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On Nonuniformly Subelliptic Equations of $Q-$sub-Laplacian Type with Critical Growth in the Heisenberg Group *

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Abstract

Let $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ be the $n-$dimensional Heisenberg group, $\nabla_{\mathbb{H}^n}$ be its subelliptic gradient operator, and $\rho(\xi) = \left(|z|^4 + t^2\right)^{1/4}$ for $\xi = (z, t) \in \mathbb{H}^n$ be the distance function in $\mathbb{H}^n$. Denote $\mathbb{H} = \mathbb{H}^n$, $Q = 2n + 2$ and $Q' = Q/(Q-1)$. Let Ω be a bounded domain with smooth boundary in $\mathbb{H}$. Motivated by the Moser-Trudinger inequalities on the Heisenberg group, we study the existence of solution to a nonuniformly subelliptic equation of the form

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}(a(\xi, \nabla_H u(\xi))) = \frac{f(\xi, u(\xi))}{\rho(\xi)^\beta} + \varepsilon h(\xi) & \text{in } \Omega \\ u \in W_0^{1,Q}(\Omega) \setminus \{0\} \\ u \geq 0 \text{ in } \Omega \end{cases},$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp\left(\alpha |u|^{Q'}\right)$ when $|u| \to \infty$. In the case of $Q-$sub-Laplacian

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u\right) = \frac{f(\xi, u)}{\rho(\xi)^\beta} + \varepsilon h(\xi) & \text{in } \Omega \\ u \in W_0^{1,Q}(\Omega) \setminus \{0\} \\ u \geq 0 \text{ in } \Omega \end{cases},$$

we will apply minimax methods to obtain multiplicity of weak solutions.

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Key words. Moser-Trudinger inequality, Heisenberg group, subelliptic equations, Q-subLaplacian, Mountain-Pass theorem, Palais-Smale sequences, existence and multiplicity of solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $W_0^{1,p}(\Omega)$ ($n \geq 2$) be the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \left(\int\limits_\Omega |\nabla u|^p \, dx + \int\limits_\Omega |u|^p \, dx \right)^{1/p}.$$

Then the Sobolev embeddings say that

$$W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega) \text{ if } 1 \leq p < n$$
$$W_0^{1,p}(\Omega) \subset C^{1-\frac{n}{p}}(\Omega) \text{ if } n < p.$$

The case $p = n$ can be seen as the limit case of these embeddings and it is known that

$$W_0^{1,n}(\Omega) \subset L^q(\Omega) \text{ for } 1 \leq q < \infty.$$

However, by some easy examples, we can conclude that $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$.

It is showed by Judovich [18], Pohozaev [38] and Trudinger [43] independently that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(|t|^{n/(n-1)}\right) - 1$. Extending this result, J. Moser [37] finds the largest positive real number $\beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit $n-$ball, such that if Ω is a domain with finite $n-$measure in Euclidean $n-$space $\mathbb{R}^n$, $n \geq 2$, then there is a constant c_0 depending only on n such that

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\beta |u|^{\frac{n}{n-1}}\right) dx \leq c_0$$

for any $\beta \leq \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n \, dx \leq 1$. Moreover, this constant β_n is sharp in the meaning that if $\beta > \beta_n$, then the above inequality can no longer hold with some c_0 independent of u. Such an inequality is nowadays known as Moser-Trudinger type inequality. However, when Ω has infinite volume, the result of J. Moser is meaningless. In this case, the sharp Moser-Trudinger type inequality was obtained by B. Ruf [39] in dimension two and Y.X. Li-Ruf [29] in general dimension.

The Moser-Trudinger type inequalities has been extended to many different settings: high order derivatives by D. Adams which is now called Adams type inequalities [1, 20, 23, 35, 40, 42]; compact Riemannian manifolds without boundary by Fontana [17] (see [28]); singular Moser-Trudinger inequalities which are the combinations of the Hardy inequalities and Moser-Trudinger inequalities are established in [3, 5, 23]. It is also worthy to note that the Moser-Trudinger type inequalities play an essential role in geometric analysis and in the study of the exponential growth partial differential equations where, roughly speaking, the nonlinearity behaves like $e^{\alpha|u|^{\frac{n}{n-1}}}$ as $|u| \to \infty$. Here we mention Atkinson-Peletier [6], Carleson-Chang [8], Flucher [16], Lin [30], Adimurthi et al. [2, 3, 4, 5], Struwe [41], de Figueiredo-Miyagaki-Ruf [12], J.M. do Ó [13], de Figueiredo- do Ó-Ruf [11], Y.X. Li [26, 27], Lu-Yang [33, 34], Lam-Lu [19, 21, 22] and the references therein.

Now, let us discuss the Moser-Trudinger type inequalities on the Heisenberg group. For some notations and preliminaries about the Heisenberg group, see the next section. In the setting of the Heisenberg group, although the rearrangement argument is not available, Cohn and the second author of this paper [9] can still set up a sharp Moser-Trudinger inequality for bounded domains on the Heisenberg group:

Theorem A *Let $\mathbb{H} = \mathbb{H}^n$ be a $n-$dimensional Heisenberg group, $Q = 2n + 2$, $Q' = Q/(Q-1)$, and $\alpha_Q = Q\left(2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$. Then there exists a constant C_0 depending only on Q such that for all $\Omega \subset \mathbb{H}^n$, $|\Omega| < \infty$,*

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha_Q |u(\xi)|^{Q'}) d\xi \leq C_0 < \infty. \tag{1.1}$$

If α_Q *is replaced by any larger number, the integral in (1.1) is still finite for any* $u \in W^{1,Q}(\mathbb{H})$*, but the supremum is infinite.*

It is clear that when $|\Omega| = \infty$, Theorem A is not meaningful. In the case, we have the following version of the Moser-Trudinger type inequality (see [10]):

Theorem B *Let* α^* *be such that* $\alpha^* = \alpha_Q / c^*$*. Then for any pair* $\beta,\ \alpha$ *satisfying* $0 \le \beta < Q,\ 0 < \alpha \le \alpha^*$*, and* $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} \le 1$*, there holds*

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H})} \le 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u) \right\} < \infty \tag{1.2}$$

where

$$S_{Q-2}(\alpha, u) = \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!} |u|^{kQ/(Q-1)}.$$

When $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} > 1$*, the integral in (1.2) is still finite for any* $u \in W^{1,Q}(\mathbb{H})$*, but the supremum is infinite if further* $\frac{\alpha}{\alpha_Q} + \frac{\beta}{Q} > 1$.

Here, c^* is defined as follows: Let $u : \mathbb{H} \to \mathbb{R}$ be a nonnegative function in $W^{1,Q}(\mathbb{H})$, and u^* be the decreasing rearrangement of u, namely

$$u^*(\xi) := \sup\{s \ge 0 : \xi \in \{u > s\}^*\}$$

where

$$\{u > s\}^* = B_r = \{\xi : \rho(\xi) \le r\}$$

such that $|\{u > s\}| = |B_r|$. It is known from a result of Manfredi and V. Vera De Serio [36] that there exists a constant $c \ge 1$ depending only on Q such that

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^*|^Q \, d\xi \le c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q \, d\xi$$

for all $u \in W^{1,Q}(\mathbb{H})$. Thus we can define

$$c^* = \inf\left\{ c^{1/(Q-1)} : \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u^*|^Q \, d\xi \le c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q \, d\xi,\ u \in W^{1,Q}(\mathbb{H}) \right\} \ge 1. \tag{1.3}$$

We notice that in Theorem B, we cannot exhibit the best constant $\alpha^*(1 - \frac{\beta}{Q})$ due to the loss of the non-optimal rearrangement argument in the Heisenberg group. Nevertheless, the first two authors recently used a completely different but much simpler approach, namely a rearrangement-free argument, to set up the sharp Moser-Trudinger type inequality on Heisenberg groups in [24]. Moreover, we have developed in [25] a rearrangement-free method to establish the sharp Adams and singular Adams inequalities on high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ with arbitrary orders (including fractional orders). This extends those results in [40] and [20, 23] in full generality. The main result on sharp Moser-Trudinger inequality on the Heisenberg group proved in [24] is as follows.

Theorem C *Let* τ *be any positive real number. Then for any pair* $\beta,\ \alpha$ *satisfying* $0 \le \beta < Q$ *and* $0 < \alpha \le \alpha_Q(1 - \frac{\beta}{Q})$ *, there holds*

$$\sup_{\|u\|_{1,\tau} \le 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u) \right\} < \infty. \tag{1.4}$$

When $\alpha > \alpha_Q(1 - \frac{\beta}{Q})$*, the integral in (1.4) is still finite for any* $u \in W^{1,Q}(\mathbb{H})$*, but the supremum is infinite. Here*

$$\|u\|_{1,\tau} = \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q \right]^{1/Q}.$$

In this paper, we will prove the critical singular Moser-Trudinger inequality on bounded domains (see Lemma 4.1) and study a class of partial differential equations of exponential growth by using the Moser-Trudinger type inequalities on the Heisenberg group. More precisely, we consider the existence of nontrivial weak solutions for the nonuniformly subelliptic equations of $Q-$sub-Laplacian type of the form:

