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Counterexamples to C^2 Boundary Estimates for a Fully Nonlinear Yamabe Problem on Manifolds with Boundary

Dedicated to Antonio Ambrosetti

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Abstract

On closed manifolds, gradient and Hessian a priori estimates for fully nonlinear Yamabe problems are known to hold. On manifolds with boundary, gradient estimates are known to hold, while Hessian estimates hold if the prescribed mean curvature is positive. Examples are given here which show that Hessian estimates can fail when the mean curvature is negative.

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1 Introduction

Fully nonlinear versions of the Yamabe problem have received much attention in recent years. Consider first a closed Riemannian manifold (M, g) of dimension $n \geq 3$. Let A_g denote the Schouten

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tensor of g, i.e.

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)}\,R_g\,g\Big),$$

where Ric_g and R_g are respectively the Ricci curvature and the scalar curvature of g. Let $\lambda(A_g) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of A_g with respect to g, and

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex symmetric cone with vertex at the origin,} \tag{1.1}$$

$$\{\lambda \in \mathbb{R}^n | \lambda_i > 0, 1 \le i \le n\} \subset \Gamma \subset \{\lambda \in \mathbb{R}^n | \lambda_1 + \ldots + \lambda_n > 0\}, \tag{1.2}$$

$$f \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma}) \text{ be concave, homogeneous of degree one, symmetric in } \lambda_i, \tag{1.3}$$

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma; \quad f_{\lambda_i} > 0 \text{ in } \Gamma\ \forall 1 \le i \le n. \tag{1.4}$$

Problem 1.1 *Let (f, Γ) satisfy (1.1)-(1.4), and let (M, g) be a compact, smooth Riemannian manifold of dimension $n \ge 3$ satisfying $\lambda(A_g) \in \Gamma$ on M. Is there a smooth positive function u on M such that $\hat{g} = u^{\frac{4}{n-2}} g$ satisfies*

$$f(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma, \quad \text{on } M? \tag{1.5}$$

See also [9, 16] for other variants.

Equation (1.5) is a second order fully nonlinear elliptic equation of u. For $1 \le k \le n$, let $\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, denote the k-th elementary symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n | \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n | \lambda_1, \cdots, \lambda_n > 0\}$. Then $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies (1.1)-(1.4). When $(f, \Gamma) = (\sigma_1, \Gamma_1)$, problem (1.5) is the Yamabe problem in the so-called positive case.

Fully nonlinear elliptic equations involving $f(\lambda(\nabla^2 u))$ were investigated in the classical paper of Caffarelli, Nirenberg and Spruck [3]. Fully nonlinear elliptic equations involving the Schouten tensor and applications to geometry and topology have been studied extensively in and after the pioneering works of Viaclovsky [35, 36, 38] and Chang, Gursky and Yang [5, 6, 7, 8]. Equation (1.5) for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ was first introduced and studied in [35]. For $(f, \Gamma) = (\sigma_2^{1/2}, \Gamma_2)$ in dimension $n = 4$, equation (1.5) was solved in [6]. Extensions, as well as developments of new methods, have been made by Ge and Wang [15], Guan and Wang [17], Gursky and Viaclovsky [19, 20], Li and Li [26, 28], Sheng, Trudinger and Wang [32], Trudinger and Wang [33, 34], among others.

Let us turn to the case of a compact Riemannian manifold (M, g) of dimension $n \ge 3$ with boundary $\partial M \neq \emptyset$. Let h_g denote the mean curvature of ∂M with respect to the inner normal (so that the mean curvature of the boundary of a Euclidean ball is positive). Instead of Problem 1.1, one considers the following.

Problem 1.2 *Let (f, Γ) satisfy (1.1)-(1.4), and let (M, g) be a compact, smooth Riemannian manifold of dimension $n \ge 3$ with boundary ∂M whose connected components are $N_1, \ldots, N_m$. Assume that $\lambda(A_g) \in \Gamma$ on M and $h_g \ge 0$ on ∂M. For given constants $c_1, \ldots, c_m$, is there a smooth positive function u on M such that $\hat{g} = u^{\frac{4}{n-2}} g$ satisfies*

$$f(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma \text{ on } M, \text{ and } h_{\hat{g}} = c_k \text{ on } N_k, k = 1, \ldots m? \tag{1.6}$$

For $(f, \Gamma) = (\sigma_1, \Gamma_1)$, equation (1.6) was studied by many authors [1, 2, 12, 13, 14, 21, 22]. It was proved, when $c_1 = \cdots = c_m = c \in \mathbb{R}$, in [21] and [22] that the problem (1.6) is solvable for $(f, \Gamma) = (\sigma_1, \Gamma_1)$ under one of the following hypotheses:

(i) $n \geq 3$, (M, g) is locally conformally flat, ∂M is umbilic;

(ii) $n \geq 5$, ∂M is not totally umbilic.

(Recall that a hypersurface is umbilic if its second fundamental form is a multiple of the metric.) Note that the case $c = 0$ was proved earlier in [13] and [14], where other cases were also studied.

