \le \varepsilon \int_{B_{\frac{2}{3}}} |Du_k - Dv_k|^p \, dx + C(\varepsilon)\left(\int_{B_{\frac{2}{3}}} |u_k - v_k|^2 \, dx\right)^{\frac{p}{2}}.$$

Integrating with respect to t and using (4.34) to get

$$\iint_{Q_{\frac{2}{3}}} |u_k - v_k|^p \, dz \le C\varepsilon \iint_{Q_1} |Du_k|^p \, dz + C(\varepsilon)\left(\sup_t \int_{B_{\frac{2}{3}}} |u_k - v_k|^2 \, dx\right)^{\frac{p}{2}}.$$

By (4.38), we see that

$$\iint_{Q_{\frac{2}{3}}} |u_k - v_k|^p \, dz \le C\varepsilon \iint_{Q_1} |Du_k|^p \, dz.$$

If we choose ε small then the above gives (4.30) with v_k being a solution to (4.31), where $\widetilde{A}(v, Dv) = A_k(c, Dv)$ for some constant vector $c \in \mathbb{R}^m$. The proof is now complete by rescaling. ∎

5 Decay estimates and the proof of Theorem 2.2

We will prove in this section that the set $\mathcal{I}$ of parameters with which the scaling decay property D) holds for bounded weak solutions to our systems is open. Let us recall the property D): For a given bounded function v in $V_p(Q_{1,1})$ we say that v satisfies a *scaling decay property* if the following holds

D) Let $M = \sup_{Q_{1,1}} |v|$. For any $R_0 > 0$ and $\eta \in (0, 1)$, there are positive numbers $A, K, L, \alpha_0, \omega_0$ depending on M, η (with K, A sufficiently large) such that there exist the following sequences

$$R_k = \frac{R_0}{K^k}, \quad \omega_{k+1} = \max\{\eta\omega_k, LR_k^{\alpha_0}\}, \quad S_k = \frac{\omega_k}{A}, \quad Q_k = B_{R_k} \times [-S_k^{2-p} R_k^p, 0] \tag{5.1}$$

such that $Q_k \subset Q_{1,1}$ for any integer $k \ge 0$ and

$$\omega_k^p \ge \fint\!\!\fint_{Q_k} |v - (v)_k|^p \, dz, \quad (v)_k = \fint\!\!\fint_{Q_k} v \, dz. \tag{5.2}$$

We now consider a family of systems of the form ($\tau \in [0, 1]$)

$$\begin{cases} v_t = \mathrm{div}(A(\tau, v, Dv)) & \text{in } Q_{1,1}, \\ v = g & \text{on } S_{1,1}. \end{cases} \tag{5.3}$$

We defined $\mathcal{I}$ to be the collection of $\tau \in [0, 1]$ such that every bounded weak solutions of the above system verifies D). Theorem 2.2 asserts that $\mathcal{I}$ is open and bounded weak solutions to (5.3) with $\tau \in \mathcal{I}$ are Hölder continuous. Its proof goes as follows.

Proof of Theorem 2.2: Fix a $\mu \in \mathcal{I}$. We will show that if $|\nu - \mu|$ is sufficiently small then $\nu \in \mathcal{I}$. That is, every bounded weak solution u of (5.3) with $\tau = \nu$ will satisfy D). Now, let u be such a solution and $M = \sup_Q |u|$.

Let η be in $(0, 1)$. The new set of constants $A, K, L, \alpha_0, R_0, \{\omega_k\}$ in D) for u will be determined in the course of our calculation and depend from that of the reference system (5.3) when $\tau = \mu$.

By induction, let us start with a positive ω_k, say $k = 0$, such that

$$\omega_k^p \geq \fint\!\!\fint_{Q_k} |u - (u)_k|^p \, dz. \tag{5.4}$$

In the sequel, for any $t > 0$ we will denote by tQ_k the cylinder with radius tR_k and being concentric with Q_k. From the Caccioppoli inequality, see Lemma 3.3 with $\rho = S_k^{2-p} R_k^p$,

$$R_k^p \fint\!\!\fint_{\frac{1}{2}Q_k} |Du|^p \, dz \leq C \fint\!\!\fint_{Q_k} |u - (u)_k|^p \, dz + C \fint\!\!\fint_{Q_k} S_k^{p-2} |u - (u)_k|^2 \, dz.$$

By an application of Young's inequality to the integrand in the second term on the right and (5.4) we can find a constant C_1 such that

$$R_k^p \fint\!\!\fint_{\frac{1}{2}Q_k} |Du|^p \, dz \leq C \fint\!\!\fint_{Q_k} |u - (u)_k|^p \, dz + C S_k^p \leq C_1 (1 + \frac{1}{A^p}) \omega_k^p. \tag{5.5}$$

For any given $\varepsilon > 0$, if $|\mu - \nu|$ is sufficiently small (depending only on M, ε) we apply the approximation result, Proposition 4.3, in $\frac{1}{2}Q_k$ to obtain a "nice" solution v satisfying (5.3) with $\tau = \mu$ or a similar system with v being replaced by a constant vector such that $\sup_{\frac{1}{4}Q_k} |v| \leq C(M)$ and (in combination with (5.4) and (5.5))

$$\begin{aligned}
\fint\!\!\fint_{\frac{1}{4}Q_k} |v - (v)_k|^p \, dz &\leq \fint\!\!\fint_{\frac{1}{4}Q_k} |v - (u)_k|^p \, dz \\
&\leq \fint\!\!\fint_{\frac{1}{4}Q_k} |v - u|^p \, dz + \fint\!\!\fint_{\frac{1}{4}Q_k} |u - (u)_k|^p \, dz \\
&\leq \varepsilon R_k^p \fint\!\!\fint_{\frac{1}{2}Q_k} |Du|^p \, dz + C_2 \omega_k^p \leq C_3 (1 + \frac{1}{A^p}) \omega_k^p \leq C_4 \omega_k^p.
\end{aligned}$$

