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Positive Solutions of the Dirichlet Problem for the One-dimensional Minkowski-Curvature Equation

Dedicated to Fabio Zanolin on the occasion of his sixtieth birthday

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Abstract

We discuss existence and multiplicity of positive solutions of the Dirichlet problem for the quasilinear ordinary differential equation

$$-\Big(u'/\sqrt{1-u'^2}\Big)' = f(t,u).$$

Depending on the behaviour of $f = f(t,s)$ near $s = 0$, we prove the existence of either one, or two, or three, or infinitely many positive solutions. In general, the positivity of f is not required. All results are obtained by reduction to an equivalent non-singular problem to which variational or topological methods apply in a classical fashion.

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1 Introduction

In this work we are concerned with the existence and the multiplicity of positive solutions of the quasilinear two-point boundary value problem

$$\begin{cases} -\Big(u'/\sqrt{1-u'^2}\Big)' = f(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases} \tag{1.1}$$

This is the one-dimensional version of the Dirichlet problem associated with the Minkowski-curvature equation

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which, as it is well-known, plays a significant role in differential geometry and in the theory of relativity. We refer, for motivations and results, to the classical papers of Bartnik and Simon [2] and of Gerhardt [6] and to the references contained therein.

It is worthy to point out that, as a consequence of the results in [2] and [6] (rediscovered in the one-dimensional case by a different and simpler approach in [3, Section 3]), problems (1.1) and (1.2) have a solution whatever f is. Nevertheless, since in our present study problem (1.1) generally admits the null solution, it may have some interest to investigate the existence of non-trivial, in particular positive, solutions: this is our aim here.

As a first step we show that problem (1.1) can always be reduced to an equivalent one, where the singularity on the left of the equation has been removed and the function on the right is bounded, actually vanishes outside the rectangle $[0,T]\times[-T,T]$, and agrees with f in a right neighbourhood of $s = 0$. Such a reduction, which is achieved by quite elementary estimates, is different depending on whether we use topological methods, in fact bifurcation theory, or variational methods. In the former case we replace the equation in (1.1) by

$$-u'' = g(t,u)\,h(u'), \tag{1.3}$$

where g is bounded and h has compact support. In the latter case, which is slightly more delicate as we have to preserve the variational structure of the problem, we substitute the equation in (1.1) with

$$-(\psi(u'))' = g(t,u), \tag{1.4}$$

where ψ is an asymptotically linear increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, such that $\psi(y) = y/\sqrt{1-y^2}$ near $y = 0$, and again g is bounded.

It is evident from this discussion that in this frame only conditions near $s = 0$ are needed for proving the existence of positive solutions.

In order to describe our results, we write the function g, appearing on the right of (1.3) and (1.4), in the form

$$g(t, s) = \lambda p(t, s) + \mu q(t, s), \tag{1.5}$$

where λ, μ are non-negative real parameters and $p, q : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions vanishing outside the rectangle $[0, T] \times [-T, T]$. The typical models for $p(t, s)$ and $q(t, s)$ are functions that behave, in a right neighbourhood of $s = 0$, as $m(t)s^p$, with $p \in]0, 1]$, and as $n(t)s^q$, with $q \in]1, +\infty[$, respectively. The coefficients $m, n : [0, T] \to \mathbb{R}$ are, say, continuous and positive somewhere, but they are allowed to change sign. With reference to these special examples, the following conclusions are obtained.

Take $\mu = 0$ in (1.5). If the exponent $p \in]0, 1[$ is fixed, we prove that the Dirichlet problem associated with (1.4), and hence (1.1), has a positive solution for every $\lambda > 0$. Such a solution is a global minimizer in $H_0^1(0, T)$, at a negative level, of the corresponding action functional, which is coercive and bounded from below. If $p = 1$, we show that the Dirichlet problem associated with (1.3), and hence (1.1), has a positive solution for $\lambda > \lambda_1(m)$, where $\lambda_1(m)$ is the positive principal eigenvalue of

$$\begin{cases} -u'' = \lambda m(t)u & \text{in }]0, T[, \\ u(0) = u(T) = 0. \end{cases}$$

Here the conclusion is achieved by applying the classical Rabinowitz bifurcation theorem. Non-existence of positive solutions is also shown to occur for small $\lambda > 0$ and additional information on the structure of the solution set are obtained. Actually, it is immediately seen that in both cases the existence of positive solutions is guaranteed, with the same choices of λ, for any given $\mu > 0$. The bifurcation diagrams depicted in Figures 1 and 2, where $\|u\|_\infty$ is plotted against λ for a fixed but generic $\mu \geq 0$, reflect the expected structure of the corresponding sets of solutions.

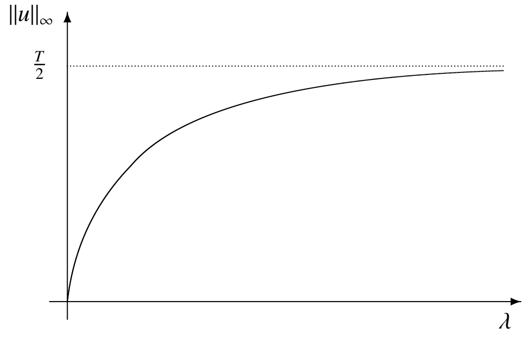

Fig. 1. Bifurcation diagram: the case $0 < p < 1$, with $\mu \geq 0$ fixed.

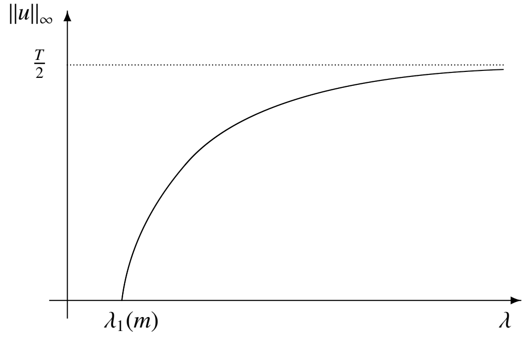

Fig. 2. Bifurcation diagram: the case $p = 1$, with $\mu \geq 0$ fixed.