For more general (f, Γ), equation (1.6) was studied in [11, 24, 25, 29, 31]. Jin, Li and Li showed in [25] that if M has umbilic boundary and is locally conformally flat near its boundary, then (1.6) is solvable when $\Gamma \subset \Gamma_j$ for some $j > \frac{n}{2}$, $c_k \geq 0$. In the same paper, they showed that the requirement $\Gamma \subset \Gamma_j$ for some $j > \frac{n}{2}$ can be relaxed if $c_1 = \ldots = c_m = 0$. A similar statement was proved independently by Chen [11]. In [31], the authors showed that if M is locally conformally flat and has umbilic boundary, then (1.6) is solvable whenever $c_k \geq 0$.

One of the key ingredients in [31] is a C^0 estimate which holds regardless of the signs of the c_k's. (The concavity assumption on f is also not needed in this estimate.) The role of the non-negativity assumption on the c_k's is to ensure a second derivative estimate at the boundary, see [25]. (Note that once a first and second derivative estimate is secured, an existence statement can be reached using a degree theory argument.) Since this is of special relevance to the present paper, let us discuss the issue in some details.

Local first and second derivative estimates are well studied in the literature. Under the assumption that $0 < u < b$ for some positive constant b (this is provided for example by the above mentioned C^0 estimate), local interior and boundary gradient estimates (for $\ln u$) were established in [30] for (f, Γ) without the need of assuming f to be concave. Different proofs for local interior gradient estimates were given in [10, 30, 39] for concave f and in [18] for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$. A different proof for local boundary gradient estimates is given in [11] under additional assumptions that f is concave, (M, g) is a locally conformally flat manifold with umbilic boundary, $c_k \geq 0$, and $\Gamma \subset \Gamma_2$ if $c_k > 0$. Under a much stronger assumption that $a < u < b$ for some positive constants a and b, local interior and boundary gradient estimates were established in [27] and [25], respectively.

Under the assumption $a < u < b$ for some positive a and b, local interior second derivative estimates were proved in [18] for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ and were extended in [26] to the general case. Local boundary second derivative estimates on locally conformally flat manifolds with umbilic boundary were established in [25] for $c_k \geq 0$. Similar results were obtained independently by different methods in [11]. The locally conformally flat assumption is unnecessary; see [25] for $(f, \Gamma) = (\sigma_n^{1/n}, \Gamma_n)$ and [24] for general (f, Γ). See also [23] for estimates for equations which are more general than (1.6).

As a side remark, we note that the local boundary second derivative estimate in [25] was derived in a more general setting where $h_g = c(x)$ with $c(x)$ being either a smooth positive function or identically zero. It is an open problem to understand whether such estimate holds for $c(x) \geq 0$.

From the foregoing discussion, it is natural to ask whether boundary second derivative estimates hold for $c_k < 0$, at least in the case where M is a locally conformally flat with umbilic boundary. For $(f, \Gamma) = (\sigma_1, \Gamma_1)$, this question is trivial – equation (1.6) is semilinear and all derivative estimates follows from an L^∞ estimate. For general (f, Γ), the answer to the above question is, somewhat surprisingly, negative. We proved in the preprint [31] that local boundary C^2 estimates for (1.6) can fail for $c_k < 0$.

In proving a local second derivative estimate, the assumption that $\lambda(A_g) \in \bar{\Gamma}$ on M and $h_g \geq 0$ on ∂M is not important, since it can always be enforced locally by replacing g by an appropriate conformal metric. One thus ponders whether it is possible to prove a "global" second derivative estimate under the above assumption. This turns out to be not possible, as announced in [31].

The aim of the current paper is to present those counterexamples to boundary C^2 estimates for (1.6). The counterexamples we constructed are radial solutions on flat annuli. Radial solutions to (1.8) were studied in detail in [4].

For $R > 1$, let A_R denote the closed annulus

$$A_R = \{x \in \mathbb{R}^n : 1 \le |x| \le R\} = \bar{B}_R \setminus B_1,$$

and g be the flat metric on $\mathbb{R}^n$. The Schouten tensor of the metric $u^{\frac{4}{n-2}} g$ is

$$A_{u^{\frac{4}{n-2}} g} = u^{\frac{4}{n-2}} A^u$$

where A^u is the conformal Hessian of u,

$$A^u = \frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Big[-\nabla^2 u + \frac{n}{n-2} u^{-1} du \otimes du - \frac{1}{n-2} |\nabla u|^2 g \Big].$$

Equation (1.6) can be rewritten as

$$\begin{cases} \sigma_k(\lambda(A^u)) = 1 \text{ in } A_R, \\ \lambda(A^u) \in \Gamma_k \text{ and } u > 0 \text{ in } A_R, \\ \frac{\partial u}{\partial r} + \frac{n-2}{2} u = -c_1 \frac{n-2}{2} u^{\frac{n}{n-2}} \text{ on } \partial B_1, \\ \frac{\partial u}{\partial r} + \frac{n-2}{2R} u = c_2 \frac{n-2}{2R} u^{\frac{n}{n-2}} \text{ on } \partial B_R. \end{cases} \tag{1.7}$$

We note that the flat metric g does not satisfy the condition that $\lambda(A_g) \in \Gamma_k$ in A_R and $h_g \ge 0$ on ∂A_R. For $1 \le k < \frac{n}{2}$, this can be by-passed since the standard cylindrical metric $g_{\rm cyl}$ on A_R, which is conformal to g, satisfies $\lambda(A_{g_{\rm cyl}}) \in \Gamma_k$ in A_R and $h_{g_{\rm cyl}} = 0$ on ∂A_R. For $k \ge \frac{n}{2}$, g has no conformal metric $\tilde{g}$ satisfying $\lambda(A_{\tilde{g}}) \in \Gamma_k$ in A_R and $h_{\tilde{g}} \ge 0$ on ∂A_R. The existence of such a metric would imply, in view of a result in [31], that A_R is a quotient of the standard half-sphere, which is absurd.

