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Beyond the Mountain Pass: Some Applications

In honor to Antonio Ambrosetti a master and a great friend, in his retirement.
"... C'è nessuno? - Nessuno, risposi. E allora addio! disse mio padre, con la voce commossa, dando un ultimo sguardo alla scuola.E mia madre ripeté: addio! E io non potei dir nulla".
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Abstract

We will survey three applications of the famous *Mountain Pass Lemma* by Antonio Ambrosetti and Paul H. Rabinowitz, [6]. Precisely, we will present some results on the following problems:

- A model of growth and roughening of surfaces, usually called Kardar-Parisi-Zhang model.
- A *supercritical problem* involving Hardy-Leray potential with the pole at the boundary.
- A fourth order nonlinear problem related to epitaxial growth.

The common characteristic of all these problems is that we have some extra difficulties in order to apply the *Mountain Pass Lemma*.

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1 Introduction

There are some mathematical results that are of special interest and are useful–Often based on some general principles and easy to check in applications.

One of these results is the celebrated *Mountain Pass Lemma*, (MPL) in short, by Antonio Ambrosetti and Paul H. Rabinowitz in [6]. The result is proved with regularity hypotheses and by means of a suitable *deformation lemma*.

Some extensions of (MPL) can be seen in [10], [24], [29], [32], [48], [50] and [51] among many others. Perhaps it is in [48] where the denomination of (MPL) is coined.

This work is mainly based on results in the references: [2], [21], [26] and [45]. We will study how to use the *mountain pass theorem* in three different frameworks:

1. A model of growth and roughening of surfaces.

2. A *supercritical problem* involving Hardy-Leray potential with the pole at the boundary.

3. A fourth order nonlinear problem related to epitaxial growth.

The philosophy of this survey is to exhibit some problems in which it is necessary to make an extra analysis of the situation in order to take advantage of the applicability of the *Mountain Pass Lemma*.

We omit the well-known applications to critical problems with lack of compactness by using the *concentration-compactness* argument by P.L. Lions in [43] and [44]. See [17]. See also [8] and [9] where, with A. Ambrosetti and J. Garca, we obtain some results in this direction.

In the applications of (MPL) that we present in this paper we find different kinds of difficulties. In Section 2 we study Example 1. where the Ambrosetti-Rabinowitz condition that guarantees that a Palais-Smale sequence necessarily must be bounded does not hold due to a slightly super-linear growth of the nonlinear term.

In Section 3 we study Example 2. in which the problem is a little more subtle to apply directly the *mountain pass* lemma. We are forced to introduce some kind of perturbation in order to be able to define a perturbed energy functional and also a term of penalization in order to control the passage to the limit in the perturbed functional. Finally, in Section 4, we consider Example 3., where a fourth order problem appears with an *exotic* nonlinearity that produces a strong dependence on the boundary data to give a variational formulation. The content of this last section seems to be the starting point for a number of problems of interest in Geometry and Physics.

2 A model of growth and roughening of surfaces

The parabolic equation

$$(E) \quad u_t - \Delta u = |\nabla u|^2 + \lambda f(x,t),$$

where $f(x,t)$ is a positive measurable function, appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation (see [39]). The equation

$$u_t - \varepsilon\Delta u = |\nabla u|^2 + \lambda f(x,t),$$

may be viewed as the viscosity approximation as $\varepsilon \to 0^+$ of Hamilton-Jacobi type equations from stochastic control theory (see [42]). Also, some related equations appear in flame propagation, see [12].

By performing the so-called Hopf-Cole change of variable,

$$v = e^u - 1,$$

(see [38]), Equation (E) is transformed into the linear equation

$$v_t - \Delta v = \lambda f(x)(v + 1).$$

Results on multiplicity of solution for the Cauchy-Dirichlet problem in a cylinder $\Omega_T = \Omega \times (0, T)$, can be found in [3].

Notice that the associated stationary problem,

$$-\Delta u = |\nabla u|^2 + \lambda f(x), \quad u > 0 \quad u \in W_0^{1,2}(\Omega),$$

can be easily solved. Indeed, as above the Hopf-Cole change of variables yields the linear problem,

$$-\Delta v = \lambda f(x)(v + 1) \quad v > 0 \quad v \in W_0^{1,2}(\Omega).$$

Therefore the solution can be obtained by minimization in $W_0^{1,2}(\Omega)$ of the functional,

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega f(x) v^2 - \lambda \int_\Omega f v,$$

when the functional is bounded from below, which means that we must require,

$$(1) \quad \lambda_1(f) = \inf_{\phi \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega f \phi^2 dx} > 0; \ (2) \quad 0 < \lambda < \lambda_1(f).$$

The first condition gives some restrictions on the summability of f and the second one on the size of the source term. These two conditions are sharp as it is proved in [2] (see also [36] for a different analysis using potential theory).

In this section we study existence of regular solutions of the more general elliptic equations of the form

$$-\Delta u = \beta(u)|\nabla u|^2 + \lambda f(x) \text{ in } \Omega, \quad u \in W_0^{1,2}(\Omega), \tag{2.1}$$

where $\Omega \subset$ is a bounded domain, $f \in L^r(\Omega)$, with $r > \frac{N}{2}$, and

$$\beta : [0, +\infty) \longrightarrow [0, +\infty)$$

is a continuous nondecreasing function such that

$$\lim_{t \to +\infty} \beta(t) = +\infty. \tag{2.2}$$

Equations of the form (2.1) have been widely studied in the literature with different hypotheses on β. Existence results start from the classic references [41] and [40]. Later on, many authors have considered elliptic equations with first-order terms having quadratic growth with respect to the gradients, see for instance [14] and the references in [2].

We try to obtain regular positive solutions to problem (2.1) by using a change in the dependent variable in such a way that the problem becomes semi-linear.

We set

$$\gamma(t) = \int_0^t \beta(s)ds, \qquad \Psi(t) = \int_0^t e^{\gamma(s)}ds, \tag{2.3}$$

and we define $v(x) = \Psi(u(x))$. Then Problem (2.1) becomes

$$-\Delta v = \lambda f(x)(1 + g(v)) \text{ in } \Omega, \; v \in W^{1,2}(\Omega) \tag{2.4}$$

where

$$g(t) = e^{\gamma(\Psi^{-1}(t))} - 1 = \int_0^t \beta(\Psi^{-1}(s))ds\,. \tag{2.5}$$

Notice that the differentiable function $g : [0, +\infty) \longrightarrow [0, +\infty)$ verifies:

1. $g(0) = 0$, and g is increasing and convex.
2. $\lim_{s\to 0} \dfrac{g(s)}{s} = g'(0) = \beta(0)$.
3. $\lim_{s\to +\infty} \dfrac{g(s)}{s} = +\infty$ (g is superlinear).
4. $\displaystyle\int_0^{+\infty} \frac{ds}{1 + g(s)} = +\infty$ (g is slightly superlinear).

It is simple to prove the following result.

Proposition 2.1 *Assume that g verifies* (1), (2), (3) *and* (4) *above. There exists λ_0 such that for $\lambda \le \lambda_0$, Problem* (2.4) *has at least a positive solution $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and then $u = \Psi^{-1}(v) \in W_0^{1,2}(\Omega)$ is a positive solution of* (2.1).

Proof. Consider $\overline{v} = tw$, where w is the solution to

$$-\Delta w = f \quad \text{in } \Omega, \quad w \in W_0^{1,2}(\Omega)\,.$$

By the growth conditions on g, the function $h(t) = \dfrac{t}{1 + g(t\|w\|_\infty)}$ admits a positive maximum in $\mathbb{R}^+$. Then, if $0 < \lambda \le \lambda_0 = \max_{\mathbb{R}^+} h(t)$ and $t \ge \lambda(1 + g(t\|w\|_\infty))$, $\overline{v}$ is a supersolution to Problem (2.4).

Since $f \in L^r(\Omega)$, $r > \frac{N}{2}$, the spectral value $\lambda_1(f)$ defined above is the principal eigenvalue of the problem

$$-\Delta\phi = \lambda_1(f)\phi, \quad \phi \in W_0^{1,2}(\Omega). \tag{2.6}$$

To have a subsolution it is sufficient to take $\underline{v} = t_1\phi_1$ where ϕ_1 is the normalized positive eigenfunction corresponding to $\lambda_1(f)$ and $\lambda_1(f)t_1\|\phi_1\|_\infty \le \lambda$. Finally using Hopf's Lemma it is easy to check that $\underline{v} \le \overline{v}$. The result is a consequence of the usual iteration argument. See [13] and [52] for more details of this kind of arguments.

Theorem 2.1 *There exists* $\Lambda > 0$ *such that, if* $\lambda > \Lambda$, *Problem* (2.4) *has no positive solution* $v \in W_0^{1,2}(\Omega)$.

Proof. Assume that for all λ there exists a positive solution. Using the properties of g, there exists a positive constant $c > 0$ such that $g(s) \geq cs - 1$. Consider ϕ_1 the normalized solution to (2.6) as a test function in the equation, we obtain that

$$\lambda_1(f) \int_\Omega f v \phi_1 \, dx = \lambda \int_\Omega f \,(g(v) + 1)\, \phi_1 \, dx \geq \lambda c \int_\Omega f v \phi_1 \, dx \, .$$

If λ such that $c\lambda > \lambda_1(f)$ we obtain that $\int_\Omega f v \phi_1 \, dx = 0$; therefore the strong maximum principle implies $v \equiv 0$. A contradiction. In [2] it is proved that the nonexistence result for λ large remain true even in the distributional framework.

Remark 2.1 In the case where β is a decreasing function, it is easy to conclude that $\frac{g(s)}{s}$ is also decreasing. In this case problem (2.4) has a unique solution for λ small enough. The existence can be proved as in Proposition 2.1, while for uniqueness we refer to [7]. If, moreover, $\beta(s) \downarrow 0$, then $\frac{g(s)}{s} \downarrow 0$ as $s \to \infty$ then there exist a unique solution for all $\lambda \in \mathbb{R}^+$. These observations motivate the hypothesis of β nondecreasing to have at least two regular solutions to Problem (2.4).

Notice that conditions on g are not sufficient to prove that the natural energy functional is defined in $W_0^{1,2}(\Omega)$. Then to prove the existence of a second positive solution $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ we need to impose some extra growth condition on g in order to obtain the natural energy functional defined in $W_0^{1,2}(\Omega)$.

Such extra hypothesis on β is

$$\lim_{t \to +\infty} \frac{\beta(t)}{e^{a \int_0^t \beta(s) ds}} = 0, \text{ for some } a < \frac{4}{N+2}. \tag{2.7}$$

We can obtain (2.7) in the equivalent form,

$$\lim_{t \to +\infty} \frac{g'(t)}{(1 + g(t))^a} = 0, \text{ for some } a < \frac{4}{N+2}.$$

Then, it is easy to check that

$$\lim_{t \to +\infty} \frac{g(t)}{t^q} = 0, \; 1 - \frac{1}{q} = a. \tag{2.8}$$

Remark 2.2 By direct calculation we check that condition (2.7) is satisfied for elementary functions such as $\beta(s) = (\log(1 + s))^\alpha, \beta(s) = s^\alpha, \beta(s) = e^s, \beta(s) = e^{e^s}$, etc.

