Advanced Nonlinear Studies **14** (2014), 137-158

Superposition Operators Between Higher-order Sobolev Spaces and a Multivariate Faà di Bruno Formula: Supercritical Case

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Received 18 January 2013
Communicated by Jean Mawhin

Abstract

This paper is a continuation of the work begun in [6] on superposition operators, $\left(N_g u\right)(x) = g(u(x))$, between two arbitrary Sobolev spaces. Sufficient conditions which ensure the well-definedness, the continuity and the validity of the higher-order chain rule for such operators are given in the supercritical case (see Remark 1.1). As a consequence of these properties, it is proved that $N_g\left(W^{m,p}(\Omega) \cap W_0^{k,p}(\Omega)\right) \subset W_0^{l,q}(\Omega)$.

2010 Mathematics Subject Classification. 47H30, 46E35, 26B05, 26A46.
Key words. Sobolev spaces, superposition operators, higher-order chain rule, homogeneous Dirichlet conditions

1 Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ be a function, let $n \in \mathbb{N}^*$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The superposition (or Nemytskij) operator generated by the function g is, by definition, the operator denoted by N_g that associates to each function $u : \Omega \to \mathbb{R}$ the function $N_g u : \Omega \to \mathbb{R}$ defined by

$$\left(N_g u\right)(x) = (g \circ u)(x) = g(u(x)), \quad x \in \Omega.$$

Throughout this paper, we will denote by ∂_i the weak derivative with respect to x_i, and by $\mathcal{L}^k$ the k-dimensional Lebesgue measure ($k \geq 1$).

The next result will be need in what follows. It is, in fact, Theorem 3.1 in [5] corresponding to the supercritical case (see Remark 1.1 below).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, having the cone property, let $m \in \mathbb{N}^*$, and let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function.*

(i) *If $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$), then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q \leq \frac{np}{n-(m-1)p}$. Moreover, N_g is bounded and the chain rule*

$$\partial_i (g \circ u) = (g' \circ u)\, \partial_i u \quad \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } i = 1, \ldots, n, \tag{1.1}$$

holds for all $u \in W^{m,p}(\Omega)$, where the product $(g' \circ u)\, \partial_i u$ is to be interpreted in the sense of de la Vallée Poussin, namely it is considered to be zero whenever $\partial_i u(x) = 0$, irrespective of whether $(g' \circ u)(x)$ is defined.

(ii) *If $p = \frac{n}{m-1}$, with $m \geq 2$, and $n \geq m-1$, then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q < \infty$. Moreover, N_g is bounded and* (1.1) *holds for all $u \in W^{m,p}(\Omega)$ (with the usual convention on the product $(g' \circ u)\, \partial_i u$).*

(iii) *If $\frac{n}{m-1} < p < \infty$, with $m \geq 2$ ($1 \leq p < \infty$ when $n \leq m-1$), then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, N_g is bounded and* (1.1) *holds for all $u \in W^{m,p}(\Omega)$ (with the usual convention on the product $(g' \circ u)\, \partial_i u$).*

(iv) *If $g^* : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function such that $g^* = g'$ $\mathcal{L}^1$-a.e. in $\mathbb{R}$, then in all cases* (i)-(iii) *the chain rule* (1.1) *can be rewritten as*

$$\partial_i (g \circ u) = (g^* \circ u)\, \partial_i u \quad \mathcal{L}^n\text{-a.e in } \Omega, \text{ for all } i = 1, \ldots, n,$$

the convention on the product $(g^* \circ u)\, \partial_i u$ being no longer necessary.

(v) *The hypotheses which ensure the well-definedness of the operator N_g from $W^{m,p}(\Omega)$ into $W^{1,q}(\Omega)$, with $1 \leq p, q < \infty$, are sufficient to ensure the continuity of N_g in each of the cases* (i)-(iii).

Remark 1.1 According to Bourdaud [3], the Sobolev space $W^{m,p}(\Omega)$ is said to be supercritical if the imbedding $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is valid.

The following result is a consequence of Theorem 1.1 and describes the image under superposition operators of the functions in $W^{m,p}(\Omega)$ vanishing on the boundary $\partial\Omega$ (see [5, Theorem 3.4]).

Theorem 1.2 *Assume that the hypotheses of Theorem* 1.1 *are satisfied with the following modification: the hypothesis "Ω has the cone property" is replaced with "Ω is of class C^1". If, in addition, $g(0) = 0$, then, $N_g\left(W^{m,p}(\Omega) \cap W_0^{1,p}(\Omega)\right) \subset W_0^{1,q}(\Omega)$, with $1 \leq p, q < \infty$, in each of the cases* (i)-(iii) *of Theorem* 1.1.

The aim of this paper is to generalize Theorems 1.1 and 1.2. The generalization of Theorem 1.1 consists in formulating sufficient conditions for a function g to generate a superposition operator N_g having the following properties: N_g is well defined from a Sobolev space $W^{m,p}(\Omega)$ into another Sobolev space $W^{l,q}(\Omega)$, with $1 \leq q, p < \infty$, $m, l \in \mathbb{N}^*$, $mp > n$, $l \leq m$, N_g is bounded, continuous, and satisfies in addition the higher-order chain rule

$$D^\alpha (g \circ u) = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \alpha \\ |\alpha^i| \neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \tag{1.2}$$

$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } \alpha \in \mathbb{N}^n \text{ with } 1 \leq |\alpha| \leq l,$$

Here and throughout this paper, $c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \in \mathbb{N}^*$ designates a combinatorial constant and D^α designates the weak derivative with respect to the multi-index $\alpha \in \mathbb{N}^n$.

Remark 1.2 Formula (1.2) is formally identical to the well-known higher-order chain rule used to compute higher partial derivatives of the composite function $g \circ u$ when $g : \mathbb{R} \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ are sufficiently smooth (see e.g. [4, Corollary 2.10, formula (2.9)]). In its turn, the result given by Corollary 2.10 in [4] is a multivariate version of the famous Faà di Bruno formula (see [8]).

2 Statements of the main results and comments

One of the main results of this article is contained in the following theorem.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, having the cone property, let $m, l \in \mathbb{N}^*$, $l \leq m$, and let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^{l-1} with $g^{(l-1)} : \mathbb{R} \to \mathbb{R}$ locally Lipschitz.*

(i) *If $\frac{n}{m} < p < \frac{n}{m-l}$, with $n \geq m-l+1$ ($1 \leq p < \frac{n}{m-l}$ when $n \in \{m-l+1, \ldots, m\}$), then $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ for all $1 \leq q \leq \frac{np}{n-(m-l)p}$. Moreover, N_g is bounded and the higher-order chain rule* (1.2) *holds for all $u \in W^{m,p}(\Omega)$, where the product $\left(g^{(l)} \circ u\right)\partial_{j_1}u \ldots \partial_{j_l}u$ is to be interpreted in the sense of de la Vallée Poussin, namely it is considered to be zero whenever one of the factors $\partial_{j_1}u(x), \ldots, \partial_{j_l}u(x)$ is zero, irrespective of whether $\left(g^{(l)} \circ u\right)(x)$ is defined.*

(ii) *If $p = \frac{n}{m-l}$, with $m \geq l+1$, and $n \geq m-l$, then $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ for all $1 \leq q < \infty$. Moreover, N_g is bounded and* (1.2) *holds for all $u \in W^{m,p}(\Omega)$* (*with the usual convention on the product $\left(g^{(l)} \circ u\right)\partial_{j_1}u \ldots \partial_{j_l}u$*).

(iii) *If $\frac{n}{m-l} < p < \infty$, with $m \geq l+1$ ($1 \leq p < \infty$ when $n \leq m-l$), then $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, N_g is bounded and* (1.2) *holds for all $u \in W^{m,p}(\Omega)$* (*with the usual convention on the product $\left(g^{(l)} \circ u\right)\partial_{j_1}u \ldots \partial_{j_l}u$*).

(iv) *If $g^* : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function such that $g^* = g^{(l)}$ $\mathcal{L}^1$-a.e. in $\mathbb{R}$, then in all cases* (i)-(iii)*, the chain rule* (1.2) *can be rewritten with g^* instead of $g^{(l)}$, the convention on the product $\left(g^{(l)} \circ u\right)\partial_{j_1}u \ldots \partial_{j_l}u$ being no longer necessary.*

(v) *The hypotheses which ensure that the operator N_g from $W^{m,p}(\Omega)$ into $W^{l,q}(\Omega)$ is well defined, with $1 \leq p, q < \infty$, are sufficient to ensure the continuity of N_g in each of the cases* (i)-(iii).