Take now $\lambda = 0$ in (1.5). If the exponent $q \in]1, +\infty[$ is fixed, we prove that the Dirichlet problem associated with (1.4), and hence (1.1), has at least two positive solutions for every sufficiently large $\mu > 0$. The action functional is again coercive and bounded from below; moreover, 0 is a local minimizer and there exists a global minimizer at a negative level. This yields the existence of a first positive solution, while a second one is a critical point of mountain pass type at a positive level. Non-existence of positive solutions is also established for small $\mu > 0$ whenever $q(t, s)$ has finite slope at $s = 0$. The corresponding bifurcation diagram, where $\|u\|_\infty$ is plotted against μ, is depicted in Figure 3.

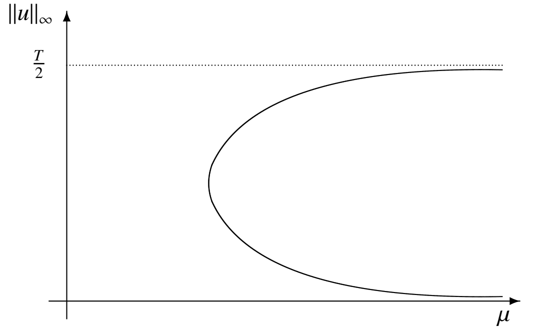

Fig. 3. Bifurcation diagram: the case $q > 1$, with $\lambda = 0$.

Next take $\lambda > 0$ and $\mu > 0$ in (1.5). Let the exponents $p \in]0, 1[$ and $q \in]1, +\infty[$ be given. Then the Dirichlet problem associated with (1.4), and hence (1.1), has at least three positive solutions for every large $\mu > 0$ and small $\lambda > 0$. This is achieved by looking at the term $\lambda p(t, s)$ as a small perturbation, that is capable of slightly modifying the geometry of the action functional by creating a non-trivial local minimum near 0, yet preserving the two other critical points whose existence was established for $\lambda = 0$ and large $\mu > 0$. The multiplicity diagram describing this situation is given in Figure 4.

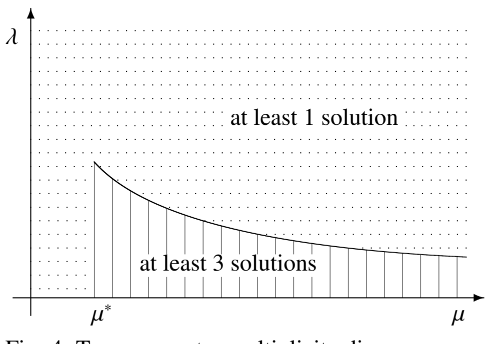

Fig. 4. Two-parameter multiplicity diagram.

The last theorem of this paper deals with the case where the function g oscillates between a sublinear and a superlinear behaviour, i.e., between two powers s^p and s^q, with $0 < p < 1 < q$, as $s \to 0^+$. Such oscillations produce the existence of infinitely many positive solutions. These are obtained by combining local minimization with the method of lower and upper solutions, which are in turn constructed via time-mapping estimates.

After completing this paper we became aware of the recent preprint [4], where the existence of a positive radial solution of (1.2) in a ball $\Omega \subset \mathbb{R}^N$ has been discussed, assuming conditions on f that guarantee the existence of a positive minimizer for the associated action functional at a negative level: the overlapping of that paper with ours is however very limited and confined to Theorem 2.1.

Notation. We list some notation that will be used throughout this paper. We set $\mathbb{R}_0^+ =]0, +\infty[$. For functions $u, v : [0, T] \to \mathbb{R}$, we write $u \leq v$ if $u(t) \leq v(t)$ a.e. in $[0, T]$, and $u < v$ if $u \leq v$ and $u(t) < v(t)$ in a subset of $[0, T]$ having positive measure. We also set $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. For functions $u, v \in C^1([0, T])$, we write $u \ll v$ if $u(t) < v(t)$ for every $t \in]0, T[$, $u'(0) < v'(0)$ in case $u(0) = v(0)$ and $u'(T) > v'(T)$ in case $u(T) = v(T)$. We set $C_0^1([0, T]) = \{u \in C^1([0, T]) : u(0) = u(T) = 0\}$.

2 Existence and multiplicity results

In this section we discuss existence and multiplicity of positive solutions of problem (1.1): the results are grouped depending on the number of solutions and on the behaviour of the function $f = f(t, s)$ near $s = 0$. Throughout we assume

(h_1) *$f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions, i.e., for a.e. $t \in [0, T]$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for every $s \in \mathbb{R}$, $f(\cdot, s) : [0, T] \to \mathbb{R}$ is measurable, and for each $r > 0$ there is $\gamma \in L^1(0, T)$ such that $|f(t, s)| \leq \gamma(t)$ for a.e. $t \in [0, T]$ and every $s \in [-r, r]$,*

and we set $F(t, s) = \displaystyle\int_0^s f(t, \xi)\, d\xi$.

We say that a function $u \in W^{2,1}(0, T)$ is a solution of (1.1) if $\|u'\|_\infty < 1$ and u satisfies the equation a.e. in $[0, T]$ and the boundary conditions in (1.1). Further, it is said to be positive if $u > 0$ and strictly positive if $u \gg 0$.

Remark 2.1 Let us define $\phi :]-1, 1[\to \mathbb{R}$ by setting

$$\phi(y) = y / \sqrt{1 - y^2}. \tag{2.1}$$

Note that, as ϕ^{-1} is globally Lipschitz in $\mathbb{R}$, for any function $u \in W^{1,1}(0, T)$ with $\|u'\|_\infty < 1$, we have $u \in W^{2,1}(0, T)$ if and only if $\phi(u') \in W^{1,1}(0, T)$.

2.1 Existence of at least one positive solution

The case of infinite slope

Theorem 2.1 *Assume* (h_1),

(h_2) *there exist a, b, with $0 \leq a < b \leq T$, such that* $\displaystyle\liminf_{s \to 0^+} \frac{F(t, s)}{s^2} > -\infty$ *uniformly a.e. in* $[a, b]$,

(h_3) *there exist c, d, with $a < c < d < b$, such that* $\displaystyle\limsup_{s \to 0^+} \int_c^d \frac{F(t, s)}{s^2}\, dt = +\infty$,

(h_4) $f(t, 0) \geq 0$ *for a.e.* $t \in [0, T]$.