Notice that in this way $q < \frac{N+2}{N-2} = 2^* - 1$, and Problem (2.4) becomes variational in nature. Moreover this variational problem has a subcritical concave-convex structure. Then we will try to apply some ideas in [7] (see too [29]).

We will look for positive solutions to Problem (2.4) as critical points of the associated energy functional

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f \, u_+ dx - \lambda \int_\Omega f \, G(u_+) dx, \tag{2.9}$$

where

$$G(s) = \int_0^s g(t)dt\,,$$

which is well defined in $W_0^{1,2}(\Omega)$.

As far as $f(x)$ is concerned, for simplicity we will prove the result in the case where it is a non-negative, bounded function. However all the results can be easily proved under the assumption that

$$f(x) \in L^r(\Omega), \qquad \text{for } r > \frac{2^*}{2^* - (q+1)},$$

where q is defined by (2.8).

The following results are in the spirit of [7] and [29]. We will prove the existence of at least two positive solutions for λ small enough, that is a local result in λ. Precisely, we have the following result.

Theorem 2.2 *Assume that* (2.2) *and* (2.7) *hold, that $f(x)$ is bounded and non-negative, and that the functional J_λ has the geometry of the mountain pass, that is, there exist two points $v_1, v_2 \in W_0^{1,2}(\Omega)$ such that, setting*

$$\Gamma = \{\gamma \in C([0,1]; W_0^{1,2}),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there holds

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}. \tag{2.10}$$

Then problem (2.4) *has a mountain pass type positive solution u.*

The difficulty is that the nonlinear term $g(u)$ has slightly super-linear growth and, in general, does not verify Ambrosetti-Rabinowitz assumption ensuring that all Palais-Smale sequences for the associated energy functional are bounded.

To overcome the difficulty we must use an alternative argument. In this direction, see for instance [5], [37] and [54]. For the proof of Theorem 2.2 we choose the following abstract result by L. Jeanjean.

Theorem 2.3 *Let X be a Banach space endowed with the norm $||.||$ and let $\mathcal{J} \subset \mathbb{R}^+$ be an interval. Let $\{J_\alpha\}_{\alpha\in\mathcal{J}}$ be a family of functionals on X of the form*

$$J_\alpha(u) = A(u) - \alpha B(u)$$

where $B(u) \geq 0$ and such that $A(u)$ or $B(u) \to +\infty$ as $||u|| \to \infty$. We assume that there exist two points $v_1, v_2 \in X$ such that, setting

$$\Gamma = \{\gamma \in C([0,1]; X),\ \gamma(0) = v_1,\ \gamma(1) = v_2\},$$

there hold, for all $\alpha \in \mathcal{J}$,

$$c(\alpha) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\alpha(\gamma(t)) > \max\{J_\alpha(v_1), J_\alpha(v_2)\}.$$

Then for almost every $\alpha \in \mathcal{J}$, there exists a sequence $\{v_k\} \subset X$ such that :

i) $\{v_k\}$ is bounded; ii) $J_\alpha(v_k) \to c(\alpha)$ and iii) $J'_\alpha(v_k) \to 0$ in X', the dual of X.

See [37] for details.

Proof of Theorem 2.2 Assume that (2.10) holds. By a continuity argument there exists $\varepsilon > 0$ such that for all $\alpha \in \mathcal{J} = [1-\varepsilon, 1+\varepsilon]$, the family of functionals $\{J_{\lambda,\alpha}\}_{\alpha\in\mathcal{J}}$ defined by

$$J_{\lambda,\alpha}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda\alpha\Big(\int_\Omega f\, u_+ dx + \int_\Omega f\, G(u_+) dx\Big)$$

have the same geometry, namely

$$c(\lambda,\alpha) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda,\alpha}(\gamma(t)) > \max\{J_{\lambda,\alpha}(v_1), J_{\lambda,\alpha}(v_2)\}$$

with v_1, v_2 independent of $\alpha \in \mathcal{J}$.

By Theorem 2.3 we obtain that for almost every $\alpha \in \mathcal{J}$ there exists a sequence $\{v_k^{(\alpha)}\}$ such that: *i*) $\{v_k^{(\alpha)}\}$ is bounded; *ii*) $J_{\lambda,\alpha}(v_k^{(\alpha)}) \to c(\lambda,\alpha)$ and *iii*) $J'_{\lambda,\alpha}(v_k^{(\alpha)}) \to 0$ in $W^{-1,2}(\Omega)$.

Since g verifies (2.7), then using the Rellich-Kondrachov compactness result, we obtain that the Palais-Smale sequence admits $v_k^{(\alpha)} \to v^{(\alpha)}$ a strongly convergent subsequence in $W_0^{1,2}(\Omega)$, where $v^{(\alpha)}$ is a positive solution to problem

$$-\Delta v^{(\alpha)} = \lambda\alpha f(x)\,(1 + g(v^{(\alpha)})) \text{ in } \Omega, \quad v^{(\alpha)} \in W_0^{1,2}(\Omega) \tag{2.11}$$

and such that $J_{\lambda,\alpha}(v^{(\alpha)}) = c(\lambda,\alpha)$.

We have to prove that the conclusion in Theorem 2.3 holds for $\alpha = 1$. Let $\{\alpha_n\}$ be a decreasing sequence in $\mathcal{J}$ such that $\alpha_n \downarrow 1$ as $n \to \infty$ and consider v_n the corresponding solution to Problem (2.11).

Claim.- $\{v_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

If $\|v_n\|_\infty \le C$ for all n, then using (2.11) and by the conditions on f and g we conclude that $\|v_n\|_{W_0^{1,2}} \le C_1$.

Assume now that $\|v_n\|_\infty \to +\infty$ as $n \to \infty$. Notice that

$$\int_\Omega f\, v_n dx \le C. \tag{2.12}$$

Indeed, take ϕ_1 the positive eigenfunction associated to the first eigenvalue in (2.6) as a test function in (2.11). We obtain that

$$\lambda_1(f)\int_\Omega f\,\phi_1 v_n dx = \lambda\alpha_n\int_\Omega f\,\phi_1 + \lambda\alpha_n\int_\Omega f\, g(v_n)\phi_1 dx.$$

Since hypothesis (3) on g holds, there exists a constant C_1 such that

$$\int_\Omega f\,\phi_1 v_n dx \le C_1 \qquad \text{and} \qquad \int_\Omega f\,\phi_1 g(v_n) dx \le C_1.$$

Let ϕ_2 be the solution to problem

$$-\Delta\phi_2 = f(x) \text{ in } \Omega,\ \phi_2 \in W_0^{1,2}(\Omega). \tag{2.13}$$

By using Hopf's Lemma, there exist $c_1, c_2 > 0$ such that $c_1\phi_1 \le \phi_2 \le c_2\phi_1$. Take ϕ_2 as a test function in (2.11), then

$$\int_\Omega f v_n dx = \lambda\alpha_n \int_\Omega f\phi_2 + \lambda\alpha_n \int_\Omega f g(v_n)\phi_2 dx. \tag{2.14}$$

Since $\phi_2 \le c_2\phi_1$ we conclude that

$$\int_\Omega f v_n dx \le \lambda\alpha_n \int_\Omega f\phi_2 + c_2\lambda\alpha_n \int_\Omega f g(v_n)\phi_1 dx.$$

Hence $\int_\Omega f v_n dx \le C$.

As $J_{\lambda,\alpha_n}(v_n) = c(\lambda, \alpha_n) \le c(\lambda) + 1$, by using (2.12) we obtain that

$$\int_\Omega f(g(v_n)v_n - 2G(v_n))dx \le C. \tag{2.15}$$

We now prove the energy estimate.

If, on the contrary, we assume that $||v_n||_{W_0^{1,2}} \to \infty$ as $n \to \infty$, one can check that (see the technical details in [2]),

$$\int_\Omega f(g(v_n)v_n - 2G(v_n))\, dx \to +\infty, \quad \text{as } n \to \infty,$$

a contradiction with (2.15). Hence

$$||v_n||_{W_0^{1,2}} \le C_1.$$

Therefore $v_n \to v$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^\theta(\Omega)$ for all $\theta < \frac{2N}{N-2}$.

By using the hypotheses on g and by a simple compactness argument we obtain that v is a weak solution to Problem (2.4) and then we get easily that $v_n \to v$ strongly in $W_0^{1,2}(\Omega)$. Therefore we conclude that v is a non-negative solution to problem (2.4) such that

$$c(\lambda, \alpha_n) = J_{\lambda,\alpha_n}(v_n) \to J_\lambda(v) \quad \text{as } n \to \infty.$$

Hence, there exists a positive solution v to Problem (2.4) with $J_\lambda(v) = c(\lambda)$ and the proof is complete. □

Corollary 2.1 *There exists λ_0 such that if $0 < \lambda \le \lambda_0$, the functional J_λ has the geometry of the mountain pass and then Problem* (2.4) *has at least two positive solutions.*

Proof. Since $J_\lambda(0) = 0$, using (2.7) one can easily prove that for λ small enough there exists a number $R = R(\lambda) > 0$ such that $J_\lambda(v) \ge \rho_0 > 0$ for every v satisfying $||v|| = R$. On the other hand, using the superlinearity at ∞ of $g(s)$, it is easy to prove for every $\lambda > 0$, the existence of a function $w \in W_0^{1,2}(\Omega)$ with norm arbitrarily large, such that $J_\lambda(w) < 0$. Therefore

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0 = \max\{J_\lambda(0), J_\lambda(w)\}.$$

Therefore, applying Theorem 2.2 to the points 0 and w, there exists a positive solution v_1 to Problem (2.4) such that $J_\lambda(v_1) = c(\lambda) > 0$. We have to prove that $v_1 \ne v$, where v is the minimal solution obtained in Proposition 2.1. It is sufficient to show that $J_\lambda(v) \leq 0$, which is a consequence of the arguments in [7]. We skip the details and refer to [2] for this precise case. To conclude this section we prove the *global* multiplicity result.

Theorem 2.4 *Under the same assumptions of Theorem 2.2, let λ^* be defined by*

$$\lambda^* = \sup\{\lambda \geq 0 \textit{ such that Problem } (2.4) \textit{ has a positive solution }\}. \tag{2.16}$$

Then for all $\lambda \in (0, \lambda^*)$, Problem (2.4) *has at least two positive solutions. If $\lambda = \lambda^*$, then problem* (2.4) *has at least one positive solution.*

Proof. Consider the case $\lambda < \lambda^*$; then Problem (2.4) has a minimal solution v_λ, and as in the Corollary 2.1 one can show that $J_\lambda(v_\lambda) \leq 0$.