Here, $C_4 = 2C_3$ if $A \geq 1$. We then take $\hat{\omega}_k = C_4 \omega_k$ $(k = 0)$ and apply the assumption D) on any solution v of (5.3) with sufficiently small $\hat{\eta}$ (in place of η in D)) to find $\hat{A}, \hat{K}, \hat{L}$ depending to $\sup_Q |v|$, and therefore M, and construct a sequence $\{\hat{\omega}_k\}$ such that the relations in (5.1) hold and

$$\fint\!\!\fint_{\hat{Q}_k} |v - (v)_k|^p \, dz \leq \hat{\omega}_k^p \quad \forall k.$$

The new constants for u will be chosen such that $K = \hat{K}, C_4 A = \hat{A}$, with $\hat{A}$ being large and $A \geq 1$. The constants L, α_0 will be determined later (using the constant C_4, A) so that $Q_{k+1} \subset \hat{Q}_{k+1}$ and $\omega_{k+1} = \max\{\eta \omega_k, L R_k^{\alpha_0}\}$.

Choose K large (or equivalently $\hat{K}$), depending on η, such that, as $\omega_{k+1} \geq \eta \omega_k$, we have

$$\frac{4^p}{K^p} \omega_k^{p-2} \leq \omega_{k+1}^{p-2} \Leftrightarrow S_{k+1}^{2-p} \left(\frac{R}{K^{n+1}} \right)^p \leq S_k^{2-p} \left(\frac{R}{4K^n} \right)^p \Rightarrow Q_{k+1} \subset \frac{1}{4} Q_k.$$

Noting that $\frac{1}{4}Q_k$ is scaled by R_k^p in the t direction, we then deduce

$$\begin{aligned}
\fint\!\!\fint_{Q_{k+1}} |u - v|^p \, dz &\leq \frac{\omega_{k+1}^{p-2}}{\omega_k^{p-2}} (2K)^{n+p} \fint\!\!\fint_{\frac{1}{4}Q_k} |u - v|^p \, dz \leq \frac{\omega_{k+1}^{p-2}}{\omega_k^{p-2}} (2K)^{n+p} \varepsilon R_k^p \fint\!\!\fint_{\frac{1}{2}Q_k} |Du|^p \, dz \\
&\leq C_5 \varepsilon (2K)^{n+p} \omega_{k+1}^{p-2} \omega_k^2.
\end{aligned}$$

Here, we have used the approximation and (5.5). Since $Q_{k+1} \subset \hat{Q}_{k+1}$, it follows that

$$\begin{aligned}\fint\!\!\fint_{Q_{k+1}} |u-(u)_{k+1}|^p\,dz &\le C_6 \fint\!\!\fint_{Q_{k+1}} |u-v|^p\,dz + C_6 \fint\!\!\fint_{Q_{k+1}} |v-(v)_{k+1}|^p\,dz \\ &\le C_6 \fint\!\!\fint_{Q_{k+1}} |u-v|^p\,dz + C_6 \frac{|\hat{Q}_{k+1}|}{|Q_{k+1}|} \fint\!\!\fint_{\hat{Q}_{k+1}} |v-(v)_{k+1}|^p\,dz.\end{aligned} \tag{5.6}$$

The first term on the right is estimated as follows. We choose ε small, depending on K and thus M, such that for any given $\varepsilon' > 0$ we have via Young's inequality

$$C_5\varepsilon(2K)^{n+p}\omega_{k+1}^{p-2}\omega_k^2 \le \frac{1}{2}\omega_{k+1}^p + C_7(\varepsilon K^{n+p})^{p/2}\omega_k^p \le \frac{1}{2}\omega_{k+1}^p + \varepsilon'\omega_k^p.$$

For the second term, since v verifies (5.2) in D) and $C_4 A = \hat{A}$ we can have

$$C_6 \frac{|\hat{Q}_{k+1}|}{|Q_{k+1}|} \fint\!\!\fint_{\hat{Q}_{k+1}} |v-(v)_{k+1}|^p\,dz = C_6 \frac{S_{k+1}^{p-2}}{\hat{S}_{k+1}^{p-2}} \fint\!\!\fint_{\hat{Q}_{k+1}} |v-(v)_{k+1}|^p\,dz = C_8\omega_{k+1}^{p-2}\hat{\omega}_{k+1}^2 \le \frac{1}{4}\omega_{k+1}^p$$

if $C_8 = C_6 C_4^{2-p}$ and $2\sqrt{C_8}\hat{\omega}_{k+1} \le \omega_{k+1}$. Recall that we also require $Q_{k+1} \subset \hat{Q}_{k+1}$. To this end, as $K = \hat{K}$, we need $S_{k+1} = \frac{\omega_{k+1}}{A} \ge \frac{\hat{\omega}_{k+1}}{\hat{A}} = \hat{S}_{k+1}$ or $\omega_{k+1} \ge C_4\hat{\omega}_{k+1}$. This and the requirement $\omega_{k+1} \ge 2\sqrt{C_8}\hat{\omega}_{k+1}$ are possible by choosing L sufficiently large or R_0 small, with $\alpha_0 < \hat{\alpha}_0$ and $\hat{\eta}$ small, so that

$$\omega_{k+1} = \max\{\eta\omega_k, LR_k^{\alpha_0}\} \ge \max\{2\sqrt{C_8}, C_4\}\max\{\hat{\eta}\hat{\omega}_k, \hat{L}R_k^{\hat{\alpha}_0}\} = \max\{2\sqrt{C_8}, C_4\}\hat{\omega}_{k+1}.$$