Theorem 1.1 *For any $2 \le k \le n$ and $c < 0$, there exist $C_0 > 0$, $R_0 > 1$ and a family $\{u_j\} \subset C^\infty(A_{R_0})$ satisfying*

$$\begin{cases} \sigma_k(\lambda(A^{u_j})) = 1 \textit{ in } A_{R_0}, \\ u_j > 0 \textit{ and } \lambda(A^{u_j}) \in \Gamma_k \textit{ in } A_{R_0}, \\ \frac{\partial u_j}{\partial r} + \frac{n-2}{2} u_j = -\frac{n-2}{2} c\, u_j^{\frac{n}{n-2}} \textit{ on } \partial B_1, \end{cases} \tag{1.8}$$

and

$$|u_j| + |u_j^{-1}| + |Du_j| \le C_0 \textit{ in } A_{R_0},$$

such that

$$\lim_{j\to\infty} \inf_{\partial B_1} |D^2 u_j| = \infty.$$

Our next result gives a counterexample for C^2 boundary estimate under a 'global' setting. Recall that the cylindrical metric $g_{\rm cyl}$ is conformal to the flat metric g, and satisfies the condition $\lambda(A_{g_{\rm cyl}}) \in \Gamma_k$ in A_R (for any $2 \le k < \frac{n}{2}$) and $h_{g_{\rm cyl}} = 0$ on ∂A_R.

Theorem 1.2 *For any $2 \le k \le n$ and $R > 1$, there exist $c_* = c_*(n,k,R) < 0$, a sequence $c_j > c_*$, $c_j \to c_*$ and a family of $\{u_j\} \subset C^\infty(A_R)$ satisfying*

$$\begin{cases} \sigma_k(\lambda(A^{u_j})) = 1 \text{ in } A_R, \\ u_j > 0 \text{ and } \lambda(A^{u_j}) \in \Gamma_k \text{ in } A_R, \\ \frac{\partial u_j}{\partial r} + \frac{n-2}{2} u_j = -\frac{n-2}{2} c_j u_j^{\frac{n}{n-2}} \text{ on } \partial B_1, \\ \frac{\partial u_j}{\partial r} + \frac{n-2}{2R} u_j = 0 \text{ on } \partial B_R, \end{cases} \tag{1.9}$$

and

$$|u_j| + |u_j^{-1}| + |Du_j| \le C_0 \text{ in } A_R,$$

such that

$$\lim_{j\to\infty} \sup_{B_R \setminus B_1} |D^2 u_j| = \infty.$$

Moreover, for fixed n and k, c_* is a continuous, strictly decreasing function of R and satisfies

$$\lim_{R\to 1} c_* = 0 \text{ and } \lim_{R\to+\infty} c_* = -\infty.$$

The rest of the paper is structured as follows. In Section 2, we begin with some preliminaries about ODE solutions of (1.7). We then give a short proof of Theorem 1.1 in Section 3. In Section 4, we give a second proof of Theorem 1.1 which generalizes to a proof of Theorem 1.2.

2 Preliminaries

We start with a simple fact from linear algebra, whose proof we omit.

Lemma 2.1 *If $M = \mu\, x \otimes x + \nu I$ then M has exactly two real eigenvalues: $\mu |x|^2 + \nu$, which is simple, and ν, which is of order $n - 1$.*

Let u be a radial function, i.e. $u(x) = u(|x|)$. Following the notation in [4], let

$$t = \ln r \text{ and } \xi = -\frac{2}{n-2} \ln u - \ln r.$$

A straightforward computation using Lemma 2.1 shows that the eigenvalues of A^u are

$$\lambda_1 = e^{2\xi}[\ddot{\xi} - \frac{1}{2}(1 - \dot{\xi}^2)] \qquad \text{and} \qquad \lambda_2 = \ldots = \lambda_n = e^{2\xi}(1 - \dot{\xi}^2).$$

Here and below, the "dot" represents differentiation in t. Therefore, if we set

$$\Theta := 2^{k-1} \binom{n-1}{k-1}^{-1},$$

then $\sigma_k(A^u) = 1$ if and only if

$$e^{2k\xi}(1 - \dot{\xi}^2)^{k-1}\Big[\ddot{\xi} + \frac{n-2k}{2k}(1 - \dot{\xi}^2)\Big] = \Theta. \tag{2.10}$$

As an example, we note that the function

$$u(x) = 2^{\frac{n-2}{4}} \binom{n}{k}^{\frac{n-2}{4k}} \Big(\frac{1}{1+|x|^2}\Big)^{\frac{n-2}{2}}$$

satisfies $\sigma_k(A^u) = 1$ in $\mathbb{R}^n$. Its corresponding ξ is given by

$$\xi(t) = \ln\cosh t + \frac{1}{2}\ln 2 - \frac{1}{2k}\ln\binom{n}{k}. \tag{2.11}$$