Using the hypothesis on g and integrating by parts we conclude that

$$\frac{g(s)}{s} \leq g'(s) \leq \frac{g(2s)}{s},$$

therefore

$$\lim_{t\to\infty} \frac{g'(t)}{t^{q_1}} = 0, \text{ for some } q_1 < \frac{4}{N-2}. \tag{2.17}$$

Fix $\lambda_1 < \lambda^*$, let $\lambda_1 < \lambda_2 < \lambda^*$ and consider $\overline{v}_1$ and $\overline{v}_2$ the minimal solutions to Problem (2.4) with $\lambda = \lambda_1,\ \lambda_2$, respectively; since $\lambda_1 < \lambda_2$, we get that $\overline{v}_2$ is a strict super-solution to problem with λ_1 and $\overline{v}_2 > \overline{v}_1$ by the strong maximum principle. We set

$$M = \{u \in W_0^{1,2}(\Omega) \,:\, 0 \leq u \leq \overline{v}_2 \text{ a.e. in } \Omega\} \qquad \text{and} \qquad I = \inf_{u\in M} J_{\lambda_1}(u).$$

Since M is a convex closed subset of $W_0^{1,2}(\Omega)$, using the fact that J_{λ_1} is bounded and weakly lower-semicontinuous in M, there exists $\vartheta \in M$ such that $J_{\lambda_1}(\vartheta) = I$. Notice that $I < 0$ and then $\vartheta \neq 0$. using a similar argument as in Theorem 2.4 of [54] we can prove that ϑ is a weak solution to Problem (2.4) with $\lambda = \lambda_1$. If $\vartheta \neq \overline{v}_1$ we obtain the existence of at least two positive solutions.

If $\vartheta = \overline{v}_{\lambda_1}$, then, by using arguments in [4] [7], one can prove that ϑ is a local minimum for J_{λ_1} (see [2] for a detailed proof).

Since now J_λ has a local minimum, then we get easily that J_{λ_1} has the geometry of the *Mountain Pass Theorem*, i.e., the existence of $(\overline{v}_1, \overline{v}_2)$ such that

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(\overline{v}_1), J_\lambda(\overline{v}_2)\}.$$

Then using Theorem 2.2 we get the multiplicity result.

Let now $\lambda = \lambda^*$ and consider a sequence of increasing numbers λ_n such that $\lambda_n \in (0, \lambda^*)$ and $\lambda_n \uparrow \lambda^*$ as $n \to \infty$. Let $\{v_{\lambda_n}\}$ be the family of minimal solution to Problem (2.4) with $\lambda = \lambda_n$. Then we obtain that $\{v_{\lambda_n}\}$ is an increasing sequence in n and $J_{\lambda_n}(v_{\lambda_n}) \leq 0$. Using the same argument as in the proof of Theorem 2.2 there exists a constant $C > 0$ such that $\int_\Omega f\, v_n dx \leq C$. Since

$$J_{\lambda_n}(v_n) = \frac{\lambda_n}{2} \int_\Omega f\Big(g(v_n)v_n - 2G(v_n)\Big)\, dx - \frac{\lambda_n}{2} \int_\Omega f\, v_n dx$$

we conclude that $|J_{\lambda_n}(v_n)| \le C_1$. Hence following again the idea of the proof of Theorem 2.2 we obtain that $\|v_{\lambda_n}\|_{W_0^{1,2}} \le C_1$ and then $v_{\lambda_n} \rightharpoonup v_0$ weakly in $W_0^{1,2}(\Omega)$. Since $\{v_{\lambda_n}\}$ is an increasing sequence we conclude that v_0 verifies

$$\begin{cases} -\Delta v_0 = \lambda^* f(x)(1+g(v_0)) & \text{in } \Omega \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

Hence we conclude.

Remark 2.3

1. The assumption $f \in L^{\frac{N}{2}}(\Omega)$ is optimal.
2. In [2] we also prove a WILD non-uniqueness result of weaker positive solutions. Indeed we prove that there exists a one to one correspondence between positive solutions to problem

$$-\Delta u = |\nabla u|^2 + \lambda f(x), \quad u > 0 \quad u \in W_0^{1,2}(\Omega),$$

and measures concentrated in subsets of Ω with zero capacity. The details can be seen in [2].

3 A problem involving Hardy-Leray potential

In this section we will study the problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.19}$$

where $p > 1$ and $\Omega \subseteq \mathbb{R}^N$, $N \ge 3$ is a bounded domain with smooth boundary, which is *supercritical* with respect to the Hardy-Sobolev embedding. We follow closely the ideas in [21]. Notice that if $0 < p < 1$ it is simple to prove existence of a solution, *independently of the location of the pole in* $\mathbb{R}^N$. By using Hardy's inequality the energy functional satisfies

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega \frac{u^{p+1}}{|x|^2} \ge \frac{1}{2}\int_\Omega |\nabla u|^2 - C(\Omega, p)(\int_\Omega |\nabla u|^2)^{\frac{p+1}{2}}$$

and then one can proceed by minimization as in [1].

In the linear problem, $p = 1$, results are as follows:

i) The problem has no positive solution if $0 \in \Omega$ (no attainability of the optimal constant $\Lambda_N = \left(\frac{N-2}{2}\right)^2$).

ii) If $0 \in \partial\Omega$ we have some results in the papers [27] and [28]. To describe such results, define

$$\mu(\Omega) = \inf_{\phi \in W^{1,2}(\Omega)\, \phi \ne 0} \frac{\displaystyle\int_\Omega |\nabla \phi|^2}{\displaystyle\int_\Omega \frac{\phi^2}{|x|^2}}.$$

It is known that $\mu(\mathbb{R}^N_+) = \dfrac{N^2}{4}$. Then the results in [27], among others, are the following:

1. If $\mu(\Omega) < \mu(\mathbb{R}^N_+) = \dfrac{N^2}{4}$, then $\mu(\Omega)$ is attained and the associated linear equation has a positive solution.
2. If $\mu(\Omega) \geq \mu(\mathbb{R}^N_+)$ there is no solution to the linear problem.

To see that the condition of attainability occurs the authors give the following geometrical condition.

Assume Ω *a bounded smooth domain such that* $0 \in \partial\Omega$ *and such that for any* $\delta > 0$ *there exists a* $\rho_\delta > 0$ *and* $\nu \in \mathbb{S}^{N-1}$ *verifying*

$$\Omega \supseteqq \{x \in \mathbb{R}^N \mid \langle x, \nu\rangle > -\delta|x|, 0 < \alpha < |x| < \beta\}$$

with $\dfrac{\beta}{\alpha} > \rho_\delta$. *Then* $\mu(\Omega)$ *is attained.*

The related problem for the Sobolev embedding $W^{1,2}(\Omega) \subset L^{p*(\alpha)}(\Omega, |x|^{-\alpha})$, $0 < \alpha < 2$, $p^*(\alpha) = \frac{2(N-\alpha)}{N-2}$, and the pole on $\partial\Omega$, has been studied by Ghoussoub-Kang in [31] and by Ghoussoub-Robert in[33], with variational methods. The authors prove that to have the attainability result it is sufficient to have a local condition on the boundary at 0, precisely a condition of negativity of the curvatures and the mean curvature at 0. The previous results are most critical.

We are interested in (3.19) with $1 < p < \frac{N+2}{N-2}$ and $0 \in \partial\Omega$ which is supercritical because when $\alpha = 2$ then $p^*(\alpha) - 1 = 1$.

If $0 \notin \overline{\Omega}$ then using the mountain pass theorem of Ambrosetti and Rabinowitz [6] one can easily show that (3.19) has a solution.

Definition 3.1 *We call a function* $u \in L^1(\Omega)$*,* $u \geq 0$ *a very weak solution of* (3.19) *if* $dist(x, \partial\Omega)\dfrac{u^p}{|x|^2} \in L^1(\Omega)$*,* $u > 0$ *in* Ω *and*

$$\int_\Omega u(-\Delta\phi) = \int_\Omega \frac{u^p}{|x|^2}\phi \quad \textit{for all } \phi \in C^\infty(\overline{\Omega}) \textit{ with } \phi|_{\partial\Omega} = 0.$$

If we assume $0 \in \Omega$ there is no *very weak solution* even locally to (3.19) as was pointed out by Brezis and Cabré [15]. That is if $0 \in \Omega$ then there is a local obstruction for the existence of solutions. On the contrary if $0 \in \partial\Omega$ we find quite a different behavior of the problem. Indeed we can exhibit at least 2 types of local solutions.

1. Consider the half space $H = \{(x_1, \ldots, x_N) : x_1 > 0\}$. Then one can find a solution of the equation in H of the form $u(x) = \varphi(\theta)$ where $\theta = x/|x|$ is on $S^{N-1}_+ = S^{N-1} \cap H$. The equation for φ becomes

$$\begin{cases} -\Delta_{S^{N-1}}\varphi = \varphi^p & \text{in } S^{N-1}_+ \\ \varphi = 0 & \text{on } \partial S^{N-1}_+. \end{cases} \tag{3.20}$$

If $1 < p < \frac{N+1}{N-3}$ ($p > 1$ if $N = 2, 3$), this problem has a positive solution and yields a solution of (3.19) in $W^{1,2}(H \cap B_R(0))$ for any $R > 0$ (if $N \geq 3$).

2. Fix $r_0 > 0$ small and define

$$D_{r_0} = \{x \in \Omega \ : \ |x| < r_0\},\ \Gamma_1 = \partial\Omega \cap \{|x| < r_0\} \text{ and } \Gamma_2 = \Omega \cap \{|x| = r_0\}.$$

Let $\lambda > 0$ and let us write $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$. By a scaling argument we construct a supersolution w to the problem

$$\begin{cases} -\Delta w = \dfrac{w^p}{|x|^2} & w > 0 \quad \text{in } D_{r_0} \\ w = 0 \qquad \text{on } \Gamma_1, & w = \lambda d_{\Gamma_1} \quad \text{on } \Gamma_2 \end{cases} \tag{3.21}$$

for any $\lambda > 0$ which is small enough, and furthermore $w(x) \le C d_{\Gamma_1}(x)$ for some constant C. Using monotone iterations this implies the existence of a solution to (3.21) for $\lambda > 0$ small, and this solution is also bounded by $C d_{\Gamma_1}(x)$.

However, we have the following nonexistence result.