We should note that once this requirement is fulfilled for $k = 0$ then $\hat{\eta}$ is fixed and the above relations continue to hold for all $k \ge 1$. By this way, we then inductively define the sequence $\{\omega_k\}$ for u. Hence, from (5.6) and the above estimates we derive

$$\fint\!\!\fint_{Q_{k+1}} |u-(u)_{k+1}|^p\,dz \le \frac{1}{2}\omega_{k+1}^p + \varepsilon'\omega_k^p + \frac{1}{4}\omega_{k+1}^p \le \omega_{k+1}^p \qquad \text{for all } k \ge 0.$$

This shows that u satisfies the same properties D) and $\mathcal{I}$ is open. In addition, an algebraic argument similar to [3, Proposition 3.1, p.44] applies to the sequence $\{\omega_k\}$ to get

$$\fint\!\!\fint_{Q_k} |u-(u)_k|^p\,dz \le \omega_k^p \le CR_k^{p\alpha_0}.$$

Moreover, since $\omega_k \le \omega_0$ when k is large and $p > 2$, we have $Q_{R_k, S_0^{2-p}R_k^p} \subset Q_k$ and the above gives

$$\fint\!\!\fint_{Q_{R_k, S_0^{2-p}R_k^p}} |u-(u)_k|^p\,dz \le \frac{S_k^{2-p}}{S_0^{2-p}} \fint\!\!\fint_{Q_k} |u-(u)_k|^p\,dz \le C^{\frac{2}{p}} S_0^{p-2} R_k^{2\alpha_0}. \tag{5.7}$$

By the Campanato imbedding [6, Theorem 2.9], (5.7) implies that u is Hölder continuous. Our proof is then complete. ∎

6 $\mathcal{I}$ is closed

In this section, we will show that the set $\mathcal{I}$ is closed in $[0,1]$ under suitable assumptions. To proceed, we take a sequence $\{\nu_k\}$ in $\mathcal{I}$ such that $\nu_k \to \mu$ and show that $\mu \in \mathcal{I}$. Thus, let us consider a bounded weak solution to (2.3) with $\nu = \mu$. By the assumption II), there is a sequence of Hölder continuous weak solutions v_k to

$$v_t = \mathrm{div}(A(v_k, \nu, Dv)) \text{ in } Q_1, \tag{6.1}$$

such that Dv_k converges weakly in $L^1(Q_1)$ to Du. Moreover, the L^∞ norms of v_k's are bounded uniformly in terms of that of u. We will derive uniform estimates for various integral norms of Dv_k in terms of the L^∞ norm of u. Once this is established, we obtain estimates for the derivatives of the limiting u and its Hölder continuity to conclude that μ is in $\mathcal{I}$.

Let v be any bounded weak solution to (6.1). We recall our assumptions here.

Let $\lambda_{\nu,v}, \Lambda_{\nu,v}$ be the ellipticity constants for the matrix $(A^{ij}_{kl}) = \frac{\partial A}{\partial \xi}(\nu, v, \xi)$, that is

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n} A^{ij}_{kl}\eta^i_k\eta^j_l \geq \lambda_{\nu,v}|\eta|^2, \quad \sum_{i,k}(\sum_{j,l} A^{ij}_{kl}\eta^j_l)^2 \leq \Lambda^2_{\nu,v}|\eta|^2 \tag{6.2}$$

for any $\eta \in \mathbb{R}^{mn}$. Moreover, for some positive constants λ_ν, Λ_ν we have

$$\lambda_{\nu,v} \geq \lambda_\nu|Dv|^{p-2}, \quad \Lambda_{\nu,v} \leq \Lambda_\nu|Dv|^{p-2}.$$

If $n > 2$, we also assume that

$$\frac{\Lambda_{\nu,v}}{\lambda_{\nu,v}} < \frac{n}{n-2}. \tag{6.3}$$

We also assume that there exists a positive constant $a_{\nu,v}$ such that

$$|\frac{\partial A}{\partial v}(\nu, v, \xi)| \leq a_{\nu,v}|\xi|^{p-1} \text{ with } 2a_{\nu,v}M_{\nu,v}(p+n-1) < \sigma_0\widehat{\lambda}_\nu, \tag{6.4}$$

where $M_{\nu,v} = \sup_{Q_{\frac{2}{3}}} |v|$ and σ_0 is a fixed number in (0, 1) and

$$\widehat{\lambda}_\nu = (1-\delta^2)\lambda_\nu \text{ and } \delta = \frac{n-2}{n}\sup\{\frac{\Lambda_{\nu,v}}{\lambda_{\nu,v}} \; : \; v \text{ is a bounded solution}\}.$$

Note that $\widehat{\lambda} > 0$ thanks to (6.3).

Fixing ν in $\mathcal{I}$ and a solution v to (6.1), we will denote $a(v,\zeta) = A(\nu, v, \zeta)$ and also omit the parameter ν in the subscripts for $\lambda_{\nu,v}, \Lambda_{\nu,v}, a_{\nu,v}$ in the sequel.