3 First proof of Theorem 1.1

Fix $c < 0$. Let $t = \ln r$ and $\xi_j = -\frac{2}{n-2}\ln u_j - \ln r$. For a radial function u_j, (1.8) becomes

$$\begin{cases} e^{2k\xi_j}(1-\dot\xi_j^2)^{k-1}[\ddot\xi_j + \frac{n-2k}{2k}(1-\dot\xi_j^2)] = \Theta, & 0 \le t \le \ln R_0, \\ -1 < \dot\xi_j < 1, \qquad 0 \le t \le \ln R_0, \\ e^{\xi_j(0)}\,\dot\xi_j(0) = c. \end{cases} \tag{3.12}$$

We will consider small constants $0 < \epsilon < \delta < \frac{1}{2}$ whose values will be specified later. For any such ϵ and δ, there is clearly a smooth function $\xi(t) \equiv \xi(t;\epsilon,\delta)$ satisfying (2.10) near $t = 0$ and

$$\xi(0) = \epsilon + \ln|c|, \qquad \dot\xi(0) = -e^{-\epsilon} = ce^{-\xi(0)}. \tag{3.13}$$

(Here we have used $1 - \dot\xi(0)^2 > 0$.) We first require that δ is small enough so that the following argument goes through (note that $0 < \epsilon < \delta$)

$$\begin{aligned} \ddot\xi(0) &= \Theta e^{-2k\xi(0)}(1-\dot\xi(0)^2)^{1-k} - \frac{n-2k}{2k}(1-\dot\xi(0)^2) \\ &= \Theta e^{-2k\ln|c|}[1+O(\epsilon)](2\epsilon + O(\epsilon^2))^{1-k} - \frac{n-2k}{2k}(2\epsilon + O(\epsilon^2)) \\ &= \frac{\Theta}{|c|^{2k}}(2\epsilon)^{1-k}[1+O(\epsilon)] \ge \frac{\Theta}{2|c|^{2k}}(2\epsilon)^{1-k} > 0. \end{aligned}$$

Therefore $\dot\xi$ is strictly increasing in t for small t.

We will only consider those values of ϵ satisfying $-e^{-\epsilon} < -1 + \frac{\delta}{2}$. It follows that for small positive t,

$$-e^{-\epsilon} < \dot\xi(t) < -1 + \delta, \tag{3.14}$$

$$\ddot\xi(t) > 0. \tag{3.15}$$

For each such ϵ, we let $(0, T(\epsilon,\delta))$, $0 < T(\epsilon,\delta) \le \infty$, be the largest open interval on which (2.10), (3.13)-(3.15) hold.

We note that, in $[0, T(\epsilon,\delta))$, one has

$$0 < (1 - (-e^{-\epsilon})^2) \le (1 - \dot\xi^2) \le 1 - (-1+\delta)^2) \le 2\delta, \tag{3.16}$$

$$\xi \le \xi(0) \le \ln|c|. \tag{3.17}$$

Thus, in view of (2.10), we have for all δ sufficiently small that

$$\ddot{\xi} \geq \Theta e^{-2k\ln|c|}(2\delta)^{1-k} - \frac{|n-2k|}{2k}(2\delta) \geq \frac{\Theta}{2}e^{-2k\ln|c|}(2\delta)^{1-k} > 0. \tag{3.18}$$

We now fix δ. By (3.14), (3.18) and the mean value theorem,

$$\delta \geq \dot{\xi}(t) - \dot{\xi}(0) \geq \frac{\Theta}{2}e^{-2k\ln|c|}(2\delta)^{1-k}\, t \text{ for } 0 < t < T(\epsilon,\delta).$$

It follows that $T(\epsilon,\delta)$ is finite and

$$T(\epsilon,\delta) \leq C(n,k,|c|,\delta), \tag{3.19}$$

where here and below $C(n,k,|c|,\delta)$ denotes some positive constant independent of ϵ. Moreover, (3.16)-(3.18) hold in $[0, T(\epsilon,\delta)]$.

We next show that

$$T(\epsilon,\delta) \geq \frac{1}{C(n,k,|c|,\delta)} > 0. \tag{3.20}$$

Evidently, this implies $\liminf_{\epsilon\to 0} T(\epsilon,\delta) > 0$, which proves the assertion by virtue of (3.12), (3.14), (3.17) and (3.19).

If $T(\epsilon,\delta) \geq 1$, (3.20) holds. Otherwise, $T(\epsilon,\delta) < 1$ and, in view of (3.14),(3.18), and the definition of $T(\epsilon,\delta)$, $\dot{\xi}(T(\epsilon,\delta)) = -1+\delta$. By (3.18) and $\dot{\xi}(0) < -1 + \frac{\delta}{2} < -1+\delta = \dot{\xi}(T(\epsilon,\delta))$, there exists some $\widehat{T} \in (0, T(\epsilon,\delta))$ such that

$$\dot{\xi}(\widehat{T}) = -1 + \frac{\delta}{2}, \quad -1 + \frac{\delta}{2} \leq \dot{\xi} \leq -1+\delta \text{ on } [\widehat{T}, T(\epsilon,\delta)].$$