Proposition 3.1 *Assume $N \ge 3$ and $1 < p < \frac{N+2}{N-2}$. Then* (3.19) *has no energy solutions if* $0 \in \partial\Omega$ *and Ω is starshaped with respect to the origin.*

Now, if u is an energy solution of (3.19) then $u \in L^\infty(\Omega)$, and there is some $C > 0$ such that

$$|\nabla u(x)| \le \frac{C}{|x|}, \quad |D^2 u(x)| \le \frac{C}{|x|^2} \quad \text{for all } x \in \Omega$$

(see [21] for details). These estimates allow us to use $\langle x, \nabla u \rangle$ as test functions in the equation and use a Pohozaev-type argument to prove the nonexistence result.

Notice that there is a gap with respect to the case $0 \in \Omega$ in which the nonexistence class is larger than the energy space and moreover no hypotheses on the geometry of Ω are needed.

Just to motivate our analysis, an elementary observation is in order. Consider the open set $\Omega = B_1((1, 0, ..., 0)) \cup B_1((3, 0, ..., 0))$ and let v be a solution to

$$\begin{cases} -\Delta v = \dfrac{v^p}{|x|^2}, & v > 0 \quad \text{in } B_1((3, 0, ..., 0)) \\ v = 0 & \text{on } \partial B_1((3, 0, ..., 0)) \end{cases}$$

obtained, for instance, using the classical Mountain Pass Theorem. Then

$$u(x) = \begin{cases} 0 & \text{if } x \in B_1((1, 0, ..., 0)) \\ v(x) & \text{if } x \in B_1((3, 0, ..., 0)) \end{cases}$$

is a solution of (3.19).

Notice that the role of the geometry of the two components is irrelevant and that in this case the domain is not connected. The idea is that a perturbation of the domain allows us to obtain connected domains for which Problem (3.19) has a solution.

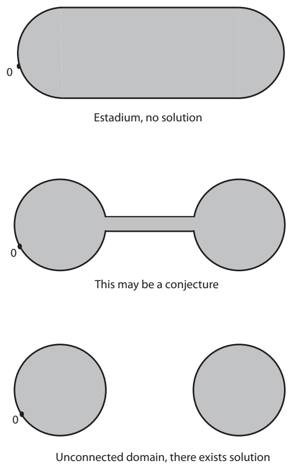

Figure 1: Motivation of the results

Remark 3.1 Perturbations of the domain have been used before to construct solutions to semilinear equations. For example, Dancer [19, p. 140] showed that the problem

$$-\Delta u = u^p \quad u > 0 \text{ in } \Omega, \quad u \in W_0^{1,2}(\Omega), \tag{3.22}$$

with $p > 1$ subcritical and Ω a dumbbell domain consisting of 2 or more balls joined by a thin tube, has multiple solutions. See the precise meaning in Definition 3.2.

For $p = \frac{N+2}{N-2}$, we recall that A. Bahri, J.M. Coron in [11] prove the existence of a positive solution to the critical problem under the hypothesis that there exists a positive integer d such that the homology group with Z_2-coefficients verifies $H_d(\Omega; Z_2) \neq 0$. However, the construction of special domains by E.N. Dancer in [20], W. Ding in [23], and D. Passaseo in [49] prove that the obstruction to the existence of positive solutions to the critical problem in not topological.

According with the previous remarks we have the conjecture sketched in Figure 1 and then we will consider domains that *are a perturbation* of the previous disconnected case. More precisely we will consider the class of domains defined below.

Definition 3.2 *We call Ω_ϵ a dumbbell domain if it is a domain with smooth boundary of the form $\Omega_\epsilon = \Omega_1 \cup C_\epsilon \cup \Omega_2$ where Ω_1 and Ω_2 are smooth bounded domains such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$, and C_ϵ is a region contained in a tubular neighborhood of radius less than $\epsilon > 0$ around a curve joining Ω_1 and Ω_2. (See Figure 2.)*

With this idea of perturbation in mind we conjecture the following result.

Theorem 3.1 *Assume that*

1. *Ω_ϵ is a dumbbell domain,*
2. *$0 \in \partial\Omega_1 \cap \partial\Omega_\epsilon$,*

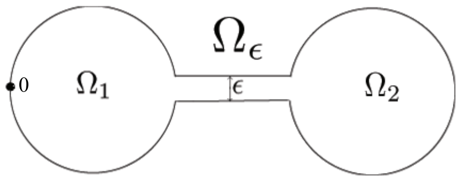

Figure 2: Dumbbell domain

3. $1 < p < \frac{N+2}{N-2}$.

Then there exist $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$*,*

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} & u > 0 \quad in\ \Omega_\epsilon \\ u = 0 & \quad on\ \partial\Omega_\epsilon \end{cases} \tag{3.23}$$

has a solution.

The natural energy functional should be

$$E(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{u^{p+1}}{|x|^2},$$

which is not well-defined in $W_0^{1,2}(\Omega_\epsilon)$. The next tentative should be to consider for $\delta > 0$ the truncated functional,

$$E_\delta(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{u^{p+1}}{|x|^2+\delta}.$$

It is clear that $E_\delta(u)$ satisfies the hypotheses of the classical *Mountain Pass Lemma* by Ambrosetti-Rabinowitz, [6], in $W_0^{1,2}(\Omega_\epsilon)$.

Consider $\phi \in W_0^{1,2}(\Omega_\epsilon)$ such that $E_1(\phi) < 0$. Then $E_\delta(\phi) < 0$ if $\delta \in (0,1)$. Define $\Gamma = \{\gamma : [0,1] \to W_0^{1,2}(\Omega) \,|\, \gamma(0) = 0,\ \gamma(1) = \phi\}$ and the mini-max level

$$c_\delta = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E_\delta(\gamma(t))$$

It is easy to check that $c_\delta \to 0$ as $\delta \to 0$. Therefore according with the example and the previous remark, the proof of the theorem relies on variational techniques with some kind of local penalization term close to the pole of the Hardy potential. The proof of the theorem is organized in several steps.

3.1 A truncated-penalized functional.

We will define the penalization term. Consider a cut-off function $\eta \in C^1(\mathbb{R})$ with the properties:

$$\begin{cases} \eta(s) = 0 & s \in [0,1] \\ 0 \le \eta(s) \le 2 \ \text{ and } \ \eta'(s) \ge 0 & s \in [1,2] \\ \eta(s) = s & s \in [2,\infty). \end{cases}$$

Given $\theta > 0$, define

$$\eta_\theta(t) = \eta(t/\theta) \quad t \geq 0. \tag{3.24}$$

Notice that in this way

$$0 < t < \theta \quad \Rightarrow \quad \eta_\theta(t) = 0. \tag{3.25}$$

The following definition is motivated by the computations in the proof and we will see that the penalization term is quite natural. Since $p < 2^* - 1$, where $2^* = \frac{2N}{N-2}$, then $(p-1)\frac{N}{2} < p+1$. Therefore we have a nonempty interval,

$$I_p = [\,\min\Big(2, (p-1)\frac{N}{2}\Big), p+1].$$

Fix $q \in I_p$, consider a function $g \in C^1(\mathbb{R})$ such that

$$\begin{cases} g(s) = 0 & s \leq 0 \\ 0 \leq g(s) \leq 1 \ \text{ and } \ 0 \leq g'(s) \leq q s^{(p-1)\frac{N}{2}-1} & s \in [0,1] \\ g(s) = s^{(p-1)\frac{N}{2}} & s \geq 1 \end{cases}$$

and given $\delta > 0$, set

$$g_\delta(t) = \delta^{(p-1)\frac{N}{2}} g(t/\delta). \tag{3.26}$$

Fixing $\epsilon > 0$ the width of the tubular neighborhood C_ϵ in Ω_ϵ, $\delta > 0$ and $\theta > 0$ we define the penalized energy functional

$$E_{\delta,\epsilon,\theta} : W_0^{1,2}(\Omega_\varepsilon) \to \mathbb{R}$$

by

$$E_{\delta,\epsilon,\theta}(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^2+\delta} + \eta_\theta(I_\delta(u)),$$

where $u^+ = \max(u, 0)$ and

$$I_\delta(u) = \int_{\Omega_1} \frac{g_\delta(u+\delta)}{(|x|^2+\delta)^{\frac{N}{2}}}.$$

It is easy to prove that $E_{\delta,\epsilon,\theta} \in C^1(W_0^{1,2}(\Omega_\varepsilon))$.

Moreover, if u is a critical point of $E_{\delta,\epsilon,\theta}$ then it satisfies the Euler equation

$$\begin{cases} -\Delta u + a(x,u) g'_\delta(u+\delta) = \dfrac{(u^+)^p}{|x|^2+\delta} & \text{in } \Omega_\epsilon \\ u = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{3.27}$$

where

$$a(x,u) = \eta'_\theta(I_\delta(u)) \frac{\chi_{\Omega_1}(x)}{(|x|^2+\delta)^{\frac{N}{2}}}$$

and χ_{Ω_1} is the characteristic function of Ω_1.

A solution $u \in W_0^{1,2}(\Omega_\varepsilon)$ to (3.27), satisfies $u \geq -\delta$ in Ω_ε but it is not necessarily positive. The following result allow us to apply the *Mountain Pass Lemma* to each functional $E_{\delta,\epsilon,\theta}$

Lemma 3.2 *Fix $\varepsilon, \theta, \delta > 0$. Then*

1. *$E_{\delta,\epsilon,\theta}$ has the geometry of the Mountain Pass Lemma.*
2. *$E_{\delta,\epsilon,\theta}$ satisfies the Palais-Smale condition.*

Proof. Proof of 1). The *mountain pass* geometry follows easily because if $||u||_{W_0^{1,2}(\Omega_\varepsilon)}$ is sufficiently small then $E_{\delta,\epsilon,\theta}(u) = \tilde{E}_\delta(u)$, where

$$\tilde{E}_\delta(u) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+)^{p+1}}{|x|^2+\delta},$$

that is, the functional without penalization.