Roughly speaking, the following result shows that the assumptions of Theorem 2.1 allow the superposition operator N_g to preserve the homogeneous Dirichlet conditions.

Theorem 2.2 *Assume that the hypotheses of Theorem* 2.1 *are satisfied with the following modification: the hypothesis "Ω has the cone property" is replaced with "Ω is of class C^1". If, in addition, $g(0) = 0$, then, $N_g\left(W^{m,p}(\Omega) \cap W_0^{k,p}(\Omega)\right) \subset W_0^{l,q}(\Omega)$, with $1 \leq p, q < \infty$, $k \in \mathbb{N}$, $l \leq k \leq m$, in each of the cases* (i)-(iii) *of Theorem* 2.1.

Theorem 2.1 generalizes some fundamental results of Marcus and Mizel. These results are contained in Theorem 2.3 in [6] which gather together Theorem 1 in [9], Lemma 3 and Theorem 2 in [10] and Theorem 8.5 in [2].

Theorem 2.1 can also be related to some fundamental results obtained by Bourdaud dealing with superposition operators $N_g : W^{m,p}(\mathbb{R}^n) \to W^{m,p}(\mathbb{R}^n)$, with $m \geq 2$. Indeed, Theorems 2, 3, and 4 in [3] (see also Subsection 5.2.4, Theorems 1 and 2 in [11]), via an extension theorem for Sobolev functions (see e.g. Theorem 4.28 in Adams [1]), lead to Corollary 2.1 in [6] which provides sufficient conditions on g such that $N_g : W^{m,p}(\Omega) \to W^{m,p}(\Omega)$ is well defined, with Ω open, bounded and smooth. In certain specific situations in which it is sufficient for the superposition operator to act from $W^{m,p}(\Omega)$ into $W^{l,q}(\Omega)$ with $l < m$, Theorem 2.1 can be more useful than this corollary since the assumptions imposed on g by Theorem 2.1 which ensure the well-definedness of $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ relaxes together with the decrease of l (the enlargement of the codomain space $W^{l,q}(\Omega)$).

3 Proofs of Theorems 2.1 and 2.2

The next theorem will be needed in the proof of Theorem 2.1. More exactly, it will be used to prove the well-definedness of N_g and the validity of the higher-order chain rule. Its proof can be found in [6].

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \leq p_1, \ldots, p_k, p \leq \infty$, and let $u_i \in W^{1,p_i}(\Omega)$, $i = 1, \ldots, k$, satisfy*

$$u_1 \cdot \ldots \cdot u_k \in L^p(\Omega),$$

$$\left(\partial_j u_1\right) u_2 \cdot \ldots \cdot u_k + \ldots + u_1 \cdot \ldots \cdot u_{k-1} \left(\partial_j u_k\right) \in L^p(\Omega) \quad \textit{for all } j = 1, \ldots, n.$$

Then $u_1 \cdot \ldots \cdot u_k \in W^{1,p}(\Omega)$ and

$$\partial_j (u_1 \cdot \ldots \cdot u_k) = \left(\partial_j u_1\right) u_2 \cdot \ldots \cdot u_k + \ldots + u_1 \cdot \ldots \cdot u_{k-1} \left(\partial_j u_k\right), \quad j = 1, \ldots, n.$$

Further on, we use the notation $p_k^* = \frac{np}{n-kp}$, provided that $k, n \in \mathbb{N}^*$, $1 \leq p < \infty$, and $kp < n$. Now, we are able to give the

Proof of Theorem 2.1. Since for $l = 1$, Theorem 2.1 reduces to Theorem 1.1, we can assume that $l \geq 2$.

Point (i) Throughout the proof of point (i), we will denote by C a positive constant which depends at most on Ω, n, m, l, p, q.

Let $u \in W^{m,p}(\Omega)$. We first show that $N_g u = g \circ u \in W^{1,q}(\Omega)$ and

$$\partial_i (g \circ u) = (g' \circ u)\, \partial_i u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ i = 1, \ldots, n. \tag{3.1}$$

To this end, we will use Theorem 1.1. We have $\frac{n}{m} < p < \frac{n}{m-l}$ ($1 \leq p < \frac{n}{m-l}$ for $n \in \{m-l+1, \ldots, m\}$) and $1 \leq q \leq p^*_{m-l} < p^*_{m-1}$. If $\frac{n}{m} < p < \frac{n}{m-1}$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$), then Theorem 1.1(i) applies and we deduce that $g \circ u \in W^{1,q}(\Omega)$ and (3.1) holds. In addition, $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is bounded. If $\frac{n}{m-1} \leq p < \frac{n}{m-l}$, then the same conclusions follows from Theorem 1.1(ii,iii).

It remains to be shown that if $g \circ u \in W^{s,q}(\Omega)$, with $1 \leq s < l$, and

$$D^{\alpha}(g \circ u) = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \alpha \\ |\alpha^i| \neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \tag{3.2}$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq s,$$

then $D^{\alpha}(g \circ u) \in W^{1,q}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$ and

$$\begin{aligned} \partial_j D^{\alpha}(g \circ u) = \sum_{k=1}^{s} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha \\ |\alpha^i|\neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \Big[\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \\ + \left(g^{(k)} \circ u\right)\left(\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u + \ldots + D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u\right)\Big] \\ \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } j = 1, \ldots, n. \end{aligned} \tag{3.3}$$

To this end, we will use Theorems 1.1 and 3.1.

By Theorem 1.1(i,ii,iii), we infer that for each $k = 1, \ldots, s$, $g^{(k)} \circ u \in W^{1,q}(\Omega)$ and

$$\partial_j\left(g^{(k)} \circ u\right) = \left(g^{(k+1)} \circ u\right)\partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ j = 1, \ldots, n,$$

with the usual convention if $k = l - 1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is bounded.

It follows from $\left|\alpha^i\right| \leq |\alpha| = s < l \leq m$ that

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \subset W^{1,p}(\Omega) \quad \text{for all } i = 1, \ldots, k.$$

In order to use Theorem 3.1, it remains to be shown that

$$\begin{gathered} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \vdots \\ \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \in L^q(\Omega). \end{gathered}$$

Let us fix $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$. Fix $k \in \{1, \ldots, s\}$ and fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. Further on, we split the proof into four cases:

1. $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$),
2. $p = \frac{n}{m-1}$, with $n \geq m - 1$,
3. $\frac{n}{m-h} < p < \frac{n}{m-h-1}$, with $n \geq m - h$, and $h \in \{1, \ldots, l-1\}$ ($1 \leq p < \frac{n}{m-h-1}$ when $n = m - h$),
4. $p = \frac{n}{m-h}$, with $n \geq m - h$, and $h \in \{2, \ldots, l-1\}$.

Case 1. $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$).

It follows from $\left|\alpha^i\right| \neq 0$ that $\left(m - \left|\alpha^i\right|\right) p \leq (m-1)p < n$ and thus the following Sobolev imbedding holds

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega). \tag{3.4}$$

According to Theorem 1.1(i), we have $g^{(k)} \circ u \in W^{1,p^*_{m-1}}(\Omega)$ and

$$\partial_j\left(g^{(k)} \circ u\right) = \left(g^{(k+1)} \circ u\right)\partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ j = 1, \ldots, n,$$

with the usual convention if $k = l - 1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,p^*_{m-1}}(\Omega)$ is bounded. We have

$$\frac{1}{p^*_{m-1}} + \sum_{i=1}^{k} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \Leftrightarrow q < \frac{np}{p(s+1) - (k+1)(mp-n)},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed

$$p^*_{m-l} < \frac{np}{p(s+1) - (k+1)(mp-n)} \Leftrightarrow p(l-s-1) + k(mp-n) > 0.$$

(The last inequality holds thanks to the hypothesis $mp > n$ and to the fact that $1 \le k \le s \le l-1$.) Therefore, $1/p^*_{m-1} + \sum_{i=1}^{k} 1/p^*_{m-|\alpha^i|} < 1/q$, and thus we can apply Hölder's inequality and imbedding (3.4) in order to obtain the following inequalities

$$\begin{aligned} &\left\|\left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{L^{p^*_{m-1}}(\Omega)} \left\|D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|}}(\Omega)} \ldots \left\|D^{\alpha^k} u\right\|_{L^{p^*_{m-|\alpha^k|}}(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{W^{1,p^*_{m-1}}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\left\|\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\le C \left\|\left(g^{(k+1)} \circ u\right) \partial_j u\right\|_{L^{p^*_{m-1}}(\Omega)} \left\|D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|}}(\Omega)} \ldots \left\|D^{\alpha^k} u\right\|_{L^{p^*_{m-|\alpha^k|}}(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{W^{1,p^*_{m-1}}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.6}$$