Then problem (1.1) *has at least one positive solution.*

Proof. **Step 1. An equivalent formulation.** Let us define $\tilde{f} : [0, T] \times \mathbb{R} \to \mathbb{R}$ by setting, for a.e. $t \in [0, T]$,

$$\tilde{f}(t, s) = \begin{cases} 0 & \text{if } |s| \geq T, \\ f(t, s) & \text{if } 0 < s \leq T/2, \\ \text{linear} & \text{if } -T < s < 0 \text{ or } T/2 < s < T. \end{cases}$$

Observe that, within the context of positive solutions, problem (1.1) is equivalent to the same problem with f replaced by $\tilde{f}$. Indeed, if u is a positive solution, then $\|u'\|_\infty < 1$ and hence $\|u\|_\infty < T/2$.

In the sequel of the proof we shall replace f with $\tilde{f}$; however, for the sake of simplicity in the notation, the modified function $\tilde{f}$ will still be denoted by f. Clearly, such a function satisfies all the properties assumed in the statement of the theorem. Furthermore, by (h_1) there exists $\gamma \in L^1(0,T)$ such that

$$|f(t,s)| \leq \gamma(t) \tag{2.2}$$

for a.e. $t \in [0,T]$ and every $s \in \mathbb{R}$. Set $\sigma = \phi'(\phi^{-1}(\|\gamma\|_{L^1}))$ and define, for $y \in \mathbb{R}$,

$$\psi(y) = \begin{cases} \sigma \cdot (y + \phi^{-1}(\|\gamma\|_{L^1})) - \|\gamma\|_{L^1} & \text{if } y < -\phi^{-1}(\|\gamma\|_{L^1}), \\ \phi(y) & \text{if } |y| \leq \phi^{-1}(\|\gamma\|_{L^1}), \\ \sigma \cdot (y - \phi^{-1}(\|\gamma\|_{L^1})) + \|\gamma\|_{L^1} & \text{if } y > \phi^{-1}(\|\gamma\|_{L^1}). \end{cases} \tag{2.3}$$

Set also

$$\Psi(y) = \int_0^y \psi(\xi)\, d\xi \tag{2.4}$$

and observe that

$$\tfrac{1}{2}y^2 \leq \Psi(y) \leq \tfrac{1}{2}\sigma y^2 \tag{2.5}$$

for every $y \in \mathbb{R}$.

Claim. A function $u \in W^{2,1}(0,T)$ is a positive solution of (1.1) *if and only if it is a positive solution of the problem*

$$\begin{cases} -(\psi(u'))' = f(t,u) & \textit{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases} \tag{2.6}$$

Suppose that u is a positive solution of (1.1). Then $u'(\tau) = 0$ for some $\tau \in [0,T]$. Integrating the equation in (1.1) between τ and $t \in [0,T]$, we obtain

$$|\phi(u'(t))| = \left| \int_\tau^t f(\xi,u)\, d\xi \right| \leq \|\gamma\|_{L^1}$$

and hence

$$|u'(t)| \leq \phi^{-1}(\|\gamma\|_{L^1}).$$

Therefore $\phi(u'(t)) = \psi(u'(t))$ in $[0,T]$ and we conclude that u is a positive solution of (2.6).

Suppose now that u is a positive solution of (2.6). Arguing as above we see that

$$|u'(t)| \leq \psi^{-1}(\|\gamma\|_{L^1}),$$

and therefore $\psi(u'(t)) = \phi(u'(t))$ in $[0,T]$. In particular $\|u'\|_\infty < 1$ and we conclude that u is a positive solution of (1.1).

Step 2. Existence of a positive solution. We define the functional $\mathcal{I} : H_0^1(0,T) \to \mathbb{R}$ by setting

$$\mathcal{I}(v) = \int_0^T \Psi(v')\, dt - \int_0^T F(t,v)\, dt. \tag{2.7}$$

$\mathcal{I}$ is C^1 and weakly lower semicontinuous. By (2.2) we can find a constant $c_f > 0$ such that

$$\int_0^T F(t,v)\, dt \leq c_f$$

for all $v \in H_0^1(0,T)$. Hence, using (2.5), we easily see that $\mathcal{I}$ is coercive and bounded from below. Consequently there exists $u \in H_0^1(0,T)$ such that

$$\mathcal{I}(u) = \min_{v \in H_0^1(0,T)} \mathcal{I}(v).$$

Clearly, $u \in W^{2,1}(0,T)$ and is a solution of problem (2.6). To check that $u \geq 0$, we test the equation in (2.6) against u^-. Using the fact that $f(t,s) \geq 0$ for a.e. $t \in [0,T]$ and every $s \leq 0$, we get

$$\int_0^T \psi((u^-)')(u^-)' \, dt \leq 0,$$

thus yielding $u^- = 0$ by the monotonicity of ψ. We finally verify that $u \neq 0$. Let $\zeta \in H_0^1(0,T)$ be such that $0 \leq \zeta \leq 1$, $\zeta(t) = 0$ for every $t \in [0,a] \cup [b,T]$ and $\zeta(t) = 1$ for every $t \in [c,d]$. By assumptions (h_2) and (h_3) there exist a constant $K > 0$ and a strictly decreasing sequence $(c_n)_n$ satisfying

$$\begin{gathered}\lim_{n\to+\infty} c_n = 0, \\ \lim_{n\to+\infty} \int_c^d \frac{F(t,c_n)}{c_n^2} \, dt = +\infty, \\ F(t, c_n\zeta(t)) \geq -Kc_n^2\zeta(t)^2 \text{ for a.e. } t \in [a,b] \text{ and all } n.\end{gathered} \tag{2.8}$$

We easily compute, using also (2.5),

$$\begin{aligned}\mathcal{I}(c_n\zeta) &= \int_0^T \Psi(c_n\zeta') \, dt - \int_0^T F(t, c_n\zeta) \, dt \\ &\leq c_n^2\Big(\tfrac{1}{2}\sigma\|\zeta'\|_{L^2}^2 - \int_c^d \frac{F(t,c_n)}{c_n^2} \, dt + K\|\zeta\|_{L^2}^2\Big).\end{aligned}$$

Hence, we infer

$$\mathcal{I}(u) \leq \mathcal{I}(c_n\zeta) < 0$$

for all large n, yielding $u \neq 0$. Therefore we conclude that u is a positive solution of (2.6) and hence, by the Claim in Step 1, u is a positive solution of (1.1).

