

On the Orbital Stability of Standing-Wave Solutions to a Coupled Non-Linear Klein-Gordon Equation

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Abstract

We show the existence of standing-wave solutions to a coupled non-linear Klein-Gordon equation. Our solutions are obtained as minimizers of the energy under a two-charges constraint. We prove that the ground state is stable and that standing-waves are orbitally stable under a non-degeneracy assumption.

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1 Introduction

This work is on the orbital stability of standing-wave solutions

$$v_j(t, x) = e^{-i\omega_j t} u_j(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad 1 \leq j \leq 2 \quad (1.1)$$

to the coupled non-linear Klein-Gordon equation

$$\square v_j + m_j^2 v_j + \partial_{z_j} F(v) = 0, \quad 1 \leq j \leq 2. \quad (\text{CNLKG})$$

The m_j 's are positive real numbers and

$$F: \mathbb{C}^2 \rightarrow \mathbb{C}$$

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is a continuously differentiable, real-valued function. Furthermore, we assume that $N \geq 3$ and

$$F(z) = -\mu|z_1 z_2|^\gamma + G(z), \quad 1 < \gamma < 1 + 2/N, \mu > 0, \quad (A_1)$$

$$|DG(z)| \leq c(|z|^{p-1} + |z|^{q-1}), \quad 2\gamma < p \leq q < 2^*, \quad G(0) = 0, \quad (A_2)$$

$$G(z) = G(|z_1|, |z_2|), \quad G \geq 0, \quad (A_3)$$

$$\int_{\mathbb{R}^N} G(u_1^*, u_2^*) \leq \int_{\mathbb{R}^N} G(u_1, u_2), \quad u_1, u_2 \geq 0, \quad (A_4)$$

$$V(z) := F(z) + \frac{1}{2}(m_1^2|z_1|^2 + m_2^2|z_2|^2) \geq 0, \quad z \in \mathbb{R}^2. \quad (A_5)$$

Finally, we assume the local existence and uniqueness of strong solutions to (CNLKG) for initial data in $H^1 \times L^2$. By definition (check T. Tao [27, Remark 3.5, p. 126]), for every

$$\Phi := (\phi, \phi_t) \in H^1 \times L^2$$

there exists, uniquely, $T := T(\phi, \phi_t) > 0$ and

$$v_j \in C_t H_x^1([0, T) \times \mathbb{R}^N, \mathbb{C}) \cap C_t^1 L_x^2([0, T) \times \mathbb{R}^N, \mathbb{C}), \quad j = 1, 2$$

such that v solves (CNLKG) and

$$(v(0, \cdot), \partial_t v(0, \cdot)) = (\phi, \phi_t).$$

We also assume that local solutions can be extended to \mathbb{R} . We use the notation

$$U(t, \Phi) := (v(t, \cdot), \partial_t v(t, \cdot)) \in H^1 \times L^2.$$

In the scalar case, existence and local uniqueness of solutions to the non-linear Klein-Gordon equation with sub-critical growth condition has been addressed in [14, 7].

In assumption (A₄), u_j^* is the Steiner symmetrization. We refer to [18] for the definition and its properties. In the scalar case, (A₄) holds for every $G: \mathbb{R}^+ \rightarrow \mathbb{R}$. A simple example of G satisfying assumptions (A₁–A₄) is given by

$$G(z) := |z|^p + |z|^q,$$

where q and p are as in (A₂). If m_j are large enough, then (A₅) also holds.

In the linear space

$$X := H^1(\mathbb{R}^N, \mathbb{C}^2) \times L^2(\mathbb{R}^N, \mathbb{C}^2)$$

we consider the metric induced by the following scalar product: given two vectors

$$\Phi = (\phi, \phi_t), \quad \Psi = (\psi, \psi_t),$$

we define

$$\langle \Phi, \Psi \rangle := \operatorname{Re} \sum_{j=1}^2 \int_{\mathbb{R}^N} (\phi_j \bar{\psi}_j + D\phi_j \cdot \overline{D\psi_j} + \phi_j^t \bar{\psi}_j^t).$$

Definition 1.1 *A subset is stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(\Phi, S) < \delta \Rightarrow d(U(t, \Phi), S) < \varepsilon$$

for every $t \geq 0$.

Given $(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2$, we define

$$(z \cdot w)_j := z_j w_j. \quad (1.2)$$

Following this notation, if v is a standing-wave as in (1.1), then

$$(v(0, \cdot), \partial_t v(0, \cdot)) = (u, -i\omega \cdot u).$$

Definition 1.2 A standing-wave is orbitally stable if the subset of X

$$\Gamma(u, \omega) = \{(\lambda \cdot u(\cdot + y), -i\omega \cdot \lambda \cdot u(\cdot + y)) \mid (\lambda, y) \in \mathbb{T}^2 \times \mathbb{R}^N\}$$

is stable.

From (A_3) , if v is a standing-wave solution to (CNLKG), then (u, ω) is a solution to the elliptic system

$$-\Delta u_j + m_j^2 u_j + \partial_{z_j} F(u) = \omega_j^2 u_j, \quad 1 \leq j \leq 2. \quad (1.3)$$

In order to solve (1.3), we follow the variational approach of [3], where the energy functional and the constraint are provided by conserved quantities: we refer to

$$X \ni (\phi, \phi_t) \mapsto \mathbf{E}(\phi, \phi_t) := \frac{1}{2} \sum_{j=1}^2 \left(\int_{\mathbb{R}^N} |\phi_t^j|^2 + |D\phi_j|^2 + 2V(\phi) \right)$$

and

$$X \ni (\phi, \phi_t) \mapsto \mathbf{C}_j(\phi, \phi_t) := -\operatorname{Im} \int_{\mathbb{R}^N} \phi_t^j \bar{\phi}_j, \quad 1 \leq j \leq 2$$

as energy and charges. By (A_3) , the functions

$$\mathbb{R} \ni t \mapsto e(t) := \mathbf{E}(v(t, \cdot), \partial_t v(t, \cdot)), \quad (1.4)$$

$$\mathbb{R} \ni t \mapsto c_j(t) := \mathbf{C}_j(v(t, \cdot), \partial_t v(t, \cdot)) \quad (1.5)$$

are constant for every solution v . In particular, if v is a standing-wave as in (1.1), then

$$e(0) = \mathbf{E}(u, -i\omega \cdot u) = \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^N} (|Du_j|^2 + m_j^2 u_j^2 + \omega_j^2 u_j^2) + \int_{\mathbb{R}^N} F(u). \quad (1.6)$$

and

$$c_j(0) = \mathbf{C}_j(u, -i\omega \cdot u) = \omega_j \int_{\mathbb{R}^N} |u_j|^2. \quad (1.7)$$

We define the energy functional

$$E: H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad E(u, \omega) := \mathbf{E}(u, -i\omega \cdot u)$$

and the constraint

$$C_j: H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad C_j(u, \omega) := \omega_j \int_{\mathbb{R}^N} |u_j|^2$$

$$M_C := \{(u, \omega) \mid C_j(u, \omega) = C_j\}.$$

The key observation made in [3, Theorem 2.6] is that critical points of E constrained to M_C are classic solutions to (1.3). In Proposition 2.2 we prove this fact for the coupled case and that each of the components u_j does not change sign.