It is then easy to see from (2.10) that

$$|\ddot{\xi}| \leq C(n,k,|c|,\delta) \text{ on } [\widehat{T}, T(\epsilon,\delta)].$$

It follows from the mean value theorem that

$$\frac{\delta}{2} = |\dot{\xi}(\widehat{T}) - \dot{\xi}(T(\epsilon,\delta)| \leq C(n,k,|c|,\delta)(T(\epsilon,\delta) - \widehat{T}) \leq C(n,k,|c|,\delta)T(\epsilon,\delta),$$

from which we deduce (3.20). □

4 Second proof of Theorem 1.1 and proof of Theorem 1.2

To proof Theorem 1.2 we need some 'global' properties of solution to (2.10). In [4] (see also [37]), it was shown that (2.10) is non-dissipative and the Hamiltonian

$$H(\xi,\dot{\xi}) = e^{(2k-n)\xi}\,(1-\dot{\xi}^2)^k - \frac{2k\Theta}{n}e^{-n\xi}$$

is constant along solutions of (2.10). To help visualizing the solution, we present in Figure 1 a phase portrait for the case $n = 7$, $k = 2$. For a complete catalog of the phase portraits of this equation, we

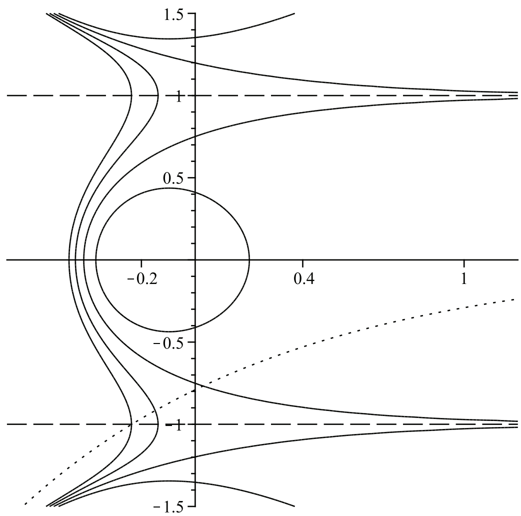

Figure 1: Integral curves for $n = 7$ and $k = 2$.

refer the readers to [4]. In the displayed figure, solid curves represent level curves of H. The strip bounded by the dashed lines represents the region where solutions satisfy $\lambda(A^u) \in \Gamma_k$. The level curve $H = 0$ is asymptotic to the line $\dot{\xi} = \pm 1$ and has three connected components. The component lying inside the strip corresponds (2.11) and its translations, i.e. standard bubbles on $\mathbb{R}^n$. The dotted curve represents the initial condition.

The idea behind our construction is as follows. We would like force the solution to escape the cone Γ_k through the boundary $\dot{\xi} = \pm 1$. Further inspection shows that this can be done through the line $\dot{\xi} = -1$ with boundary data satisfying $c < 0$ (i.e. negative constant mean curvature). The rest of the argument is to verify this statement.

The location of points in the $(\xi, \dot{\xi})$-plane satisfying the boundary condition in (1.8) and (1.9) will play an important role in our analysis. Such points lie along level sets of $e^{\xi}\,\dot{\xi}$. We continue with a brief analysis of the location of such point(s) on a fixed level curve of H.

For $c < 0$, let

$$h_c := H(\ln|c|, -1) = -\frac{2k\Theta}{n}|c|^{-n} < 0.$$

Fix some $h \in [h_c, 0]$ for the moment and consider the equation

$$H(\xi, c\, e^{-\xi}) = h, \tag{4.21}$$

which is equivalent to

$$G(c, h; \xi) := (e^{2\xi} - c^2)^k + |h|\, e^{n\xi} = \frac{2k\Theta}{n}. \tag{4.22}$$

For $\xi \in [\ln|c|, \infty)$, we have

$$\partial_\xi G(c, h; \xi) = 2ke^{2\xi}(e^{2\xi} - c^2)^{k-1} + n|h|e^{n\xi} \geq 0, \tag{4.23}$$

where equality holds if and only if $h = 0$ and $\xi = \ln|c|$. In addition,

$$G(c, h; \ln|c|) = |h||c|^n \le |h_c||c|^n = \frac{2k\Theta}{n}, \tag{4.24}$$

and

$$G(c, h; \xi) > \frac{2k\Theta}{n} \quad \text{if } e^{2\xi} > c^2 + \Big(\frac{2k\Theta}{n}\Big)^{1/k} =: e^{2\beta^*(c)}.$$

Thus, for each $h \in [h_c, 0]$, (4.22) has a unique solution $\beta(c, h) \in [\ln|c|, \beta^*(c)]$, i.e.

$$G(c, h; \beta(c, h)) = \frac{2k\Theta}{n}. \tag{4.25}$$

We list here some further properties of β.