Example 2.1 Take $p \in]0,1[$ and $q \in]1,+\infty[$. Let $m, n : [0,T] \to \mathbb{R}$ be continuous functions, with $m^+ > 0$. Then Theorem 2.1 yields the existence of a positive solution of the problem

$$\begin{cases} -\big(u'/\sqrt{1-u'^2}\big)' = m(t)u^p + n(t)u^q & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases}$$

The case of finite non-zero slope

Let us introduce the weighted eigenvalue problem

$$\begin{cases} -u'' = \lambda m(t)u & \text{in }]0,T[, \\ u(0) = u(T) = 0, \end{cases} \tag{2.9}$$

where it is assumed that

(h_5) *$m \in L^\infty(0,T)$ is such that $m^+ > 0$.*

Denote by $\mathcal{K} : L^1(0,T) \to C_0^1([0,T])$ the operator which sends any function $v \in L^1(0,T)$ onto the unique solution $w \in W^{2,1}(0,T)$ of

$$\begin{cases} -w'' = v & \text{in }]0,T[, \\ w(0) = w(T) = 0. \end{cases}$$

Let $\mathcal{L} : L^1(0,T) \to C_0^1([0,T])$ be defined by $\mathcal{L}(u) = \mathcal{K}(mu)$. Both $\mathcal{K}$ and $\mathcal{L}$ are completely continuous and (2.9) is equivalent to

$$u = \lambda\mathcal{L}(u),$$

so that the eigenvalues of (2.9) are precisely the characteristic values of $\mathcal{L}$.

Lemma 2.1 *Assume* (h_5). *Then the non-negative eigenvalues of problem* (2.9) *form a sequence* $(\lambda_n(m))_n$ *such that*

$$(0 <)\, \lambda_1(m) < \cdots < \lambda_n(m) < \ldots .$$

The eigenspace corresponding to the minimum eigenvalue $\lambda_1(m)$ *is spanned by an eigenfunction* φ_1 *with* $\varphi_1 \gg 0$. *Moreover, the algebraic multiplicity of* $\lambda_1(m)$ *as a characteristic value of* $\mathcal{L}$ *is* 1. *Finally, all eigenfunctions corresponding to any eigenvalue* $\lambda_n(m)$ *with* $n > 1$ *change sign.*

This result is standard: its proof essentially follows from [8] and [7]. Note that in [7] the weight function m was supposed to be continuous, but turning to the current hypothesis requires only minor changes.

Theorem 2.2 *Assume* (h_5),

(h_6) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies the* L^∞*-Carathéodory conditions, i.e., for a.e.* $t \in [0,T]$, $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ *is continuous, for every* $s \in \mathbb{R}$, $f(\cdot, s) : [0,T] \to \mathbb{R}$ *is measurable, and for each* $r > 0$ *there exists* $\gamma \in L^\infty(0,T)$ *such that* $|f(t,s)| \le \gamma(t)$ *for a.e.* $t \in [0,T]$ *and every* $s \in [-r, r]$,

(h_7) $\displaystyle\lim_{s\to 0^+} \frac{f(t,s)}{s} = m(t)$ *uniformly a.e. in* $[0,T]$.

Then there exists $\lambda_* \in\,]0, \lambda_1(m)]$ *such that, for all* $\lambda \in\,]0, \lambda_*[$, *the problem*

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda f(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0 \end{cases} \tag{2.10}$$

has no positive solutions and, for all $\lambda > \lambda_1(m)$, *it has at least one strictly positive solution.*

Proof. **Step 1. An equivalent formulation.** Let us define a function $\tilde f : [0,T] \times \mathbb{R} \to \mathbb{R}$ by setting, for a.e. $t \in [0,T]$,

$$\tilde f(t,s) = \begin{cases} f(t,s) & \text{if } 0 \le s \le T/2, \\ 0 & \text{if } s \ge T, \\ \text{linear} & \text{if } T/2 < s < T, \\ -\tilde f(t,-s) & \text{if } s < 0. \end{cases}$$

Like in Step 1 of the proof of Theorem 2.1, we see that, within the context of positive solutions, problem (2.10) is equivalent to the same problem with f replaced by $\tilde f$. Clearly, $\tilde f$ satisfies all the properties assumed in the statement of the theorem. Furthermore, $\tilde f(t,\cdot)$ is an odd function for a.e. $t \in [0,T]$. In the sequel of the proof we shall replace f with $\tilde f$; however, for the sake of simplicity, the modified function $\tilde f$ will still be denoted by f. Next, let us define $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(y) = \begin{cases} (1-y^2)^{3/2} & \text{if } |y| \le 1, \\ 0 & \text{if } |y| > 1. \end{cases}$$

Claim. A function $u \in W^{2,1}(0,T)$ *is a positive solution of* (2.10) *if and only if it is a positive solution of the problem*

$$\begin{cases} -u'' = \lambda f(t,u) h(u') & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases} \tag{2.11}$$

To verify this it is enough to assume (h_1) in place of (h_6). It is clear that a positive solution $u \in W^{2,1}(0,T)$ of (2.10) is a positive solution of (2.11) as well. Conversely, suppose that $u \in W^{2,1}(0,T)$ is a positive solution of (2.11). We aim to show that $\|u'\|_\infty < 1$. Assume by contradiction that this is not the case. Then we can easily find an interval $[a,b] \subseteq [0,T]$ such that, either $u'(a) = 0$, $0 < |u'(t)| < 1$ in $]a,b[$ and $|u'(b)| = 1$, or $|u'(a)| = 1$, $0 < |u'(t)| < 1$ in $]a,b[$ and $u'(b) = 0$. Suppose the former case occurs (in the latter one the argument would be similar). The function u satisfies the equation

$$-(\phi(u'))' = \lambda f(t,u)$$

in $[a,b[$. For each $t \in]a,b[$, integrating over the interval $[a,t]$ and using (h_1), we obtain

$$|\phi(u'(t))| \leq \lambda \int_a^t \gamma(\xi)\, d\xi,$$

with $\gamma \in L^1(0,T)$ such that $|f(t,s)| \leq \gamma(t)$ for a.e. $t \in [0,T]$ and every s such that $|s| \leq \|u\|_\infty$. Hence we get

$$|u'(t)| \leq \phi^{-1}(\lambda \|\gamma\|_{L^1})$$

for every $t \in [a,b[$. Since $\phi^{-1}(\lambda\|\gamma\|_{L^1}) < 1$, taking the limit as $t \to b^-$ we obtain the contradiction $|u'(b)| < 1$. Therefore $\|u'\|_\infty < 1$ and, as a consequence, u is a positive solution of (2.10).