It follows easily from $\frac{n}{m} < p < \frac{n}{m-1}$ that $p^*_{m-1} > n$. Hence, the following Sobolev imbedding holds

$$g^{(k)} \circ u \in W^{1,p^*_{m-1}}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{3.7}$$

Also, by $\left(m - \left|\alpha^i\right| - 1\right) p < (m-1) p < n$, the Sobolev imbedding

$$\partial_j D^{\alpha^i} u \in W^{m-|\alpha^i|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^i|-1}}(\Omega) \tag{3.8}$$

is valid. We have

$$\frac{1}{p^*_{m-|\alpha^i|-1}} + \sum_{i=2}^{k} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \Leftrightarrow q \le \frac{np}{p(s+1) - k(mp-n)},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed, by $mp > n$, we infer that

$$\min_{1 \le k \le s \le l-1} \frac{np}{p(s+1) - k(mp-n)} = \frac{np}{n - (m-l)p} = p^*_{m-l},$$

the minimum being obtained for $s = l-1$ and $k = 1$. Therefore, $1/p^*_{m-|\alpha^1|-1} + \sum_{i=2}^{k} 1/p^*_{m-|\alpha^i|} \le 1/q$, and Hölder's inequality can be applied together with imbeddings (3.4), (3.7), (3.8) in order to obtain the following inequality

$$\begin{aligned} &\left\|\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \\ &\times \left\|\partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \left\|D^{\alpha^2} u\right\|_{L^{p^*_{m-|\alpha^2|}}(\Omega)} \ldots \left\|D^{\alpha^k} u\right\|_{L^{p^*_{m-|\alpha^k|}}(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{W^{1,p^*_{m-1}}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty. \end{aligned} \tag{3.9.1}$$

An analogous argument allows us to derive other $k-1$ formulas similar to (3.9.1), the last of them being

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)D^{\alpha^1}u\dots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\right\|_{L^q(\Omega)}\leq C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\\&\times\left\|D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|}}(\Omega)}\dots\left\|D^{\alpha^{k-1}}u\right\|_{L^{p^*_{m-|\alpha^{k-1}|}}(\Omega)}\left\|\partial_j D^{\alpha^k}u\right\|_{L^{p^*_{m-|\alpha^k|-1}}(\Omega)}\\&\leq C\left\|g^{(k)}\circ u\right\|_{W^{1,p^*_{m-1}}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\end{aligned}\tag{3.9.k}$$

It follows from (3.2), (3.5), (3.6), (3.9.1), . . . , (3.9.k), and Theorem 3.1 that $D^\alpha(g\circ u)\in W^{1,q}(\Omega)$ for all $\alpha\in\mathbb{N}^n$ with $|\alpha|=s$, and formula (3.3) holds. Consequently, $N_g u\in W^{l,q}(\Omega)$, and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g: W^{m,p}(\Omega)\to W^{l,q}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g: W^{m,p}(\Omega)\to W^{1,q}(\Omega)$ and inequalities (3.5), (3.6), (3.9.1), . . . , (3.9.k). Point (i) is proved in the first case.

Case 2. $p=\frac{n}{m-1}$, with $n\geq m-1$.

Denote $I_1=\left\{i\in\{1,\dots,k\}:\left|\alpha^i\right|=1\right\}$, $I_2=\left\{i\in\{1,\dots,k\}:2\leq\left|\alpha^i\right|\leq s\right\}$. We have $\left(m-\left|\alpha^i\right|\right)p=n$ if $i\in I_1$, and $\left(m-\left|\alpha^i\right|\right)p<n$ if $i\in I_2$, whence we deduce the following Sobolev imbeddings

$$D^{\alpha^i}u\in W^{m-|\alpha^i|,p}(\Omega)\hookrightarrow L^r(\Omega)\ \text{ for all } 1\leq r<\infty,\quad\text{if } i\in I_1,\tag{3.10}$$

$$D^{\alpha^i}u\in W^{m-|\alpha^i|,p}(\Omega)\hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega)\quad\text{if } i\in I_2.\tag{3.11}$$

According to Theorem 1.1(ii), we have $g^{(k)}\circ u=N_{g^{(k)}}u\in W^{1,r}(\Omega)$ for all $1\leq r<\infty$, and

$$\partial_j\left(g^{(k)}\circ u\right)=\left(g^{(k+1)}\circ u\right)\partial_j u\quad\mathcal{L}^n\text{-a.e. in }\Omega,\ j=1,\dots,n,$$

with the usual convention if $k=l-1$. In addition, $N_{g^{(k)}}: W^{m,p}(\Omega)\to W^{1,r}(\Omega)$ is bounded for all $1\leq r<\infty$.

We have

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}=\sum_{i\in I_2}\frac{n-\left(m-\left|\alpha^i\right|\right)p}{np}=\frac{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1}{np},$$

and this sum is to be interpreted as zero when $I_2=\phi$. Thus, if $I_2=\phi$, then

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}=0<\frac{1}{q},$$

and if $I_2\neq\phi$, then

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}<\frac{1}{q}\Leftrightarrow q<\frac{np}{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1},$$

which is true, because $1\leq q\leq p^*_{m-l}$. Indeed,

$$\begin{aligned}&p^*_{m-l}<\frac{np}{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1}\\&\quad\Leftrightarrow p\left(l-\sum_{i\in I_2}\left|\alpha^i\right|\right)+(mp-n)\left(\sum_{i\in I_2}1-1\right)>0.\end{aligned}$$

(The last inequality is justified by the hypothesis $mp > n$ and by the obvious inequalities $1 \le \sum_{i\in I_2} \left|\alpha^i\right| \le s \le l-1$, $\sum_{i\in I_2} 1 \ge 1$.) Consequently, $\sum_{i\in I_2} 1/p^*_{m-|\alpha^i|} < 1/q$, and thus there is $1 \le r_4 < \infty$ such that

$$\sum_{i\in I_2} \frac{1}{p^*_{m-|\alpha^i|}} + \frac{1}{r_4} = \frac{1}{q}.$$

Let $k_1 = |I_1|$ be the cardinality of I_1. By applying Hölder's inequality and imbeddings (3.10), (3.11), we obtain

$$\begin{aligned} &\left\|\left(g^{(k)} \circ u\right) D^{\alpha^1} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\quad \le C \left\|g^{(k)} \circ u\right\|_{L^{r_4(k_1+1)}(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^{r_4(k_1+1)}(\Omega)} \prod_{i\in I_2} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \\ &\quad \le C \left\|g^{(k)} \circ u\right\|_{W^{1,r_4(k_1+1)}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\left\|\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\le C \left\|\left(g^{(k+1)} \circ u\right) \partial_j u\right\|_{L^{r_4(k_1+1)}(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^{r_4(k_1+1)}(\Omega)} \prod_{i\in I_2} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{W^{1,r_4(k_1+1)}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.13}$$

where the products $\prod_{i\in I_1}$ and $\prod_{i\in I_2}$ are to be interpreted as 1 when $I_1 = \phi$ and $I_2 = \phi$, respectively.

We have seen above that $g^{(k)} \circ u \in W^{1,r}(\Omega)$ for all $1 \le r < \infty$. It follows that the Sobolev imbedding

$$g^{(k)} \circ u \in W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{for all } n < r < \infty, \tag{3.14}$$

is valid. Since $\left(m - \left|\alpha^1\right| - 1\right) p < (m-1)\, p = n$, the Sobolev imbedding

$$\partial_j D^{\alpha^1} u \in W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \tag{3.15}$$

is also valid. We have

$$\begin{aligned} &\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \\ &\quad = \frac{p\left(\sum_{i\in I_2\setminus\{1\}} \left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_2\setminus\{1\}} 1 + 1\right)}{np}. \end{aligned} \tag{3.16}$$

We intend to show that

$$\left\|\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \tag{3.17.1}$$

with $r \in (n, \infty)$ arbitrarily fixed. To this end, we will study two situations: $1 \in I_2$ and $1 \in I_1$, respectively.