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}(a(\xi, \nabla_{\mathbb{H}} u(\xi))) = \frac{f(\xi, u(\xi))}{\rho(\xi)^{\beta}} + \varepsilon h(\xi) \quad \text{in } \Omega \\ u \in W_0^{1,Q}(\Omega) \setminus \{0\} \\ u \geq 0 \text{ in } \Omega \end{cases} \tag{NU}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{H}$,

$$|a(\xi, \tau)| \leq c_0 \left(h_0(\xi) + h_1(\xi) |\tau|^{Q-1}\right)$$

for any τ in $\mathbb{R}^{Q-2}$ and a.e. ξ in Ω, $h_0 \in L^{Q'}(\Omega)$ and $h_0 \in L^{\infty}_{loc}(\Omega)$, $0 \leq \beta < Q$, , $f : \Omega \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp\left(\alpha |u|^{Q'}\right)$ when $|u| \to \infty$, $h \in \left(W_0^{1,Q}(\Omega)\right)^*$, $h \neq 0$ and ε is a positive parameter. The main features of this class of problems are that it involves critical growth and the nonlinear operator $Q-$sub-Laplacian type. In spite of a possible failure of the Palais-Smale (PS) compactness condition, in this article we apply minimax method, in particular, the mountain-pass theorem to obtain the weak solution of (NU) in a suitable subspace of $W_0^{1,Q}(\Omega)$. Moreover, in the case of $Q-$sub-Laplacian, i.e.,

$$a(\xi, \nabla_{\mathbb{H}} u) = |\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u,$$

we will apply minimax methods, more precisely, the mountain-pass theorem combined with minimization and the Ekeland variational principle, to obtain the existence of at least two weak solutions to the nonhomogeneous problem

$$\begin{cases} -\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u\right) = \frac{f(\xi, u)}{\rho(\xi)^{\beta}} + \varepsilon h(\xi) \quad \text{in } \Omega \\ u \in W_0^{1,Q}(\Omega) \setminus \{0\} \\ u \geq 0 \text{ in } \Omega. \end{cases} . \tag{NH}$$

Our paper is organized as follows: In Section 2, we give some notations and preliminaries about the Heisenberg group. We also provide the assumptions on the nonlinearity f in this section. We will discuss the variational framework and state our main results in Section 3. In Section 4, we proof critical singular Moser-Trudinger inequality on bounded domains and also, some basic lemmas that are useful in our paper. We will investigate the existence of nontrivial solution to Eq. (NU) (Theorem 3.1) in Section 5. The last section (Section 6) is devoted to the study of multiplicity of solutions to equation (NH) (Theorems 3.2 and 3.3).

2 Preliminaries and assumptions

First, we provide some notations and preliminary results. Let $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ be the $n-$dimensional Heisenberg group. Recall that the Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x' \rangle - \langle x, y' \rangle)),$$

where $x, y, x', y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ denotes the standart inner product in $\mathbb{R}^n$. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector files

$$T = \frac{\partial}{\partial t}, \; X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \; Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \; i = 1, ..., n.$$

We will fix some notations:

$$z = (x, y) \in \mathbb{R}^{2n}, \xi = (z, t) \in \mathbb{H}^n,\ \rho(\xi) = \left(|z|^4 + t^2\right)^{1/4},$$

where $\rho(\xi)$ denotes the Heisenberg distance between ξ and the origin. Denote $\mathbb{H} = \mathbb{H}^n,\ Q = 2n+2$, $\alpha_Q = Q\sigma_Q^{1/(Q-1)},\ \sigma_Q = \int_{\rho(z,t)=1} |z|^Q\, d\mu$. We now use $|\nabla_{\mathbb{H}} u|$ to express the norm of the subelliptic gradient of the function $u : \mathbb{H} \to \mathbb{R}$:

$$|\nabla_{\mathbb{H}} u| = \left(\sum_{i=1}^{n}\left((X_i u)^2 + (Y_i u)^2\right)\right)^{1/2}.$$

Now, we will provide conditions on the nonlinearity of Eq. (NU) and (NH). Motivated by the Moser-Trudinger inequalities (Theorems A and B), we consider here the maximal growth on the nonlinear term $f(\xi, u)$ which allows us to treat Eq.(NU) and (NH) variationally in a suitable subspace of $W_0^{1,Q}(\Omega)$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(\xi, 0) = 0$ and f behaves like $\exp\left(\alpha |u|^{Q/(Q-1)}\right)$ as $|u| \to \infty$. More precisely, we assume the following growth conditions on the nonlinearity $f(\xi, u)$:

$(f1)$There exists $\alpha_0 > 0$ such that $\lim_{|s|\to\infty} \frac{|f(\xi,s)|}{\exp\left(\alpha|s|^{Q'}\right)} = 0$ uniformly on $\xi \in \Omega,\ \forall \alpha > \alpha_0$ and $\lim_{|s|\to\infty} \frac{|f(\xi,s)|}{\exp\left(\alpha|s|^{Q'}\right)} = \infty$ uniformly on $\xi \in \Omega,\ \forall \alpha < \alpha_0$.

$(f2)$ There exist constant $R_0,\ M_0 > 0$ such that for all $\xi \in \Omega$ and $s \geq R_0$,

$$F(\xi, s) = \int_0^s f(\xi, \tau) d\tau \leq M_0 f(\xi, s).$$

Since we are interested in nonnegative weak solutions, it is convenient to define

$$f(\xi, u) = 0 \text{ for all } (\xi, u) \in \Omega \times (-\infty, 0].$$

We note that conditions $(f1)$, $(f2)$ imply that:

(a) $F(\xi, s) \geq 0,\ \forall (\xi, s) \in \Omega \times \mathbb{R}$.

(b) There is a positive constant C such that

$$\forall s \geq R_0, \forall \xi \in \Omega :\ F(\xi, s) \geq C \exp\left(\frac{1}{M_0} u\right).$$

$(f3)$ There exists $p > Q$ and $s_1 > 0$ such that for all $\xi \in \Omega$ and $s > s_1$,

$$0 < pF(\xi, s) \leq s f(\xi, s).$$

Let A be a measurable function on $\Omega \times \mathbb{R}^{Q-2}$ such that $A(\xi, 0) = 0$ and $a(\xi, \tau) = \frac{\partial A(\xi,\tau)}{\partial \tau}$ is a Caratheodory function on $\Omega \times \mathbb{R}^{Q-2}$. Assume that there are positive real numbers $c_0,\ c_1,\ k_0,\ k_1$ and two nonnegative measurable functions $h_0,\ h_1$ on Ω such that $h_1 \in L^\infty_{loc}(\Omega)$, $h_0 \in L^{Q/(Q-1)}(\Omega)$, $h_1(\xi) \geq 1$ for a.e. ξ in Ω and the following condition holds:

$(A1)$ $|a(\xi, \tau)| \leq c_0\left(h_0(\xi) + h_1(\xi)|\tau|^{Q-1}\right),\ \forall \tau \in \mathbb{R}^{Q-2},\ a.e. \xi \in \Omega$

$(A2)$ $c_1 |\tau - \tau_1|^Q \leq \langle a(\xi, \tau) - a(\xi, \tau_1), \tau - \tau_1\rangle,\ \forall \tau, \tau_1 \in \mathbb{R}^{Q-2},\ a.e. \xi \in \Omega$

$(A3)$ $0 \leq a(\xi, \tau) \cdot \tau \leq QA(\xi, \tau)\ , \forall \tau \in \mathbb{R}^{Q-2},\ a.e. \xi \in \Omega$

$(A4)$ $A(\xi, \tau) \geq k_0 h_1(\xi)|\tau|^Q,\ \forall \tau \in \mathbb{R}^{Q-2},\ a.e. \xi \in \Omega.$

Then A verifies the growth condition:

$$A(\xi, \tau) \leq c_0\left(h_0(\xi)|\tau| + h_1(\xi)|\tau|^Q\right),\ \forall \tau \in \mathbb{R}^{Q-2},\ a.e. \xi \in \Omega. \tag{2.5}$$

For examples of A, we can consider $A(\xi, \tau) = h(\xi)\frac{|\tau|^Q}{Q}$ where $h \in L^\infty_{loc}(\Omega)$.

3 Variational framework and main results

We introduce some notations:

$$E = \left\{u \in W_0^{1,Q}(\Omega) : \int_\Omega h_1(\xi)\,|\nabla_{\mathbb{H}} u|^Q\,d\xi < \infty\right\}$$

$$\mathcal{M} = \lim_{k\to\infty} k\int_0^1 \exp\left(k\left(t^{Q'} - t\right)\right)dt$$

d is the radius of the largest open ball centered at 0 contained in Ω.

We notice that $\mathcal{M}$ is well-defined and is a real number greater than or equal to 2 (see [13]).

Under our conditions, we can see that E is a reflexive Banach space when endowed with the norm

$$||u|| = \left(\int_\Omega h_1(\xi)\,|\nabla_{\mathbb{H}} u|^Q\,d\xi\right)^{1/Q}.$$

Furthermore, since $h_1(\xi) \geq 1$ for a.e. ξ in Ω :

$$\lambda_1(Q) = \inf\left\{\frac{||u||^Q}{\int_\Omega \frac{|u|^Q}{\rho(\xi)^\beta}d\xi} : u \in E \setminus \{0\}\right\} > 0 \text{ for any } 0 \leq \beta < Q. \tag{3.6}$$

Now, from $(f1)$, we obtain for all $(\xi, u) \in \Omega \times \mathbb{R}$,

$$|f(\xi,u)|,\ \ |F(\xi,u)| \leq b_3 \exp\left(\alpha_1 |u|^{Q/(Q-1)}\right)$$

for some constants α_1, $b_3 > 0$. Thus, by the Moser-Trudinger type inequalities, we have $F(\xi,u) \in L^1(\Omega)$ for all $u \in W_0^{1,Q}(\Omega)$. Define the functional $E,\ T,\ J_\varepsilon : E \to \mathbb{R}$ by

$$E(u) = \int_\Omega A(\xi, \nabla_{\mathbb{H}} u)d\xi$$
$$T(u) = \int_\Omega \frac{F(\xi,u)}{\rho(\xi)^\beta}d\xi$$
$$J_\varepsilon(u) = E(u) - T(u) - \varepsilon\int_\Omega hud\xi$$

then the functional J_ε is well-defined. Moreover, J_ε is a C^1 functional on E with

$$DJ_\varepsilon(u)\,v = \int_\Omega a(\xi, \nabla_{\mathbb{H}} u)\,\nabla_{\mathbb{H}} v d\xi - \int_\Omega \frac{f(\xi,u)v}{\rho(\xi)^\beta}d\xi - \varepsilon\int_\Omega hvd\xi,\ \forall u, v \in E.$$

We next state our main results.