Let $\phi_2 > 0$ be the eigenfunction corresponding to the principal eigenvalue

$$-\Delta\phi_2 = \lambda_1(\Omega_2)\phi_2, \text{ in } \Omega_2, \ \phi_2 \in W_0^{1,2}(\Omega_2), \tag{3.28}$$

normalized such that $||\phi_2||_{L^2(\Omega_2)} = 1$. We conclude taking $A > 0$ large such that

$$E_{\delta,\epsilon,\theta}(A\phi_2) = \frac{1}{2}A^2\int_{\Omega_2} |\nabla\phi_2|^2 - \frac{1}{p+1}A^{p+1}\int_{\Omega_2} \frac{\phi_2^{p+1}}{|x|^2+\delta} < 0.$$

Proof of 2). Consider a Palais-Smale sequence for the functional $E_{\delta,\epsilon,\theta}$, that is, a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,2}(\Omega_\varepsilon)$ such that $E_{\delta,\epsilon,\theta}(u_n) \le C$ and $E'_{\delta,\epsilon,\theta}(u_n) \to 0$ in $W^{-1,2}(\Omega_\varepsilon)$. Then

$$C + o(1)||u_n||_{W_0^{1,2}} \ge E_{\delta,\epsilon,\theta}(u_n) - \frac{1}{p+1}E'_{\delta,\epsilon,\theta}(u_n)u_n$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)||u_n||^2_{W_0^{1,2}} + \eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}}.$$

By direct calculation we have that

$$g'_\delta(u+\delta)u \le q g_\delta(u+\delta) \quad \text{for all } u \in \mathbb{R}. \tag{3.29}$$

i) Assume first $I_\delta(u_n) \ge 2\theta$. Then

$$\begin{aligned}\eta_\theta(I_\delta(u_n)) \quad &- \quad \frac{1}{p+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \\ &= \quad \frac{1}{\theta}\left(I_\delta(u) - \frac{1}{p+1}\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}}\right).\end{aligned}$$

Using (3.29) we deduce that

$$\int_{\Omega_1} \frac{g'_\delta(u+\delta)u}{(|x|^2+\delta)^{\frac{N}{2}}} \le qI_\delta(u) \tag{3.30}$$

and therefore

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \ge \frac{1}{\theta}\left(1 - \frac{q}{p+1}\right)I_\delta(u) \ge 0.$$

Hence, if $I_\delta(u_n) \ge 2\theta$ we find $||u_n||_{W_0^{1,2}} \le C$ for all n.

ii) If $I_\delta(u_n) \le \theta$ we obtain the same conclusion because

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta'_\theta(I_\delta(u_n)) \int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} = 0.$$

iii) If $\theta \le I_\delta(u_n) \le 2\theta$ then using (3.30)

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{p+1}\eta'_\theta(I_\delta(u_n)) \int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^2+\delta)^{\frac{N}{2}}} \ge -\frac{C}{\theta} I_\delta(u_n) \ge -C.$$

Therefore, again in this case, we prove that $\|u_n\|_{W_0^{1,2}} \le C$ for all n for some constant C.

The compactness is a consequence of the Rellich-Kondrachov Theorem.

Define the mini-max level corresponding to the mountain pass,

$$c_{\delta,\epsilon,\theta} = \inf_{\gamma} \max_{t\in[0,1]} E(\gamma(t)) \tag{3.31}$$

where the infimum ranges over all continuous paths $\gamma : [0,1] \to W_0^{1,2}(\Omega_\varepsilon)$ such that $\gamma(0) = 0$ and $\gamma(1) = A\phi_2$ where A as above, i.e., $E_{\delta,\epsilon,\theta}(A\phi_2) < 0$.

Corollary 3.1 *There exists a mountain pass critical point* $u_{\delta,\epsilon,\theta} \in W_0^{1,2}(\Omega_\epsilon)$ *of* $E_{\delta,\epsilon,\theta}$ *with critical value* $c_{\delta,\epsilon,\theta}$.

3.2 Estimates from above and from below and the mini-max levels.

We will prove estimates from above of the mini-max level defined by (3.31) and we also will prove that the penalization term gives a lower estimate. Notice that the estimate from below is a key point in the proof, describing how the penalization term avoids the zero limit.

Lemma 3.3 *Let* $c_{\delta,\epsilon,\theta}$ *be defined by* (3.31). *Then*

1. *There exists a constant C independent of* $\varepsilon, \theta, \delta$ *such that*

$$c_{\delta,\epsilon,\theta} \le C.$$

2. *There exist* $\theta_0 > 0$ *and* $c_0 > 0$ *independent of* $\varepsilon, \theta, \delta$ *such that*

$$c_{\delta,\epsilon,\theta} \ge c_0$$

for $0 < \theta \le \theta_0$.

Proof.

1) Take the path $\gamma(t) = tA\phi_2$ with ϕ_2 as above. Since $\max_{t\in[0,1]} E_{\delta,\epsilon,\theta}(tA\phi_2)$ is bounded uniformly in δ, ϵ, θ we obtain the upper bound for $c_{\delta,\epsilon,\theta}$.

2) Since in $\Omega_\varepsilon \setminus \Omega_1$ the weight $\frac{1}{|x|^2+\delta}$ is bounded uniformly in ε, δ, we can fix $\rho > 0$ independently of ε, δ and θ such that

$$\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon\setminus\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \ge \frac{1}{4}\|u\|^2_{W_0^{1,2}} \quad \text{for all } \|u\|_{W_0^{1,2}} \le \rho.$$

By Hölder's inequality

$$\int_{\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \le \left(\int_{\Omega_1} (u^+)^{2^*}\right)^{\frac{N-2}{N}} \left(\int_{\Omega_1} \frac{(u^+)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}\right)^{\frac{2}{N}}$$

(this last inequality motivates the definition of I_δ for the penalization).

Using Sobolev's inequality we obtain

$$\int_{\Omega_1} \frac{(u^+)^{p+1}}{|x|^2+\delta} \le C\|u\|^2_{W_0^{1,2}} I_\delta(u)^{\frac{2}{N}}.$$

It follows that

$$E_{\delta,\epsilon,\theta}(u) \ge \frac{1}{4}\|u\|^2_{W_0^{1,2}} - C\rho^2 I_\delta(u)^{\frac{2}{N}} + \eta_\theta(I_\delta(u)) \quad \text{for all } \|u\|_{W_0^{1,2}} \le \rho. \tag{3.32}$$

Defining $h(t) = \eta_\theta(t) - C\rho^2 t^{\frac{2}{N}}$ there exists $\theta_0 > 0$ such that if $\theta < \theta_0$

a) $h(t) \ge t^{\frac{2}{N}}$ for $t \ge 2\theta$.

b) $h(t) \ge -C\rho^2(2\theta)^{\frac{2}{N}}$ for $0 < t < 2\theta$.

Let $\gamma : [0,1] \to W_0^{1,2}(\Omega_\varepsilon)$ be an arbitrary continuous path such that $\gamma(0) = 0$ and $\gamma(1) = A\phi_2$. Take $\rho > 0$ smaller than the previous one such that $A\|\phi_2\|_{W_0^{1,2}} > \rho$. Let t^* be defined by

$$t^* = \min\{\, t \in [0,1] \;:\; \|\gamma(t)\|_{W_0^{1,2}} \ge \rho \text{ or } I_\delta(\gamma(t)) \ge 1 \,\}.$$

Then t^* is well-defined and we have

$$\|\gamma(t)\|_{W_0^{1,2}} \le \rho, \quad I_\delta(\gamma(t)) \le 1 \text{ for } 0 \le t \le t^*$$

and one of the following cases: either *i*) $\|\gamma(t^*)\|_{W_0^{1,2}} = \rho$ or *ii*) $I_\delta(\gamma(t^*)) = 1$.

Assume first that $\|\gamma(t^*)\|_{W_0^{1,2}} = \rho$. Then using (3.32), *a*) and *b*)

$$E_{\delta,\epsilon,\theta}(\gamma(t^*)) \ge \frac{1}{4}\|\gamma(t^*)\|^2_{W_0^{1,2}} - C\rho^2(2\theta)^{\frac{2}{N}} = \frac{1}{4}\rho^2 - C\rho^2(2\theta)^{\frac{2}{N}}.$$

Choosing θ_0 smaller we can achieve

$$\frac{1}{4}\rho^2 - C\rho^2(2\theta)^{\frac{2}{N}} \ge \frac{1}{8}\rho^2 \quad \text{for } 0 < \theta \le \theta_0. \tag{3.33}$$

Then we obtain

$$E_{\delta,\epsilon,\theta}(\gamma(t^*)) \ge \frac{1}{8}\rho^2.$$

Assume now that that $I_\delta(\gamma(t^*)) = 1$. We may also assume that $\theta_0 \le \frac{1}{2}$. Then by (3.32) and *a*),

$$E_{\delta,\epsilon,\theta}(\gamma(t^*)) \ge \|\gamma(t^*)\|^2_{W_0^{1,2}} + I_\delta(\gamma(t^*))^{\frac{2}{N}} \ge 1.$$

It follows that the mountain pass level $c_{\delta,\epsilon,\theta}$ satisfies

$$c_{\delta,\epsilon,\theta} \ge \min\left(\frac{\rho^2}{8}, 1\right)$$

provided $0 < \theta \le \theta_0$, and $0 < \theta_0 \le \frac{1}{2}$ is such that (3.33) hold.

3.3 Uniform estimates for the *mountain pass* critical points.

We will prove that that there exists C independent of $\delta, \theta, \varepsilon$ such that for all $\delta > 0$, $\theta > 0$ and $\epsilon > 0$, if $u_{\delta,\epsilon,\theta} \in W_0^{1,2}(\Omega_\varepsilon)$ is a mountain pass critical point of the functional $E_{\delta,\epsilon,\theta}$ then

$$\|u_{\delta,\epsilon,\theta}\|_{W_0^{1,2}(\Omega_\varepsilon)} \le C \tag{3.34}$$

and

$$I_\delta(u_{\delta,\epsilon,\theta}) \le C\theta. \tag{3.35}$$

The argument is similar to the proof of Lemma 3.2. Indeed, since

$$E_{\delta,\epsilon,\theta}(u_{\delta,\epsilon,\theta}) = c_{\delta,\epsilon,\theta} \le C$$

we have

$$C \ge E_{\delta,\epsilon,\theta}(u_{\delta,\epsilon,\theta}) - \frac{1}{p+1}\langle E'_{\delta,\epsilon,\theta}(u_{\delta,\epsilon,\theta}), u_{\delta,\epsilon,\theta}\rangle$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_{\delta,\epsilon,\theta}\|^2_{W_0^{1,2}} + \eta_\theta(I_\delta(u_{\delta,\epsilon,\theta})) - \frac{q}{p+1}\eta'_\theta(I_\delta(u_{\delta,\epsilon,\theta}))I_\delta(u_{\delta,\epsilon,\theta}).$$

Therefore if

1. $I_\delta(u_{\delta,\epsilon,\theta}) \ge 2\theta$, $\eta_\theta(I_\delta(u_{\delta,\epsilon,\theta})) - \frac{q}{p+1}\eta'_\theta(I_\delta(u_{\delta,\epsilon,\theta}))I_\delta(u_{\delta,\epsilon,\theta}) = \frac{1}{\theta}\left(1 - \frac{q}{p+1}\right)I_\delta(u_{\delta,\epsilon,\theta})$.
2. $I_\delta(u_{\delta,\epsilon,\theta}) \le \theta$, $\eta_\theta(I_\delta(u_{\delta,\epsilon,\theta})) - \frac{q}{p+1}\eta'_\theta(I_\delta(u_{\delta,\epsilon,\theta}))I_\delta(u_{\delta,\epsilon,\theta}) = 0$.
3. $I_\delta(u_{\delta,\epsilon,\theta}) \in (\theta, 2\theta)$, $\eta_\theta(I_\delta(u_{\delta,\epsilon,\theta})) - \frac{q}{p+1}\eta'_\theta(I_\delta(u_{\delta,\epsilon,\theta}))I_\delta(u_{\delta,\epsilon,\theta}) \ge -\frac{C}{\theta}I_\delta(u_{\delta,\epsilon,\theta}) \ge -C$.