Let us consider the situation $1 \in I_2$. Then, in view of (3.16), we have

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_2} \left|\alpha^i\right| + 1\right) - (mp-n)\sum_{i\in I_2} 1},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + 1\right) - (mp-n)\sum_{i\in I_2} 1} = p^*_{m-l},$$

the minimum being obtained only for $I_1 = \phi$, $s = l-1$, $I_2 = \{1\}$. Consequently,

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \quad \text{if } I_1 = \phi,$$
$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \quad \text{if } I_1 \ne \phi.$$

If $I_1 = \phi$, then by applying Hölder's inequality and imbeddings (3.11), (3.14), (3.15), we obtain

$$\begin{aligned}
&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}u D^{\alpha^2}u\dots D^{\alpha^k}u\right\|_{L^q(\Omega)} \\
&\le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\left\|\partial_j D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \prod_{i\in I_2\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \\
&\le C\left\|g^{(k)}\circ u\right\|_{W^{1,r}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

If $I_1 \ne \phi$, then there is $1 < r_5 < \infty$ such that

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} + \frac{1}{r_5} = \frac{1}{q}.$$

By applying Hölder's inequality and imbeddings (3.10), (3.11), (3.14), (3.15), we obtain

$$\begin{aligned}
&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}u D^{\alpha^2}u\dots D^{\alpha^k}u\right\|_{L^q(\Omega)} \\
&\le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\left\|\partial_j D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \prod_{i\in I_1}\left\|D^{\alpha^i}u\right\|_{L^{r_5 k_1}(\Omega)} \\
&\quad\times \prod_{i\in I_2\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{W^{1,r}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.17.1) is proved in the situation $1 \in I_2$.

Now, we consider the situation $1 \in I_1$. Then, in view of (3.16), we have

$$\begin{aligned}
&\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \\
&\Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_2} 1 + 1\right)},
\end{aligned}$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_2} 1 + 1\right)} = p^*_{m-l}$$

the minimum being obtained only for $I_1 = \{1\}$, $s = l-1$, $I_2 = \phi$. Consequently,

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \quad \text{if } I_1 = \{1\},$$
$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \quad \text{if } I_1 \supsetneq \{1\}.$$

If $I_1 = \{1\}$, then $I_2 = \{2, \ldots, k\}$ or $I_2 = \phi$, and thus, by applying Hölder's inequality and imbeddings (3.11), (3.14), (3.15), we obtain

$$\begin{aligned}
&\left\| \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \\
&\leq C \left\| g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \left\| \partial_j D^{\alpha^1} u \right\|_{L^{p^*_{m-|\alpha^1|-1}}} \prod_{i \in I_2} \left\| D^{\alpha^i} u \right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \\
&\leq C \left\| g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

If $I_1 \supsetneq \{1\}$, then there is $1 < r_6 < \infty$ such that

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} + \frac{1}{r_6} = \frac{1}{q}.$$

By applying Hölder's inequality and imbeddings (3.10), (3.11), (3.14), (3.15), we obtain

$$\begin{aligned}
&\left\| \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \\
&\leq C \left\| g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \left\| \partial_j D^{\alpha^1} u \right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \prod_{i \in I_1 \setminus \{1\}} \left\| D^{\alpha^i} u \right\|_{L^{r_6(k_1-1)}(\Omega)} \\
&\times \prod_{i \in I_2} \left\| D^{\alpha^i} u \right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \leq C \left\| g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.17.1) is proved in the situation $1 \in I_1$ as well.

In conclusion, inequality (3.17.1) is valid in any situation of the second case. An analogous argument allows us to derive other $k-1$ formulas similar to (3.17.1), the last of them being

$$\left\| \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \right\|_{L^q(\Omega)} \leq C \left\| g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \tag{3.17.k}$$

with $r \in (n, \infty)$ arbitrarily fixed.

It follows from (3.2), (3.12), (3.13), (3.17.1), ..., (3.17.k), and Theorem 3.1 that $D^\alpha (g \circ u) \in W^{1,q}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, and formula (3.3) holds. Consequently, $N_g u \in W^{l,q}(\Omega)$ and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ and inequalities (3.12), (3.13), (3.17.1), ..., (3.17.k). Point (i) is proved in the second case.

Case 3. $\frac{n}{m-h} < p < \frac{n}{m-h-1}$, with $n \geq m-h$, and $h \in \{1, \ldots, l-1\}$ ($1 \leq p < \frac{n}{m-h-1}$ when $n = m-h$).

Denote $I_1 = \left\{i \in \{1, \ldots, k\} : 1 \leq \left|\alpha^i\right| \leq h\right\}$, $I_2 = \left\{i \in \{1, \ldots, k\} : h+1 \leq \left|\alpha^i\right| \leq s\right\}$. We have $\left(m - \left|\alpha^i\right|\right) p > n$ if $i \in I_1$, and $\left(m - \left|\alpha^i\right|\right) p < n$ if $i \in I_2$, whence we deduce the following Sobolev imbeddings

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if } i \in I_1, \tag{3.18}$$

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega) \quad \text{if } i \in I_2. \tag{3.19}$$

According to Theorem 1.1(iii), we have $g^{(k)} \circ u = N_{g^{(k)}} u \in W^{1,\infty}(\Omega)$ and

$$\partial_j \left(g^{(k)} \circ u\right) = \left(g^{(k+1)} \circ u\right) \partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ j = 1, \ldots, n,$$

with the usual convention if $k = l-1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,\infty}(\Omega)$ is bounded.

We have

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}=\frac{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1}{np},$$

and this sum is to be interpreted as zero when $I_2=\phi$. Thus, if $I_2=\phi$, then

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}=0<\frac{1}{q},$$

and if $I_2\neq\phi$, then

$$\sum_{i\in I_2}\frac{1}{p^*_{m-|\alpha^i|}}<\frac{1}{q}\Leftrightarrow q<\frac{np}{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1},$$

which is true, because $1\leq q\leq p^*_{m-l}$. Indeed,

$$\begin{aligned}p^*_{m-l}&<\frac{np}{p\sum_{i\in I_2}\left|\alpha^i\right|-(mp-n)\sum_{i\in I_2}1}\\&\Leftrightarrow p\left(l-\sum_{i\in I_2}\left|\alpha^i\right|\right)+(mp-n)\left(\sum_{i\in I_2}1-1\right)>0.\end{aligned}$$

(The last inequality is justified by the hypothesis $mp>n$ and by the obvious inequalities $1\leq\sum_{i\in I_2}\left|\alpha^i\right|\leq s\leq l-1$, $\sum_{i\in I_2}1\geq 1$.) Consequently, $\sum_{i\in I_2}1/p^*_{m-|\alpha^i|}<1/q$, and thus we can apply Hölder's inequality and imbeddings (3.18), (3.19) in order to obtain

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)D^{\alpha^1}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\\&\leq C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\prod_{i\in I_1}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\prod_{i\in I_2}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)}\\&\leq C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty,\end{aligned}\tag{3.20}$$

$$\begin{aligned}&\left\|\left(g^{(k+1)}\circ u\right)\partial_j uD^{\alpha^1}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\\&\leq C\left\|\left(g^{(k+1)}\circ u\right)\partial_j u\right\|_{L^\infty(\Omega)}\prod_{i\in I_1}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\prod_{i\in I_2}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)}\\&\leq C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty,\end{aligned}\tag{3.21}$$

where the products $\prod_{i\in I_1}$ and $\prod_{i\in I_2}$ are to be interpreted as 1 when $I_1=\phi$ and $I_2=\phi$, respectively.

We intend to thow that

$$\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\leq C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\tag{3.22.1}$$

In order to use the best possible Sobolev imbedding for the function $\partial_j D^{\alpha^1}u\in W^{m-|\alpha^1|-1,p}(\Omega)$, we will study three situations: $1\in I_2$, $1\in I_1$ and $\left|\alpha^1\right|=h$, and $1\in I_1$ and $\left|\alpha^1\right|\leq h-1$, respectively.