The main theorems of this work are the following:

Theorem 1.1 Given a minimising sequence $(u_n, \omega_n)_{n \geq 1}$ for E over M_C , there exists a minimiser (u, ω) and $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that, up to extract a subsequence,

$$(u_n, \omega_n) = (u(\cdot + y_n), \omega) + o(1).$$

The proof is carried out by proving a concentration behaviour of the minimising sequences of the functional

$$J(u) = \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^N} |Du_j|^2 + \int_{\mathbb{R}^N} F(u)$$

on the constraint

$$N_\rho := \{u \mid \|u_j\|_{L^2(\mathbb{R}^N)}^2 = \rho_j\}.$$

In turn, such behaviour follows from the sub-additivity property of the function $I(\rho) := \inf_{N_\rho} J$

$$I(\rho) < I(\tau) + I(\rho - \tau), \quad 0 < \tau_j \leq \rho_j, \quad \tau \neq \rho. \quad (1.8)$$

Such property plays a crucial role in the proof of the orbital stability of standing-wave solutions to a variety of evolution problems: the non-linear Schrödinger equation, [11, 4], coupled NLS in dimension $N = 1$, in [21], and KdV-NLS systems [1]. In these references, (1.8) is obtained through rescaling argument (as in [4]) or symmetries arising from the choice of the non-linear term (as in [21]). Due to the lack of suitable rescaling arguments for non-linearities satisfying (A_1) , we obtain (1.8) from considerations on the gradient terms. We exploit an idea carried out by J. Byeon in [10, Proposition 1.4] which is based on the symmetric rearrangement and we prove that, if

$$(u, v) \in N_\tau \times N_{\rho-\tau}$$

have disjoint support and are a good approximation of $I(\tau)$ and $I(\rho - \tau)$, respectively, then there exists $D = D(\rho, \tau) > 0$ such that

$$\|Dw^*\|^2 \leq \|Du\|^2 + \|Dv\|^2 - D,$$

where $w = u + v$ and w^* is the Steiner symmetrization of w . We prove this fact in Lemma 3.1.

Preliminary notation is required to introduce the next results. To C in \mathbb{R}^2 , we can associate the subset

$$m_C := \inf_{M_C} E, \quad K_C := \{(u, \omega) \in M_C \mid E(u, \omega) = m_C\}$$

and

$$\Gamma_C = \bigcup_{(u, \omega) \in K_C} \Gamma(u, \omega).$$

Theorem 1.2 *Let $C \in \mathbb{R}^2$ be such that $C_j \neq 0$ for $j = 1, 2$. Given a sequence*

$$(\Phi_n)_{n \geq 1} \subset X$$

then $d(\Phi_n, \Gamma_C) \rightarrow 0$ if and only if

$$E(\Phi_n) \rightarrow m_C \text{ and } C_j(\Phi_n) \rightarrow C_j.$$

A proof of this theorem in the scalar case can be found in [3, §3.1]. We included a proof which does not use the local well-posedness of (CNLKG) (implicitly used in [3, Lemma 3.5]). Our proof relies on an improved version of the Convexity Inequality for Gradients, [18, Theorem 7.8, p., 177], outlined in Lemma 5.1. Theorem 1.2 implies that

$$X \ni \Phi \mapsto \mathbf{V}(\Phi) := (E(\Phi) - m_C)^2 + \sum_{j=1}^2 (C_j(\Phi) - C_j)^2 \quad (1.9)$$

is a Lyapunov function for Γ_C (see [3, Definition 2.4]).

Given a subset $S \subset H^1 \times \mathbb{R}^2$ and (u, ω) in K_C , we define the following subsets of $H^1 \times \mathbb{R}^2$:

$$B_\delta(S) := \{(w, \alpha) \mid d((w, \alpha), S) < \delta\}, \quad G(u, \omega) := \{(u(\cdot + y), \omega) \mid y \in \mathbb{R}^N\}.$$

We say that (u, ω) satisfies the condition (D) if there exists $\delta > 0$ such that, for every

$$(u', \omega') \in K_C \setminus \{(u, \omega)\}$$

such that

$$\Gamma(u', \omega') \neq \Gamma(u, \omega),$$

there holds

$$B_\delta(G(u, \omega)) \cap G(u', \omega') = \emptyset. \quad (\text{D})$$

Theorem 1.3 *The subset $\Gamma_C \subset X$ is stable. For every minimiser (u, ω) fulfilling condition (D), $\Gamma(u, \omega)$ is stable.*

We intentionally restricted our work to the higher dimensional case $N \geq 3$ and to $C_1 C_2 \neq 0$. We address to further works the treatment of the semi-trivial case ($C_j = 0$ for some $1 \leq j \leq 2$), and the lower dimensions $N = 1, 2$.

Results on the orbital stability of standing-wave solutions to coupled non-linear Klein-Gordon equations, with a different variational characterisation, have been obtained in [30]. Numerical results on the existence of coupled standing-waves have been obtained in [8] when $N = 3$ and the non-linearity has a critical exponent.

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2 Basic properties of the functional J

Proposition 2.1 *For every ρ in \mathbb{R}^2 with $\rho_j > 0$,*

(i) *J attains negative values on N_ρ ;*

(ii) *J is bounded from below and minimising sequences of J over N_ρ are bounded;*

moreover,

(iii) *J is continuous;*

(iv) *given a weakly converging sequence $u_n \rightharpoonup u$, up to extract a subsequence*

$$J(u_n - u) = J(u_n) - J(u) + o(1).$$

Proof. (i) By choosing a test function in a neighbourhood of the origin we can write $F = F_0 + F_\infty$, where

$$|F_0(z)| \leq c|z|^p, \quad |F_\infty(z)| \leq c|z|^q.$$

A sequence $(u_n)_{n \geq 1}$ such that $u_n \rightarrow u$ in H^1 , converges in $L^p(\mathbb{R}^N)$ by the Sobolev inequality

$$\|u\|_{L^p} \leq S \|u\|_{L^2}^{1 - \frac{N}{2} + \frac{N}{p}} \|Du\|_{L^2}^{\frac{N}{2} - \frac{N}{p}}. \quad (3.1)$$

There exists g in $L^p(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k \geq 1}$ such that

$$|u_{n_k}^j| \leq g$$

and $u_{n_k} \rightarrow u$ pointwise a.e., by [6, Théorème IV.9, p. 58]. Then

$$\int_{\mathbb{R}^N} F_0(u_{n_k}) \rightarrow \int_{\mathbb{R}^N} F_0(u)$$

by the dominated convergence theorem. We can extract a subsequence alike from every subsequence of $(u_n)_{n \geq 1}$. Then, the map $u \mapsto \int F_0 \circ u$ is continuous. Similarly, $u \mapsto \int F_\infty \circ u$ is continuous, and the gradient part of J is smooth. An adaptation of the technique used in [2, Theorem 2.6, p. 17] would allow to conclude that J is $C^1(H^1, \mathbb{R})$.

(ii) We refer to Step II of the Appendix of [4], which addresses the scalar case.