Hence we deduce (3.34) and (3.35) with C independent of $\delta, \theta, \varepsilon$.

3.4 There is no local obstruction: Existence of a local supersolution.

The behavior of the Hardy potential with the pole at the boundary is very different from the behavior when the pole is in Ω. The biggest difference is that in the case where $0 \in \partial\Omega$ we are able to find a local supersolution. This supersolution allows us to control uniformly the mountain pass solutions near the singularity.

Fix $r_0 > 0$ small and define $D = \{x \in \Omega_\epsilon \ : \ |x| < r_0\}$, $\Gamma_1 = \partial\Omega_\epsilon \cap \{|x| < r_0\}$ and $\Gamma_2 = \Omega_\epsilon \cap \{|x| = r_0\}$ (see Figure 3). Since we assume that the curve that joins Ω_1 and Ω_2 along which runs C_ε, is fixed and $0 \in \partial\Omega_1 \cap \partial\Omega_\epsilon$, if we take $r_0 > 0$ small then D is independent of ε.

Call $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$ and consider ζ the solution to

$$\begin{cases} -\Delta\zeta = \dfrac{d_{\Gamma_1}^p}{|x|^2} & \text{in } D \\ \zeta = 0 \quad \text{on } \Gamma_1, & \zeta = d_{\Gamma_1} \quad \text{on } \Gamma_2. \end{cases}$$

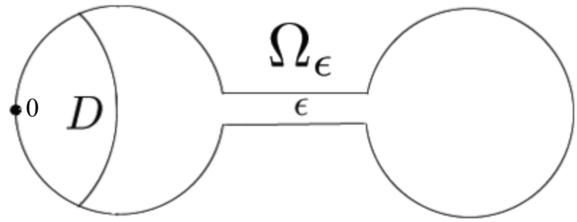

Figure 3: Subdomain D to find a supersolution

The function $d_{\Gamma_1}^p/|x|^2$ belongs to $L^q(D)$ for any $1 \le q < \frac{N}{2-p}$ if $p < 2$ and for any $q \ge 1$ if $p \ge 2$. Therefore, in any case, there exists a $q > N$ such that $d_{\Gamma_1}^p/|x|^2 \in L^q(D)$. By elliptic L^p estimates and the Morrey embedding $\zeta \in C^{1,\beta}(\overline{D})$, for some $\beta > 0$. Hence there is some constant $C > 0$ such that $\zeta \le C d_{\Gamma_1}$. Setting $\lambda_0 = C^{-\frac{p}{p-1}} > 0$ and $w = \lambda\zeta$, the function w satisfies

$$\begin{cases} -\Delta w \ge \dfrac{w^p}{|x|^2} & w > 0 \quad \text{in } D \\ \quad w = 0 \quad \text{on } \Gamma_1 & w \ge \lambda d_{\Gamma_1} \quad \text{on } \Gamma_2 \end{cases} \tag{3.36}$$

for any $0 \le \lambda \le \lambda_0$ and furthermore $w(x) \le C d_{\Gamma_1}(x)$ for some constant C. In the sequel we fix $\lambda = \lambda_0$ and $w = \lambda_0\zeta$.

3.5 Uniform estimates to eliminate the penalization term.

We need to eliminate the penalization term in order to find a solution to Problem (3.19). The main goal of this section is to prove that for values of ϵ small enough we find that $I_\delta(u_{\varepsilon,\delta}) < \theta$, and then, by the definition of η_θ, we obtain that the penalization term is zero is such domains Ω_ϵ, that is,

$$E_{\delta,\epsilon,\theta}(u_{\varepsilon,\delta,\theta}) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u_{\varepsilon,\delta,\theta}|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} \frac{(u^+_{\varepsilon,\delta,\theta})^{p+1}}{|x|^2+\delta}.$$

In order to control the penalization term we will prove first the following comparison result, that is also needed to pass to the limit.

Lemma 3.4 *Let w be the local supersolution defined by* (3.36) *and D as in the Figure 2. Then, there is $\theta_1 > 0$ such that if $0 < \theta \le \theta_1$ then any mountain pass critical point $u_{\delta,\varepsilon,\theta}$ of $E_{\delta,\varepsilon,\theta}$ satisfies*

$$u_{\delta,\varepsilon,\theta} \le w \quad \text{in } D.$$

Proof. Since (3.35) and (3.34) hold, we have $I_\delta(u) \le C\theta$ and $||u_{\delta,\epsilon,\theta}||_{W_0^{1,2}(\Omega_\varepsilon)} \le C$. The classical L^∞ and C^β-estimates gives us that for any compact

$$K \subset (\Omega_1 \cup \Gamma_1) \setminus \{0\}, \ ||u_{\delta,\epsilon,\theta}||_{L^\infty(K)} \to 0 \quad \text{as } \theta \to 0, \text{ uniformly in } \varepsilon, \delta.$$

Then by bootstrapping $||u_{\delta,\epsilon,\theta}||_{C^{1,\beta}(K)} \le C$ uniformly in ε, δ. Thus there is $\theta_1 > 0$ independent of ε, δ, such that for $0 < \theta \le \theta_1$ we have

$$u_{\delta,\epsilon,\theta} \le \lambda d_{\Gamma_1} \text{ on } \Gamma_2.$$

From (3.27) we have

$$-\Delta u_{\delta,\epsilon,\theta} \le \frac{u^p}{|x|^2+\delta} \quad \text{in } D,$$

and therefore

$$-\Delta(u_{\delta,\epsilon,\theta} - w) \le \frac{u^p_{\delta,\epsilon,\theta} - w^p}{|x|^2+\delta} \quad \text{in } D.$$

Multiplying by $(u_{\delta,\epsilon,\theta} - w)^+$, integrating on D and using the Hölder inequality, we find

$$\int_D |\nabla(u_{\delta,\epsilon,\theta} - w)^+|^2 \le \left(\int_D |(u_{\delta,\epsilon,\theta} - w)^+|^{2^*}\right)^{\frac{N-2}{N}} \left(\int_D \frac{|u_{\delta,\epsilon,\theta}|^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}\right)^{\frac{2}{N}}$$

$$\le \left(\int_D |(u_{\delta,\epsilon,\theta} - w)^+|^{2^*}\right)^{\frac{N-2}{N}} I_\delta(u_{\delta,\epsilon,\theta})^{\frac{2}{N}}.$$

Hence, using (3.35)

$$\int_D |\nabla(u_{\delta,\epsilon,\theta} - w)^+|^2 \le C\theta^{\frac{2}{N}} \int_D |\nabla(u_{\delta,\epsilon,\theta} - w)^+|^2.$$

Taking $\theta_1 > 0$ smaller if necessary we conclude that $(u_{\delta,\epsilon,\theta} - w)^+ \equiv 0$ in D, that is, $u_{\delta,\epsilon,\theta} \le w$ in D.

Fix $0 < \theta < \min\{\theta_0, \theta_1\}$. Now we are able to prove the following key result.

Lemma 3.5 *Let θ be such that* $0 < \theta < \min\{\theta_0, \theta_1\}$.

There exists $\varepsilon_0 > 0$ such that

$$I_\delta(u_{\varepsilon,\delta,\theta}) < \theta \quad \textit{for all } 0 < \varepsilon \le \varepsilon_0 \textit{ and all } 0 < \delta \le \varepsilon_0.$$

Proof. We argue by contradiction. Assume that for such θ fixed, there are sequences of positive numbers $\varepsilon_n \to 0$, $\delta_n \to 0$ such that $I_{\delta_n}(u_{\varepsilon_n,\delta_n,\theta}) \ge \theta$. Let us write $u_n = u_{\varepsilon_n,\delta_n,\theta}$. By (3.34) up to a subsequence $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega_1)$ and strongly in $L^{p+1}(\Omega_1)$. Moreover $u_n \le w$ in D for all $n \in \mathbb{N}$.

Claim.- Consider $I_{\delta_n}(u_n) = \displaystyle\int_{\Omega_1} \frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}$, then

$$I_{\delta_n}(u_n) \to I_0(u) \quad \text{as } n \to \infty \text{ where } I_0 = \int_{\Omega_1} \frac{(u^+)^{(p-1)\frac{N}{2}}}{|x|^N}. \tag{3.37}$$

Proof of the Claim.- We have that

1. $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \to \dfrac{(u^+)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta)^{\frac{N}{2}}}$ pointwise in Ω_1.

2. $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}$ is uniformly bounded in $\Omega_1 \setminus D$.

3. $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le \dfrac{g_{\delta_n}(w+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C\dfrac{(w+\delta_n)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta_n)^{\frac{N}{2}}}$ in D

Since the uniform local supersolution satisfies $w \le C|x|$, we find that

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C\frac{(|x|+\delta_n)^{(p-1)\frac{N}{2}}}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C(|x|+\delta_n^{1/2})^{(p-1)\frac{N}{2}-N}.$$

If $(p-1)\frac{N}{2}-N \ge 0$ then this quantity is uniformly bounded and if $(p-1)\frac{N}{2}-N < 0$ then

$$\frac{g_{\delta_n}(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}} \le C|x|^{(p-1)\frac{N}{2}-N},$$

which is integrable. By the dominated convergence theorem we deduce the validity of (3.37). As a consequence of (3.37) and (3.35) we obtain that

$$I_0(u) \le C\theta. \tag{3.38}$$

Moreover u satisfies

$$\begin{cases} -\Delta u + (p-1)\dfrac{N}{2}\eta'_\theta(I_0(u))\dfrac{\chi_{[u>0]}}{|x|^N}u^{(p-1)\frac{N}{2}-1} \le \dfrac{u^p}{|x|^2}, & \text{in } \Omega_1 \\ u = 0 \quad \text{on } \partial\Omega_1 \end{cases} \tag{3.39}$$

Indeed, take $\varphi \in C^1(\overline{\Omega}_1)$, $\varphi \ge 0$ with $\varphi = 0$ on $\partial\Omega_1$ as a test function in (3.27) then,

$$\int_{\Omega_1} \nabla u_n \nabla\varphi + \eta'_\theta(I_{\delta_n}(u_n))\int_{\Omega_1}\frac{g'_{\delta_n}(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}\varphi = \int_{\Omega_1}\frac{(u_n^+)^p}{|x|^2+\delta_n}\varphi.$$

As before

$$\int_{\Omega_1}\frac{(u_n^+)^p}{|x|^2+\delta_n}\varphi \to \int_{\Omega_1}\frac{(u^+)^p}{|x|^2}\varphi \quad \text{as } n\to\infty.$$

Using Fatou's lemma

$$\int_{\Omega_1}\frac{\chi_{[u>0]}u^{(p-1)\frac{N}{2}-1}}{(|x|^2)^{\frac{N}{2}}}\varphi \le \int_{\Omega_1}\frac{g'_{\delta_n}(u_n)}{(|x|^2+\delta_n)^{\frac{N}{2}}}\varphi$$

and this proves (3.39).