(i) By (4.23) and the implicit function theorem, β is smooth in $\{c < 0, h_c \le h < 0\}$. It is also easy to see that $\beta(c, h)$ is continuous in $\{c \le 0, h_c \le h < 0\}$, after defining $\beta(0, h)$ to be the unique solution of

$$e^{2k\beta(0,h)} + |h|e^{n\beta(0,h)} = \frac{2k\Theta}{n}.$$

(ii) As equality in (4.24) holds if and only if $h = h_c$, we have

$$\beta(c, h) > \beta(c, h_c) = \ln|c| \text{ for } h_c < h < 0, \tag{4.26}$$

and, since $c < 0$,

$$0 > c\,e^{-\beta(c,h)} > c\,e^{-\ln|c|} = -1 \text{ for } h_c < h < 0. \tag{4.27}$$

(iii) By the equivalence of (4.21) and (4.22), we have, for $c \le 0$,

$$H(\beta(c, h), ce^{-\beta(c,h)}) = h \quad \text{for } h \in [h_c, 0]. \tag{4.28}$$

(iv) Differentiating (4.25) with respect to c, we get

$$\begin{aligned} 0 &= \partial_c G(c, h; \beta(c, h)) + \partial_\xi G(c, h; \beta(c, h))\,\partial_c\beta(c, h) \\ &= -2k\,c(e^{2\beta(c,h)} - c^2)^{k-1} + \partial_\xi G(c, h; \beta(c, h))\partial_c\beta(c, h). \end{aligned}$$

Hence, in view of (4.23) and (4.26), we infer for $h_c < h < 0$ and $c < 0$ that,

$$\partial_c\beta(c, h) = \frac{2k\,c(e^{2\beta(c,h)} - c^2)^{k-1}}{\partial_\xi G(c, h; \beta(c, h))} < 0, \tag{4.29}$$

(v) Similarly, differentiating (4.25) in h leads to, in view of (4.23),

$$\partial_h\beta(c, h) = \frac{e^{n\beta(c,h)}}{\partial_\xi G(c, h; \beta(c, h))} > 0 \quad \text{for } h_c \le h < 0,\ c < 0. \tag{4.30}$$

We next consider solutions of (2.10) which satisfy the desired boundary condition at $t = 0$ and "belong" to a given level curve of H. For $c < 0$ and $h \in (h_c, 0)$, consider the initial value problem

$$\begin{cases} H(\xi, \dot\xi) &= h, \\ \xi(0) &= \beta(c,h), \quad \dot\xi(0) = c e^{-\beta(c,h)} \in (-1,0), \\ -1 &< \dot\xi < 0. \end{cases} \tag{4.31}$$

Because of (4.27), (4.28) and the explicit form of H, (4.31) has a unique solution for small t. Let $\xi(c,h;t)$ denote the solution. Note that (4.31) implies $e^{\xi(c,h;0)}\,\dot\xi(c,h;0) = c$.

Let $(0, T(c,h))$ be the maximal interval on which (4.31) has a (unique) solution. Then $0 < T(c,h) \le \infty$.

Lemma 4.1 *The function T has the following properties.*

(a) T is finite and positive in $\{c < 0, h_c < h < 0\}$.

(b) T is smooth in $\{c < 0, h_c < h < 0\}$ and, for $c < 0$ and $h_c < h < 0$,

$$\partial_c T(c,h) < 0 \text{ and } \partial_h T(c,h) > 0.$$

(c) T extends to a (finite) positive continuous function on $\{c < 0, h_c \le h < 0\}$.

(d) The extended value $T(c,h_c)$ is strictly decreasing in c and satisfies

$$\lim_{c\to 0} T(c,h_c) = 0 \text{ and } \lim_{c\to -\infty} T(c,h_c) = +\infty. \tag{4.32}$$

Proof. (a): We first show that T is finite. Using the expression for H and (4.31), we find

$$(1 - \dot\xi(c,h;t)^2)^k = e^{-2k\xi(c,h;t)}\Big(- |h|\, e^{n\xi(c,h;t)} + \frac{2k\Theta}{n}\Big) =: \Psi(h, \xi(c,h;t)). \tag{4.33}$$

We know that

$$-1 < \dot\xi(c,h;t) < 0 \text{ and } \xi(c,h;\cdot) \text{ is strictly decreasing in } (0, T(c,h)). \tag{4.34}$$

Differentiating (4.33) in t gives

$$-2k\dot\xi\ddot\xi(1 - \dot\xi^2)^{k-1} = -2k\Psi(h,\xi)\dot\xi - n|h|e^{(n-2k)\xi}\dot\xi > 0 \quad \text{in } (0, T(c,h)).$$

Hence

$$\ddot\xi(c,h;t) > 0 \text{ and } \dot\xi(c,h;\cdot) \text{ is strictly increasing in } (0, T(c,h)). \tag{4.35}$$

Set

$$A := \lim_{t\to T(c,h)^-} \xi(c,h;t) < \xi(c,h;0) = \beta(c,h),$$
$$B := \lim_{t\to T(c,h)^-} \dot\xi(c,h;t) \in (-1, 0].$$

By continuity, we have $H(A,B) = h$. Hence, by the maximality of $T(c,h)$, one of the following two cases must occur:

(i) $T(c,h) = \infty$,

(ii) $T(c,h) < \infty$ and $B = 0$.

To finish (a), we need to show that case (i) cannot hold.