Theorem 3.1 *Suppose that (f1)-(f3) are satisfied. Furthermore, assume that*

$$\textit{(f4)}\quad \limsup_{s\to 0+} \frac{F(\xi,s)}{k_0\,|s|^Q} < \lambda_1(\Omega) \text{ uniformly in } \xi \in \Omega.$$

Then there exists $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$*, (NU) has a weak solution of mountain-pass type.*

Theorem 3.2 *In addition to the hypotheses in Theorem 3.1, assume that*

$$\textit{(f5)}\quad \lim_{s\to\infty} sf(\xi,s)\exp\left(-\alpha_0 |s|^{Q/(Q-1)}\right) \geq \beta_0 > \frac{(Q-\beta)^Q}{Q^{Q-1}r^{Q-\beta}}\frac{1}{\mathcal{M}\alpha_0^{Q-1}}\frac{\alpha_Q^{Q-1}}{\omega_{Q-1}}$$

uniformly on Ω*. Then, there exists* $\varepsilon_2 > 0$*, such that for each* $0 < \varepsilon < \varepsilon_2$*, problem (NH) has at least two weak solutions and one of them has a negative energy.*

In the case where the function h does not change sign, we have

Theorem 3.3 *Under the assumptions in Theorems 3.1 and 3.2, if* $h(\xi) \geq 0$ $(h(\xi) \leq 0)$ *a.e., then the solutions of problem (NH) are nonnegative (nonpositive).*

4 Some lemmas

First, we will prove the following critical singular Moser-Trudinger inequality:

Lemma 4.1 (The critical singular Moser-Trudinger inequality) *Let* $\mathbb{H} = \mathbb{H}^n$ *be a* $n-$*dimensional Heisenberg group,* $\Omega \subset \mathbb{H}^n$, $|\Omega| < \infty$, $Q = 2n+2$, $Q' = Q/(Q-1)$, $0 \le \beta < Q$, *and* $\alpha_Q = Q\sigma_Q^{1/(Q-1)}$, $\sigma_Q = \int_{\rho(z,t)=1} |z|^Q \, d\mu$. *Then there exists a constant* C_0 *depending only on* Q *and* β *such that*

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \le 1} \frac{1}{|\Omega|^{1-\frac{\beta}{Q}}} \int_\Omega \frac{\exp(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u(\xi)|^{Q'}) d\xi}{\rho(\xi)^\beta} \le C_0 < \infty.$$

If $\alpha_Q \left(1 - \frac{\beta}{Q}\right)$ *is replaced by any larger number, then the supremum is infinite.*

Recall in [9] that for $0 < \alpha < Q$, we will say that a non-negative function g on $\mathbb{H}$ is a kernel of order α if g has the form $g(\xi) = \rho(\xi)^{\alpha - Q} g(\xi^*)$ where $\xi^* = \frac{\xi}{\rho(\xi)}$ is a point on the unit sphere. We are also assuming that for every $\delta > 0$ and $0 < M < \infty$ there are constants $C(\delta, M)$ such that

$$\int_{\Sigma_\delta} \int_0^M \left| g\left(\xi^* (s\zeta^*)^{-1}\right) - g(\xi^*)\right| \frac{ds}{s} d\mu(\xi^*) \le C(\delta, M)$$

for all $\zeta^* \in \Sigma = \{\xi \in \mathbb{H} : \rho(\xi) = 1\}$, where Σ_δ is the subset of the unit sphere given by

$$\Sigma_\delta = \left\{\xi^* \in \Sigma : \delta \le g(\xi^*) \le \delta^{-1}\right\}.$$

Note that we will choose $g(\xi) = \frac{|z|^{Q-1}}{|\xi|^{2Q-2}}$ in the proof of Lemma 4.1. First, we will prove the following result:

Lemma 4.2 *Suppose* g *is an allowed kernel of order* α, $Q - p\alpha = 0$, $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$, $f \in L^p$ *and* $0 \le \beta < Q$. *Let*

$$A(g,p) = A_\alpha(g) = \frac{Q}{\int_\Sigma |g(\xi^*)|^{p'} d\mu}.$$

Here $p' = p/(p-1)$. *Then there exists a constant* C_0 *depending only on* Q *and* β *such that*

$$\frac{1}{|\Omega|^{1-\frac{\beta}{Q}}} \int_\Omega \frac{\exp\left[A(g,p)\left(1 - \frac{\beta}{Q}\right)\left(\frac{f * g(\xi)}{\|f\|_p}\right)^{p'}\right]}{\rho(\xi)^\beta} d\mu \le C_0.$$

Proof. Set

$$u(\xi) = f * g(\xi)$$
$$\phi(s) = |\Omega|^{1/p} f^*(|\Omega| e^{-s}) e^{-s/p}.$$

Then

$$\int_\Omega f(\xi)^p d\xi = \int_0^{|\Omega|} f^*(t)^p dt = \int_0^\infty \phi(s)^p ds.$$

By the Hardy-Littlewood inequality, the O'Neil's lemma, noting that with $h(\xi) = \frac{1}{\rho(\xi)^\beta}$, then $h^*(t) = \left(\frac{C_Q}{t}\right)^{\frac{\beta}{Q}}$, where C_Q is the volume of the unit ball, we have

$$
\begin{aligned}
&\int_{\Omega} \frac{\exp\left(A\left(g,p\right)\left(1-\frac{\beta}{Q}\right)|u(\xi)|^{p'}\right)}{\rho\left(\xi\right)^{\beta}} d\xi \\
&\leq C_Q^{\frac{\beta}{Q}} \int_0^{|\Omega|} \frac{e^{A(g,p)\left(1-\frac{\beta}{Q}\right)u^*(t)^{p'}}}{t^{\frac{\beta}{Q}}} dt \\
&= C_Q^{\frac{\beta}{Q}} |\Omega|^{1-\frac{\beta}{Q}} \int_0^{\infty} \exp\left[\left(1-\frac{\beta}{Q}\right) A\left(g,p\right) u^*\left(|\Omega|\, e^{-s}\right)^{p'} - \left(1-\frac{\beta}{Q}\right) s\right] ds \\
&\leq C_Q^{\frac{\beta}{Q}} |\Omega|^{1-\frac{\beta}{Q}} \times \\
&\int_0^{\infty} \frac{\exp\left[\left(1-\frac{\beta}{Q}\right)\left(p\left(|\Omega|\, e^{-s}\right)^{-\frac{1}{p'}} \int_0^{|\Omega|e^{-s}} f^*(z)dz + \int_{|\Omega|e^{-s}}^{|\Omega|} f^*(z) z^{-\frac{1}{p'}} dz\right)^{p'}\right]}{\exp\left[\left(1-\frac{\beta}{Q}\right) s\right]} ds \\
&= C_Q^{\frac{\beta}{Q}} |\Omega|^{1-\frac{\beta}{Q}} \int_0^{\infty} \frac{\exp\left[\left(1-\frac{\beta}{Q}\right)\left(p e^{s/p'} \int_s^{\infty} \phi(w) e^{-\frac{w}{p'}} dw + \int_0^{s} \phi(w)\right)^{p'}\right]}{\exp\left[\left(1-\frac{\beta}{Q}\right) s\right]} ds \\
&= C_Q^{\frac{\beta}{Q}} |\Omega|^{1-\frac{\beta}{Q}} \int_0^{\infty} \exp\left[-F_{\left(1-\frac{\beta}{Q}\right)}(s)\right] ds
\end{aligned}
$$

where

$$
\begin{aligned}
F_{\left(1-\frac{\beta}{Q}\right)}(s) &= \left(1-\frac{\beta}{Q}\right) t - \left(1-\frac{\beta}{Q}\right)\left(\int_{-\infty}^{\infty} a(s,t)\phi(s) ds\right)^{p'}, \\
a(s,t) &= \begin{cases} 1 \ \text{ for } 0 < s < t \\ p e^{(t-s)/p'} \ \text{ for } t < s < \infty \\ 0 \ \text{ for } -\infty < s \leq 0. \end{cases}
\end{aligned}
$$

Using Lemma 3.1 in [23], we get our desired result.

Proof. (of Lemma 4.1) Now, using Lemma 4.2, noting that by Theorem 1.2 in [9], we have that

$$
|u\left(\xi\right)| \leq \left(\sigma_Q\right)^{-1} |\nabla_{\mathbb{H}} u| * g\left(\xi\right),
$$

where $g\left(\xi\right) = \frac{|z|^{Q-1}}{|\xi|^{2Q-2}}$, we can derive the result of Lemma 4.1. Note that the sharpness of the constant $\alpha_Q\left(1-\frac{\beta}{Q}\right)$ comes from the test functions in [9].