Let us consider the situation $1\in I_2$. Then, $\left(m-\left|\alpha^1\right|-1\right)p=\left(m-\left|\alpha^1\right|\right)p-p<n-p<n$, and thus the Sobolev imbedding

$$\partial_j D^{\alpha^1}u\in W^{m-|\alpha^1|-1,p}(\Omega)\hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega)\tag{3.23}$$

is valid. In view of (3.16), we have

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + 1\right) - (mp-n)\sum_{i\in I_2} 1},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + 1\right) - (mp-n)\sum_{i\in I_2} 1} = p^*_{m-l},$$

the minimum being obtained for $I_1 = \phi$, $s = l-1$, $I_2 = \{1\}$. Consequently, $1/p^*_{m-|\alpha^1|-1} + \sum_{i\in I_2\setminus\{1\}} 1/p^*_{m-|\alpha^i|} \le 1/q$; thus, we can apply Hölder's inequality and imbeddings (3.18), (3.19), (3.23) in order to obtain

$$\begin{aligned}
&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\quad \le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)} \left\|\partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \prod_{i\in I_1}\left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \\
&\quad\quad \times \prod_{i\in I_2\setminus\{1\}}\left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.22.1) is proved in the situation $1 \in I_2$.

Now, we consider the situation $1 \in I_1$ and $\left|\alpha^1\right| = h$. Then, $1 \notin I_2$, $\left(m - \left|\alpha^1\right| - 1\right)p = (m-h-1)p < n$, and thus the Sobolev imbedding (3.23) holds. In view of (3.16), we have

$$\begin{aligned}
&\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_2} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \\
&\quad \Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_2} 1 + 1\right)},
\end{aligned}$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_2}\left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_2} 1 + 1\right)} = p^*_{m-l},$$

the minimum being obtained for $I_2 = \phi$, $s = l-1$, $I_1 = \{1\}$. Consequently, $1/p^*_{m-|\alpha^1|-1} + \sum_{i\in I_2} 1/p^*_{m-|\alpha^i|} \le 1/q$. Hence, we can apply Hölder's inequality and imbeddings (3.18), (3.19), (3.23) in order to obtain

$$\begin{aligned}
&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\quad \le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)} \left\|\partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \prod_{i\in I_1\setminus\{1\}}\left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \\
&\quad\quad \times \prod_{i\in I_2}\left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.22.1) is proved in the situation $1 \in I_1$ and $\left|\alpha^1\right| = h$.

Finally, we consider the situation $1 \in I_1$ and $\left|\alpha^1\right| \leq h-1$. Then, $1 \notin I_2$, $\left(m-\left|\alpha^1\right|-1\right)p \geq (m-h)\,p > n$, and thus the Sobolev imbedding

$$\partial_j D^{\alpha^1} u \in W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \tag{3.24}$$

is valid. We have $\sum_{i\in I_2} 1/p^*_{m-\left|\alpha^i\right|} < 1/q$. Hence, we can apply Hölder's inequality and imbeddings (3.18), (3.19), (3.24) in order to obtain

$$\begin{aligned}
&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\quad\leq C\left\|g^{(k)}\circ u\right\|_{L^{\infty}(\Omega)}\left\|\partial_j D^{\alpha^1} u\right\|_{L^{\infty}(\Omega)}\prod_{i\in I_1\setminus\{1\}}\left\|D^{\alpha^i} u\right\|_{L^{\infty}(\Omega)} \\
&\qquad\times\prod_{i\in I_2}\left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)} \leq C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.22.1) is proved in the situation $1 \in I_1$ and $\left|\alpha^1\right| \leq h-1$.

In conclusion, inequality (3.22.1) is valid in any situation of the third case. An analogous argument allows us to derive other $k-1$ formulas similar to (3.22.1), the last of them being

$$\left\|\left(g^{(k)}\circ u\right)D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u\partial_j D^{\alpha^k} u\right\|_{L^q(\Omega)} \leq C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty. \tag{3.22.k}$$

It follows from (3.2), (3.20), (3.21), (3.22.1), …, (3.22.k), and Theorem 3.1 that $D^{\alpha}(g\circ u) \in W^{1,q}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, and formula (3.3) holds. Consequently, $N_g u \in W^{l,q}(\Omega)$ and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ and inequalities (3.20), (3.21), (3.22.1), …, (3.22.k). Point (i) is proved in the third case.

Case 4. $p = \frac{n}{m-h}$, with $n \geq m-h$, and $h \in \{2,\ldots,l-1\}$.

Denote $I_1 = \left\{i \in \{1,\ldots,k\} : 1 \leq \left|\alpha^i\right| < h\right\}$, $I_2 = \left\{i \in \{1,\ldots,k\} : \left|\alpha^i\right| = h\right\}$, $I_3 = \left\{i \in \{1,\ldots,k\} : h < \left|\alpha^i\right| \leq s\right\}$. We have $\left(m-\left|\alpha^i\right|\right)p > n$ if $i \in I_1$, $\left(m-\left|\alpha^i\right|\right)p = n$ if $i \in I_2$, and $\left(m-\left|\alpha^i\right|\right)p < n$ if $i \in I_3$, whence we deduce the following Sobolev imbeddings

$$D^{\alpha^i} u \in W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad \text{if } i \in I_1, \tag{3.25}$$

$$D^{\alpha^i} u \in W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^{r}(\Omega) \quad \text{for all } 1 \leq r < \infty, \text{ if } i \in I_2, \tag{3.26}$$

$$D^{\alpha^i} u \in W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^{p^*_{m-\left|\alpha^i\right|}}(\Omega) \quad \text{if } i \in I_3. \tag{3.27}$$

According to Theorem 1.1(iii), we have $g^{(k)}\circ u = N_{g^{(k)}}u \in W^{1,\infty}(\Omega)$ and

$$\partial_j\left(g^{(k)}\circ u\right) = \left(g^{(k+1)}\circ u\right)\partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ j = 1,\ldots,n,$$

with the usual convention if $k = l-1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,\infty}(\Omega)$ is bounded.

We have

$$\sum_{i\in I_3}\frac{1}{p^*_{m-\left|\alpha^i\right|}} = \frac{p\sum_{i\in I_3}\left|\alpha^i\right| - (mp-n)\sum_{i\in I_3} 1}{np},$$

and this sum is to be interpreted as zero when $I_3 = \phi$. Thus, if $I_3 = \phi$, then

$$\sum_{i\in I_3}\frac{1}{p^*_{m-\left|\alpha^i\right|}} = 0 < \frac{1}{q},$$

and if $I_3 \neq \phi$, then

$$\sum_{i\in I_3} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \Leftrightarrow q < \frac{np}{p\sum_{i\in I_3}\left|\alpha^i\right| - (mp-n)\sum_{i\in I_3} 1},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$p^*_{m-l} < \frac{np}{p\sum_{i\in I_3}\left|\alpha^i\right| - (mp-n)\sum_{i\in I_3} 1}$$
$$\Leftrightarrow p\left(l - \sum_{i\in I_3}\left|\alpha^i\right|\right) + (mp-n)\left(\sum_{i\in I_3} 1 - 1\right) > 0.$$

(The last inequality is justified by the hypothesis $mp > n$ and by the obvious inequalities $1 \le \sum_{i\in I_3}\left|\alpha^i\right| \le s \le l-1$, $\sum_{i\in I_3} 1 \ge 1$.) Consequently, $\sum_{i\in I_3} 1/p^*_{m-|\alpha^i|} < 1/q$, and thus there is $1 \le r_7 < \infty$ such that

$$\sum_{i\in I_3} \frac{1}{p^*_{m-|\alpha^i|}} + \frac{1}{r_7} = \frac{1}{q}.$$

Let $k_2 = |I_2|$ be the cardinality of I_2. By applying Hölder's inequality and imbeddings (3.25), (3.26), (3.27), we obtain

$$\begin{aligned} &\left\|\left(g^{(k)} \circ u\right) D^{\alpha^1} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \prod_{i\in I_2} \left\|D^{\alpha^i} u\right\|_{L^{k_2 r_7}(\Omega)} \\ &\times \prod_{i\in I_3} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\left\|\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\ &\le C \left\|\left(g^{(k+1)} \circ u\right) \partial_j u\right\|_{L^\infty(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \prod_{i\in I_2} \left\|D^{\alpha^i} u\right\|_{L^{k_2 r_7}(\Omega)} \\ &\times \prod_{i\in I_3} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.29}$$

where the products $\prod_{i\in I_1}$, $\prod_{i\in I_2}$, and $\prod_{i\in I_3}$ are to be interpreted as 1 when $I_1 = \phi$, $I_2 = \phi$, and $I_3 = \phi$, respectively.