By (4.34) and (4.35), for any $\bar t < T(c,h)$, the map $\xi(c,h;\cdot)$ is an (orientation-reversing) diffeomorphism between $[0,\bar t]$ and $[\underline{a} := \xi(c,h;\bar t), \beta(c,h)]$, and, in light of (4.33),

$$\dot\xi(c,h;t) = -\sqrt{1 - \Psi(h,\xi(c,h;t))^{1/k}} \in (-1,0) \text{ for } t \in [0,\bar t]. \tag{4.36}$$

Let $\tau(c,h;\cdot)$ denote the inverse function of $\xi(c,h;\cdot)$. Then

$$\tau_\xi(c,h;\xi) = -\frac{1}{\sqrt{1-\Psi(h,\xi)^{1/k}}},$$

which implies

$$\bar t = -\int_{\underline{a}}^{\beta(c,h)} \tau_\xi(c,h;s)\,ds = \int_{\underline{a}}^{\beta(c,h)} \frac{1}{\sqrt{1-\Psi(h,s)^{1/k}}}\,ds.$$

Letting $\bar t \to T(c,h)$, we get

$$T(c,h) = \int_A^{\beta(c,h)} \frac{1}{\sqrt{1-\Psi(h,s)^{1/k}}}\,ds. \tag{4.37}$$

Note that, by (4.36), $\Psi(h,s) \in (0,1)$ for $s \in (A, \beta(c,h)]$.

From the above, if case (i) occurs, i.e. $T(c,h) = \infty$, we must have

$$\Psi(h,A) = 1,$$

which is equivalent to $G(0,h;A) = \frac{2k\Theta}{n}$, i.e. $A = \beta(0,h)$. On the other hand, for $h < 0$ and $\xi \in [\beta(0,h), \beta(c,h)]$, we estimate using (4.33),

$$\partial_\xi \Psi(h,\xi) = -n\,|h|\,e^{(n-2k)\xi} - 2k\,\Psi(h,\xi). \tag{4.38}$$

Hence $\partial_\xi \Psi(h,A) < 0$ and there is some $\delta > 0$ such that

$$\sqrt{1-\Psi(h,s)^{1/k}} = \delta\,\sqrt{s - \beta(0,h)} + o(|s-\beta(0,h)|) \text{ for } s \text{ close to } A.$$

This implies that the integral on the right hand side of (4.37) converges, contradicting our assumption that $T(c,h) = \infty$.

We have shown that case (i) cannot occur. Hence (ii) takes place, i.e.

$$T(c,h) \text{ is finite and } B = 0. \tag{4.39}$$

As $H(A,B) = h$, this implies that $G(0,h;A) = \frac{2k\Theta}{n}$ and so $A = \beta(0,h)$. Hence, (4.37) gives

$$T(c,h) = \int_{\beta(0,h)}^{\beta(c,h)} \frac{1}{\sqrt{1-\Psi(h,s)^{1/k}}}\,ds. \tag{4.40}$$

(b) and (c): By (4.40) and (4.29), $\partial_c T < 0$.

We next show that $\partial_h T > 0$. For this, we introduce the change of variable

$$s = \beta(\lambda, h).$$

This is a valid change of variables in view of (4.29). Then (4.40) can be rewritten as

$$T(c,h) = \int_0^c \frac{1}{\sqrt{1 - \Psi(h, \beta(\lambda,h))^{1/k}}}\, \partial_\lambda \beta(\lambda,h)\, d\lambda.$$

Using (4.23), (4.25) and (4.29), we arrive at

$$T(c,h) = \int_c^0 \frac{2k\, e^{\beta(\lambda,h)}(e^{2\beta(\lambda,h)} - \lambda^2)^{k-1}}{2k\, e^{2\beta(\lambda,h)}(e^{2\beta(\lambda,h)} - \lambda^2)^{k-1} + n|h|\, e^{n\beta(\lambda,h)}}\, d\lambda. \tag{4.41}$$

Using again (4.25), we get

$$T(c,h) = \int_c^0 \Phi(\lambda, \beta(\lambda,h))\, d\lambda, \tag{4.42}$$

where

$$\Phi(\lambda, s) = \frac{e^s}{\frac{2k-n}{2k} e^{2s} + \frac{n}{2k}\lambda^2 + \frac{\Theta}{(e^{2s}-\lambda^2)^{k-1}}} =: \frac{e^s}{Q}.$$

Differentiating Φ in s we get

$$\partial_s \Phi(\lambda, s) = Q^{-2}\, e^s \Big\{\frac{n-2k}{2k} e^{2s} + \frac{n}{2k}\lambda^2 + \frac{\Theta}{(e^{2s}-\lambda^2)^{k-1}} + \frac{2(k-1)\Theta\, e^{2s}}{(e^{2s}-\lambda^2)^{k}}\Big\}.$$

Note that $\beta(\lambda,h) > \ln|\lambda|$ and, by (4.25), $\Theta > \frac{n}{2k}(e^{2\beta(\lambda,h)} - \lambda^2)^k$. We thus have

$$\begin{aligned} \partial_s \Phi(\lambda, \beta(\lambda,h)) &> Q^{-2}\, e^{\beta(\lambda,h)} \Big\{\frac{n-2k}{2k} e^{2\beta(\lambda,h)} + \frac{n}{2k}\lambda^2 + \frac{n}{2k}(e^{2\beta(\lambda,h)} - \lambda^2)\Big\} \\ &= \frac{n-k}{k} Q^{-2}\, e^{3\beta(\lambda,h)} \geq 0, \end{aligned}$$

and $Q > e^{2\beta(\lambda,h)}$. Hence, differentiating $T(c,h)$ with respect to h in (4.42), we get

$$\partial_h T(c,h) = \int_c^0 \partial_s \Phi(\lambda, \beta(\lambda,h))\, \partial_h \beta(\lambda,h)\, d\lambda > 0.$$