The proof of Theorem 2.3 relies mainly on the following two lemmas which establish uniform bounds for the L^q norms of Dv. First of all, we need the following simple consequence of Sobolev's inequality. For any $q, r > 0$ such that $q + r\frac{2}{n} = \frac{1}{2}q\frac{2n}{n-2}\frac{n-2}{n} + r\frac{2}{n}$, assuming $n > 2$ as the case $n = 2$ is easy, we have by Hölder and Sobolev's inequalities the following for V, ϕ in suitable Sobolev spaces and $\phi = 0$ on the boundary of B_1.

$$\begin{aligned}\int_{B_1} |V|^{q+r\frac{2}{n}}\phi^{2+\frac{4}{n}}\,dx &\leq \left(\int_{B_1} [|V|^{\frac{1}{2}q}\phi]^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\left(\int_{B_1} |V|^r\phi^2\,dx\right)^{\frac{2}{n}} \\ &\leq C\int_{B_1} |D(|V|^{\frac{1}{2}q}\phi)|^2\,dx\left(\int_{B_1} |V|^r\phi^2\,dx\right)^{\frac{2}{n}}.\end{aligned}$$

Therefore, if Q_1 is the cylinder with base B_1 then by integrating in t

$$\iint_{Q_1} |V|^{q+r\frac{2}{n}}\phi^{2+\frac{4}{n}}\,dz \leq C\sup_{t\in(-1,0)}\left(\int_{B_1} |V|^r\phi^2\,dx\right)^{\frac{2}{n}}\iint_{Q_1} |D(|V|^{\frac{1}{2}q}\phi)|^2\,dz. \tag{6.5}$$

In the sequel, we will make use of difference quotients. For any vector valued function f, $i = 1, \dots, n$ and real number $h \neq 0$, we denote

$$\delta_h^{(i)} f(x,t) = \frac{1}{h}(f(x+he_i,t) - f(x,t)), \quad e_i \text{ is the unit vector in the } i^{th} \text{ direction of } \mathbb{R}^n.$$

If an argument holds for any i, we will simply omit the superscript (i) in the above notation.

For any v being a weak solution to a nice system, v is Hölder continuous and the difference $\delta_h v$ weakly solves

$$(\delta_h v)_t = \operatorname{div}(\delta_h a(v, Dv)). \tag{6.6}$$

We first have the following estimate for such "nice" solutions.

Lemma 6.1 *Let v be a Hölder continuous weak solution to (6.1). For any $\phi \in C_0^1(Q_{\frac{3}{4}})$ there exists a constant C depending on $M = \sup_{Q_{\frac{2}{3}}} |v|$ such that*

$$\sup_{t\in(-1,0)} \int_{B_1} |Dv|^2\phi^2\, dx + \lambda \iint_{Q_1} |Dv|^{p-2}|D^2v|^2\phi^2\, dz \text{ and } \iint_{Q_1} |Dv|^{p+\frac{4}{n}}\phi^{2+\frac{4}{n}}\, dz \le C. \tag{6.7}$$

Proof. Let ϕ be in $C_0^1(Q_{\frac{2}{3}})$. For any function f in (x,t), $h \neq 0$ and $e = e_i$ $(i = 1, \dots, n)$, we will write $\delta_h^+ f(x,t) = (f(x+he,t) - f(x,t))/h$ and $\delta_h^- f(x,t) = (f(x,t) - f(x+he,t))/h$. Testing (6.6) with $\delta_h^+ v\phi^2$ and integrating by parts in x, we get

$$\sup_{t\in(-1,0)} \int_{B_1} |\delta_h^+ v|^2\phi^2\, dx + \iint_{Q_1} \langle \delta_h^+ a(v,Dv), D(\delta_h^+ v\phi^2)\rangle\, dz \le \iint_{Q_1} |\delta_h^+ v|^2\phi_t\, dz. \tag{6.8}$$

We then set

$$E_0 = \{(x,t) \in Q_1 \,:\, |Dv(x,t)| \le |Dv(x+h,t)|\}, \quad E_1 = Q_1 \setminus E_0.$$

We now split the integral of $\langle \delta_h^+ a(v,Dv), D(\delta_h^+ v\phi^2)\rangle$ on Q_1 into those on E_0, E_1. On E_1, we have

$$\langle \delta_h^+ a(v,Dv), D(\delta_h^+ v\phi^2)\rangle = \langle \delta_h^- a(v,Dv), D(\delta_h^- v\phi^2)\rangle = \langle \delta_h^- a(v,Dv), D(\delta_h^- v)\phi^2 + \delta_h^- v D(\phi^2)\rangle.$$

Concerning the term $\delta_h^- a(v,Dv)$, we write

$$\begin{aligned}
\delta_h^- a(v,Dv) &= \tfrac{1}{h}[a(v(x,t),Dv(x,t)) - a(v(x,t),Dv(x+h,t))] + \\
&\quad \tfrac{1}{h}[a(v(x,t),Dv(x+h,t)) - a(v(x+h,t),Dv(x+h,t))] \\
&= \int_0^1 \frac{\partial a}{\partial \xi}(v, sDv(x,t) + (1-s)Dv(x+h,t))D\delta_h^- v ds + \\
&\quad \int_0^1 \frac{\partial a}{\partial v}(sv(x,t) + (1-s)v(x+h,t), Dv(x+h,t))\delta_h^- v ds.
\end{aligned}$$

Using the fact that $|sDv(x,t) + (1-s)Dv(x+h,t)| \ge |Dv(x+h,t)|$ on E_1 and the ellipticity condition of $\partial a/\partial\xi$ we get

$$\langle \frac{\partial a}{\partial \xi}(v, sDv(x,t) + (1-s)Dv(x+h,t))D\delta_h^- v ds, D\delta_h^- v\rangle \ge \lambda |Dv(x+h,t)|^{p-2}|D\delta_h^- v|^2.$$