We intend to show that

$$\left\|\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty. \tag{3.30.1}$$

In order to use the best possible Sobolev imbedding for the function $\partial_j D^{\alpha^1} u \in W^{m-|\alpha^1|-1,p}(\Omega)$, we will study three situations: $1 \in I_3$, $1 \in I_2$, and $1 \in I_1$, respectively

Let us consider the situation $1 \in I_3$. Then, $\left(m - \left|\alpha^1\right| - 1\right) p = \left(m - \left|\alpha^1\right|\right) p - p < n - p < n$, and thus the Sobolev imbedding

$$\partial_j D^{\alpha^1} u \in W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \tag{3.31}$$

is valid. In view of (3.16), we have

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_3\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_3}|\alpha^i|+1\right)-(mp-n)\sum_{i\in I_3}1},$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_3}|\alpha^i|+1\right)-(mp-n)\sum_{i\in I_3}1} = p^*_{m-l},$$

the minimum being obtained only for $I_1 = I_2 = \phi$, $s = l-1$, $I_3 = \{1\}$. Consequently,

$$\begin{aligned}\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_3\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &< \frac{1}{q} \quad \text{if } I_2 \ne \phi,\\ \frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_3\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &\le \frac{1}{q} \quad \text{if } I_2 = \phi.\end{aligned}$$

If $I_2 \ne \phi$, then there is $1 < r_8 < \infty$ such that

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_3\setminus\{1\}} \frac{1}{p^*_{m-|\alpha^i|}} + \frac{1}{r_8} = \frac{1}{q}.$$

By applying Hölder's inequality and imbeddings (3.25), (3.26), (3.27), (3.31), we obtain

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\\ &\quad\times\left\|\partial_j D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)}\prod_{i\in I_1}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\prod_{i\in I_2}\left\|D^{\alpha^i}u\right\|_{L^{k_2r_8}(\Omega)}\\ &\quad\times\prod_{i\in I_3\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty.\end{aligned}$$

If $I_2 = \phi$, then we can apply Hölder's inequality and imbeddings (3.25), (3.27), (3.31) in order to obtain

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\\ &\quad\le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\left\|\partial_j D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)}\prod_{i\in I_1}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\\ &\quad\times\prod_{i\in I_3\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)} < \infty.\end{aligned}$$

Consequently, (3.30.1) is proved in the situation $1 \in I_3$.

Now, we consider the situation $1 \in I_2$. Then, $\left(m-|\alpha^1|-1\right)p = \left(m-|\alpha^1|\right)p - p = n-p < n$, and thus the Sobolev imbedding (3.31) holds. In view of (3.16), we have

$$\begin{aligned}&\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i\in I_3} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q}\\ &\Leftrightarrow q \le \frac{np}{p\left(\sum_{i\in I_3}|\alpha^i|+|\alpha^1|+1\right)-(mp-n)\left(\sum_{i\in I_3}1+1\right)},\end{aligned}$$

which is true, because $1 \le q \le p^*_{m-l}$. Indeed,

$$\min_{1\le k\le s\le l-1} \frac{np}{p\left(\sum_{i\in I_3}\left|\alpha^i\right| + \left|\alpha^1\right| + 1\right) - (mp-n)\left(\sum_{i\in I_3} 1 + 1\right)} = p^*_{m-l},$$

the minimum being obtained only for $I_1 = I_3 = \phi$, $s = l-1$, $I_2 = \{1\}$. Consequently,

$$\frac{1}{p^*_{m-\left|\alpha^1\right|-1}} + \sum_{i\in I_3} \frac{1}{p^*_{m-\left|\alpha^i\right|}} < \frac{1}{q} \quad \text{if } I_2 \supsetneq \{1\},$$
$$\frac{1}{p^*_{m-\left|\alpha^1\right|-1}} + \sum_{i\in I_3} \frac{1}{p^*_{m-\left|\alpha^i\right|}} \le \frac{1}{q} \quad \text{if } I_2 = \{1\}.$$

If $I_2 \supsetneq \{1\}$, there is $1 < r_9 < \infty$ such that

$$\frac{1}{p^*_{m-\left|\alpha^1\right|-1}} + \sum_{i\in I_3} \frac{1}{p^*_{m-\left|\alpha^i\right|}} + \frac{1}{r_9} = \frac{1}{q}.$$

By applying Hölder's inequality and imbeddings (3.25), (3.26), (3.27), (3.31), we obtain

$$\begin{aligned}
&\left\|\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \\
&\times \left\|\partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-\left|\alpha^1\right|-1}}(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \prod_{i\in I_2\setminus\{1\}} \left\|D^{\alpha^i} u\right\|_{L^{(k_2-1)r_9}(\Omega)} \\
&\times \prod_{i\in I_3} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

If $I_2 = \{1\}$, then we can apply Hölder's inequality and imbeddings (3.25), (3.27), (3.31) in order to obtain

$$\begin{aligned}
&\left\|\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \left\|\partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-\left|\alpha^1\right|-1}}(\Omega)} \prod_{i\in I_1} \left\|D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \\
&\quad \times \prod_{i\in I_3} \left\|D^{\alpha^i} u\right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.30.1) is proved in the situation $1 \in I_2$ as well.

Finally, we consider the situation $1 \in I_1$. If $\left|\alpha^1\right| = h-1$, then $\left(m - \left|\alpha^1\right| - 1\right) p = (m-h)\, p = n$ and thus the following Sobolev imbedding holds

$$\partial_j D^{\alpha^1} u \in W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } 1 \le r < \infty. \tag{3.32}$$

We have $\sum_{i\in I_3} 1/p^*_{m-\left|\alpha^i\right|} < 1/q$. Hence, there is $1 \le r_{10} < \infty$ such that

$$\sum_{i\in I_3} \frac{1}{p^*_{m-\left|\alpha^i\right|}} + \frac{1}{r_{10}} = \frac{1}{q}. \tag{3.33}$$

By applying Hölder's inequality and imbeddings (3.25), (3.26), (3.27), (3.32), we obtain

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\\ &\times\left\|\partial_j D^{\alpha^1}u\right\|_{L^{(k_2+1)r_{10}}(\Omega)}\prod_{i\in I_1\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\prod_{i\in I_2}\left\|D^{\alpha^i}u\right\|_{L^{(k_2+1)r_{10}}(\Omega)}\\ &\times\prod_{i\in I_3}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)}\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\end{aligned}$$

If $\left|\alpha^1\right|\le h-2$, then $\left(m-\left|\alpha^1\right|-1\right)p\ge(m-h+1)\,p=(m-h)\,p+p=n+p>n$, and thus the Sobolev imbedding

$$\partial_j D^{\alpha^1}u\in W^{m-|\alpha^1|-1,p}(\Omega)\hookrightarrow L^\infty(\Omega) \tag{3.34}$$

holds. By using (3.33), Hölder's inequality, and imbeddings (3.25), (3.26), (3.27), (3.34), we obtain

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\le C\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\left\|\partial_j D^{\alpha^1}u\right\|_{L^\infty(\Omega)}\\ &\times\prod_{i\in I_1\setminus\{1\}}\left\|D^{\alpha^i}u\right\|_{L^\infty(\Omega)}\prod_{i\in I_2}\left\|D^{\alpha^i}u\right\|_{L^{k_2r_{10}}(\Omega)}\prod_{i\in I_3}\left\|D^{\alpha^i}u\right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)}\\ &\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\end{aligned}$$

Consequently, (3.30.1) is proved in the situation $1\in I_1$ as well.

In conclusion, inequality (3.30.1) is valid in any situation of the fourth case. An analogous argument allows us to derive other $k-1$ formulas similar to (3.30.1), the last of them being

$$\left\|\left(g^{(k)}\circ u\right)D^{\alpha^1}u\ldots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\right\|_{L^q(\Omega)}\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty. \tag{3.30.k}$$

It follows from (3.2), (3.28), (3.29), (3.30.1), ..., (3.30.k), and Theorem 3.1 that $D^\alpha(g\circ u)\in W^{1,q}(\Omega)$ for all $\alpha\in\mathbb{N}^n$ with $|\alpha|=s$, and formula (3.3) holds. Consequently, $N_gu\in W^{l,q}(\Omega)$ and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g:W^{m,p}(\Omega)\to W^{l,q}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g:W^{m,p}(\Omega)\to W^{1,q}(\Omega)$ and inequalities (3.28), (3.29), (3.30.1), ..., (3.30.k). Point (i) is proved in the fourth case, and thus it is completely proved.

Point (ii) Throughout the proof of point (ii), we will denote by C a positive constant which depends at most on Ω,n,m,l,q.