Using the critical singular Moser-Trudinger inequality, we can prove the following two lemmas (see [14] and [10]):

Lemma 4.3 *For* $\kappa > 0$, $q > 0$ *and* $\|u\| \leq M = M(\beta, \kappa)$ *with* M *sufficiently small, we have*

$$\int_\Omega \frac{\exp\left(\kappa |u|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} |u|^q \, d\xi \leq C(Q, \kappa) \|u\|^q .$$

Proof. By Holder inequality and Lemma 4.1, we have

$$\begin{aligned} &\int_\Omega \frac{\exp\left(\kappa |u|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} |u|^q \, d\xi \\ &\leq \left(\int_\Omega \frac{\exp\left(\kappa r \|u\|^{Q'} \left(\frac{|u|}{\|u\|}\right)^{Q'}\right)}{\rho(\xi)^{r\beta}} d\xi \right)^{1/r} \left(\int_\Omega |u|^{r'q} \, d\xi \right)^{1/r'} \\ &\leq C(Q) \left(\int_\Omega |u|^{r'q} \, d\xi \right)^{1/r'} \end{aligned}$$

if $r > 1$ sufficiently close to 1. Here $r' = r/(r-1)$. By the Sobolev embedding, we get the result.

Lemma 4.4 *If* $u \in E$ *and* $\|u\| \leq N$ *with* N *sufficiently small* ($\kappa N^{Q'} < \alpha_Q \left(1 - \frac{\beta}{Q}\right)$)*, then*

$$\int_\Omega \frac{\exp\left(\kappa |u|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} |v| \, d\xi \leq C(Q, M, \kappa) \|v\|_s$$

for some $s > 1$.

Proof. The proof is similar to Lemma 4.3.

We also have the following lemma (for Euclidean case, see [31]):

Lemma 4.5 *Let* $\{w_k\} \subset W_0^{1,Q}(\Omega)$, $\|\nabla_{\mathbb{H}} w_k\|_{L^Q(\Omega)} \leq 1$. *If* $w_k \to w \neq 0$ *weakly and almost everywhere,* $\nabla_{\mathbb{H}} w_k \to \nabla_{\mathbb{H}} w$ *almost everywhere, then* $\frac{\exp\{\alpha |w_k|^{Q/(Q-1)}\}}{\rho(\xi)^\beta}$ *is bounded in* $L^1(\Omega)$ *for*

$$0 < \alpha < \left(1 - \frac{\beta}{Q}\right) \alpha_Q \left(1 - \|\nabla_{\mathbb{H}} w\|_{L^Q(\Omega)}^Q\right)^{-1/(Q-1)} .$$

Proof. Using Brezis-Lieb Lemma in [7], we deduce that

$$\|\nabla_{\mathbb{H}} w_k\|_{L^Q(\Omega)}^Q - \|\nabla_{\mathbb{H}} w_k - \nabla_{\mathbb{H}} w\|_{L^Q(\Omega)}^Q \to \|\nabla_{\mathbb{H}} w\|_{L^Q(\Omega)}^Q .$$

Thus for k large enough and $\delta > 0$ small enough:

$$0 < \alpha (1 + \delta) \|\nabla_{\mathbb{H}} w_k - \nabla_{\mathbb{H}} w\|_{L^Q(\Omega)}^{Q/(Q-1)} < \left(1 - \frac{\beta}{Q}\right) \alpha_Q. \tag{4.7}$$

Now, noting that for some $C(\delta) > 0$:

$$|w_k|^{Q/(Q-1)} \leq (1 + \frac{\delta}{2}) |w_k - w|^{Q/(Q-1)} + C(\delta) |w|^{Q/(Q-1)}$$

by Holder inequality, we have

$$\begin{aligned}
&\int_\Omega \frac{\exp\left\{\alpha\,|w_k|^{Q/(Q-1)}\right\}}{\rho(\xi)^\beta}d\xi \\
&\le \int_\Omega \frac{\exp\left\{\alpha(1+\frac{\delta}{2})\,|w_k-w|^{Q/(Q-1)}+\alpha C(\delta)\,|w|^{Q/(Q-1)}\right\}}{\rho(\xi)^\beta}d\xi \\
&\le \left(\int_\Omega \frac{\exp\left\{\alpha q(1+\frac{\delta}{2})\,|w_k-w|^{Q/(Q-1)}\right\}}{\rho(\xi)^\beta}d\xi\right)^{1/q}\left(\int_\Omega \frac{\exp\left\{\alpha q' C(\delta)\,|w|^{Q/(Q-1)}\right\}}{\rho(\xi)^\beta}d\xi\right)^{1/q'} \\
&\le C\left(\int_\Omega \frac{\exp\left\{\alpha(1+\delta)\,|w_k-w|^{Q/(Q-1)}\right\}}{\rho(\xi)^\beta}d\xi\right)^{1/q}
\end{aligned}$$

where we choose $q = \frac{1+\delta}{1+\frac{\delta}{2}}$ and $q' = \frac{q}{q-1}$. Now, by (4.7) and Lemma 4.1, we have that $\frac{\exp\{\alpha|w_k|^{Q/(Q-1)}\}}{\rho(\xi)^\beta}$ is bounded in $L^1(\Omega)$.

5 The existence of solution for the problem (NU)

The existence of nontrivial solution to Eq. (NU) will be proved by a mountain-pass theorem without a compactness condition such like the one of the (PS) type. This version of the mountain-pass theorem is a consequence of the Ekeland's variational principle. First, we will check that the functional J_ε satisfies the geometric conditions of the mountain-pass theorem.

Lemma 5.1 *Suppose that* $(f1)$ *and* $(f4)$ *hold. Then there exists* $\varepsilon_1 > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$, *there exists* $\rho_\varepsilon > 0$ *such that* $J_\varepsilon(u) > 0$ *if* $\|u\| = \rho_\varepsilon$. *Furthermore,* ρ_ε *can be chosen such that* $\rho_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. From $(f4)$, there exist $\tau,\ \delta > 0$ such that $|u| \le \delta$ implies

$$F(\xi,u) \le k_0\,(\lambda_1(Q)-\tau)\,|u|^Q \tag{5.8}$$

for all $\xi \in \Omega$. Moreover, using $(f1)$ for each $q > Q$, we can find a constant $C = C(q,\delta)$ such that

$$F(\xi,u) \le C\,|u|^q \exp\left(\kappa\,|u|^{Q/(Q-1)}\right) \tag{5.9}$$

for $|u| \ge \delta$ and $\xi \in \Omega$. From (5.8) and (5.9) we have

$$F(\xi,u) \le k_0\,(\lambda_1(Q)-\tau)\,|u|^Q + C\,|u|^q \exp\left(\kappa\,|u|^{Q/(Q-1)}\right)$$

for all $(\xi,u) \in \Omega\times\mathbb{R}$. Now, by $(A4)$, Lemma 4.3, (3.6) and the continuous embedding $E \hookrightarrow L^Q(\Omega)$, we obtain

$$\begin{aligned}
J_\varepsilon(u) &\ge k_0\,\|u\|^Q - k_0\,(\lambda_1(Q)-\tau)\int_\Omega \frac{|u|^Q}{\rho(\xi)^\beta}d\xi - C\,\|u\|^q - \varepsilon\,\|h\|_*\,\|u\| \\
&\ge k_0\left(1-\frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right)\|u\|^Q - C\,\|u\|^q - \varepsilon\,\|h\|_*\,\|u\|.
\end{aligned}$$

Thus

$$J_\varepsilon(u) \ge \|u\|\left[k_0\left(1-\frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right)\|u\|^{Q-1} - C\,\|u\|^{q-1} - \varepsilon\,\|h\|_*\right] \tag{5.10}$$

Since $\tau > 0$ and $q > Q$, we may choose $\rho > 0$ such that $k_0\left(1-\frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right)\rho^{Q-1} - C\rho^{q-1} > 0$. Thus, if ε is sufficiently small then we can find some $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\| = \rho_\varepsilon$ and even $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$.

Lemma 5.2 *There exists $e \in E$ with $\|e\| > \rho_\varepsilon$ such that $J_\varepsilon(e) < \inf_{\|u\|=\rho_\varepsilon} J_\varepsilon(u)$.*

Proof. Let $u \in E \setminus \{0\}$, $u \geq 0$ with compact support $\Omega' = supp(u)$. By $(f2)$ and $(f3)$, we have that for $p > Q$, there exists a positive constant $C > 0$ such that

$$\forall s \geq 0,\ \forall \xi \in \Omega : \ F(\xi, s) \geq cs^p - d. \tag{5.11}$$

Then by (2.5), we get

$$J_\varepsilon(\gamma u) \leq C\gamma \int_\Omega h_0(\xi) |\nabla_{\mathbb{H}} u| \, d\xi + C\gamma^Q \|u\|^Q - C\gamma^p \int_\Omega \frac{|u|^p}{\rho(\xi)^\beta} d\xi + C + \varepsilon\gamma \left| \int_\Omega hud\xi \right|.$$

Since $p > Q$, we have $J_\varepsilon(\gamma u) \to -\infty$ as $\gamma \to \infty$. Setting $e = \gamma u$ with γ sufficiently large, we get the conclusion.

In studying this class of sub-elliptic problems involving critical growth, the loss of the (PS) compactness condition raises many difficulties. In the following lemmas, we will analyze the compactness of (PS) sequences of J_ε.