The last term will stay on the left of (6.8). On the other hand, bty (6.4), we have

$$\left|\frac{\partial a}{\partial v}(sv(x,t) + (1-s)v(x+h,t), Dv(x+h,t))\delta_h^- v\right| \le |a_v||Dv(x+h,t)|^{p-1}|\delta_h^- v|.$$

Thus, by Young's inequality, we have for any positive ε the following

$$\left|\langle \frac{\partial a}{\partial v}(sv(x,t)+(1-s)v(x+h,t), Dv(x+h,t))\delta_h^- v, D(\delta_h^- v)\rangle\right| \le \\ \le \quad \varepsilon |Dv(x+h,t)|^{p-2}|D(\delta_h^- v)|^2 + C(\varepsilon)|a_v|^2|Dv(x+h,t)|^p|\delta_h^- v|^2.$$

The last term will be kept on the right hand side of (6.8).

Similar argument will apply to the set E_0. We then choose ε sufficiently small and derive from (6.8) and the above estimates the following.

$$\sup_{t\in(-1,0)}\int_{B_1} |\delta_h v|^2\phi^2\,dx + \frac{\lambda}{2}\iint_{Q_1} |v_d|^{p-2}(|D(\delta_h^+ v)|^2 + |D\delta_h^- v|^2)\phi^2\,dz \le \\ \iint_{Q_1} \frac{|a_v|^2}{\lambda}|v_D|^p|V_h|^2\phi^2 + |v_D|^2(|\phi_t| + \Lambda|v_D|^{p-2}|D\phi|^2 + 1)\,dz, \tag{6.9}$$

where $v_D = \max\{|Dv(x,t)|, |Dv(x+h,t)|\}$, $v_d = \min\{|Dv(x,t)|, |Dv(x+h,t)|\}$ and $V_h = \max\{|\delta_h^+ v|, |\delta_h^- v|\}$.

Sending h to zero, we get

$$\sup_{t\in(-1,0)}\int_{B_1} |Dv|^2\phi^2\,dx + \frac{\lambda}{2}\iint_{Q_1} |Dv|^{p-2}|D^2v|^2\phi^2\,dz \le \\ \iint_{Q_1} \frac{|a_v|^2}{\lambda}|Dv|^{p+2}\phi^2 + |Dv|^2(|\phi_t| + \Lambda|Dv|^{p-2}|D\phi|^2 + 1)\,dz. \tag{6.10}$$

Of course the above argument is justified if $Dv \in L^{p+2}_{loc}$. This fact will be established in Lemma 6.2 following this proof.

We now estimate the integral of $|Dv|^{p+2}$ in (6.10). By integrating by parts in x

$$\iint_{Q_1} |Dv|^{p+2}\phi^2\,dz = \iint_{Q_1} vD(|Dv|^{p+1}\phi^2)\,dz \le M\iint_{Q_1} (|D^2v||Dv|^p\phi^2 + |Dv|^{p+1}\phi|D\phi|)\,dz,$$

where $M = M_{v,v} = \sup_{Q_{\frac{3}{4}}} |v|$. Young's inequality applying to the right then gives

$$\iint_{Q_1} |Dv|^{p+2}\phi^2\,dz \le \varepsilon\iint_{Q_1} |Dv|^{p+2}\phi^2\,dz + C(\varepsilon)M^2\iint_{Q_1} (|D^2v|^2|Dv|^{p-2}\phi^2 + |Dv|^p|D\phi|^2)\,dz.$$

Thus, for $\varepsilon = \frac{1}{2}$, we obtain

$$\frac{a_v^2}{\lambda}\iint_{Q_1} |Dv|^{p+2}\phi^2\,dz \le 4\frac{a_v^2M^2}{\lambda}\iint_{Q_1} |Dv|^{p-2}|D^2v|^2\phi^2 + |Dv|^p|D\phi|^2\,dz.$$

Using this in (6.10) and the assumption on the smallness of a_vM in (6.4), we obtain

$$\sup_{t\in(-1,0)}\int_{B_1} |Dv|^2\phi^2\,dx + \lambda\iint_{Q_1} |Dv|^{p-2}|D^2v|^2\phi^2\,dz \le \\ C\iint_{Q_1} |Dv|^2(|\phi_t| + (|Dv|^{p-2}\Lambda + \lambda)|D\phi|^2 + 1)\,dz. \tag{6.11}$$

By Caccioppoli's inequality we note that

$$\iint_{Q_1} |Dv|^2(|\phi_t| + |Dv|^{p-2}|D\phi|^2)\,dz \le C(|\phi_t|, |D\phi|)\iint_{Q_1} |v|^2\,dz \le C(M). \tag{6.12}$$

The above and (6.11) then imply

$$\sup_{t\in(-1,0)} \int_{B_1} |Dv|^2\phi^2\, dx + \iint_{Q_1} |Dv|^{p-2}|D^2v|^2\phi^2\, dz \le C(M).$$

We now make use of (6.5), with $V = Dv$ and $q = p, r = 2$, and combine with the above to get the second estimate in (6.7) and complete the proof. ∎

To justify the calculation leading to (6.10), we now show

Lemma 6.2 *Spatial derivatives of Hölder continuous weak solutions v to (6.1) are in L^{p+2}_{loc}.*

Proof. Let v be a Hölder continuous weak solution to (6.1). We will show that at almost every point $z_0 = (x_0, t_0) \in Q_1$, with $Dv(z_0) \neq 0$ and R is sufficiently small, there is a constant C, which may depend on z_0, such that

$$R^{p+2}⨌_{Q_R} |\delta_h v|^{p+2}\, dz \le C⨌_{Q_{4R}} |Dv|^p\, dz \quad \text{for } h \in (0,1). \tag{6.13}$$