(iii) Following the proof of [4, Lemma 5], we can show that J attains negative values on N_ρ for every choice of ρ : setting

$$\lambda := (\rho_1^{-1} \rho_2)^{1/2}$$

and

$$u := (w, \lambda w), \quad w \in N_{\rho_1},$$

we have

$$J(u) = (1 + \lambda^2)^{-1} J_1(w),$$

where

$$\begin{aligned} J_1(w) &:= \frac{1}{2} \int_{\mathbb{R}^N} |Dw|^2 + \int_{\mathbb{R}^N} F_1(w) \\ J_1(s) &:= (1 + \lambda^2)^{-1} \left(-\mu \lambda^\gamma |s|^{2\gamma} + G(s, \lambda s) \right). \end{aligned}$$

By (A_1) and (A_2) the non-linearity F_1 fulfils hypotheses (F_ρ) and (F_2) of [4]. Then, by [4, Lemma 5], there exists w such that $J_1(w) < 0$. Then $J(u) < 0$.

(iv) By the Hölder inequality and (A_3) , we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} \sum_{j=1}^2 \|Du_j\|_{L^2}^2 - 2\mu (\|u_1\|_{L^{2\gamma}} \|u_2\|_{L^{2\gamma}})^\gamma \\ &\geq \frac{1}{2} \sum_{j=1}^2 \left(\|Du_j\|_{L^{2\gamma}}^2 - \mu \|u_j\|_{L^{2\gamma}}^{2\gamma} \right). \end{aligned} \tag{3.2}$$

From (3.1), there exists a constant c' such that

$$J(u) \geq c' \sum_{j=1}^2 \|Du_j\|_{L^2}^2 - \|Du_j\|_{L^2}^{2\gamma \left(\frac{N}{2} - \frac{N}{2\gamma} \right)}. \tag{3.3}$$

By the hypotheses on γ in (A_1) , the right member of the above inequality is bounded from below, as J is. Given a minimising sequence, $(u_n)_{n \geq 1}$ in N_ρ , for n large we have $J(u_n) < 0$, by (i). Then, $\|Du_n\|_{L^2}$ is bounded by (3.3). Because $\|u_n^j\|_{L^2}^2 = \rho_j$, the H^1 -norm is bounded too.

Proposition 2.2 *Given C in \mathbb{R}^2 such that $C_1 C_2 \neq 0$, the following properties hold:*

(i) *E is coercive;*

(ii) *critical points of E over M_C are solutions to the elliptic system (1.3);*

(iii) if (u, ω) is a minimiser, then for $j = 1, 2$, u_j is either positive or negative.

Proof. The proof of (i) follows from the arguments used in Step I of [3, Proof of Lemma 2.7].

(ii) If (u, ω) is a critical point, there are two Lagrange multipliers λ_1 and λ_2 such that

$$DE = \lambda_1 DC_1 + \lambda_2 DC_2.$$

Taking the projection on $H^1(\mathbb{R}^N, \mathbb{R}^2) \times \{0\}$, and on $\{0\} \times \mathbb{R}^2$, we obtain

$$\begin{aligned} -\Delta u_j + m_j^2 u_j + \partial_{z_j} F(u) &= 2\lambda_j \omega_j u_j, \\ \omega_j \|u_j\|_{L^2}^2 &= \lambda_j \|u_j\|_{L^2}^2 \end{aligned}$$

for $j = 1, 2$. Because $u_j \neq 0$ we obtain $\lambda_j = \omega_j$ and thus (1.3). By local regularity results, [13, §8], u is a classic solution.

(iii) We define

$$w_j := |u_j| \geq 0.$$

From (A_3) it follows that $(w, \omega) \in M_C$ and $E(u, \omega) = E(w, \omega)$. By (ii),

$$-\Delta w_j = (\omega_j^2 - m_j^2)w_j + \gamma \mu w_j^{\gamma-1} w_{\sigma(j)}^\gamma - \partial_{z_j} G(w)$$

where $\sigma(1) = 2$ and $\sigma(2) = 1$. Hence,

$$-\Delta w_j + \lambda_j w_j + \partial_{z_j} G(w) \geq 0,$$

where $\lambda_j := m_j^2 - \omega_j^2$. Let us define

$$A_j(x) = \begin{cases} \lambda_j + \partial_{z_j} G(w) w_j^{-1} & \text{if } w_j(x) \neq 0 \\ \lambda_j & \text{otherwise.} \end{cases}$$

By (A_2) , and the continuity of w_j and $\partial_{z_j} G$, we have A_j^+ is $C_+(\mathbb{R}^N)$. Therefore,

$$-\Delta w_j + A_j^+(x) w_j \geq 0.$$

Now, we can apply the strong maximum principle. Hence, $w_j > 0$.

3 The sub-additivity property of I

Given a non-negative function f , we denote with f^{*e} the Steiner symmetrization with respect to the direction e in \mathbb{R}^N (with $|e| = 1$), [18, §3.7, p. 87]. We denote with f^* the symmetric rearrangement, [18, §3.3, p. 80]. The next lemma addresses the one-dimensional case of [10, Proposition 1.4]. The argument goes back to B. Kawohl [17, Lemma 2.6, p. 33].

Lemma 3.1 *Let u, v be $H^1(\mathbb{R})$ non-negative functions with compact support, symmetric and radially decreasing with respect to the origin, and such that $u(0) \leq v(0)$. Let T be such that*

$$\text{supp}(u) \cap \text{supp}(v(\cdot - T)) = \emptyset.$$

Then

$$\|w^{*'}\|_{L^2}^2 \leq \|w'\|_{L^2}^2 - \frac{3}{4} \|u'\|_{L^2}^2$$

where $w(t) := u(t) + v(t - T)$.

Proof. We denote with $[-c, c]$ and $[-d, d]$ the support of u and v , respectively. Firstly, we prove the estimate under the additional assumptions that u and v are continuously differentiable and

$$tu'(t) < 0 \text{ on } \{t \in (-c, c), t \neq 0\} \quad (5.1)$$

$$tv'(t) < 0 \text{ on } \{t \in (-d, d), t \neq 0\}. \quad (5.2)$$

We set $a := \sup(u)$ and $b := \sup(v)$. The functions

$$u: [0, c] \rightarrow [0, a], \quad v: [0, d] \rightarrow [0, b]$$

are invertible because they are strictly decreasing. Their inverses, y_u and y_v , are continuously differentiable on $(0, a)$ and $(0, b)$, respectively. Thus,

$$u(y_u(s)) = s \text{ on } [0, a], \quad v(y_v(s)) = s \text{ on } [0, b]. \quad (5.3)$$

Because w^* is symmetric and decreasing, the level set $\{w^* \geq s\}$ is an interval. We denote its length by $2z(s)$. We have

$$2z(s) = |\{w^* \geq s\}| = \begin{cases} 2y_u(s) + 2y_v(s) & \text{if } s \in [0, a] \\ 2y_v(s) & \text{if } s \in [a, b]. \end{cases} \quad (5.4)$$