Here we have used (4.30). We have shown that $\partial_h T > 0$, so (b) is established. We see from the above expression of $\partial_h T$ that for c in a compact subset of $(-\infty, 0)$ and for $h > h_c$ and close to h_c, $|\partial_h T|$ is uniformly bounded. So T extends to a continuous function on $\{c < 0, h_c \leq h < 0\}$. The positivity of $T(c, h_c)$ can be seen from (4.41), since for each fixed $c \in (-\infty, 0)$, the integrand is clearly bounded below by a nonnegative function which vanishes only if $h = h_c$ and $\lambda = c$, so (c) is established.

(d) The monotonicity of $T(c, h_c)$ follows from (b). It remains to show (4.32). By (4.40),

$$T(c,h) \geq \beta(c,h) - \beta(0,h), \quad \forall\, c < 0, h_c < h < 0.$$

Also as $c \to -\infty$, one has $h_c \to 0$, and so $\lim_{c\to-\infty} \lim_{h\to h_c^+} \beta(0,h) = \frac{1}{2k}\ln\frac{2k\Theta}{n}$, while $\beta(c,h_c) = \ln|c| \to +\infty$. Thus, sending h to h_c and then c to $-\infty$, we obtain the second half of (4.32).

For the other limit, we use (4.25) to rewrite (4.41) as

$$T(c,h) = \int_c^0 \frac{2k\, e^{\beta(\lambda,h)}(e^{2\beta(\lambda,h)} - \lambda^2)^{k-1}}{k\,(e^{2\beta(\lambda,h)} + \lambda^2)(e^{2\beta(\lambda,h)} - \lambda^2)^{k-1} + (n-k)|h|\, e^{n\beta(\lambda,h)} + \frac{2k^2\Theta}{n}}\, d\lambda.$$

Notice that for a fixed $\delta > 0$ and $-\delta \le c \le \lambda \le 0$ and $h_c \le h \le 0$,

$$\beta(\lambda,h) \le \beta^*(-\delta) = \frac{1}{2}\ln\Big[\delta^2 + \big(\frac{2k\Theta}{n}\big)^{1/k}\Big].$$

We infer for $-\delta \le c \le 0$ and $h_c \le h \le 0$ that

$$T(c,h) \le \int_c^0 \frac{2k\, e^{\beta(\lambda,h)}(e^{2\beta(\lambda,h)} - \lambda^2)^{k-1}}{\frac{2k^2\Theta}{n}}\, d\lambda \le C(\delta,n,k)|c|.$$

The first half of (4.32) follows. □

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. To prove the result, it suffices to produce a family $\{\xi^j\}$ of solutions to (3.12) such that $\dot\xi^j(0)$ stays as close to -1 as one wishes. For (2.10) implies that when $\dot\xi_j$ tends to -1, $\ddot\xi_j$ goes to infinity.

For $h_c := H(\ln|c|, -1)$, select a sequence $\{h_j\} \subset (h_c, 0)$ converging to h_c from the right. Let $\xi(c,h_j;\cdot)$ be the solution to (4.31). Then for $t \in [0, T(c,h_j)]$,

$$\dot\xi(c,h_j;\cdot) \in (-1,0],$$

which implies that $\xi(c,h_j;\cdot)$ is a C^2 solution of (3.12) in $[0, T(c,h_j)]$.

On the other hand, as $h_j \to h_c$, we must have $\xi(c,h_j;0) \to \ln|c|$ and $\dot\xi(c,h_j;0) \to -1$. The assertion follows in view of Lemma 4.1(c). □

Proof of Theorem 1.2. By Lemma 4.1(c) and (d), we can select $c_* < 0$ such that

$$T(c_*, h_{c_*}) = \ln R.$$

The last assertion follows from Lemma 4.1(b)(c) and (d).

As before, assuming radial symmetry, (1.9) is equivalent to

$$\begin{cases} e^{2k\xi_j}(1-\dot\xi^2)^{k-1}[\ddot\xi_j + \frac{n-2k}{2k}(1-\dot\xi_j^2)] = \Theta, & 0 \le t \le \ln R, \\ -1 < \dot\xi < 1, \quad 0 \le t \le \ln R, \\ e^{\xi_j(0)}\,\dot\xi_j(0) = c, \\ e^{\xi_j(\ln R)}\,\dot\xi_j(\ln R) = 0, \end{cases} \tag{4.43}$$

Pick $h_j \in (h_{c_*}, 0)$ such that $h_j \to h_{c_*}$. Since T is decreasing in c and increasing in h (see Lemma 4.1(c)), we can pick $c_j > c$ such that

$$T(c_j, h_j) = \ln R.$$

The family $\{\xi_j(t) = \xi(c_j,h_j;t) : 0 \le t \le \ln R = T(c_j,h_j)\}$ satisfies (4.43) and gives rise to a family of u_j satisfying (1.9) and the required C^0 and C^1 estimates. As $j \to \infty$, $\dot\xi(c_j,h_j;0) \to -1$, (2.10) implies that $\ddot\xi(c_j,h_j;0)$ blows up, which proves the result. □

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