Lemma 5.3 *Let $(u_k) \subset E$ be an arbitrary (PS) sequence of J_ε, i.e.,*

$$J_\varepsilon(u_k) \to c,\ DJ_\varepsilon(u_k) \to 0 \text{ in } E' \text{ as } k \to \infty.$$

Then there exists a subsequence of (u_k) (still denoted by (u_k)) and $u \in E$ such that

$$\begin{cases} \frac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \frac{f(\xi,u)}{\rho(\xi)^\beta} & \text{strongly in } L^1_{loc}(\Omega) \\ \nabla_{\mathbb{H}} u_k(\xi) \to \nabla_{\mathbb{H}} u(\xi) & \text{almost everywhere in } \Omega \\ a(\xi, \nabla_{\mathbb{H}} u_k) \rightharpoonup a(\xi, \nabla_{\mathbb{H}} u) & \text{weakly in } \left(L^{Q/(Q-1)}_{loc}(\Omega)\right)^{Q-2} \\ u_k \rightharpoonup u & \text{weakly in } E. \end{cases}$$

Furthermore u is a weak solution of (NU).

In order to prove this lemma, we need the following two lemmas that can be found in [10], [13], [14] and [32]:

Lemma 5.4 *Let $B_r(\xi^*)$ be a Heisenberg ball centered at $(\xi^*) \in \Omega$ with radius r. Then there exists a positive ε_0 depending only on Q such that*

$$\sup_{\int_{B_r(\xi^*)} |\nabla_{\mathbb{H}} u|^Q d\xi \leq 1,\ \int_{B_r(\xi^*)} ud\xi = 0} \frac{1}{|B_r(\xi^*)|} \int_{B_r(\xi^*)} \exp\left(\varepsilon_0 |u|^{Q/(Q-1)}\right) d\xi \leq C_0$$

for some constant C_0 depending only on Q.

Lemma 5.5 *Let (u_k) in $L^1(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ and let f be a continuous function. Then $\frac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \frac{f(\xi,u)}{\rho(\xi)^\beta}$ in $L^1(\Omega)$, provided that $\frac{f(\xi,u_k(\xi))}{\rho(\xi)^\beta} \in L^1(\Omega)$ $\forall k$ and $\int_\Omega \frac{|f(\xi,u_k(\xi))u_k(\xi)|}{\rho(\xi)^\beta} d\xi \leq C_1$.*

Now we are ready to prove Lemma 5.3.

Proof. The proof is similar to Lemma 3.4 in [10]. For the completeness, we sketch the proof here. By the assumption, we have

$$\int_\Omega A(\xi, \nabla_{\mathbb{H}} u_k) d\xi - \int_\Omega \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi - \varepsilon \int_\Omega hu_k d\xi \stackrel{k\to\infty}{\to} c \tag{5.12}$$

and

$$\left|\int_{\Omega} a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \nabla_{\mathbb{H}} v d\xi - \int_{\Omega} \frac{f(\xi, u_k) v}{\rho(\xi)^{\beta}} d\xi - \varepsilon \int_{\Omega} h v d\xi\right| \leq \tau_k \|v\| \tag{5.13}$$

for all $v \in E$, where $\tau_k \to 0$ as $k \to \infty$. Choosing $v = u_k$ in (5.13) and by $(A3)$, we get

$$\int_{\Omega} \frac{f(\xi, u_k) u_k}{\rho(\xi)^{\beta}} d\xi + \varepsilon \int_{\Omega} h u_k d\xi - Q \int_{\Omega} A\left(\xi, \nabla_{\mathbb{H}} u_k\right) \leq \tau_k \|u_k\|.$$

This together with (5.12), $(f3)$ and $(A4)$ leads to

$$\left(\frac{p}{Q} - 1\right) \|u_k\|^{Q} \leq C\left(1 + \|u_k\|\right)$$

and hence $\|u_k\|$ is bounded and thus

$$\int_{\Omega} \frac{f(\xi, u_k) u_k}{\rho(\xi)^{\beta}} d\xi \leq C, \quad \int_{\Omega} \frac{F(\xi, u_k)}{\rho(\xi)^{\beta}} d\xi \leq C. \tag{5.14}$$

Note that the embedding $E \hookrightarrow L^{q}(\Omega)$ is compact for all $q \geq 1$, by extracting a subsequence, we can assume that

$$u_k \to u \text{ weakly in } E \text{ and for almost all } \xi \in \Omega.$$

Thanks to Lemma 5.5, we have

$$\frac{f(\xi, u_k)}{\rho(\xi)^{\beta}} \to \frac{f(\xi, u)}{\rho(\xi)^{\beta}} \text{ in } L^{1}(\Omega). \tag{5.15}$$

Now, similar to that in [10], up to a subsequence, we define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_{\delta} = \left\{\xi \in \Omega : \lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(\xi)} \left(|u_k|^{Q} + |\nabla_{\mathbb{H}} u_k|^{Q}\right) d\xi' \geq \delta\right\}.$$

Since (u_k) is bounded in E, Σ_{δ} must be a finite set. For any $\xi^{*} \in \overline{\Omega} \setminus \Sigma_{\delta}$, there exist $r : 0 < r < dist\left(\xi^{*}, \Sigma_{\delta}\right)$ such that

$$\lim_{k \to \infty} \int_{B_r(\xi^{*})} \left(|u_k|^{Q} + |\nabla_{\mathbb{H}} u_k|^{Q}\right) d\xi < \delta$$

so for large k :

$$\int_{B_r(\xi^{*})} \left(|u_k|^{Q} + |\nabla_{\mathbb{H}} u_k|^{Q}\right) d\xi < \delta. \tag{5.16}$$

By results in [10], we can prove that

$$\begin{aligned} &\int_{B_r(\xi^{*})} \frac{|f(\xi, u_k)|\,|u_k - u|}{\rho(\xi)^{\beta}} d\xi \\ &\leq \left\|\frac{f(\xi, u_k)}{\rho(\xi)^{\beta/q}}\right\|_{L^{q}} \left\|\frac{1}{\rho(\xi)^{\beta}}\right\|_{L^{s}}^{1/q'} \|u_k - u\|_{L^{q's'}} \leq C\|u_k - u\|_{L^{q's'}} \to 0 \end{aligned} \tag{5.17}$$

and for any compact set $K \subset\subset \Omega \setminus \Sigma_{\delta}$,

$$\lim_{k \to \infty} \int_{K} \frac{|f(\xi, u_k) u_k - f(\xi, u) u|}{\rho(\xi)^{\beta}} d\xi = 0. \tag{5.18}$$

So now, we will prove that for any compact set $K \subset\subset \Omega \setminus \Sigma_{\delta}$,

$$\lim_{k \to \infty} \int_{K} |\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u|^{Q} d\xi = 0. \tag{5.19}$$

It is enough to prove for any $\xi^* \in \Omega \setminus \Sigma_\delta$, and r given by (5.16), there holds

$$\lim_{k\to\infty} \int_{B_{r/2}(\xi^*)} |\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u|^Q \, d\xi = 0. \tag{5.20}$$

For this purpose, we take $\phi \in C_0^\infty(B_r(\xi^*))$ with $0 \le \phi \le 1$ and $\phi = 1$ on $B_{r/2}(\xi^*)$. Obviously ϕu_k is a bounded sequence. Choose $v = \phi u_k$ and $v = \phi u$ in (5.13), we have:

$$\begin{aligned}
&\int_{B_r(\xi^*)} \phi \left(a(\xi, \nabla_{\mathbb{H}} u_k) - a(\xi, \nabla_{\mathbb{H}} u)\right)\left(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u\right) d\xi \\
&\le \int_{B_r(\xi^*)} a(\xi, \nabla_{\mathbb{H}} u_k) \nabla_{\mathbb{H}} \phi \,(u - u_k)\, d\xi \\
&+ \int_{B_r(\xi^*)} \phi a(\xi, \nabla_{\mathbb{H}} u)(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k)\, d\xi + \int_{B_r(\xi^*)} \phi\,(u_k - u) \frac{f(\xi, u_k)}{\rho(\xi)^\beta} d\xi \\
&+ \tau_k \|\phi u_k\| + \tau_k \|\phi u\| + \varepsilon \int_{B_r(\xi^*)} \phi h\,(u_k - u)\, d\xi.
\end{aligned}$$

Note that by Holder inequality and the compact embedding of $E \hookrightarrow L^Q(\Omega)$, we get

$$\lim_{k\to\infty} \int_{B_r(\xi^*)} a(\xi, \nabla_{\mathbb{H}} u_k) \nabla_{\mathbb{H}} \phi\,(u - u_k)\, d\xi = 0 \tag{5.21}$$

Since $\nabla_{\mathbb{H}} u_k \rightharpoonup \nabla_{\mathbb{H}} u$ and $u_k \rightharpoonup u$, there holds

$$\lim_{k\to\infty} \int_{B_r(\xi^*)} \phi a(\xi, \nabla_{\mathbb{H}} u)(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k)\, d\xi = 0 \text{ and } \lim_{k\to\infty} \int_{B_r(\xi^*)} \phi h\,(u_k - u)\, d\xi = 0. \tag{5.22}$$

The Holder inequality and (5.17) implies that

$$\lim_{k\to\infty} \int_{B_r(\xi^*)} \phi\,(u_k - u) f(\xi, u_k)\, d\xi = 0.$$

So we can conclude that

$$\lim_{k\to\infty} \int_{B_r(\xi^*)} \phi \left(a(\xi, \nabla_{\mathbb{H}} u_k) - a(\xi, \nabla_{\mathbb{H}} u)\right)(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u)\, d\xi = 0$$

and hence we get (5.20) by $(A2)$. So we have (5.19) by a covering argument. Since Σ_δ is finite, it follows that $\nabla_{\mathbb{H}} u_k$ converges to $\nabla_{\mathbb{H}} u$ almost everywhere. This immediately implies, up to a subsequence, $a(\xi, \nabla_{\mathbb{H}} u_k) \rightharpoonup a(\xi, \nabla_{\mathbb{H}} u)$ weakly in $\left(L_{loc}^{Q/(Q-1)}(\Omega)\right)^{Q-2}$. Let k tend to infinity in (5.13) and combine with (5.15), we obtain

$$\langle DJ_\varepsilon(u), h\rangle = 0 \;\forall h \in C_0^\infty(\Omega).$$

This completes the proof of the Lemma.