Now, multiplying (3.39) by u^+, integrating in Ω_1 and applying the Hölder and Sobolev inequalities, yields

$$\int_{\Omega_1}|\nabla u^+|^2 \le \int_{\Omega_1}\frac{(u^+)^{p+1}}{|x|^2} \le C\int_{\Omega_1}|\nabla u^+|^2 I_0(u)^{\frac{2}{N}},$$

so that by (3.38)

$$\int_{\Omega_1}|\nabla u^+|^2 \le C\theta^{\frac{2}{N}}\int_{\Omega_1}|\nabla u^+|^2.$$

Since $\theta > 0$ is small, we conclude that $u^+ \equiv 0$ in Ω_1 and this implies that $I_0(u) = 0$. But $I_0(u) = \lim_{n\to\infty} I_{\delta_n}(u_n) \ge \theta$, which is a contradiction.

3.6 Passing to the limit: end of the proof.

If $0 < \varepsilon \le \varepsilon_0$, $0 < \delta \le \varepsilon_0$ and $0 < \theta < \min\{\theta_0, \theta_1\}$, the mountain pass solution $u_{\varepsilon,\delta,\theta}$ of Lemma 3.3 satisfies

$$\begin{cases} -\Delta u = \dfrac{(u^+)^p}{|x|^2+\delta} & \text{in } \Omega_\epsilon \\ \quad u = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{3.40}$$

Then $u_{\varepsilon,\delta,\theta} \ge 0$. By Lemma 3.3 we have that $E_{\delta,\varepsilon,\theta}(u_{\varepsilon,\delta}) \ge c_0 > 0$. Thus $u_{\varepsilon,\delta,\theta} \not\equiv 0$ and then $u_{\varepsilon,\delta,\theta} > 0$ in Ω_ε by the strong maximum principle.

Now, for fixed $0 < \varepsilon \le \varepsilon_0$ we let $\delta \to 0$. By (3.34) there is a sequence $\delta_n \to 0$ such that $u_{\varepsilon,\delta_n,\theta}$ converges weakly in $W_0^{1,2}(\Omega_\varepsilon)$, and in the C^1 norm on compact sets of $\overline{\Omega_\epsilon} \setminus \{0\}$. Since we have $u_{\varepsilon,\delta_n,\theta} \le w$ we can use dominated convergence to show that

$$\int_{\Omega_\varepsilon} \frac{u^p_{\varepsilon,\delta_n}}{|x|^2+\delta_n}\varphi \to \int_{\Omega_\varepsilon} \frac{u^p_\varepsilon}{|x|^2}\varphi \quad \text{as } \delta_n \to 0 \quad \text{for any } \varphi \in C^1(\overline{\Omega}_\varepsilon).$$

Thus $u_{\varepsilon,0,\theta}$ satisfies

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, \quad u \ge 0 \quad \text{in } \Omega_\epsilon \\ u = 0 \quad \text{on } \partial\Omega_\varepsilon \end{cases}$$

Multiplying (3.40) by $u_{\varepsilon,\delta_n,\theta}$ we find that

$$E_{\delta_n,\epsilon,\theta}(u_{\varepsilon,\delta_n,\theta}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} \frac{u^p_{\varepsilon,\delta_n,\theta}}{|x|^2+\delta_n}$$

and by dominated convergence, using $u_{\varepsilon,\delta_n,\theta} \le w$ in D, we see that

$$E_{\delta_n,\epsilon,\theta}(u_{\varepsilon,\delta_n,\theta}) \to \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} \frac{u^p_{\varepsilon,0,\theta}}{|x|^2} \quad \text{as } n \to \infty.$$

Since $E_{\delta_n,\epsilon,\theta}(u_{\varepsilon,\delta_n,\theta}) \ge c_0 > 0$ by Lemma 3.3 we deduce that $u_{\varepsilon,0,\theta} > 0$. Therefore, we have concluded the proof of Theorem 3.1.

3.7 A note on the regularizing effect of a concave term

In this section we give a remark on the existence of a solution to the problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda u^q, \quad u \ge 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.41}$$

without restriction on the shape of a smooth domain Ω with $0 \in \partial\Omega$. Precisely, we are able to prove the following result.

Theorem 3.2 *Let* $0 \le q < 1$*, and* $p > 1$*. Then there exists* $\Lambda_0 > 0$*, such that*

a) $\forall \lambda \in (0, \Lambda_0)$ *Problem* (3.41) *admits a solution* $u_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$*.*

b) If $\lambda > \Lambda_0$ *Problem* (3.41) *has no solution.*

Moreover, if $\lambda_i < \Lambda_0$*,* $i = 1, 2$*,*

1. $u_{\lambda_i} \in C^{2,\alpha}(\Omega)$*.*
2. *If* $0 < \lambda_1 < \lambda_2 \le \Lambda_0$*, then* $u_{\lambda_1} \le u_{\lambda_2}$*.*

This analysis is in some way in the spirit of the paper by Brezis and Nirenberg [17]. Instead of considering a perturbation of the domain, we consider a perturbation of the nonlinear term.

We kill *the enemy*, the trivial solution, with the concave part of the nonlinearity. We skip the details and refer to the paper [45].

4 A nonlinear elliptic fourth order equation

The mathematical description of epitaxial growth uses the function

$$u : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}, \tag{4.42}$$

which describes the height of the growing interface at the spatial point $x \in \Omega \subset \mathbb{R}^2$ at time $t \in \mathbb{R}^+$.

The macroscopic description of the growing interface is given by a partial differential equation for u which is postulated using phenomenological and symmetry arguments as in [35]

We will focus on one such equation that was derived in the context of non-equilibrium surface growth. It reads

$$u_t = 2K_1 \det\left(D^2 u\right) - K_2 \Delta^2 u + \xi(x,t), \tag{4.43}$$

where $\xi(x,t)$ is a stochastic process.

The derivation of this equation was actually geometric: it arose as a gradient flow pursuing the minimization of the functional

$$\mathcal{V}(u) = \int_\Omega \left(K_1 H + \frac{K_2}{2} H^2\right) \sqrt{1 + |\nabla u|^2}\, dx, \tag{4.44}$$

where H is the mean curvature of the graph of u, in the context of non-equilibrium statistical mechanics of surface growth. The actual terms on the right hand side of

$$u_t = 2K_1 \det\left(D^2 u\right) - K_2 \Delta^2 u + \xi(x,t)$$

are found after formally expanding the Euler-Lagrange equation corresponding to this functional for small values of $|\nabla u|$ and retaining only linear and quadratic terms. See [25].

In this survey we consider the associated stationary problem, with a null source and with Dirichlet boundary conditions. That is, we will study the fourth order problem,

$$\begin{cases} \Delta^2 u = \det\left(D^2 u\right), & x \in \Omega \subset \mathbb{R}^2, \\ \text{u=0}, \quad \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.45}$$

in order to find a non-trivial solution.

We try to find a Lagrangian $L(\nabla u, D^2 u)$ such that the critical points of the functional

$$\begin{cases} J: & W_0^{2,2}(\Omega) & \to \mathbb{R} \\ & u & \to J(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\Delta u|^2\,dx - \int_\Omega L(\nabla u, D^2 u)\,dx \end{cases} \tag{4.46}$$

are solutions to Problem (4.45).

Remark 4.1 *We will use the following result based on papers by S. Müller, [46] and by R. Coifman, P. L. Lions, Y. Meyer, and S. Semmes, [18].*

Lemma 4.1 *Let $v \in W^{2,2}(\mathbb{R}^2)$. Then,*

$$det\left(D^2 v\right),$$

$$(v_{x_1} v_{x_2 x_2})_{x_1} - (v_{x_1} v_{x_2 x_1})_{x_2}$$

and

$$(v_{x_1} v_{x_2})_{x_1 x_2} - \frac{1}{2}(v_{x_2}^2)_{x_1 x_1} - \frac{1}{2}(v_{x_1}^2)_{x_2 x_2}$$

belong to the space $\mathcal{H}^1(\mathbb{R}^2)$ and are equal in $\mathcal{D}(\mathbb{R}^2)$, where $\mathcal{H}^1(\mathbb{R}^2)$ is the Hardy space.

See for instance [53] to have details about $\mathcal{H}^1(\mathbb{R}^2)$.

All the expressions involving third derivatives are understood in the distributional sense; then the result in the lemma is highly non-trivial. It is interesting to point out that this result deeply depends on Luc Tartar and François Murat's arguments on *compactness by compensation* [55, 56] and [47] respectively.

D. C. Chang, G. Dafni, and E. M. Stein in [22] give the definition of Hardy space in a bounded domain Ω in order to have the regularity theory for the Laplacian similar to the one in $\mathbb{R}^N$.

Before we proceed looking for a variational formulation of the problem we formulate the following result.

We use the following consequence which is a byproduct of the results in [18] and in [22].

Lemma 4.2 *Let $u \in W_0^{2,2}(\Omega)$. Then we have the following identity in $\mathcal{D}'(\Omega)$*

$$Det\left(D^2 u\right) = (u_{x_1} u_{x_2})_{x_1 x_2} - \frac{1}{2}(u_{x_2}^2)_{x_1 x_1} - \frac{1}{2}(u_{x_1}^2)_{x_2 x_2}. \tag{4.47}$$

Moreover both term in the distributional identity belong to $L^1(\Omega) \cap h_r^1(\Omega)$. Here $h_r^1(\Omega)$ is the class of function restrictions of $\mathcal{H}^1(\mathbb{R}^N)$ to Ω.

To find the Lagrangian, our tools will be the distributional identity (4.47) and the fact that $C_0^\infty(\Omega)$

is dense in $W_0^{2,2}(\Omega)$. Consider $\phi \in C_0^\infty(\Omega)$ and

$$\begin{aligned}\int_\Omega \det\left(D^2u\right)\phi\,dx, &= \\ &= \int_\Omega \left[-\frac{1}{2}(u_{x_2}^2)_{x_1x_1} - \frac{1}{2}(u_{x_1}^2)_{x_2x_2} + (u_{x_1}u_{x_2})_{x_1x_2}\right]\phi\,dx \\ &= \int_\Omega \left[\frac{1}{2}\phi_{x_1}(u_{x_2}^2)_{x_1} + \frac{1}{2}\phi_{x_2}(u_{x_1}^2)_{x_2} + u_{x_1}u_{x_2}\phi_{x_1x_2}\right]dx \\ &= \frac{d}{dt}G(u+t\phi)\Big|_{t=0},\end{aligned}$$

where

$$G(u) := \int_\Omega u_{x_1}u_{x_2}u_{x_1x_2}\,dx. \tag{4.48}$$

Notice that by density we can take $\phi \in W_0^{2,2}(\Omega)$ and by direct application of Lemma 4.2 we find that the first variation of $G(u)$ on $W_0^{2,2}(\Omega)$ is

$$\frac{\delta G(u)}{\delta u} = \det\left(D^2u\right).$$

Then we will consider as *energy functional* for problem (4.45) the following one

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \int_\Omega u_{x_1}u_{x_2}u_{x_1x_2}\,dx, \tag{4.49}$$

defined in $W_0^{2,2}(\Omega)$.