Step 2. A bifurcation result. By (h_6) and (h_7) we can write, for a.e. $t \in [0,T]$ and every $s \in \mathbb{R}$,

$$f(t,s) = (m(t) + l(t,s))s,$$

where $l : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^∞-Carathéodory conditions and

$$\lim_{s\to 0} l(t,s) = 0 \tag{2.12}$$

uniformly a.e. in $[0,T]$. Let us set, for convenience, $k(y) = h(y) - 1$, for $y \in \mathbb{R}$. We have

$$\lim_{y\to 0} \frac{k(y)}{y} = 0. \tag{2.13}$$

Define the operator $\mathcal{H} : \mathbb{R} \times C^1_0([0,T]) \to C^1_0([0,T])$ by

$$\mathcal{H}(\lambda,u) = \lambda\mathcal{K}\Big((l(\cdot,u) + (m + l(\cdot,u))k(u'))u\Big).$$

Clearly, $\mathcal{H}$ is completely continuous and, by (2.12) and (2.13),

$$\lim_{\|u\|_{C^1}\to 0} \frac{\|\mathcal{H}(\lambda,u)\|_{C^1}}{\|u\|_{C^1}} = 0,$$

uniformly with respect to λ varying in bounded intervals. Observe that, for any λ, the couple $(\lambda,u) \in \mathbb{R} \times C^1_0([0,T])$, with $u > 0$, is a solution of the equation

$$u = \lambda\mathcal{L}(u) + \mathcal{H}(\lambda,u) \tag{2.14}$$

if and only if u is a positive solution of (2.10). We say that a solution $(\lambda,u) \in \mathbb{R}\times C^1_0([0,T])$ of (2.14) is non-trivial if $u \neq 0$. Denote by $\mathcal{S}$ the closure in $\mathbb{R}\times C^1_0([0,T])$ of the set of all non-trivial solutions (λ,u) of (2.14) with $\lambda > 0$. Note that the set $\{u \in C^1_0([0,T]) : (\lambda,u) \in \mathcal{S}\}$ is bounded in $C^1_0([0,T])$. Moreover, for every $\bar{\lambda} > 0$ there exists a constant $M > 0$ such that

$$\|u\|_{W^{2,\infty}} \leq M \tag{2.15}$$

for all $(\lambda,u) \in \mathcal{S}$ with $\lambda \in [0,\bar{\lambda}]$. Theorem 1.3 in [10] yields the existence of a maximal closed connected set C in $\mathcal{S}$ such that $(\lambda_1(m),0) \in C$ and at least one of the following conditions holds:

(i) C is unbounded in $\mathbb{R} \times C^1_0([0,T])$,

(ii) there exists a characteristic value $\hat{\lambda}(m)$ of $\mathcal{L}$, with $\hat{\lambda}(m) \neq \lambda_1(m)$, such that $(\hat{\lambda}(m),0) \in C$.

In what follows we prove several properties which will eventually lead to the conclusion.

Claim 1. Suppose $(\hat{\lambda}, 0) \in \mathcal{S}$, then $\hat{\lambda}$ is a characteristic value of $\mathcal{L}$. Let $((\lambda_n, u_n))_n$ be a sequence of non-trivial solutions of (2.14), converging to $(\hat{\lambda}, 0)$ in $\mathbb{R}\times C^1_0([0,T])$. Setting, for all n, $v_n = u_n/\|u_n\|_{C^1}$, we have

$$v_n = \lambda_n \mathcal{L}(v_n) + \frac{\mathcal{H}(\lambda_n, u_n)}{\|u_n\|_{C^1}}. \tag{2.16}$$

As $(v_n)_n$ is bounded in $C^1_0([0,T])$ and $\mathcal{L}$ is completely continuous, there exist $w \in C^1_0([0,T])$ and a subsequence of $(v_n)_n$, that we denote in the same way, such that $\lim_{n\to+\infty} \mathcal{L}(v_n) = w$ in $C^1_0([0,T])$. Hence we conclude by (2.16) that $\lim_{n\to+\infty} v_n = \hat{\lambda} w$ in $C^1_0([0,T])$. Therefore we have $w = \hat{\lambda}\mathcal{L}(w)$, with $\|\hat{\lambda} w\|_{C^1} = 1$, and in particular $w \neq 0$. Accordingly, $\hat{\lambda}$ is a characteristic value of $\mathcal{L}$.

Claim 2. There exists $\varepsilon > 0$ such that $\mathcal{S} \subset [\varepsilon, +\infty[\times C^1_0([0,T])$. By contradiction, we can suppose that there exists a sequence $((\lambda_n, u_n))_n$ of non-trivial solutions of (2.14), converging in $\mathbb{R}\times C^1_0([0,T])$ to some $(0, u) \in \mathbb{R}\times C^1_0([0,T])$. Arguing as in the proof of Claim 1, we set $v_n = u_n/\|u_n\|_{C^1}$ and conclude that, possibly passing to a subsequence, $\lim_{n\to+\infty} v_n = 0$ in $C^1_0([0,T])$, which contradicts $\|v_n\|_{C^1} = 1$.

Claim 3. $(\lambda, u) \in C$ if and only if $(\lambda, -u) \in C$. This follows from the fact that f, and hence $\mathcal{H}$, is odd with respect to the second variable.

In the sequel we denote by P the positive cone in $C^1_0([0,T])$, i.e., $P = \{u \in C^1_0([0,T]) : u \geq 0\}$, by int$P$ its interior and by ∂P its boundary.