5.1 The proof of Theorem 3.1

Proposition 5.1 *Under the assumptions (f1)-(f4), there exists* $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$*, the problem* (NU) *has a solution* u_M *via mountain-pass theorem.*

Proof. For ε sufficiently small, by Lemma 5.1 and 5.2, J_ε satisfies the hypotheses of the mountain-pass theorem except possibly for the (PS) condition. Thus, using the mountain-pass theorem without the (PS) condition, we can find a sequence (u_k) in E such that

$$J_\varepsilon(u_k) \to c_M > 0 \text{ and } \|DJ_\varepsilon(u_k)\| \to 0$$

where c_M is the mountain-pass level of J_ε. Now, by Lemma 5.3, the sequence (u_k) converges weakly to a weak solution u_M of (NU) in E. Moreover, $u_M \neq 0$ since $h \neq 0$.

6 The multiplicity results to the problem (NH)

In this section, we study the problem (NH). Note that Eq. (NH) is a special case of the problem (NU) where

$$A(\xi,\tau)=\frac{|\tau|^{Q}}{Q},$$
$$\|u\|=\left(\int_{\Omega}|\nabla_{\mathbb{H}}u|^{Q}\,d\xi\right)^{1/Q}.$$

As a consequence, there exists a nontrivial solution of standard "mountain-pass" type as in Theorem 3.1. Now, we will prove the existence of the second solution.

Lemma 6.1 *There exists* $\eta>0$ *and* $v\in E$ *with* $\|v\|=1$ *such that* $J_{\varepsilon}(tv)<0$ *for all* $0<t<\eta$. *In particular,* $\inf\limits_{\|u\|\le\eta}J_{\varepsilon}(u)<0$.

Proof. Let $v\in E$ be a solution of the problem

$$\begin{cases}-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}}v|^{Q-2}\,\nabla_{\mathbb{H}}v\right)=h\text{ in }\Omega\\ v=0\text{ on }\partial\Omega.\end{cases}$$

Then, for $h\neq 0$, we have $\int_{\Omega}hv=\|v\|^{Q}>0$. Moreover,

$$\frac{d}{dt}J_{\varepsilon}(tv)=t^{Q-1}\,\|v\|^{Q}-\int_{\Omega}\frac{f(\xi,tv)\,v}{\rho(\xi)^{\beta}}d\xi-\varepsilon\int_{\Omega}hv d\xi$$

for $t>0$. Since $f(\xi,0)=0$, by continuity, it follows that there exists $\eta>0$ such that $\frac{d}{dt}J_{\varepsilon}(tv)<0$ for all $0<t<\eta$ and thus $J_{\varepsilon}(tv)<0$ for all $0<t<\eta$ since $J_{\varepsilon}(0)=0$.

Next, we define the Moser Functions (see [10, 19]):

$$\widetilde{m}_k(\xi,r)=\frac{1}{\sigma_Q^{1/Q}}\begin{cases}(\log k)^{(Q-1)/Q} & \text{if }\rho(\xi)\le\frac{r}{k}\\ \frac{\log\frac{r}{\rho(\xi)}}{(\log k)^{1/Q}} & \text{if }\frac{r}{k}\le\rho(\xi)\le r\\ 0 & \text{if }\rho(\xi)\ge r.\end{cases}$$

Using the fact that $|\nabla_{\mathbb{H}}\rho(\xi)|=\frac{|z|}{\rho(\xi)}$ where $\xi=(z,t)\in\Omega$, we can conclude that $\widetilde{m}_k(.,r)\in W^{1,Q}(\Omega)$, the support of $\widetilde{m}_k(\xi,r)$ is the ball B_r, and

$$\int_{\Omega}|\nabla_{\mathbb{H}}\widetilde{m}_k(\xi,r)|^{Q}\,d\xi=1. \tag{6.23}$$

Lemma 6.2 *Suppose that (f1)-(f5) hold. Then there exists* $k\in\mathbb{N}$ *such that*

$$\max_{t\ge 0}\left\{\frac{t^{Q}}{Q}-\int_{\Omega}\frac{F(\xi,tm_k)}{\rho(\xi)^{\beta}}d\xi\right\}<\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Proof. Let $r>0$ and $\beta_0>0$ be such that

$$\lim_{s\to\infty}sf(\xi,s)\exp\left(-\alpha_0\,|s|^{Q'}\right)\ge\beta_0>\frac{(Q-\beta)^{Q}}{Q^{Q-1}r^{Q-\beta}}\frac{1}{\mathcal{M}\alpha_0^{Q-1}}\frac{\alpha_Q^{Q-1}}{\omega_{Q-1}}$$

uniformly for almost every $\xi \in \Omega$ and $B(0,r) \subset \Omega$. Let $m_k(\xi) = \widetilde{m}_k(\xi, r)$. Then we have

$$m_k \in W_0^{1,Q}(B_r) \text{ and } \|m_k\| = 1.$$

Suppose, by contradiction, that for all k we get

$$\max_{t\geq 0}\left\{\frac{t^Q}{Q} - \int_\Omega \frac{F(\xi, tm_k)}{\rho(\xi)^\beta} d\xi\right\} \geq \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

For each k, we can find $t_k > 0$ such that

$$\frac{t_k^Q}{Q} - \int_\Omega \frac{F(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi = \max_{t\geq 0}\left\{\frac{t^Q}{Q} - \int_\Omega \frac{F(\xi, tm_k)}{\rho(\xi)^\beta} d\xi\right\}.$$

Thus

$$\frac{t_k^Q}{Q} - \int_\Omega \frac{F(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi \geq \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

From $F(\xi, u) \geq 0$, we obtain

$$t_k^Q \geq \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}. \tag{6.24}$$

Since at $t = t_k$ we have

$$\frac{d}{dt}\left(\frac{t^Q}{Q} - \int_\Omega \frac{F(\xi, tm_k)}{\rho(\xi)^\beta} d\xi\right) = 0$$

it follows that

$$t_k^Q = \int_\Omega \frac{t_k m_k f(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi = \int_{\rho(\xi)\leq r} \frac{t_k m_k f(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi. \tag{6.25}$$

Using hypothesis $(f5)$, given $\tau > 0$ there exists $R_\tau > 0$ such that for all $u \geq R_\tau$ and $\rho(\xi) \leq r$, we have

$$uf(\xi, u) \geq (\beta_0 - \tau)\exp\left(\alpha_0 |u|^{Q/(Q-1)}\right). \tag{6.26}$$

From (6.25) and (6.26), for large k, we obtain

$$\begin{aligned} t_k^Q &\geq (\beta_0 - \tau)\int_{\rho(\xi)\leq \frac{r}{k}} \frac{\exp\left(\alpha_0 |t_k m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi \\ &= (\beta_0 - \tau)\frac{\omega_{Q-1}}{Q-\beta}\left(\frac{r}{k}\right)^{Q-\beta} \exp\left(\alpha_0 t_k^{Q/(Q-1)} \sigma_Q^{-1/(Q-1)} \log k\right) \\ &= (\beta_0 - \tau)\frac{\omega_{Q-1}}{Q-\beta} r^{Q-\beta} \exp\left(\frac{\alpha_0}{\alpha_Q} t_k^{Q/(Q-1)} Q\log k - Q\log k + \beta \log k\right). \end{aligned}$$

Thus, setting

$$L_k = \frac{\alpha_0 Q \log k}{\alpha_Q} t_k^{Q/(Q-1)} - Q\log t_k - (Q-\beta)\log k$$

we have

$$1 \geq (\beta_0 - \tau)\frac{\omega_{Q-1}}{Q-\beta} r^{Q-\beta} \exp L_k.$$

Consequently, the sequence (t_k) is bounded. Moreover, by (6.24) and

$$t_k^Q \geq (\beta_0 - \tau)\frac{\omega_{Q-1}}{Q-\beta} r^{Q-\beta} \exp\left[\left(Q\frac{\alpha_0 t_k^{Q/(Q-1)}}{\alpha_Q} - (Q-\beta)\right)\log k\right]$$

it follows that

$$t_k^Q \overset{k\to\infty}{\to} \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}. \tag{6.27}$$

Set

$$A_k = \{\xi \in B_r : t_k m_k \geq R_\tau\} \text{ and } B_k = B_r \setminus A_k.$$

From (6.25) and (6.26) we have

$$\begin{aligned} t_k^Q \geq (\beta_0-\tau)\int_{\rho(\xi)\leq r} \frac{\exp\left(\alpha_0 |t_k m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi + \int_{B_k} \frac{t_k m_k f(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi \\ -(\beta_0-\tau)\int_{B_k} \frac{\exp\left(\alpha_0 |t_k m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi. \end{aligned} \tag{6.28}$$

Notice that $m_k(\xi) \to 0$ and the characteristic functions $\chi_{B_k} \to 1$ for almost everywhere ξ in B_r. Therefore the Lebesgue dominated convergence theorem implies

$$\int_{B_k} \frac{t_k m_k f(\xi, t_k m_k)}{\rho(\xi)^\beta} d\xi \to 0$$

and

$$\int_{B_k} \frac{\exp\left(\alpha_0 |t_k m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi \to \frac{\omega_{Q-1}}{Q-\beta} r^{Q-\beta}.$$