Here, $Q_R = B_R(x_0)\times(t_0 - R^p, t_0)$ and $Q_R \subset Q_{4R} \subset Q_1$. If this is not true then there will be a sequence $R_k \to 0$ such that

$$R_k^{p+2}⨌_{Q_{R_k}} |\delta_h v|^{p+2}\, dz > k⨌_{Q_{4R_k}} |Dv|^p\, dz \quad \text{for some positive small } h.$$

We then use the scaling $v_k(X,T) = v(x_0 + R_kX, t_0 + R_k^pT)$ and get a sequence of functions v_k on Q_1 such that

$$\iint_{Q_{\frac14}} |\delta_h v_k|^{p+2}\, dz \ge \frac{k}{R_k^p}\iint_{Q_1} |Dv_k|^p\, dz. \tag{6.14}$$

Since v is Hölder continuous and $R_k \to 0$ we see that v_k can be arbitrarily close to $(v_k)_{Q_1}$ on Q_1. Thus, v_k approximately solves the following system

$$U_t = \operatorname{div}(a((v_k)_{Q_1}, DU)). \tag{6.15}$$

For such system, which does not explicitly depend on U and has the same ellipticity constants λ, Λ, we can find (see Remark 6.4 at the end of this section) a function $C(x)$ which is bounded if x is bounded such that

$$\iint_{Q_{\frac14}} |DU|^{p+2}\, dz \le C\left(\iint_{Q_{\frac12}} |DU|^p\, dz\right). \tag{6.16}$$

For sufficiently small $h > 0$, the above yields

$$\iint_{Q_{\frac14}} |\delta_h U|^{p+2}\, dz \le C\left(\iint_{Q_{\frac12}} |DU|^p\, dz\right).$$

Our approximation results then give a sequence $\{U_k\}$ of weak solutions to (6.15) satisfying

$$\iint_{Q_{\frac12}} |DU_k|^p\, dz \le c\iint_{Q_{\frac12}} |Dv_k|^p\, dz,$$

and $U_k - v_k \to 0$ in $L^p(Q_{\frac{1}{4}})$ (as well as in $L^q(Q_{\frac{1}{4}})$ for any $q > 1$ because U_k, v_k are bounded). Moreover, $DU_k - Dv_k \to 0$ weakly in $L^p(Q_{\frac{1}{4}})$.

Combining the above estimates, we obtain

$$\frac{k}{R_k^p}\iint_{Q_1} |Dv_k|^p\, dz \le \iint_{Q_{\frac{1}{4}}} |\delta_h v_k|^{p+2}\, dz, \quad \iint_{Q_{\frac{1}{4}}} |\delta_h U_k|^{p+2}\, dz \le C\left(\iint_{Q_{\frac{1}{2}}} |Dv_k|^p\, dz\right).$$

With h being fixed and U_k, v_k being bounded, we have $\delta_h(U_k) - \delta_h(v_k) \to 0$ in L^q for all $q > 1$ as $k \to \infty$. Moreover, since $R_k^{-p}\iint_{Q_{\frac{1}{2}}} |Dv_k|^p\, dz = \fint\!\!\fint_{Q_{R_k}} |Dv|^p\, dz \to |Dv(z_0)|^p \ne 0$, the above gives a contradiction when $k \to \infty$ because its left most term becomes arbitrarily large, the right most one stays finite and the middle ones are getting close to each other. Thus, (6.13) holds almost everywhere on the set where $Dv \ne 0$. Finally, by sending h to 0 it is easy to see that (6.13) implies $Dv \in L^{p+2}_{loc}$. ∎

To get estimates for higher powers of $|Dv|$, we need the following lemma.

Lemma 6.3 *Let v be a Hölder continuous weak solution to (6.1) and α be a positive number. Assume that $\frac{\alpha}{2+\alpha} = \delta_{\alpha,v}\frac{\lambda_v}{\Lambda_v}$ for some $\delta_{\alpha,v} \in (0,1)$ and*

$$2a_v M(p+\alpha+1) < \sigma_0\widehat{\lambda_v}, \quad \text{with } \widehat{\lambda_v} = (1-\delta_{\alpha,v}^2)\lambda_v, \quad \sigma_0 \in (0,1). \tag{6.17}$$

If $\phi \in C_0^1(Q_1)$ and $supp(\phi) \subset Q_{\frac{3}{4}}$ then

$$\iint_{Q_1} |Dv|^{p+\alpha+(2+\alpha)\frac{2}{n}}\phi^{2+\frac{4}{n}}\, dz \le C\left(\iint_{Q_1} |Dv|^{2+\alpha}(|\phi_t| + |Dv|^{p-2}|D\phi|^2 + 1)\, dz\right). \tag{6.18}$$

Proof. To proceed, we recall the following facts from [9]. From the ellipticity condition of $\frac{\partial a}{\partial \xi} = (A_{kl}^{ij})$ we have for $\kappa_v = \lambda_v/\Lambda_v^2$ and $\nu_v = \lambda_v/\Lambda_v$ that

$$\sum_{i,k}(\eta_i^k - \kappa A_{kl}^{ij}\eta_l^j)^2 \le (1 - 2\kappa_v\lambda_v + \kappa_v^2\Lambda_v^2)|\eta|^2 = (1-\nu_v^2)|\eta|^2.$$