Thus, z is strictly decreasing and continuously differentiable for every $s \notin \{0, a, b\}$. Moreover,

$$w^*(z(s)) = s \text{ on } [0, b]. \quad (5.5)$$

Taking the derivative with respect to s in (5.5) and in (5.3), we have

$$w^{*'}(z(s))z'(s) = 1, \quad u'(y_u(s))y'_u(s) = 1, \quad v'(y_v(s))y'_v(s) = 1 \quad (5.6)$$

on the complement of a finite set. Hence,

$$\begin{aligned} \int_{\mathbb{R}} |w^{*'}|^2 dt &= 2 \int_0^{c+d} |w^{*'}|^2 dt = -2 \int_0^b |w^{*'}(z(s))|^2 z'(s) ds = -2 \int_0^b (z'(s))^{-1} ds \\ &= -2 \int_0^a (y'_u(s) + y'_v(s))^{-1} ds - 2 \int_a^b (y'_v(s))^{-1} ds. \end{aligned} \quad (5.7)$$

The second equality follows from a change of variable and the first of (5.6). The fourth equality follows from (5.4). From the inequality

$$2(x+y)^{-1} \leq x^{-1} + y^{-1} - \max\{x^{-1}, y^{-1}\}, \quad x, y > 0$$

the first integration of the second line of (5.7) can be estimated from above with

$$- \int_0^a ((y'_u(s))^{-1} + (y'_v(s))^{-1}) ds + \int_0^a \max\{y'_u(s)^{-1}, y'_v(s)^{-1}\} ds. \quad (5.8)$$

Using the estimate $2 \max\{t, s\} \geq t + s$, (5.8) and (5.7), the left member of the first equality in (5.7) is bounded by

$$\begin{aligned} & -\frac{1}{2} \int_0^a (y'_u(s))^{-1} ds - \frac{1}{2} \int_0^a (y'_v(s))^{-1} ds - 2 \int_a^b (y'_v(s))^{-1} ds \\ & \leq -\frac{1}{2} \int_0^a (y'_u(s))^{-1} ds - 2 \int_0^b (y'_v(s))^{-1} ds \\ & = \frac{1}{4} \cdot \left(-2 \int_0^a (y'_u(s))^{-1} \right) + \left(-2 \int_0^b (y'_v(s))^{-1} \right) ds. \end{aligned}$$

From a change of variable and (5.6), it follows that

$$\|u'\|_{L^2}^2 = -2 \int_0^a (y'_u(s))^{-1} ds, \quad \|v'\|_{L^2}^2 = -2 \int_0^b (y'_v(s))^{-1} ds.$$

Thus, from (5.7), we obtain

$$\|w^{*'}\|_{L^2}^2 \leq \frac{1}{4} \|u'\|_{L^2}^2 + \|v'\|_{L^2}^2 = \|w'\|_{L^2}^2 - \frac{3}{4} \|u'\|_{L^2}^2. \quad (5.9)$$

In the general case, we can approximate u and v with functions satisfying (5.1) and (5.2): firstly, we consider

$$\sigma_u: [0, c] \rightarrow \mathbb{R}^+, \quad \sigma'_u(t) < 0 \text{ on } (0, c), \quad \sigma'_u(0) = 0 \quad (5.10)$$

smooth, and extend it to \mathbb{R} as $\sigma_u(-t) = \sigma_u(t)$. We define

$$U := u + \|u - v\|_{L^\infty(0, \delta)} \sigma_u, \quad u_\delta := \rho_\delta * U \quad (5.11)$$

where ρ_δ is a symmetric mollifier. Thus, u_δ is an even function. Because U is strictly decreasing, given $t \geq 0$, we have

$$u'_\delta(t) = \int_0^\delta \rho'_\delta(y)(U(t-y) - U(t+y))dy < 0,$$

unless $t = 0$. Similarly, we define σ_v as in (5.10) with the additional hypothesis

$$\sigma_u(0) < \sigma_v(0) - 1.$$

V and v_δ are defined as in (5.11), by replacing σ_u with σ_v . Thus, if $\delta > 0$ is sufficiently small,

$$\sup(u_\delta) \leq \sup(v_\delta)$$

and the supports of u_δ and $v_\delta(\cdot - T)$ are disjoint. Therefore, we can apply estimate (5.9) to

$$w_\delta = u_\delta + v_\delta(\cdot - T)$$

and obtain

$$\|w_\delta^{*'}\|_{L^2}^2 \leq \|w'_\delta\|_{L^2}^2 - \frac{3}{4} \|u'_\delta\|_{L^2}^2.$$

By the continuity of the symmetric rearrangement in $H^1(\mathbb{R})$, [12], we can take the limit as $\delta \rightarrow 0$ in the above inequality.

Proposition 3.1 *Let ρ, τ be such that $\rho_j \geq \tau_j > 0$ and $\tau \neq \rho$. Then,*

$$I(\rho) < I(\tau) + I(\rho - \tau).$$

Proof. Define $\sigma := \rho - \tau$, and let

$$(u_n)_{n \geq 1} \subset N_\tau, \quad (v_n)_{n \geq 1} \subset N_\sigma \quad (5.12)$$

be minimising sequences of J over N_τ and N_σ , respectively. By (iii) of Proposition 2.1, we can suppose that each of the sequences have compact support, that u_n^j and v_n^j are non-negative, from (A_3) , and symmetrically decreasing, by (A_1) , (A_4) , [26, Lemma 1] and [18, Theorem 3.4, p. 82].

We set $e_N := (0, \dots, 0, 1)$. Let $(T_n)_{n \geq 1}$ be a real sequence such that the two functions

$$u_n^i, v_n^j(\cdot + T_n e_N)$$

have disjoint support for every i, j in $\{1, 2\}$. Then,

$$w_n := u_n + v_n(\cdot + T_n e_N) \in N_\rho \quad (5.13)$$

$$J(w_n) = J(u_n) + J(v_n). \quad (5.14)$$

We denote the Steiner symmetrization of w_n with respect to e_N with $w_n^{*e_N}$. By [17, (C), p. 22], $w_n^{*e_N} \in N_\rho$. From [18, (v), p. 81], and [18, Eq. (1), p. 82],

$$-\int_{\mathbb{R}^N} |w_n^{1*e_N} w_n^{2*e_N}|^\gamma dx \leq -\int_{\mathbb{R}^N} |w_n^1 w_n^2|^\gamma dx.$$

Along with (A_4) , the above inequality yields

$$\int_{\mathbb{R}^N} F(w_n^{*e_N}) \leq \int_{\mathbb{R}^N} F(w_n).$$

By [26, Lemma 1],

$$\|Dw_n^{j*e_N}\|_{L^2} \leq \|Dw_n^j\|_{L^2}. \quad (5.15)$$

Thus $J(w_n^{*e_N}) \leq J(w_n)$. Given $x' \in \mathbb{R}^{N-1}$,

$$\partial_{x_N} w_n^{j*e_N}(x', t) = w_n^{j*}(x', \cdot)'(t).$$

Then, we can write

$$\begin{aligned} \int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N}|^2 dx &= \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' \\ &= \int_{U_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' \\ &\quad + \int_{V_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' =: A_1^j + A_2^j \end{aligned} \quad (5.16)$$

where

$$\begin{aligned} U_n^j &= \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} u_n^j(x', \cdot) \leq \sup_{\mathbb{R}} v_n^j(x', \cdot)\} \\ V_n^j &= \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} v_n^j(x', \cdot) < \sup_{\mathbb{R}} u_n^j(x', \cdot)\}. \end{aligned}$$