Moreover, using

$$t_k^Q \underset{\geq}{\overset{k\to\infty}{\to}} \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1},$$

we have

$$\begin{aligned} &\int_{\rho(\xi)\leq r} \frac{\exp\left(\alpha_0 |t_k m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi \\ &\geq \int_{\rho(\xi)\leq r} \frac{\exp\left(\frac{Q-\beta}{Q}\alpha_Q |m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi \\ &= \int_{\rho(\xi)\leq r/k} \frac{\exp\left(\frac{Q-\beta}{Q}\alpha_Q |m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi + \int_{r/k\leq\rho(\xi)\leq r} \frac{\exp\left(\frac{Q-\beta}{Q}\alpha_Q |m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi \end{aligned}$$

and

$$\begin{aligned} \int_{\rho(\xi)\leq r/k} \frac{\exp\left(\frac{Q-\beta}{Q}\alpha_Q |m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} d\xi &= \int_{\rho(\xi)\leq r/k} \exp\left[\frac{Q-\beta}{Q}\alpha_Q \sigma_Q^{-1/(Q-1)} \log k\right] d\xi \\ &= \frac{\omega_{Q-1}}{Q-\beta}\left(\frac{r}{k}\right)^{Q-\beta} k^{(Q-\beta)} \\ &= \frac{\omega_{Q-1}}{Q-\beta} r^{Q-\beta}. \end{aligned}$$

Now, using the change of variable

$$\zeta = \frac{\log\left(\frac{r}{s}\right)}{\log k}$$

by straightforward computation, we have

$$\int_{r/k\le\rho(\xi)\le r}\frac{\exp\left(\frac{Q-\beta}{Q}\alpha_Q\,|m_k|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta}d\xi$$
$$=\omega_{Q-1}r^{Q-\beta}\log k\int_0^1\exp\left[(Q-\beta)\log k\left(\zeta^{Q/(Q-1)}-\zeta\right)\right]d\zeta$$

which converges to $\frac{\omega_{Q-1}}{Q-\beta}r^{Q-\beta}\mathcal{M}$ as $k\to\infty$. Finally, taking $k\to\infty$ in (6.28) and using (6.27), we obtain

$$\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\ge(\beta_0-\tau)\,\omega_{Q-1}r^{Q-\beta}\frac{\mathcal{M}}{Q-\beta}$$

which implies that

$$\beta_0\le\frac{(Q-\beta)^Q}{Q^{Q-1}r^{Q-\beta}}\frac{1}{\mathcal{M}\alpha_0^{Q-1}}\frac{\alpha_Q^{Q-1}}{\omega_{Q-1}},$$

but it is a contradiction and the proof is complete.

Corollary 6.1 *Under the hypotheses (f1)-(f5), if ε is sufficiently small then*

$$\max_{t\ge0}J_\varepsilon(tm_k)=\max_{t\ge0}\left\{\frac{t^Q}{Q}-\int_\Omega\frac{F(\xi,tm_k)}{\rho(\xi)^\beta}d\xi-t\int_\Omega\varepsilon hm_kd\xi\right\}<\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Proof. Since $\left|\int_\Omega\varepsilon hm_kd\xi\right|\le\varepsilon\,\|h\|_*$, taking ε sufficiently small, the result follows.

Note that we can conclude by inequality (5.10) and Lemma 6.1 that

$$-\infty<c_\varepsilon=\inf_{\|u\|\le\rho_\varepsilon}J_\varepsilon(u)<0. \tag{6.29}$$

Next, we will prove that this infimum is achieved and generate a solution. In order to obtain convergence results, we need to improve the estimate of Lemma 6.2.

Corollary 6.2 *Under the hypotheses (f1)-(f5), there exist $\varepsilon_2\in(0,\varepsilon_1]$ and $u\in W_0^{1,Q}(\Omega)$ with compact support such that for all $0<\varepsilon<\varepsilon_2$,*

$$J_\varepsilon(tu)<c_\varepsilon+\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\quad\text{for all } t\ge0.$$

Proof. It is possible to raise the infimum c_ε by reducing ε. By Lemma 5.1, $\rho_\varepsilon\overset{\varepsilon\to0}{\to}0$. Consequently, $c_\varepsilon\overset{\varepsilon\to0}{\to}0$. Thus there exists $\varepsilon_2>0$ such that if $0<\varepsilon<\varepsilon_2$ then, by Corollary 6.1, we have

$$\max_{t\ge0}J_\varepsilon(tm_k)<c_\varepsilon+\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Taking $u=m_k\in W_0^{1,Q}(\Omega)$, the result follows.

Lemma 6.3 *If (u_k) is a (PS) sequence for J_ε at any level with*

$$\liminf_{k\to\infty}\|u_k\|<N \tag{6.30}$$

with N sufficiently small, then (u_k) possesses a subsequence which converges strongly to a solution u_0 of (NH).

Proof. Extracting a subsequence of (u_k) if necessary, we can suppose that

$$\liminf_{k\to\infty} \|u_k\| = \lim_{k\to\infty} \|u_k\| < N.$$

By Lemma 5.3, we have that $u_k \rightharpoonup u_0$ where u_0 is a weak solution of (NH). Writing $u_k = u_0 + w_k$, it follows that $w_k \rightharpoonup 0$ in E. Thus $w_k \to 0$ in $L^q(\Omega)$ for all $1 \le q < \infty$. Using the Brezis-Lieb Lemma in [7], we have

$$\|u_k\|^Q = \|u_0\|^Q + \|w_k\|^Q + o_k(1). \tag{6.31}$$

We claim that

$$\int_\Omega \frac{f(\xi, u_k)u_0}{\rho(\xi)^\beta} d\xi \overset{k\to\infty}{\to} \int_\Omega \frac{f(\xi, u_0)u_0}{\rho(\xi)^\beta} d\xi. \tag{6.32}$$

In fact, using $u_0 \in E$, given $\tau > 0$, there exists $\varphi \in C_0^\infty(\Omega)$ such that $\|\varphi - u_0\| < \tau$. We have that

$$\begin{aligned}\left|\int_\Omega \frac{f(\xi, u_k)u_0}{\rho(\xi)^\beta} d\xi - \int_\Omega \frac{f(\xi, u_0)u_0}{\rho(\xi)^\beta} d\xi\right| &\le \left|\int_\Omega \frac{f(\xi, u_k)(u_0 - \varphi)}{\rho(\xi)^\beta} d\xi\right| \\ + \|\varphi\|_\infty \int_{\mathrm{supp}\varphi} \frac{|f(\xi, u_k) - f(\xi, u_0)|}{\rho(\xi)^\beta} d\xi &+ \left|\int_\Omega \frac{f(\xi, u_0)(u_0 - \varphi)}{\rho(\xi)^\beta} d\xi\right|.\end{aligned}$$

Since $|DJ_\varepsilon(u_k)(u_0 - \varphi)| \le \tau_k \|u_0 - \varphi\|$ with $\tau_k \to 0$, we have

$$\begin{aligned}\left|\int_\Omega \frac{f(\xi, u_k)(u_0 - \varphi)}{\rho(\xi)^\beta} d\xi\right| &\le \tau_k \|u_0 - \varphi\| + \|\nabla_{\mathbb{H}} u_k\|^{Q-1} \|u_0 - \varphi\| \\ &+ \|\varepsilon h\|_* \|u_0 - \varphi\| \\ &\le C\|u_0 - \varphi\| < C\tau,\end{aligned}$$

where C is independent of k and τ. Similarly, using that $DJ_\varepsilon(u_0)(u_0 - \varphi) = 0$, we have

$$\left|\int_\Omega \frac{f(\xi, u_0)(u_0 - \varphi)}{\rho(\xi)^\beta} d\xi\right| < C\tau.$$

Since $\frac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \frac{f(\xi,u_0)}{\rho(\xi)^\beta}$ strongly in $L^1(\Omega)$ and by the previous inequalities, we conclude that

$$\lim_{k\to\infty} \left|\int_\Omega \frac{f(\xi, u_k)u_0}{\rho(\xi)^\beta} d\xi - \int_\Omega \frac{f(\xi, u_0)u_0}{\rho(\xi)^\beta} d\xi\right| < 2C\tau$$

and this shows the convergence (6.32) because τ is arbitrary. From (6.31) and (6.32), we can write

$$DJ_\varepsilon(u_k) u_k = DJ_\varepsilon(u_0) u_0 + \|w_k\|^Q - \int_\Omega \frac{f(\xi, u_k) w_k}{\rho(\xi)^\beta} d\xi + o_k(1)$$

that is

$$\|w_k\|^Q = \int_\Omega \frac{f(\xi, u_k) w_k}{\rho(\xi)^\beta} d\xi + o_k(1).$$

From $(f1)$, Lemma 4.1, Lemma 4.4 and the compact embedding $E \hookrightarrow L^t(\Omega)$ for $t \ge 1$, we have

$$\int_\Omega \frac{f(\xi, u_k) w_k}{\rho(\xi)^\beta} d\xi \to 0$$

from which $\|w_k\| \to 0$ and the result follows.

6.1 Proof of Theorem 3.2

The proof of the existence of the second solution of (NH) follows by a minimization argument and Ekeland's variational principle.