Lemma [9, p.677] then gives $D\zeta D(\zeta|\zeta|^\alpha) \ge \mu^{\frac{1}{2}}(\alpha)|D\zeta||D(\zeta|\zeta|^\alpha)|$ for any $\zeta : \mathbb{R}^n \to \mathbb{R}^{nm}$ and $\mu(\alpha) = 1 - (\frac{\alpha}{2+\alpha})^2$. Therefore, with $\zeta = Dv$, we have

$$\begin{aligned}\kappa_v \sum \tfrac{\partial a}{\partial \xi} D^2 v D(Dv|Dv|^\alpha) &= \sum(\kappa_v \tfrac{\partial a}{\partial \xi} D^2 v - D^2 v)D(Dv|Dv|^\alpha) + D(Dv)D(Dv|Dv|^\alpha)\\ &\ge (\mu^{\frac{1}{2}}(\alpha) - (1-\nu_v^2)^{\frac{1}{2}})|D(Dv)||D(Dv|Dv|^\alpha)|.\end{aligned}$$

Thus, if $\frac{\alpha}{2+\alpha} = \delta_{\alpha,v}\frac{\lambda_v}{\Lambda_v}$ for some $\delta_{\alpha,v} \in (0,1)$ then the constant in the right hand side is

$$\mu(\alpha)^{\frac{1}{2}} - (1-\nu_v^2)^{\frac{1}{2}} = \frac{\nu_v^2 - \frac{\alpha^2}{(2+\alpha)^2}}{(\mu(\alpha)^{\frac{1}{2}} + (1-\nu_v^2)^{\frac{1}{2}})} \ge (1-\delta_{\alpha,v}^2)\nu_v^2 = \kappa_v(1-\delta_{\alpha,v}^2)\lambda_v.$$

Hence, by the assumption on λ_v and the definition of $\widehat{\lambda_v}$, we get

$$\begin{aligned}\sum \tfrac{\partial a}{\partial \xi} D^2 v D(Dv|Dv|^\alpha) &\ge \tfrac{1}{\kappa_v}(\mu^{\frac{1}{2}}(\alpha) - (1-\nu_v^2)^{\frac{1}{2}})|D(Dv)||D(Dv|Dv|^\alpha)\\ &\ge (1-\delta_{\alpha,v}^2)\lambda_v|Dv|^\alpha|D^2v|^2 = \widehat{\lambda_v}|Dv|^{p-2+\alpha}|D^2v|^2.\end{aligned} \tag{6.19}$$

The following calculation could be rigorously justified by using difference quotient operator δ_h, as in the previous lemmas, in place of the differentiation D below. However, in order to be more suggestive, we will write (6.6) formally as

$$(Dv)_t = \mathrm{div}(\frac{\partial a}{\partial \xi}(v, Dv)D^2 v + \frac{\partial a}{\partial v}(v, Dv)Dv). \tag{6.20}$$

Testing (6.20) with $Dv|Dv|^\alpha\phi^2$ to obtain (compare with (6.9))

$$\sup_t \int_{B_1} |Dv|^{2+\alpha}\phi^2\, dx + \iint_{Q_1} \langle\frac{\partial a}{\partial \xi}(v, Dv)D^2 v, D(Dv|Dv|^\alpha)\phi^2\rangle\, dz \le \iint_{Q_1} |Dv|^{2+\alpha}|\phi_t|\, dz$$
$$+ \iint_{Q_1} |\langle\frac{\partial a}{\partial \xi}(v, Dv)D^2 v, Dv|Dv|^\alpha D\phi\phi\rangle| + \left|\frac{\partial a}{\partial v}\right|(|Dv|^{2+\alpha}\phi^2 + |Dv|^{1+\alpha}|\phi D\phi|)\, dz.$$

Using (6.19) and Young's inequality, we deduce

$$\begin{aligned} &\sup_t \int_{B_1} |Dv|^{2+\alpha}\phi^2\, dx + \widehat{\lambda}_v \iint_{Q_1} |D^2 v|^2 |Dv|^{p-2+\alpha}\phi^2\, dz \le \\ &\frac{|a_v|^2}{\widehat{\lambda}_v} \iint_{Q_1} |Dv|^{p+2+\alpha}\phi^2\, dz + C\iint_{Q_1} |Dv|^{2+\alpha}(|\phi_t| + |Dv|^{p-2}|D\phi|^2 + 1)\, dz. \end{aligned} \tag{6.21}$$

Again, since v is Hölder continuous, similar argument as that of Lemma 6.2 shows that $Dv \in L^{p+2+\alpha}_{loc}$ and justifies our calculation. We now estimate the integral of $|Dv|^{p+2+\alpha}\phi^2$. By integrating by parts in x, we have

$$\begin{aligned} \iint_{Q_1} |Dv|^{p+2+\alpha}\phi^2\, dz &= \iint_{Q_1} vD(Dv|Dv|^{p+\alpha}\phi^2)\, dz \\ &\le M\iint_{Q_1} (p + \alpha + 1)|D^2 v||Dv|^{p+\alpha}\phi^2 + |Dv|^{p+1+\alpha}\phi|D\phi|\, dz, \end{aligned}$$

where $M = \sup_{Q_{\frac{2}{3}}} |v|$. Young's inequality applying to the right then gives

$$\iint_{Q_1} |Dv|^{p+2+\alpha}\phi^2\, dz \le (\frac{1}{2} + \varepsilon)\iint_{Q_1} |Dv|^{p+2+\alpha}\phi^2\, dz +$$
$$[2M(p + \alpha + 1)]^2 \iint_{Q_1} |D^2 v|^2|Dv|^{p-2+\alpha}\phi^2\, dz + C(\varepsilon)M^2 \iint_{Q_1} |Dv|^{p+\alpha}|D\phi|^2\, dz.$$