For every $x' \in \mathbb{R}^{N-1}$, $u_n^j(x', \cdot)$ and $v_n^j(x', \cdot)$ satisfy the hypotheses of Lemma 3.1 with $T = T_n$. Thus,

$$\begin{aligned} A_1^j &\leq \int_{U_n^j} \left(\|w_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4} \|u_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 \right) dx' \\ A_2^j &\leq \int_{V_n^j} \left(\|w_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4} \|v_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 \right) dx'. \end{aligned}$$

Taking the sum, we obtain

$$\begin{aligned} A_1^j + A_2^j &\leq \|\partial_{x_N} w_n^j\|_{L^2}^2 \\ &\quad - \frac{3}{4} \left(\|\partial_{x_N} u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|\partial_{x_N} v_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2 \right). \end{aligned}$$

Because u_n^j and $|\partial_{x_i} u^j|$ are radially symmetric, we have

$$\|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 = N \|\partial_{x_N} u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2.$$

From (5.13), it follows that

$$N\|\partial_{x_N} w_n^j\|_{L^2}^2 = \|Dw_n^j\|_{L^2}^2.$$

Thus,

$$\begin{aligned} N(A_1^j + A_2^j) &\leq \|Dw_n^j\|_{L^2}^2 \\ &\quad - \frac{3}{4} \left(\|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|Dv_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2 \right). \end{aligned} \quad (5.17)$$

We define

$$d_n^j = \|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|Dv_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2.$$

We prove that $(d_n^j)_{n \geq 1}$ is bounded from below. On the contrary, up to extract a subsequence, we can suppose that $d_n^j \rightarrow 0$ for some $1 \leq j \leq 2$. Because u_n and v_n are minimising sequences, by (ii) of Proposition 2.1, they are also bounded in H^1 . By construction, u_n and v_n are radially decreasing. Then, by [5, Theorem A.I'], up to extract a subsequence, we can suppose that

$$u_n^j \rightarrow u_j, v_n^j \rightarrow v_j \text{ in } L^{2\gamma}(\mathbb{R}^N), \text{ a.e.}$$

By (i) of Proposition 2.1 and the first inequality in (3.2)

$$\|u_n^j\|_{L^{2\gamma}}, \|v_n^j\|_{L^{2\gamma}} \geq c = c(\rho, \tau) > 0, \quad (5.18)$$

whence $u_j, v_j \neq 0$. We fix $R > 0$ and consider the domains

$$E_n^j := (U_n^j \times \mathbb{R}) \cap B_R, \quad F_n^j := (V_n^j \times \mathbb{R}) \cap B_R. \quad (5.19)$$

Because the two domains are bounded,

$$\begin{aligned} d_n^j &\geq \frac{1}{m(E_n^j)} \cdot \|Du_n^j\|_{L^1(E_n^j)}^2 + \frac{1}{m(F_n^j)} \cdot \|Dv_n^j\|_{L^1(F_n^j)}^2 \\ &\geq \frac{1}{\omega_N R^N} \left(\|Du_n^j\|_{L^1(E_n^j)}^2 + \|Dv_n^j\|_{L^1(F_n^j)}^2 \right). \end{aligned} \quad (5.20)$$

Up to extract a subsequence there are two sets $U_j, V_j \subset \mathbb{R}^{N-1}$ such that the convergence

$$\chi_{U_n^j} \rightarrow \chi_{U_j}, \quad \chi_{V_n^j} \rightarrow \chi_{V_j}$$

is strong in $L^2(B_R^{N-1})$, where $B_R^{N-1} := B_R \cap (\mathbb{R}^{N-1} \times 0)$. Moreover, U_j and V_j are radially symmetric and the convergence

$$\chi_{E_n^j} \rightarrow \chi_{E_j}, \quad \chi_{F_n^j} \rightarrow \chi_{F_j}$$

is strong in $L^2(B_R)$, where

$$E_j = (U_j \times \mathbb{R}) \cap B_R, \quad F_j = (V_j \times \mathbb{R}) \cap B_R.$$

Taking the limit in (5.20), we obtain

$$Du_j \equiv 0, \quad E_j \text{ a.e.}, \quad Dv_j \equiv 0, \quad \text{on } F_j \text{ a.e.}$$

whence

$$Du_j \equiv 0 \text{ on } U_j, \quad Dv_j \equiv 0 \text{ on } V_j \quad (5.21)$$

and

$$u_j \leq v_j \text{ on } U_j, \quad v_j \leq u_j \text{ on } V_j. \quad (5.22)$$

By the Ekeland Principle, we can suppose that the sequences in (5.12) are Palais-Smale. Therefore, u_j and v_j are weak solutions to an elliptic system and, by local regularity results, continuously differentiable. Thus, we can suppose that U_j is open and V_j is closed. Because such sets are radially symmetric, we can write

$$U_j = \{x' \in B_R^{N-1} \mid |x'| \in \Omega\}, \quad V_j = \{x' \in B_R^{N-1} \mid |x'| \in G\}$$

where Ω and G are open and closed subsets of $\langle e_1 \rangle$. We set

$$\Omega_1 := \Omega \cap \{te_1 \mid t > 0\}, \quad G_1 := G \cap \{te_1 \mid t > 0\}.$$

Then

$$\Omega_1 = \bigcup_{i \in \mathbb{Z}} (a_i, b_i), \quad a_i \leq b_i, \quad G_1 = \bigcup_{i \in \mathbb{Z}} [b_i, a_{i+1}].$$

For every $i \in \mathbb{Z}$, v_j is constant on $[b_i, a_{i+1}]$ by (5.21). Thus,

$$v_j(b_i) = v_j(a_{i+1}). \quad (5.23)$$

In the case $b_i = a_{i+1}$ the above equality is obviously true. By the continuity of u_j and v_j , and (5.22) and (5.21), it follows

$$u_j(b_i) = v_j(b_i), \quad u_j(a_{i+1}) = v_j(a_{i+1}) \quad (5.24)$$

$$u_j \equiv c_i \text{ on } (a_i, b_i) \quad (5.25)$$

for some constant $c_i \in \mathbb{R}$. From (5.23) and (5.24) we have

$$c_i = u_j(b_i) = v_j(b_i) = v_j(a_{i+1}) = u_j(a_{i+1}) = c_{i+1}.$$

Given $x \in [b_i, a_{i+1}]$

$$c_i \geq u_j(x) \geq c_{i+1} = c_i,$$

because u_j is monotonically non-increasing. Then, u_j is constant on $\{te_1 \mid t > 0\}$. Because u_j is radially symmetric, u_j is constant on B_R . By applying the same argument for every $R > 0$, we obtain that u_j is constant on \mathbb{R}^N . Because u_j is L^2 , we have $u_j \equiv 0$ obtaining a contradiction with (5.18). The contradiction follows from the assumption that $d_n^j \rightarrow 0$. So, we proved that each of the sequences $(d_n^j)_{n \geq 1}$ is bounded from away from zero. Let d be such that

$$d_n^j \geq d \text{ for all } n.$$

Then, from (5.16), (5.17) we obtain

$$N \int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N}|^2 dx \leq \|Dw_n^j\|_{L^2}^2 - \frac{3d_n^j}{4} \leq \|Dw_n^j\|_{L^2}^2 - \frac{3d}{4}. \quad (5.26)$$