Let $u\in W^{m,p}(\Omega)$. We first show that $N_gu=g\circ u\in W^{1,q}(\Omega)$ and

$$\partial_i(g\circ u)=(g'\circ u)\,\partial_i u\quad\mathcal{L}^n\text{-a.e. in }\Omega,\ i=1,\ldots,n. \tag{3.35}$$

To this end, we will use Theorem 1.1. We have $p=\frac{n}{m-l}>\frac{n}{m-1}$ and $1\le q<\infty$. Therefore, Theorem 1.1(iii) applies and we deduce that $g\circ u\in W^{1,\infty}(\Omega)\subset W^{1,q}(\Omega)$ and (3.35) holds. In addition, $N_g:W^{m,p}(\Omega)\to W^{1,q}(\Omega)$ is bounded.

It remains to be shown that if $g\circ u\in W^{s,q}(\Omega)$, with $1\le s<l$, and

$$D^\alpha(g\circ u)=\sum_{k=1}^{|\alpha|}\sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha\\ |\alpha^i|\ne0}}c_{\alpha,k,\alpha^1,\ldots,\alpha^k}\left(g^{(k)}\circ u\right)D^{\alpha^1}u\ldots D^{\alpha^k}u \tag{3.36}$$
$$\mathcal{L}^n\text{-a.e. in }\Omega,\ \text{for all }\alpha\in\mathbb{N}^n\text{ with }|\alpha|\le s,$$

then $D^{\alpha}(g \circ u) \in W^{1,q}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, and

$$\begin{aligned} \partial_j D^{\alpha}(g \circ u) = \sum_{k=1}^{s} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha \\ |\alpha^i| \neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \Big[\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \\ + \left(g^{(k)} \circ u\right) \left(\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u + \ldots + D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \right) \Big] \\ \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } j = 1, \ldots, n. \end{aligned} \tag{3.37}$$

To this end, we will use Theorems 1.1 and 3.1.

By Theorem 1.1(iii), we infer that for each $k = 1, \ldots, s$, $N_{g^{(k)}} u = g^{(k)} \circ u \in W^{1,\infty}(\Omega) \subset W^{1,q}(\Omega)$ and

$$\partial_j \left(g^{(k)} \circ u \right) = \left(g^{(k+1)} \circ u \right) \partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega, \; j = 1, \ldots, n,$$

with the usual convention if $k = l - 1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,\infty}(\Omega)$ is bounded.

It follows from $\left|\alpha^i\right| \leq |\alpha| = s < l \leq m$ that

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \subset W^{1,p}(\Omega) \quad \text{for all } i = 1, \ldots, k.$$

In order to use Theorem 3.1, it remains to be shown that

$$\begin{gathered} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \in L^q(\Omega), \\ \vdots \\ \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \in L^q(\Omega). \end{gathered}$$

Let us fix $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$. Fix $k \in \{1, \ldots, s\}$ and fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

Since

$$\left(m - \left|\alpha^i\right|\right) p \geq (m - s) p > (m - l) p = n,$$

the following Sobolev imbedding holds

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^{\infty}(\Omega). \tag{3.38}$$

By using this imbedding, the following inequalities can be derived

$$\begin{aligned} &\left\| \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \\ &\quad \leq C \left\| g^{(k)} \circ u \right\|_{L^{\infty}(\Omega)} \left\| D^{\alpha^1} u \right\|_{L^{\infty}(\Omega)} \ldots \left\| D^{\alpha^k} u \right\|_{L^{\infty}(\Omega)} \\ &\quad \leq C \left\| g^{(k)} \circ u \right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty, \end{aligned} \tag{3.39}$$

$$\begin{aligned} &\left\| \left(g^{(k)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \\ &\quad \leq C \left\| \left(g^{(k)} \circ u\right) \partial_j u \right\|_{L^{\infty}(\Omega)} \left\| D^{\alpha^1} u \right\|_{L^{\infty}(\Omega)} \ldots \left\| D^{\alpha^k} u \right\|_{L^{\infty}(\Omega)} \\ &\quad \leq C \left\| g^{(k)} \circ u \right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty. \end{aligned} \tag{3.40}$$

Denote $I_1 = \left\{i \in \{1,\dots,k\} : 1 \le \left|\alpha^i\right| \le l-2\right\}$ and $I_2 = \left\{i \in \{1,\dots,k\} : \left|\alpha^i\right| = l-1\right\}$. We intend to show that

$$\left\|\left(g^{(k)} \circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty. \tag{3.41.1}$$

In order to use the best possible Sobolev imbedding for the function $\partial_j D^{\alpha^1} u \in W^{m-\left|\alpha^1\right|-1,p}(\Omega)$, we will study two situations: $1 \in I_1$ and $1 \in I_2$, respectively.

If $1 \in I_1$, then $\left(m - \left|\alpha^1\right| - 1\right)p \ge (m-l+1)p = (m-l)p + p = n + p > n$, and thus the Sobolev imbedding

$$\partial_j D^{\alpha^1} u \in W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \tag{3.42}$$

is valid. By using imbeddings (3.38) and (3.42), we get

$$\begin{aligned}
&\left\|\left(g^{(k)} \circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \left\|\partial_j D^{\alpha^1} u\right\|_{L^\infty(\Omega)} \left\|D^{\alpha^2} u\right\|_{L^\infty(\Omega)} \dots \left\|D^{\alpha^k} u\right\|_{L^\infty(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.41.1) is valid in the first situation.

If $1 \in I_2$, then $\left(m - \left|\alpha^1\right| - 1\right)p = (m-l)p = n$, and thus the following Sobolev imbedding holds

$$\partial_j D^{\alpha^1} u \in W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } 1 \le r < \infty. \tag{3.43}$$

By using imbeddings (3.38) and (3.43), we get

$$\begin{aligned}
&\left\|\left(g^{(k)} \circ u\right)\partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u\right\|_{L^q(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \left\|\partial_j D^{\alpha^1} u\right\|_{L^q(\Omega)} \left\|D^{\alpha^2} u\right\|_{L^\infty(\Omega)} \dots \left\|D^{\alpha^k} u\right\|_{L^\infty(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty.
\end{aligned}$$

Consequently, (3.41.1) is valid in the second situation.

In conclusion, inequality (3.41.1) is valid in any situation of the fourth point. An analogous argument allows us to derive other $k-1$ formulas similar to (3.41.1), the last of them being

$$\left\|\left(g^{(k)} \circ u\right) D^{\alpha^1} u \dots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u\right\|_{L^q(\Omega)} \le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty. \tag{3.41.k}$$

It follows from (3.36), (3.39), (3.40), (3.41.1), ..., (3.41.k), and Theorem 3.1 that $D^\alpha (g \circ u) \in W^{1,q}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, and formula (3.37) holds. Consequently, $N_g u \in W^{l,q}(\Omega)$ and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ and inequalities (3.39), (3.40), (3.41.1), ... (3.41.k). Point (ii) is proved.

Point (iii) Throughout the proof of point (iii), we will denote by C a positive constant which depends at most on Ω, n, m, l, p, q.

Let $u \in W^{m,p}(\Omega)$. We first show that $N_g u = g \circ u \in W^{1,\infty}(\Omega)$ and

$$\partial_i (g \circ u) = (g' \circ u)\,\partial_i u \quad \mathcal{L}^n\text{-a.e. in } \Omega,\ i = 1,\dots,n. \tag{3.44}$$

To this end, we will use Theorem 1.1. We have $p > \frac{n}{m-l} > \frac{n}{m-1}$. Therefore, Theorem 1.1(iii) applies and we deduce that $g \circ u \in W^{1,\infty}(\Omega)$ and (3.44) holds. In addition, $N_g : W^{m,p}(\Omega) \to W^{1,\infty}(\Omega)$ is bounded.

It remains to be shown that if $g \circ u \in W^{s,\infty}(\Omega)$, with $1 \le s < l$, and

$$D^\alpha (g \circ u) = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha \\ |\alpha^i|\neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \tag{3.45}$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \le s,$$

then $D^\alpha (g \circ u) \in W^{1,\infty}(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$, and

$$\partial_j D^\alpha (g \circ u) = \sum_{k=1}^{s} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\alpha \\ |\alpha^i|\neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \left[\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \right.$$
$$\left. + \left(g^{(k)} \circ u\right)\left(\partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u + \ldots + D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u\right)\right] \tag{3.46}$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } j = 1, \ldots, n.$$

To this end, we will use Theorems 1.1 and 3.1.