Claim 4. There exists a neighbourhood U of $(\lambda_1(m), 0)$ in $\mathbb{R}\times C^1_0([0,T])$ such that, for all $(\lambda, u) \in C \cap U$, either $(\lambda, u) = (\lambda_1(m), 0)$, or $u \in \mathrm{int}P$, or $-u \in \mathrm{int}P$. Assume, by contradiction, that there is a sequence $((\lambda_n, u_n))_n$ in $C \setminus \{(\lambda_1(m), 0)\}$ converging to $(\lambda_1(m), 0)$ in $\mathbb{R}\times C^1_0([0,T])$ and such that, for every n, both $u_n \notin \mathrm{int}P$ and $-u_n \notin \mathrm{int}P$. By Lemma 2.1 and Claim 1, without loss of generality we can assume that $u_n \neq 0$ for all n. Then, setting $v_n = u_n/\|u_n\|_{C^1}$ and arguing as in the proof of Claim 1, we conclude that, possibly passing to a subsequence, $\lim_{n\to+\infty} \mathcal{L}(v_n) = w$ in $C^1_0([0,T])$, with $w = \lambda_1(m)\mathcal{L}(w)$ and $w \neq 0$. By Lemma 2.1, we conclude that either $w \in \mathrm{int}P$, or $-w \in \mathrm{int}P$, and hence either $u_n \in \mathrm{int}P$ or $-u_n \in \mathrm{int}P$ for infinitely many n, which is a contradiction.

Claim 5. Assume $(\lambda, u) \in C$ and $u \in \partial P$. Suppose further that (λ, u) is the limit of a sequence $((\lambda_n, u_n))_n$ in C, with $u_n > 0$ for all n. Then $(\lambda, u) = (\lambda_1(m), 0)$. We first show that $u = 0$. Suppose, by contradiction, that $u > 0$. By (h_6) we can take $c > 0$ such that

$$\lambda(m(t) + l(t,u))h(u') + c \geq 1$$

a.e. in $[0,T]$. Hence we get

$$-u'' + cu = (\lambda(m(t) + l(t,u))h(u') + c)u$$

a.e. in $[0,T]$. As $(\lambda(m + l(\cdot,u))h(u') + c)u > 0$, the strong maximum principle yields $u \gg 0$, contradicting $u \in \partial P$. Therefore we conclude that $u = 0$. We next show that $\lambda = \lambda_1(m)$. By Claim 1, λ is a characteristic value of $\mathcal{L}$. Setting $v_n = u_n/\|u_n\|_{C^1}$ and arguing as in the proof of Claim 1, we conclude that, possibly passing to a subsequence, $\lim_{n\to+\infty} \mathcal{L}(v_n) = w$ in $C^1_0([0,T])$, where w is an eigenfunction of (2.9) associated with λ. Since $w > 0$, we conclude that $\lambda = \lambda_1(m)$.

Claim 6. For all $(\lambda, u) \in C$, either $u \in \mathrm{int}P$, or $-u \in \mathrm{int}P$, or $(\lambda, u) = (\lambda_1(m), 0)$. Set

$$\mathcal{E} = \{(\lambda, u) \in C : u \notin \mathrm{int}P, -u \notin \mathrm{int}P, (\lambda, u) \neq (\lambda_1(m), 0)\}.$$

By Claim 4, $\mathcal{E}$ is a closed subset of C. Let us verify that $\mathcal{E}$ is open in C. Suppose this is not the case. Then there exist $(\lambda, u) \in \mathcal{E}$ and a sequence $((\lambda_n, u_n))_n$ in $C \setminus \mathcal{E}$ converging to (λ, u). We may assume that $u_n \in \mathrm{int}P$ for all n; hence, by Claim 5, we obtain $(\lambda, u) = (\lambda_1(m), 0)$, contradicting the fact that $(\lambda, u) \in \mathcal{E}$. As C is connected and $(\lambda_1(m), 0) \in C \setminus \mathcal{E}$, we conclude that $\mathcal{E} = \emptyset$.

We are now in position of getting the conclusions of the theorem. By Claim 6 we have that, if $(\hat{\lambda}(m), 0) \in C$, then $\hat{\lambda}(m) = \lambda_1(m)$ and hence condition (*ii*) above does not hold. Consequently, condition (*i*) is valid and therefore, by (2.15), C is unbounded with respect to λ. Hence, using Claim 2, we infer that for all $\lambda > \lambda_1(m)$ problem (2.10) has at least one non-trivial solution. From Claim 3 and Claim 6 we deduce that at least one of those solutions belongs to int$P$. Thus we conclude that, for all $\lambda > \lambda_1(m)$, problem (2.10) has at least one strictly positive solution.

Finally, let Λ be the set of all $\lambda > 0$ such that problem (2.10) has at least one positive solution and define $\lambda_* = \inf \Lambda$. By Claim 2 we obtain $\lambda_* > 0$. Then we conclude that, for all $\lambda \in]0, \lambda_*[$, problem (2.10) has no positive solutions.

Remark 2.2 If in addition to all the assumptions of Theorem 2.2 we suppose that

(h_8) $0 \leq f(t,s) \leq m(t)s$ *for a.e.* $t \in [0,T]$ *and every* $s \in [0, +\infty[$,

then we see that $\lambda_* = \lambda_1(m)$. Indeed, fix $\lambda > 0$ for which there exists a positive, and hence strictly positive, solution u of (2.10). Set $\tilde{m} = \frac{f(t,u)}{u} h(u')$ a.e. in $[0,T]$. By (h_6), (h_8) and the definition of h, we have that $\tilde{m} \in L^\infty(0,T)$ and $\tilde{m} > 0$. Notice that u is a positive solution of the problem

$$\begin{cases} -u'' = \lambda \tilde{m}(t) u & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases}$$

Lemma 2.1 implies that $\lambda = \lambda_1(\tilde{m})$. On the other hand, from (h_8) we have $\tilde{m} \leq m$ a.e. in $[0,T]$. Hence, the monotone dependence of eigenvalues on weights (cf. [8, Proposition 4]) yields $\lambda_1(\tilde{m}) \geq \lambda_1(m)$, i.e. $\lambda \geq \lambda_1(m)$.

Remark 2.3 Assume (h_1) and

(h_9) $\limsup_{s \to 0^+} \dfrac{f(t,s)}{s} < +\infty$ *uniformly a.e. in* $[0,T]$.