We choose $\varepsilon < 1/2$ in the above to obtain an estimate for the integral of $|Dv|^{p+2+\alpha}\phi^2$. Using this in (6.21) and the assumption (6.17) on $|a_v M|$ and $\widehat{\lambda}_v$, we obtain

$$\begin{aligned} &\sup_t \int_{B_1} |Dv|^{2+\alpha}\phi^2\, dx + (1 - \sigma_0^2)\widehat{\lambda}_v \iint_{Q_1} |D^2 v|^2|Dv|^{p-2+\alpha}\phi^2\, dz \le \\ &\qquad C\iint_{Q_1} |Dv|^{2+\alpha}(|\phi_t| + \widehat{\lambda}|Dv|^{p-2}|D\phi|^2 + 1)\, dz. \end{aligned} \tag{6.22}$$

The above also gives similar estimate for $\||Dv|^{(p+\alpha)/2}\phi\|_{V(Q_1)}$. Applying (6.5), with $V = Dv$ and $q = p + \alpha, r = 2 + \alpha$, we get the lemma. ∎

We are now ready to give

Proof of Theorem 2.3. By M.2) (see (6.4)), we can choose a number $\beta > n-2$ and some $\delta' \in (0,1)$ such that for $M = \sup_{Q_{\frac{3}{4}}} |v|$

$$2a_v M(p+\beta+1) < \sigma_0 \widehat{\lambda_v}, \text{ and } \frac{\beta}{\beta+2} = \delta' \frac{\lambda_v}{\Lambda_v} > \frac{n-2}{n}.$$

Clearly, starting with $\alpha_0 \le 2$ (thus $p + \alpha_0 \le p+2$), we can find finitely many numbers $\alpha_0, \dots, \alpha_K$ such that

$$2a_v M(p+\alpha_k+1) < \sigma_0 \widehat{\lambda} \quad \text{and} \quad \frac{\alpha_k}{\alpha_k+2} = \delta_{s,v}^{(k)} \frac{\lambda_v}{\Lambda_v}, \quad \delta_{s,v}^{(k)} \le \delta'$$

for any integer $k \le K$. Moreover, it is easy to see that we can also choose $\delta_{s,v}^{(k)}$ such that $\alpha_k \le \alpha_{k+1} \le \alpha_k + (2+\alpha_k)\frac{2}{n}$ and $\alpha_K = \beta$. We also define $q_k = p + \alpha_k + (2+\alpha_k)\frac{2}{n}$.

Since $Q_{\frac{5}{7}} \subset Q_{\frac{3}{4}}$, by the Caccioppoli inequality we obtain $\|Du\|_{L^p(Q_{\frac{5}{7}})} \le C(M)$. Using the estimate for $|Dv|^{p+2}$ in Lemma 6.1 and a cut-off function ϕ for $Q_{\frac{2}{3}}, Q_{\frac{5}{7}}$ as in Lemma 6.3, we see that

$$\iint_{Q_{\frac{2}{3}}} |Dv|^{p+2}\, dz \le C(M).$$

Let the function ϕ in Lemma 6.3 be the cut-off functions for Q_{R_k} and $Q_{R_{k-1}}$ with $R_k = \frac{2}{3} - k\frac{1}{6K}$, $k = 1, \dots, K$. It is clear that the above choice of α_k and the fact that $p + \alpha_{k+1} \le q_k$ allow us to apply (6.18) inductively in finite steps to obtain the following

$$\iint_{Q_{\frac{1}{2}}} |Dv|^{q_K}\, dz \le C(K, M). \tag{6.23}$$

Note that $q_K = p + \alpha_K + (2+\alpha_K)\frac{2}{n} > n + p$ because $\alpha_K = \beta > n-2$.

Now, let u be a weak solution which is, by II), approximated by a sequence $\{v_k\}$ of weak solutions to nice systems and $Dv_k \rightharpoonup Du$ weakly in $L^1(Q_{\frac{1}{2}})$. By the semicontinuity of seminorms and (6.23), we have for any $Q_R \subset Q_{\frac{1}{2}}$ that

$$\iint_{Q_R} |Du|^{q_K}\, dz \le \liminf_{k\to\infty} \iint_{Q_R} |Dv_k|^{q_K}\, dz \le C(M).$$

Hence, with $q = q_K/p$,

$$\iint_{Q_R} |Du|^p\, dz \le \left(\iint_{Q_R} |Du|^{q_K}\, dz\right)^{\frac{1}{q}} |Q_R|^{1-\frac{1}{q}} \le C(M) R^{n+p-(n+p)\frac{1}{q}} = C(M) R^{n+\alpha}.$$

Here, $\alpha = p - (n+p)\frac{1}{q} = \frac{p}{q_K}(q_K - n - p)$ is positive. Hölder continuity for u then follows from the above estimate and the Poincaré inequality in Lemma 3.1. Thus, u can be approximated by solutions to systems that do not explicitly depend on u and satisfy the property D). This implies that u satisfies D) and $\mathcal{I}$ is closed. ∎

Remark 6.4 We should note that the estimate (6.16) for a solution U to systems with coefficients independent of their solutions could be derived directly from our proof. Indeed, for such systems, $\frac{\partial a}{\partial U} = 0$. Therefore, in this case, our arguments which lead to (6.10) and (6.21) would not yield the integral of $|DU|^{p+2+\alpha}$ on the right hand sides of our estimates. Therefore, our proof provides a similar bound as in (6.16) for the integral of $|DU|^{q_K}$ for some $q_K > n + p$. Hence, the calculation in our lemmas is justified.

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