By Theorem 1.1(iii), we infer that for each $k = 1, \ldots, s$, $N_{g^{(k)}} u = g^{(k)} \circ u \in W^{1,\infty}(\Omega)$ and

$$\partial_j \left(g^{(k)} \circ u\right) = \left(g^{(k+1)} \circ u\right) \partial_j u \quad \mathcal{L}^n\text{-a.e. in } \Omega, \ j = 1, \ldots, n,$$

with the usual convention if $k = l - 1$. In addition, $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,\infty}(\Omega)$ is bounded.

It follows from $\left|\alpha^i\right| \le |\alpha| = s < l \le m$ that

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \subset W^{1,p}(\Omega) \quad \text{for all } i = 1, \ldots, k.$$

In order to use Theorem 3.1, it remains to be shown that

$$\begin{gathered}
\left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^\infty(\Omega), \\
\left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \in L^\infty(\Omega), \\
\left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \in L^\infty(\Omega), \\
\vdots \\
\left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \in L^\infty(\Omega).
\end{gathered}$$

Let us fix $\alpha \in \mathbb{N}^n$ with $|\alpha| = s$. Fix $k \in \{1, \ldots, s\}$ and fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

Since

$$\left(m - \left|\alpha^i\right|\right) p \ge (m - s) p > (m - l) p > n,$$

the following Sobolev imbedding holds

$$D^{\alpha^i} u \in W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{3.47}$$

By using this imbedding, the following inequalities can be derived

$$\begin{aligned}
&\left\|\left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u\right\|_{L^\infty(\Omega)} \\
&\le \left\|g^{(k)} \circ u\right\|_{L^\infty(\Omega)} \left\|D^{\alpha^1} u\right\|_{L^\infty(\Omega)} \cdots \left\|D^{\alpha^k} u\right\|_{L^\infty(\Omega)} \\
&\le C \left\|g^{(k)} \circ u\right\|_{W^{1,\infty}(\Omega)} \|u\|^k_{W^{m,p}(\Omega)} < \infty,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j u D^{\alpha^1}u\dots D^{\alpha^k}u\right\|_{L^\infty(\Omega)}\\&\le\left\|\left(g^{(k)}\circ u\right)\partial_j u\right\|_{L^\infty(\Omega)}\left\|D^{\alpha^1}u\right\|_{L^\infty(\Omega)}\dots\left\|D^{\alpha^k}u\right\|_{L^\infty(\Omega)}\\&\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\end{aligned}\tag{3.49}$$

Since

$$\left(m-\left|\alpha^1\right|-1\right)p\ge(m-s-1)\,p\ge(m-l)\,p>n,$$

the following Sobolev imbedding holds

$$\partial_j D^{\alpha^1}u\in W^{m-|\alpha^1|-1,p}(\Omega)\hookrightarrow L^\infty(\Omega).\tag{3.50}$$

By using imbeddings (3.47) and (3.50), we get

$$\begin{aligned}&\left\|\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\dots D^{\alpha^k}u\right\|_{L^\infty(\Omega)}\\&\le\left\|g^{(k)}\circ u\right\|_{L^\infty(\Omega)}\left\|\partial_j D^{\alpha^1}u\right\|_{L^\infty(\Omega)}\left\|D^{\alpha^2}u\right\|_{L^\infty(\Omega)}\dots\left\|D^{\alpha^k}u\right\|_{L^\infty(\Omega)}\\&\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\end{aligned}\tag{3.51.1}$$

An analogous argument allows us to derive other $k-1$ formulas similar to (3.51.1), the last of them being

$$\left\|\left(g^{(k)}\circ u\right)D^{\alpha^1}u\dots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\right\|_{L^\infty(\Omega)}\le C\left\|g^{(k)}\circ u\right\|_{W^{1,\infty}(\Omega)}\|u\|^k_{W^{m,p}(\Omega)}<\infty.\tag{3.51.k}$$

It follows from (3.45), (3.48), (3.49), (3.51.1), . . . , (3.51.k), and Theorem 3.1 that $D^\alpha(g\circ u)\in W^{1,\infty}(\Omega)$ for all $\alpha\in\mathbb{N}^n$ with $|\alpha|=s$, and formula (3.46) holds. Consequently, $N_g u\in W^{l,\infty}(\Omega)$ and the higher-order chain rule (1.2) is valid. The boundedness of the operator $N_g: W^{m,p}(\Omega)\to W^{l,\infty}(\Omega)$ is an immediate consequence of the boundedness of the operator $N_g: W^{m,p}(\Omega)\to W^{1,\infty}(\Omega)$ and inequalities (3.48), (3.49), (3.51.1), . . . , (3.51.k). Point (iii) is proved.

Point (iv) It is a consequence of Theorem 1.1(iv) and of the first three points.

Point (v) The continuity of $N_g: W^{m,p}(\Omega)\to W^{l,q}(\Omega)$, with $1\le p,q<\infty$, can be proved in each of the cases (i)-(iii) by using a similar reasoning to that used in order to prove the continuity of $N_g: W^{m,p}(\Omega)\to W^{l,q}(\Omega)$, with $1\le p,q<\infty$, in the subcritical case (just adapt the proof of point (iv) of Theorem 2.1 in [6] to the particularities of the supercritical case). Besides, a detailed proof of point (v) can be found in [7].

Theorem 2.1 is completely proved. □

Remark 3.1 Theorem 2.1 remains valid provided that the hypothesis "Ω has the cone property" is replaced with the hypothesis "Ω is of class C^1" since this replacement does not affect the validity of the Sobolev imbedding theorem used in the proof of Theorem 2.1.

The proof of Theorem 2.2 relies on Theorem 2.1, Remark 3.1, and the following lemma whose proof can be found in [6].

Lemma 3.1 *If the superposition operator N_g is well defined and continuous from $W^{m,p}(\Omega)$ into $W^{l,q}(\Omega)$, with $m,l\in\mathbb{N}^*$, $1\le p,q<\infty$, and, in addition, $g(0)=0$, then $N_g\left(W_0^{m,p}(\Omega)\right)\subset W_0^{l,q}(\Omega)$.*

Proof of Theorem 2.2. Let $u \in W^{m,p}(\Omega) \cap W_0^{k,p}(\Omega)$. By Theorem 2.1 and Remark 3.1, we have $N_g u \in W^{l,q}(\Omega)$. We will only show that $N_g u \in W_0^{l,q}(\Omega)$ under the hypotheses of point (i) of Theorem 2.1, namely $\frac{n}{m} < p < \frac{n}{m-l}$, with $n \geq m-l+1$ ($1 \leq p < \frac{n}{m-l}$ when $n \in \{m-l+1,\ldots,m\}$), and $1 \leq q \leq \frac{np}{n-(m-l)p}$. The proof for the cases (ii) and (iii) is similar.

By using a Sobolev imbedding and $(m-k)\,p \leq (m-l)\,p < n$, we get

$$u \in W^{m,p}(\Omega) \hookrightarrow W^{k,p^*_{m-k}}(\Omega).$$

On the other hand, $u \in W_0^{k,p}(\Omega) = W^{k,p}(\Omega) \cap W_0^{k,1}(\Omega)$, and thus $u \in W_0^{k,1}(\Omega)$. Therefore, $u \in W^{k,p^*_{m-k}}(\Omega) \cap W_0^{k,1}(\Omega) = W_0^{k,p^*_{m-k}}(\Omega)$.

Simple computations show that

$$\frac{n}{m} < p < \frac{n}{m-l} \Leftrightarrow \frac{n}{k} < p^*_{m-k} < \frac{n}{k-l},$$

$$\frac{np}{n-(m-l)\,p} = \frac{np^*_{m-k}}{n-(k-l)\,p^*_{m-k}}.$$

Therefore, we have $\frac{n}{k} < p^*_{m-k} < \frac{n}{k-l}$ and $1 \leq q \leq \frac{np^*_{m-k}}{n-(k-l)p^*_{m-k}}$. Consequently, Theorem 2.1(i,v) apply and we deduce that N_g is well defined and continuous from $W^{k,p^*_{m-k}}(\Omega)$ into $W^{l,q}(\Omega)$. By combining this fact with $u \in W_0^{k,p^*_{m-k}}(\Omega)$, we infer by Lemma 3.1 that $N_g u \in W_0^{l,q}(\Omega)$. □

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