Then there exists $\lambda_* > 0$ such that, for all $\lambda \in]0, \lambda_*[$, problem (2.10) has no positive solutions. By contradiction, suppose that there exists a decreasing sequence $(\lambda_n)_n$ such that $\lim_{n \to +\infty} \lambda_n = 0$ and, for each n, problem (2.10), with $\lambda = \lambda_n$, has at least one positive solution u_n. We easily see, using (h_1) too, that

$$\lim_{n \to +\infty} \|u_n\|_{W^{2,1}} = 0.$$

Assumption (h_9) yields the existence of constants $\delta > 0$ and $c > 0$ such that $f(t,s) \leq cs$, for a.e. $t \in [0,T]$ and every $s \in [0, \delta]$. Take n such that $\|u_n\|_\infty \leq \delta$ and $c\lambda_n < (\pi/T)^2$. Then we obtain

$$-u_n''(t) = \lambda_n f(t, u_n(t)) h(u_n'(t)) \leq \lambda_n c u_n(t),$$

for a.e. $t \in [0,T]$, which in turn yields

$$\int_0^T |u_n'|^2 dt \leq \lambda_n c \int_0^T |u_n|^2 dt.$$

A contradiction follows from the Poincaré inequality.

2.2 Existence of at least two positive solutions

The case of zero slope

Theorem 2.3 *Suppose that*

(h_{10}) $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies the* L^1*-Carathéodory conditions* *and set* $G(t,s) = \int_0^s g(t,\xi)\,d\xi$. *Assume further*

(h_{11}) *there exists* $w \in H_0^1(0,T)$, *with* $w > 0$ *and* $\|w'\|_\infty < 1$, *such that* $\int_0^T G(t,w)\,dt > 0$,

(h_{12}) $\limsup_{s\to 0^+} \frac{G(t,s)}{s^2} \le 0$ *uniformly a.e. in* $[0,T]$,

(h_{13}) $g(t,0) = 0$ *for a.e.* $t \in [0,T]$.

Then there exists $\mu^* > 0$ *such that, for all* $\mu > \mu^*$, *the problem*

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \mu g(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0, \end{cases} \tag{2.17}$$

has at least two positive solutions.

Proof. Arguing as in Step 1 of the proof of Theorem 2.1, we can replace g with the function $\tilde{g}$: $[0,T]\times\mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}(t,s) = \begin{cases} 0 & \text{if } s \le 0 \text{ or } s \ge T, \\ g(t,s) & \text{if } 0 < s \le T/2, \\ \text{linear} & \text{if } T/2 < s < T, \end{cases}$$

and observe once more that, within the context of positive solutions of (2.17), problem (2.17) is equivalent to the same problem with g replaced by $\tilde{g}$. In the sequel of the proof we shall replace g with $\tilde{g}$, however, for the sake of simplicity in the notation, the modified function $\tilde{g}$ will still be denoted by g. Clearly, such a function satisfies all the properties assumed in the statement of the theorem.

Let $w \in H_0^1(0,T)$ be the function with the properties described in (h_{11}) and let μ^* be such that

$$\int_0^T \Phi(w')\,dt - \mu^* \int_0^T G(t,w)\,dt = 0, \tag{2.18}$$

where, for $y \in]-1,1[$, $\Phi(y) = \int_0^y \phi(\xi)\,d\xi$ and ϕ has been defined in (2.1). Let us fix now $\mu > \mu^*$. By (h_{10}) there exists $\gamma \in L^1(0,T)$ such that

$$\mu|g(t,s)| \le \gamma(t)$$

for a.e. $t \in [0,T]$ and every $s \in \mathbb{R}$. Without restriction we may also assume that

$$\phi(\|w'\|_\infty) < \|\gamma\|_{L^1}.$$

We define ψ as in (2.3), Ψ as in (2.4), and $\mathcal{I}_\mu : H_0^1(0,T) \to \mathbb{R}$ by setting

$$\mathcal{I}_\mu(v) = \int_0^T \Psi(v')\,dt - \mu \int_0^T G(t,v)\,dt.$$

$\mathcal{I}_\mu$ is C^1 and weakly lower semicontinuous; moreover, it is coercive and bounded from below.

Let $u_1 \in H_0^1(0,T)$ be such that

$$\mathcal{I}_\mu(u_1) = \min_{v\in H_0^1(0,T)} \mathcal{I}_\mu(v)$$

and observe that, by (2.18),

$$\mathcal{I}_\mu(u_1) < 0. \tag{2.19}$$

Note that $u_1 \in W^{2,1}(0,T)$ and is a non-trivial solution of the problem

$$\begin{cases} -(\psi(u'))' = \mu g(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases} \tag{2.20}$$

Using the fact that $g(t,s) = 0$, for a.e. $t \in [0,T]$ and every $s \le 0$, and arguing as in the proof of Theorem 2.1, we see that any solution u of (2.20) satisfies $u \ge 0$. In particular u_1 is a positive solution of (2.20).

A second solution u_2 can be found using the mountain pass theorem (see, e.g., [1]). Note that the coercivity of $\mathcal{I}_\mu$ implies that the Palais-Smale condition holds. Let us check that the functional has a mountain pass geometry near the origin. Take $\varepsilon > 0$ such that

$$\tfrac{1}{2} - \mu\varepsilon(\tfrac{T}{\pi})^2 > 0.$$

By assumption (h_{12}) there exists r such that $0 < r < \|u_1\|_{H_0^1}$ and

$$G(t,s) \le \varepsilon s^2$$

for a.e. $t \in [0,T]$ and every $s \in [0,r]$. Let $c_\infty > 1$ be such that

$$\|v\|_\infty \le c_\infty \|v\|_{H_0^1} \tag{2.21}$$

for all $v \in H_0^1(0,T)$. Therefore, for all $v \in H_0^1(0,T)$, with $0 < \|v\|_{H_0^1} \le r/c_\infty$, we have, using also (2.5),

$$\begin{aligned} \mathcal{I}_\mu(v) &= \int_0^T \Psi(v')\,dt - \mu \int_0^T G(t,v)\,dt \\ &\ge \tfrac{1}{2}\int_0^T |v'|^2\,dt - \mu\varepsilon \int_0^T |v|^2\,dt \ge \|v\|^2_{H_0^1}(\tfrac{1}{2} - \mu\varepsilon(\tfrac{T}{\pi})^2) > 0. \end{aligned}$$

Since (2.19) also holds, we conclude that the functional $\mathcal{I}_\mu$ has a critical point u_2, with $\mathcal{I}_\mu(u_2) > 0$. Therefore u_2 is a positive solution of (2.20), which is different from u_1. By the Claim in Step 1 of Theorem 2.1, we finally conclude that u_1 and u_2 are actually solutions of problem (2.17).