As we will see J is unbounded from below and then we cannot use standard minimization results but the general theory of critical points of functionals.

Remark 4.2 Notice that this Lagrangian is not useful for other boundary conditions. Indeed, consider $\phi \in \mathcal{X} = \{\phi \in C^\infty(\Omega) \,|\, \phi(x) = 0 \text{ on } \partial\Omega\}$ and u a smooth funtion, then

$$\begin{aligned}\frac{d}{dt}G(u+t\phi)|_{t=0} &= \int_\Omega (u_{x_1}u_{x_2}\phi_{x_1x_2} + u_{x_1}\phi_{x_2}u_{x_1x_2} + \phi_{x_1}u_{x_2}u_{x_1x_2})\,dx \\ &= \int_\Omega \det\left(D^2u\right)\phi\,dx - \frac{1}{2}\int_{\partial\Omega} u_{x_1}u_{x_2}(\phi_{x_1}\nu_{x_2} + \phi_{x_2}\nu_{x_1})ds,\end{aligned}$$

and therefore for $\phi \in \mathcal{X}$ the boundary term does not cancel.

This observation justifies the dependence of the variational formulation of the problem on the boundary conditions.

4.1 The geometry of J

Notice that by Hölder and Sobolev inequalities we find the following estimate

$$J(u) \geq$$

$$\frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \left(\int_\Omega |u_{x_1x_2}|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_{x_1}|^4\,dx\right)^{\frac{1}{4}} \left(\int_\Omega |u_{x_2}|^4\,dx\right)^{\frac{1}{4}}$$

$$\frac{1}{2}\int_\Omega |\Delta u|^2\,dx - c_1\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac{3}{2}} \equiv g\left(||\Delta u||_2\right),$$

where

$$g(s) = \frac{1}{2}\,s^2 - c_1\,s^3.$$

Therefore we easily prove that the *radial lower estimate (in the Sobolev space),* given by g has a positive local maximum.

Remark 4.3 Assuming a source term $\lambda f \neq 0$, $f \in L^1(\Omega)$ then the radial lower profile becomes

$$g(s) = \frac{1}{2}\,s^2 - c_1\,s^3 - \lambda\,c_2\,||f||_1\,s.$$

The *radial profile* and the properties proven in the text show that the mountain pass geometry holds for J_λ if $\lambda > 0$ is small enough. Moreover a *local* minimum could be found. See details and other results in [26].

The Nehari manifold is defined by

$$\mathfrak{N} := \left\{v \in H_0^2(\Omega)\setminus\{0\};\ \langle J'(v), v\rangle = ||v||^2 - 3\int_\Omega v_xv_yv_{xy} = 0\right\}.$$

Consider $\mathcal{B} := \{v \in H_0^2(\Omega);\ \int_\Omega v_xv_yv_{xy} = 1\}$. It is clear that $v \in \mathfrak{N}$ if and only if $\alpha\,v \in \mathcal{B}$ for some $\alpha > 0$. Notice that not on all the straight directions arising from 0 in the phase space $H_0^2(\Omega)$ there exists an intersection with $\mathfrak{N}$.

Hence, $\mathfrak{N}$ is an unbounded manifold (of codimension 1) which separates the two regions

$$\mathfrak{N}_+ = \left\{v \in H_0^2(\Omega);\ ||v||^2 > 3\int_\Omega v_xv_yv_{xy}\right\} \text{ and } \mathfrak{N}_- = \left\{v \in H_0^2(\Omega);\ ||v||^2 < 3\int_\Omega v_xv_yv_{xy}\right\}.$$

According to this ideas if $\phi \in \mathfrak{N}_-$,

$$J(t\phi) > 0 \text{ for } t \text{ small enough and } J(s\psi) < 0 \text{ for } s \text{ large enough.}$$

Then we will try to apply the *mountain pass lemma* by A. Ambrosetti and P. H. Rabinowitz, [6].

4.2 Palais-Smale condition for J

As usual, we call $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ a Palais-Smale sequence for J to the level c if

i) $J(u_k) \to c$ as $k \to \infty$

ii) $J'(u_k) \to 0$ in $W^{-2,2}(\Omega)$.

We say that J satisfies the *local Palais-Smale condition to the level c* if each Palais-Smale sequence to the level c, $\{u_k\}_{k\in\mathbb{N}}$, admits a strongly convergent subsequence in $W_0^{2,2}(\Omega)$. We are able to prove the following compactness result.

Lemma 4.3 *Assume a bounded Palais-Smale sequence for J, that is $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ verifying*

1. $J(u_k) \to c$ as $k \to \infty$,

2. $J'(u_k) \to 0$ in $W^{-2,2}$.

Then there exists a subsequence $\{u_k\}_{k\in\mathbb{N}}$ that converges in $W_0^{2,2}(\Omega)$.

Proof. Since $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is bounded, up to passing to a subsequence, we have:

i) $u_k \rightharpoonup u$ weakly in $W_0^{2,2}(\Omega)$,

ii) $\nabla u_k \to \nabla u$ strongly in $[L^p(\Omega)]^2$ for all $p < \infty$,

iii) $u_k \to u$ uniformly in Ω.

We could write the condition $J'(u_k) \to 0$ in $W^{-2,2}$ as

$$\Delta^2 u_k = \det(D^2 u_k) + y_k, \quad \text{with } y_k \to 0 \text{ in } W^{-2,2}(\Omega). \tag{4.50}$$

Notice that multiplying (4.50) by $(u_k - u)$, we have for all fixed k

$$\int_\Omega \Delta(u_k)\Delta(u_k - u)\,dx = \int_\Omega (u_k - u)\det\left(D^2 u_k\right)\,dx + y_k(u_k - u). \tag{4.51}$$

The three terms on the right-hand side go to zero as $k \to \infty$ by the convergence properties *i*) and *iii*). Moreover adding-up to both terms of (4.51)

$$-\int_\Omega \Delta u\,\Delta(u_k - u)\,dx = o(1), \quad k \to \infty,$$

we obtain,

$$\int_\Omega |\Delta(u_k - u)|^2\,dx = \int_\Omega (u_k - u)\det\left(D^2 u_k\right)\,dx + y_k(u_k - u) - \int_\Omega \Delta u\,\Delta(u_k - u)\,dx.$$

As a consequence

$$\int_\Omega |\Delta(u_k - u)|^2\,dx \to 0 \text{ as } k \to \infty, \tag{4.52}$$

that is, J satisfies the Palais-Smale condition to the level c.

4.3 Existence of nontrivial solutions

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then there exists at least a nontrivial solution, which is a mountain pass critical point for J.*

Proof. By the estimates in Subsection 4.1, J verifies the geometrical requirements of the Mountain Pass Theorem (see [6] and [10]). Consider $v \in W_0^{2,2}(\Omega)$ with $\|\Delta v\|_2 > r_{max}$ and such that $J(v) < J(0) = 0$. We define

$$\Gamma = \{\gamma \in C\left([0,1], W_0^{2,2}(\Omega)\right) \mid \gamma(0) = 0,\ \gamma(1) = v\},$$

and the minimax value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J[\gamma(t)].$$

Applying the Ekeland variational principle (see [24]), there exists a Palais-Smale sequence to the level c, i. e., there exists $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ such that

1. $J(u_k) \to c$ as $k \to \infty$,
2. $J'(u_k) \to 0$ in $W^{-2,2}$.

Claim.- If $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is a Palais-Smale sequence for J at the level c, then there exists $C > 0$ such that $\|\Delta u_k\|_2 < C$.

Notice that if $\varphi \in C_0^\infty(\Omega)$, integrating by parts we find that

$$\begin{aligned}
\int_\Omega \varphi \det\left(D^2\varphi\right) dx &= \int_\Omega \varphi\left[(\varphi_{x_1}\varphi_{x_2x_2})_{x_1} - (\varphi_{x_1}\varphi_{x_2x_1})_{x_2}\right] dx \\
&- \int_\Omega (\varphi_{x_1})^2 \varphi_{x_2x_2}\, dx + \int_\Omega \varphi_{x_1}\varphi_{x_2x_1}\varphi_{x_2}\, dx = \\
&= 2\int_\Omega \varphi_{x_1}\varphi_{x_1x_2}\varphi_{x_2}\, dx + \int_\Omega \varphi_{x_1}\varphi_{x_2x_1}\varphi_{x_2}\, dx = \\
&= 3\int_\Omega \varphi_{x_1}\varphi_{x_2x_1}\varphi_{x_2}\, dx.
\end{aligned} \tag{4.53}$$

An argument of density proves the same identity for $\varphi \in W_0^{2,2}(\Omega)$.

Then if $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is a Palais-Smale sequence for J at the level c, and calling $\langle y_k, u_k\rangle = \langle J'(u_k), u_k\rangle$, we have

$$\begin{aligned}
c + o(1) &= J(u_k) - \frac{1}{3}\langle J'(u_k), u_k\rangle + \frac{1}{3}\langle y_k, u_k\rangle \geq \\
&\geq \left(\frac{1}{2} - \frac{1}{3}\right)\int_\Omega |\Delta u_k|^2\, dx - \frac{1}{3}\|y_k\|_{W^{-2,2}}\left(\int_\Omega |\Delta u_k|^2\, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

This inequality implies that the sequence is bounded.

By using Lemma 4.3, J satisfies the Palais-Smale condition to level c. Therefore there exists $u_* \in W_0^{2,2}(\Omega)$ such that,

1. $J(u_*) = \lim_{k\to\infty} J(u_k) = c$.

2. $J'_\lambda(u_*) = 0$, thus
$$\Delta^2 u_* = \det(D^2 u_*),\ \ u_* \in W_0^{2,2}(\Omega).$$

In other words u_* is a *mountain pass type* solution to the Problem (4.45).

Remark 4.4 Notice that we cannot directly conclude that a bounded Palais-Smale sequence gives a solution in the distributional sense; indeed, we would need the convergence property $\det(D^2 u_k) \rightharpoonup \det(D^2 u_*)$ at least in $L^1(\Omega)$. To have this property up to passing to a subsequence we need almost everywhere convergence (see the result by Jones and Journé in [34]). And the a.e. convergence for the second derivatives is only known after the proof of Lemma 4.3.

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