Finally, we consider the decreasing rearrangement of $w_n^{j*e_N}$. By applying (5.15) in dimension $N = 1$, we have

$$\begin{aligned} \|\partial_{x_N} w_n^{j*e_N*}\|_{L^2}^2 &= \int_{\mathbb{R}^{N-1}} \|w_n^{j*e_N*}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' \\ &\leq \int_{\mathbb{R}^{N-1}} \|w_n^{j*e_N}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' = \|\partial_{x_N} w_n^{j*e_N}\|_{L^2}^2. \end{aligned}$$

From (5.26), we note

$$N \int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N}|^2 \leq \|Du_n^j\|_{L^2}^2 + \|Dv_n^j\|_{L^2}^2 - \frac{3d}{4}.$$

Because $w_n^{j*e_{N^*}}$ is radially symmetric, from (5.26) it follows that

$$\int_{\mathbb{R}^N} |Dw_n^{j*e_{N^*}}|^2 \leq \|Du_n^j\|_{L^2}^2 + \|Dv_n^j\|_{L^2}^2 - \frac{3d}{4}$$

and

$$J(w_n^{j*e_{N^*}}) \leq J(w_n^{j*e_N}), \quad w_n^{j*e_{N^*}} \in N_\rho.$$

Hence,

$$I(\rho) \leq J(w_n^{j*e_{N^*}}) \leq J(u_n) + J(v_n) - \frac{3d}{4}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$I(\rho) \leq I(\tau) + I(\sigma) - \frac{3d}{4}.$$

We set $D := 3d/4 > 0$.

4 Minimising sequences of (J, N_ρ) and (E, M_C)

Lemma 4.1 *Let $(u_n)_{n \geq 1}$ be a bounded sequence in H^1 such that*

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n^1 u_n^2|^\gamma > 0$$

where $1 < \gamma < 2^/2$. Then, there exist $u \in H^1$ and a sequence $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that*

$$u_n^j(\cdot - y_n) \rightharpoonup u_j, \quad u_1 u_2 \not\equiv 0.$$

Proof. Let $w_n = u_n^1 u_n^2$. From the Schwarz inequality, we have

$$w_n \in L^1(\mathbb{R}^N);$$

by applying the Hölder inequality with the pair of exponents

$$\left(\frac{2(N-1)}{N}, \frac{2(N-1)}{N-2} \right),$$

we obtain

$$Dw_n \in L^{N/(N-1)}(\mathbb{R}^N).$$

We use [20, Lemma I.1] with $q = 1$ and $p = N/(N-1)$. Hence, given $R > 0$, either there exists a sequence $(y_n)_{n \geq 1}$ such that

$$\liminf_{n \rightarrow \infty} \int_{B(-y_n, R)} |w_n| > 0 \tag{6.1}$$

or

$$w_n \rightarrow 0 \text{ in } L^\alpha(\mathbb{R}^N), \quad \alpha \in (1, N/(N-2)).$$

The latter is ruled out by the hypothesis on γ . Hence, (6.1) holds. We set

$$v_n^j := u_n^j(\cdot - y_n)$$

and obtain

$$\liminf_{n \rightarrow \infty} \int_{B_R} |v_n^1 v_n^2| > 0. \tag{6.2}$$

Because v_n^j are bounded in H^1 , we can suppose that they converge weakly to some limits u_1 and u_2 , respectively. By the Rellich-Kondrakhov Theorem, we can suppose that such convergence is strong in $L^2(B_R)$. Thus, (6.2) yields

$$\int_{B_R} u_1 u_2 > 0$$

which implies $u_1 u_2 \not\equiv 0$.

Theorem 4.1 *Let $(u_n)_{n \geq 1}$ be a minimising sequence for J over N_ρ . Then, there exists $u \in N_\rho$ and a sequence $(y_n)_{n \geq 1}$ such that*

$$u_n = u(\cdot + y_n) + o(1) \text{ in } H^1$$

$$J(u) = \inf_{N_\rho} J.$$

Proof. By (i) and (ii) of Proposition 2.1, $I(\rho) < 0$ and the sequence $(u_n)_{n \geq 1}$ is bounded. Because $G \geq 0$, $(u_n)_{n \geq 1}$ fulfils the hypothesis of Lemma 4.1 if $\gamma < N/(N-2)$ holds. This, in turn, follows from (A_1) and

$$1 + \frac{2}{N} < \frac{N}{N-2}.$$

Then, we consider the sequence $(y_n)_{n \geq 1}$ and $u \in H^1$ given by Lemma 4.1. We define

$$v_n := u_n(\cdot - y_n) - u, \quad \tau := (\|u_1\|_{L^2}^2, \|u_2\|_{L^2}^2).$$

Note that $\tau_j \leq \rho_j$ by the weak lower semi-continuity property of the L^2 -norm and that $\tau_j > 0$, from Lemma 4.1. Suppose that $\tau \neq \rho$. By (iv) of Proposition 2.1, up to extract a subsequence, we can suppose that

$$J(v_n) = J(u_n(\cdot - y_n)) - J(u) + o(1).$$

After a change of variable, the first term of the right member equals $J(u_n)$, which converges to $I(\rho)$. Hence, by Proposition 3.1

$$I(\rho - \tau) \leq I(\rho) - I(\tau) < I(\rho - \tau).$$

Thus, we obtain a contradiction with the assumption that $\tau \neq \rho$. Then $\tau = \rho$ and $u \in N_\rho$. Thus,

$$u_n^j(\cdot - y_n) - u_j \rightarrow 0 \text{ in } L^2(\mathbb{R}^N).$$

Up to extract a subsequence, we can suppose that the above convergence is weak in H^1 . We set $w_n := u_n(\cdot - y_n)$. By (3.1), the above convergence holds in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. Therefore, as in the proof of (iii) of Proposition 2.1, we conclude that

$$\int_{\mathbb{R}^N} F(w_n) \rightarrow \int_{\mathbb{R}^N} F(u).$$

We have

$$J(w_n) = \int_{\mathbb{R}^N} F(w_n) + \frac{1}{2} \int_{\mathbb{R}^N} |Dw_n|^2 \geq \int_{\mathbb{R}^N} F(u) + \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 = J(u).$$

Because $(w_n)_{n \geq 1}$ is a minimising sequence, taking the limit, we obtain

$$I(\rho) = \int_{\mathbb{R}^N} F(u) + \lim_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} |Dw_n|^2 \geq \int_{\mathbb{R}^N} F(u) + \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 = J(u) \geq I(\rho).$$

Then, the two above inequalities are equalities:

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |Dw_n|^2 = \int_{\mathbb{R}^N} |Du|^2, \quad J(u) = I(\rho).$$

Thus, $Dw_n \rightarrow Du$ strongly in L^2 and u is a minimiser.