Proposition 6.1 *There exists* $\varepsilon_2 > 0$ *such that for each* ε *with* $0 < \varepsilon < \varepsilon_2$*, Eq. (NH) has a minimum type solution* u_0 *with* $J_\varepsilon(u_0) = c_\varepsilon < 0$*, where* c_ε *is defined in (6.29).*

Proof. Let $\rho_\varepsilon,\ N$ be as in Lemma 5.1 and Lemma 6.3. Note that we can choose $\varepsilon_2 > 0$ sufficiently small such that

$$\rho_\varepsilon < N.$$

Since $\overline{B}_{\rho_\varepsilon}$ is a complete metric space with the metric given by the norm of E, convex and the functional J_ε is of class C^1 and bounded below on $\overline{B}_{\rho_\varepsilon}$, by the Ekeland's variational principle there exists a sequence (u_k) in $\overline{B}_{\rho_\varepsilon}$ such that

$$J_\varepsilon(u_k) \to c_\varepsilon = \inf_{\|u\| \le \rho_\varepsilon} J_\varepsilon(u) \text{ and } \|DJ_\varepsilon(u_k)\|_* \to 0$$

Observing that

$$\|u_k\| \le \rho_\varepsilon < N$$

by Lemma 6.3, it follows that there exists a subsequence of (u_k) which converges to a solution u_0 of (NH). Therefore, $J_\varepsilon(u_0) = c_\varepsilon < 0$.

Remark 6.1 *By Corollary 6.2, we can conclude that*

$$0 < c_M < c_\varepsilon + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Proposition 6.2 *If* $\varepsilon_2 > 0$ *is enough small, then the solutions of (NH) obtained in Propositions 5.1 and 6.1 are distinct.*

Proof. By Proposition 5.1 and 6.1, there exist sequences (u_k), (v_k) in E such that

$$u_k \to u_0,\ J_\varepsilon(u_k) \to c_\varepsilon < 0,\ DJ_\varepsilon(u_k)\, u_k \to 0$$

and

$$v_k \rightharpoonup u_M,\ J_\varepsilon(v_k) \to c_M > 0,\ DJ_\varepsilon(v_k)\, v_k \to 0,\ \nabla_{\mathbb{H}} v_k(\xi) \to \nabla_{\mathbb{H}} u_M(\xi) \text{ a.e. } \Omega.$$

Now, suppose by contradiction that $u_0 = u_M$. Then we have that $v_k \nrightarrow u_0$. As in the proof of Lemma 5.3 we obtain

$$\frac{f(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{f(\xi, u_0)}{\rho(\xi)^\beta} \text{ in } L^1(\Omega). \tag{6.33}$$

From this, we have by $(f2)$, $(f3)$ and the Generalized Lebesgue's Dominated Convergence Theorem:

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{F(\xi, u_0)}{\rho(\xi)^\beta} \text{ in } L^1(\Omega).$$

Now, we have that

$$J_\varepsilon(u_k) = \frac{1}{Q}\|u_k\|^Q - \int_\Omega \frac{F(\xi, u_0)}{\rho(\xi)^\beta} - \varepsilon \int_\Omega h u_0 + o(1) = c_\varepsilon + o(1)$$

and

$$J_\varepsilon(v_k) = \frac{1}{Q}\|v_k\|^Q - \int_\Omega \frac{F(\xi, u_0)}{\rho(\xi)^\beta} - \varepsilon \int_\Omega h u_0 + o(1) = c_M + o(1)$$

which implies

$$\|u_k\|^Q - \|v_k\|^Q \to Q\,(c_\varepsilon - c_M) < 0. \tag{6.34}$$

Also, since (u_k), (v_k) are both bounded Palais-Smale sequences, we have

$$DJ_\varepsilon(u_k)\,u_k = \|u_k\|^Q - \int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^\beta}u_k - \varepsilon\int_\Omega hu_k \to 0$$

$$DJ_\varepsilon(v_k)\,v_k = \|v_k\|^Q - \int_\Omega \frac{f(\xi,v_k)}{\rho(\xi)^\beta}v_k - \varepsilon\int_\Omega hv_k \to 0$$

and then

$$\left(\|u_k\|^Q - \|v_k\|^Q\right) - \int_\Omega\left[\frac{f(\xi,u_k)}{\rho(\xi)^\beta}(u_k - v_k) + \left(\frac{f(\xi,u_k) - f(\xi,v_k)}{\rho(\xi)^\beta}\right)v_k\right]$$
$$-\,\varepsilon\int_\Omega\left[h\,(u_k - u_0) - h\,(v_k - u_0)\right]$$

tends to 0 as $k \to \infty$. But it is clear that

$$\int_\Omega\left[h\,(u_k - u_0) - h\,(v_k - u_0)\right] \to 0 \tag{6.35}$$

since $h \in \left(W_0^{1,Q}(\Omega)\right)^*$ and $u_k \to u_0,\ v_k \rightharpoonup u_0$ in $W_0^{1,Q}(\Omega)$. Also, notice that

$$\int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^\beta}(u_k - v_k) \to 0. \tag{6.36}$$

Indeed, let $w_k = v_k - u_0$. Thus $w_k \rightharpoonup 0$ and $\lim\limits_{k\to\infty}\|w_k\| > 0$ since $w_k \nrightarrow 0$. Again, using Holder inequality, Lemma 4.1 and Theorem A, note that $\|u_k\|$ is small, we have

$$\left|\int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^\beta}(u_k - v_k)\right| \le \left|\int_\Omega\left(\frac{f(\xi,u_k)}{\rho(\xi)^\beta}\right)^q d\xi\right|^{1/q}\left|\int_\Omega (u_k - v_k)^{q'}\right|^{1/q'}$$
$$\le C\,\|u_k - v_k\|_{q'} \to 0.$$

So, it remains to show that

$$\int_\Omega\left(\frac{f(\xi,u_k) - f(\xi,v_k)}{\rho(\xi)^\beta}\right)v_k d\xi \to 0. \tag{6.37}$$

The left hand side of (6.37) can be written as

$$\int_\Omega\left(\frac{f(\xi,u_k) - f(\xi,v_k)}{\rho(\xi)^\beta}\right)u_0 d\xi + \int_\Omega\left(\frac{f(\xi,u_k) - f(\xi,v_k)}{\rho(\xi)^\beta}\right)w_k d\xi.$$

Arguing as in Lemma 6.3, we can conclude that

$$\int_\Omega\left(\frac{f(\xi,u_k) - f(\xi,v_k)}{\rho(\xi)^\beta}\right)u_0 d\xi \to 0. \tag{6.38}$$

Now, again by Holder inequality, Theorem A and the Sobolev embedding, we get

$$\int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^\beta}w_k d\xi \le C\,\|w_k\|_{q'} \to 0. \tag{6.39}$$

So now, we just need to prove

$$\int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^\beta}w_k d\xi \to 0. \tag{6.40}$$

Indeed, for large k, we have

$$\begin{aligned} c_M - c_\varepsilon &= J_\varepsilon(v_k) - J_\varepsilon(u_k) + o(1) \\ &= \frac{1}{Q}\|v_k\|^Q - \frac{1}{Q}\|u_0\|^Q + o(1) \\ &< \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}. \end{aligned}$$

Therefore, we can find $s > 1$ sufficiently close to 1 such that for large k,

$$\|v_k\|^Q - \|u_0\|^Q < \left(\frac{Q-s\beta}{Q}\frac{\alpha_Q}{s\alpha_0}\right)^{Q-1}.$$

Thus,

$$s\alpha_0\|v_k\|^{Q'} < \alpha_Q\frac{Q-s\beta}{Q}\left(1-\frac{\|u_0\|^Q}{\|v_k\|^Q}\right)^{-1/(Q-1)}.$$

Define $V_k = \frac{v_k}{\|v_k\|}$. Thus $\|V_k\| = 1$, $V_k \rightharpoonup V_0 = \frac{u_0}{\lim\|v_k\|}$ and $\|V_0\| < 1$. By Lemma 4.5 and the information that $\|w_k\|_{s'} \to 0$, we have

$$\begin{aligned} \left|\int_\Omega \frac{f(\xi,v_k)}{\rho(\xi)^\beta}w_k d\xi\right| &\le C\left[\int_\Omega \frac{\exp\left[s\alpha_0\|v_k\|^{Q'}\left|\frac{v_k}{\|v_k\|}\right|^{Q'}\right]}{\rho(\xi)^{s\beta}}\right]^{1/s}\|w_k\|_{s'} \\ &\to 0 \end{aligned}$$

and then we have (6.40). From (6.35), (6.36), (6.37), (6.38), (6.39) and (6.40) we have $\|u_k\|^Q - \|v_k\|^Q \to 0$ which is a contradiction to (6.34). The proof is completed now.

6.2 Proof of Theorem 3.3

Corollary 6.3 *There exists* $\varepsilon_3 > 0$ *such that if* $0 < \varepsilon < \varepsilon_3$ *and* $h(\xi) \ge 0$ *for all* $\xi \in \Omega$*, then the weak solutions of (NH) are nonnegative.*

Proof. Let u be a weak solution of (NH), that is,

$$\int_\Omega |\nabla_{\mathbb{H}}u|^{Q-2}\nabla_{\mathbb{H}}u\nabla_{\mathbb{H}}v d\xi - \int_\Omega \frac{f(\xi,u)\,v}{\rho(\xi)^\beta}d\xi - \int_\Omega \varepsilon h v d\xi = 0$$

for all $v \in E$. Taking $v = u^- \in E$ and observing that $f(\xi,u(\xi))\,u^-(\xi) = 0$ a.e., we have

$$\left\|u^-\right\|_E^Q = -\int_H \varepsilon h u^- d\xi \le 0.$$

Consequently, $u = u^+ \ge 0$.

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