Proof of Theorem 1.1 From (i) of Proposition 2.2, given a minimising sequence (u_n, ω_n) , there exists ρ such that

$$\|u_n^j\|_{L^2} \rightarrow \sqrt{\rho_j} > 0, \quad \omega_n \rightarrow \omega.$$

As in Step II of the proof of [3, Lemma 2.7], it can be shown that

$$v_n^j = \frac{\sqrt{\rho_j} u_n^j}{\|u_n^j\|_{L^2}}$$

is a minimising sequence for J over N_ρ (notice that, unlike stated in [3, p. 13], their proof requires only a combined power-type estimate on DF , as in (A_2) , rather than the condition (H_3) of [3]). Then, by Theorem 4.1, there exists a sequence $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that

$$v_n(\cdot + y_n) \rightarrow u \text{ in } H^1$$

for some $u \in H^1$. Then, $(u, \omega) \in M_C$ is a minimiser of E over M_C .

5 Stability results

Lemma 5.1 *Let ϕ be a $H^1(\mathbb{R}^N, \mathbb{R}^k)$ function. Then $|\phi|$ is $H^1(\mathbb{R}^N)$ and*

$$\|D\phi\|_{L^2} \geq \|D|\phi|\|_{L^2}. \quad (7.1)$$

Suppose that for every bounded subset $S \subset \mathbb{R}^N$ $\text{ess inf}_S |\phi| > 0$. If equality holds between the two above norms, then there exists λ in \mathbb{R}^k such that $|\lambda| = 1$ and

$$\phi(x) = \lambda |\phi(x)|. \quad (7.2)$$

Proof. The proof of the fact that $|\phi|$ is $H^1(\mathbb{R}^N, \mathbb{R}^k)$ follows the same steps of the case $k = 2$ in [18, Theorem 6.17, p. 152]. Then

$$\partial_{x_i} |\phi| = \begin{cases} \frac{\langle \phi, \partial_{x_i} \phi \rangle}{|\phi|} & \text{if } \phi \neq 0 \\ 0 & \text{if } \phi = 0 \end{cases}$$

for every $1 \leq i \leq N$. By the Schwarz inequality,

$$\|D|\phi|\|^2 = \sum_{i=1}^N \|\partial_{x_i} |\phi|\|^2 = \frac{1}{|\phi|^2} \sum_{i=1}^N |\langle \phi, \partial_{x_i} \phi \rangle|^2 \leq \sum_{i=1}^N \|\partial_{x_i} \phi\|^2 = \|D\phi\|^2 \quad (7.3)$$

if $\phi \neq 0$. On the region $\{\phi = 0\}$, the same inequality follows easily. Then $D|\phi|$ is L^2 . By integrating (7.3), we prove the first part of the statement. Now, we suppose that in (7.1) the equality holds and $|\phi|$ is essentially bounded from below on every bounded subset of \mathbb{R}^N . From (7.3) we obtain

$$|\phi| \|\partial_{x_i} \phi\| = |\langle \phi, \partial_{x_i} \phi \rangle|.$$

Because $\phi(x) \neq 0$ a.e., there exists $\mu_i: \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\partial_{x_i} \phi = \mu_i \phi \text{ a.e.} \quad (7.4)$$

We claim that each of the functions

$$\Lambda_j: \mathbb{R}^N \rightarrow \mathbb{R}, \quad x \mapsto \frac{\phi_j(x)}{|\phi(x)|}$$

is constant. From the same approximation argument as [18, Theorem 6.16, p. 178], it follows that Λ_j is $H_{loc}^1(\mathbb{R}^N)$ and

$$|\phi|^3 \partial_{x_i} \Lambda_j = \partial_{x_i} \phi_j |\phi|^2 - \phi_j \langle \phi, \partial_{x_i} \phi \rangle = \sum_{h=1}^k \partial_{x_i} \phi_j \phi_h^2 - \phi_j \phi_h \partial_{x_i} \phi_h = \mu_i \sum_{h=1}^k \phi_j \phi_h^2 - \phi_j \phi_h^2 = 0.$$

The last equality follows from (7.4). So, there exists λ_j in \mathbb{R} with $\Lambda_j \equiv \lambda_j$ a.e. which satisfies (7.2).

A similar result has been proved in [18, Theorem 7.8] in the case $k = 2$, under the assumption that one of the components of ϕ is positive almost everywhere.

Let C be such that $C_j \neq 0$ for $j = 1, 2$. For every (ϕ, ϕ_t) in X such that $\phi_j \neq 0$, we define the map

$$X \ni (\phi, \phi_t) \mapsto \mathbf{P}(\phi, \phi_t) := \left(|\phi_1|, |\phi_2|, \frac{C_1}{\|\phi_1\|_{L^2}^2}, \frac{C_2}{\|\phi_2\|_{L^2}^2} \right) \in M_C. \quad (7.5)$$

Proposition 5.1 *For every $\Phi := (\phi, \phi_t)$ such that $\phi_j \neq 0$, for $j = 1, 2$, there holds*

$$\mathbf{E}(\Phi) \geq E(\mathbf{P}(\Phi)), \quad \mathbf{C}_j(\Phi) = C_j(\mathbf{P}(\Phi)).$$

In the proposition \mathbf{E} and \mathbf{C}_j are the energy and charges defined in (1.6) and (1.7).

Proof. From the Schwarz inequality, we obtain

$$\frac{|\mathbf{C}_j(\phi, \phi_t)|}{\|\phi_j\|_{L^2}} \leq \|\phi_t^j\|_{L^2}. \quad (7.6)$$

By Lemma 5.1 and (7.6),

$$\begin{aligned} \mathbf{E}(\phi, \phi_t) &= \frac{1}{2} \int_{\mathbb{R}^N} |D\phi|^2 + |\phi_t|^2 + 2V(\phi) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} |D\phi|^2 + 2V(|\phi_1|, |\phi_2|) + \frac{1}{2} \sum_{j=1}^2 \frac{\mathbf{C}_j(\phi, \phi_t)^2}{\|\phi_j\|_{L^2}^2} \\ &= E(\mathbf{P}(\Phi)), \end{aligned}$$

and

$$C_j(\mathbf{P}(\Phi)) = \frac{\mathbf{C}_j(\Phi)}{\|\phi_j\|_{L^2}^2} \int_{\mathbb{R}^N} |\phi_j|^2 = \mathbf{C}_j(\Phi).$$

Proof of Theorem 1.2. Given Φ in Γ_C there exists (u, ω) in K_C and (λ, y) in $\mathbb{T}^2 \times \mathbb{R}^N$ such that

$$\Phi = (\lambda \cdot u(\cdot + y), -i\omega \cdot \lambda \cdot u(\cdot + y)).$$

We used the notation introduced in (1.2). Then

$$\mathbf{E}(\Phi) = E(u, \omega) = m_C, \quad \mathbf{C}_j(\Phi) = \omega_j \|u_j\|_{L^2}^2 = C_j.$$

Because \mathbf{E} and \mathbf{C}_j are continuous, if $d(\Phi_n, \Gamma_C) \rightarrow 0$, then

$$\mathbf{E}(\Phi_n) \rightarrow m_C, \quad \mathbf{C}_j(\Phi_n) \rightarrow C_j. \quad (7.7)$$

We prove the converse and suppose that (7.7) holds. We set

$$\Phi_n := (\phi_n, \phi_n^t).$$

Because $C_j \neq 0$ for $j = 1, 2$, $\phi_n^j \not\equiv 0$ for n large enough. Then, it makes sense to define

$$(u_n, \omega_n) := \mathbf{P}(\Phi_n). \quad (7.8)$$

From Proposition 5.1 and (7.7), $(u_n, \omega_n)_{n \geq 1}$ is a minimising sequence of E over M_C . By Theorem 1.1, there are

$$(u, \omega) \in K_C, \quad (y_n)_{n \geq 1} \subset \mathbb{R}^N$$

such that

$$u_n = u(\cdot + y_n) + o(1), \quad \omega_n = \omega + o(1). \quad (7.9)$$

We set

$$\psi_n := \phi_n(\cdot - y_n), \quad \psi_n^t := \phi_n^t(\cdot - y_n).$$

By a change of variable, we have

$$\mathbf{E}(\psi_n, \psi_n^t) = \mathbf{E}(\phi_n, \phi_n^t), \quad \mathbf{C}_j(\psi_n, \psi_n^t) = \mathbf{C}_j(\phi_n, \phi_n^t). \quad (7.10)$$

Up to extract a subsequence, we can suppose that there exists (ψ, ψ_t) in X such that

$$\psi_n \rightharpoonup \psi \text{ in } H^1(\mathbb{R}^N, \mathbb{C}^2), \quad \psi_n^t \rightharpoonup \psi_t \text{ in } L^2(\mathbb{R}^N, \mathbb{C}^2). \quad (7.11)$$

By the weak lower semi-continuity of the norm, the strong convergence of $|\psi_n|$, (7.6) and Lemma 5.1, we have

$$\begin{aligned} \mathbf{E}(\psi_n, \psi_n^t) &= \frac{1}{2} \int_{\mathbb{R}^N} |D\psi_n|^2 + |\psi_n^t|^2 + 2V(\psi_n) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} |D\psi|^2 + |\psi_t|^2 + 2V(\psi) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} |D\psi|^2 + 2V(|\psi_1|, |\psi_2|) + \frac{1}{2} \sum_{j=1}^2 \frac{\mathbf{C}_j(\psi, \psi_t)^2}{\|\psi_j\|_{L^2}^2} \geq m_C. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, by (7.10), the first of (7.7) and the first of the above inequalities, we obtain

$$\lim_{n \rightarrow \infty} \|\psi_n^t\|_{L^2} = \|\psi_t\|_{L^2}, \quad \lim_{n \rightarrow \infty} \|D\psi_n\|_{L^2} = \|D\psi\|_{L^2}. \quad (7.12)$$

From the second inequality, we obtain

$$\int_{\mathbb{R}^N} |D\psi_j|^2 = \int_{\mathbb{R}^N} |D|\psi_j||^2, \quad \frac{\mathbf{C}_j(\psi, \psi_t)}{\|\psi_j\|_{L^2}} = \|\psi_t^j\|_{L^2}. \quad (7.13)$$

The weak limit in (7.11) and the strong convergence of $|\psi_n^j|$ to u_j implies that

$$|\psi_j| = u_j \text{ a.e.} \quad (7.14)$$

and

$$\psi_n^j \rightarrow \psi_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}). \quad (7.15)$$

Because (u, ω) is a minimiser of E over M_C , u_j are regular, by (ii), and positive, by (iii) of Proposition 2.2 and (7.14). Thus, ψ_j fulfils the hypotheses of Lemma 5.1. From (7.13) there are λ_j in \mathbb{C} such that $|\lambda_j| = 1$ and

$$\psi_j = \lambda_j |\psi_j| = \lambda_j u_j.$$

The second limit in (7.12) and the first in (7.11) yield

$$D\psi_n^j \rightarrow D\psi_j.$$

By (7.15),

$$\psi_n^j \rightarrow \lambda_j u_j \text{ in } H^1(\mathbb{R}^N, \mathbb{C}). \quad (7.16)$$

The second equality in (7.13) can be written as

$$\operatorname{Re} \int_{\mathbb{R}^N} \overline{-i\psi_j} \cdot \psi_t^j = \|\psi_t^j\|_{L^2} \|\psi_j\|_{L^2}.$$

Thus, we have an equality between the scalar product and the product of norms. Then

$$\psi_t^j = -i \frac{C_j}{\|\psi_j\|_{L^2}^2} \psi_j. \quad (7.17)$$

From (7.5) and (7.8), we have

$$\omega_n^j = \frac{C_j}{\|\phi_n^j\|_{L^2}^2}.$$

Taking the limit, we obtain

$$\omega_j = \frac{C_j}{\|\psi_j\|_{L^2}^2}.$$

Then (7.17) can be written as

$$\psi_t^j = -i\omega_j \psi_j.$$

By the second limit in (7.11) and the first limit of (7.12)

$$\psi_n^{t,j} \rightarrow \psi_t^j = -i\omega_j \lambda_j u_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}). \quad (7.18)$$

Thus, (7.16) and (7.18) yield

$$d((\psi_n, \psi_n^t), \Gamma_C) \rightarrow 0$$

so that $d((\phi_n, \phi_n^t), \Gamma_C) \rightarrow 0$.

Proof of Theorem 1.3. The proof of the stability of Γ_C follows from the fact that \mathbf{V} , defined in (1.9), is a Lyapunov function (see [3, Definition 2.4]) and from the definition of orbital stability. We prove that $\Gamma(u, \omega)$ is stable if condition (D) is satisfied. We argue by contradiction and suppose that there exists $\varepsilon_0 > 0$ and $(t_n, \Phi_n)_{n \geq 1}$ such that

$$d(\Phi_n, \Gamma(u, \omega)) \rightarrow 0, \quad d(U(t_n, \Phi_n), \Gamma(u, \omega)) \geq \varepsilon_0.$$

Thus, there exists (u', ω') in K_C such that

$$\Gamma(u', \omega') \neq \Gamma(u, \omega)$$

and

$$d(U(t_n, \Phi_n), \Gamma(u', \omega')) \rightarrow 0 \quad (7.19)$$

By Theorem 1.2, $\mathbf{E}(U(t_n, \Phi_n)) \rightarrow m_C$ and

$$(\mathbf{P}(U(t_n, \Phi_n)))_{n \geq 1}$$

is a minimising sequence of E over M_C . By Theorem 1.1, up to extract a subsequence,

$$\mathbf{P}(U(t_n, \Phi_n)) \rightarrow G(u'', \omega''). \quad (7.20)$$

for some (u'', ω'') in K_C . By (7.19),

$$\Gamma(u'', \omega'') = \Gamma(u', \omega').$$

Now we set

$$E_\delta := \inf_{\partial B_\delta} E > m_C.$$

The inequality follows from Theorem 1.1 and condition (D). For n large enough,

$$\mathbf{E}(U(t, \Phi_n)) = \mathbf{E}(\Phi_n) < E_\delta$$

for every $t \in \mathbb{R}$. By Proposition 5.1,

$$E_\delta > E(\mathbf{P}(U(t, \Phi_n))).$$

Our assumption on the regularity of the solutions of (CNLKG), ensures that $U(\cdot, \Phi_n)$ is continuous in H^1 . Then,

$$\mathbf{P}(U(t_n, \Phi_n)) \in B_\delta(G(u, \omega))$$

otherwise the path $\mathbf{P}(U(\cdot, \Phi_n))$ intersects the boundary of B where $E \geq E_\delta$. By (7.20), $G(u', \omega') \cap B_\delta \neq \emptyset$, so contradicting (D).

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