Remark 2.4 From the first part of the proof of Theorem 2.3 we see that conditions (h_{10}), (h_{11}) and (h_{13}) yield the existence of $\mu^* > 0$ such that, for all $\mu > \mu^*$, problem (2.17) has at least one positive solution. In this way we get the existence of positive solutions under a set of assumptions slightly different from those considered in Theorem 2.2, but we loose all the additional information therein obtained.

Remark 2.5 In addition to (h_{10}), (h_{11}), (h_{12}) and (h_{13}) assume

(h_{14}) $\limsup_{s\to 0^+} \dfrac{g(t,s)}{s} < +\infty$ *uniformly a.e. in* $[0,T]$.

Then, by Remark 2.3 there exists $\mu_* \in]0,\mu^*]$ such that, for all $\mu \in]0,\mu_*[$, problem (2.17) has no positive solutions.

Observe that, in case $\mu_* = \mu^*$, problem (2.17) has at least one positive solution for $\mu = \mu^*$. Indeed, take a decreasing sequence $(\mu_n)_n$, with $\lim_{n\to+\infty} \mu_n = \mu^*$, and let u_n be a solution of (2.17) for $\mu = \mu_n$, satisfying

$$\mathcal{I}_{\mu_n}(u_n) \ge r^2(\tfrac{1}{2} - \mu^*\varepsilon(\tfrac{T}{\pi})^2) > 0. \tag{2.22}$$

Since $(u_n)_n$ is bounded in $W^{2,1}(0,T)$, it has a subsequence converging in $C^1([0,T])$ to a function u, which is a solution of (2.17) for $\mu = \mu^*$. By (2.22) we conclude that $u > 0$.

In case $\mu_* < \mu^*$, if we further assume

(h_{15}) *$g(t,s) \geq 0$ for a.e. $t \in [0,T]$ and every $s \in [0,T/2]$,*

then we can also conclude that problem (2.17) has at least one positive solution for all $\mu \in]\mu_*, \mu^*]$. To prove this last statement we consider problem (2.20), where ψ, depending on μ^*, is fixed and μ can vary in $]0,\mu^*]$. Let $\mathcal{M}$ be the set of all $\mu \in]0,\mu^*]$ such that problem (2.20) has at least one positive solution. Let us verify that $\mathcal{M}$ is an interval. Set $\mu_* = \inf \mathcal{M}$ and pick any $\mu_1 \in]\mu_*, \mu^*]$. Then there exists $\mu_2 \leq \mu_1$ such that problem (2.20), with $\mu = \mu_2$, has at least one positive solution α. By (h_{15}), α is a lower solution of problem (2.20) with $\mu = \mu_1$. On the other hand, by the way we modified g, any constant $\beta \geq T$ is an upper solution of problem (2.20) for every μ. Then there exists a solution u of (2.20) with $\mu = \mu_1$, such that $\alpha \leq u \leq \beta$ (see, e.g., [9, Lemma 2.1]). Thus we conclude that, for all $\mu \in]\mu_*, \mu^*]$ problem (2.20) has at least one positive solution u. By the Claim in Step 1 of Theorem 2.1, u is a solution of (2.17) as well. Accordingly, for all $\mu \in]\mu_*, \mu^*]$, problem (2.17) has at least one positive solution.

Example 2.2 Take $q \in]1, +\infty[$. Let $n : [0,T] \to \mathbb{R}$ be a continuous function, with $n^+ > 0$. Then Theorem 2.3 and Remark 2.5 yield the existence of μ_* and μ^*, with $0 < \mu_* \leq \mu^*$ such that the problem

$$\begin{cases} -\left(u' / \sqrt{1-u'^2}\right)' = \mu n(t) u^q & \text{in }]0,T[, \\ u(0) = u(T) = 0, \end{cases}$$

has no positive solutions if $\mu < \mu_*$, while it has at least two positive solutions if $\mu > \mu^*$.

2.3 Existence of at least three positive solutions

A two-parameter problem

Theorem 2.4 *Assume (h_1), (h_2), (h_3), (h_4), (h_{10}) (h_{11}), (h_{12}), (h_{13}) and*

(h_{16}) $\liminf_{s\to 0^+} \frac{G(t,s)}{s^2} > -\infty$ *uniformly a.e. in $[a,b]$, with a and b defined in (h_2).*

Then there exists $\mu^* > 0$ and a function $\lambda :]\mu^*, +\infty[\to \mathbb{R}_0^+ \cup \{+\infty\}$ such that, for all $\mu > \mu^*$ and all $\lambda \in]0, \lambda(\mu)[$, the problem

$$\begin{cases} -\left(u' / \sqrt{1-u'^2}\right)' = \lambda f(t,u) + \mu g(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0, \end{cases} \tag{2.23}$$

has at least three positive solutions.

Proof. As in the proofs of Theorem 2.1 and Theorem 2.3 we replace f and g with functions, we still denote by f and g, which satisfy all the assumptions of the theorem, agree with the original functions in $[0,T] \times [0,T/2]$, vanish outside the rectangle $[0,T] \times [-T,T]$, and such that $f(t,s) \geq 0$ and $g(t,s) = 0$ for a.e. $t \in [0,T]$ and every $s \leq 0$. Hence, in particular, we can find constants $c_f, c_g > 0$ such that

$$\int_0^T |F(t,v)|\, dt \leq c_f \quad \text{and} \quad \int_0^T G(t,v)\, dt \leq c_g$$

for all $v \in H_0^1(0,T)$. The proof will follow closely the lines of the proof of Theorem 2.3. The properties of g yield, for large μ, a first solution as a global minimizer and a second solution as a mountain pass critical point. Next, for small λ, the properties of f produce an additional local minimum point close to the origin.

As in the proof of Theorem 2.3 let $w \in H_0^1(0,T)$ be the function with the properties described in (h_{11}) and let μ^* be such that

$$\int_0^T \Phi(w')\, dt - \mu^* \int_0^T G(t,w)\, dt + 2c_f = 0. \tag{2.24}$$

We fix now $\mu > \mu^*$. By (h_1) and (h_{10}) there exists $\gamma \in L^1(0,T)$ such that

$$|f(t,s)| + \mu|g(t,s)| \le \gamma(t)$$

for a.e. $t \in [0,T]$ and every $s \in \mathbb{R}$. Without restriction we can also suppose that

$$\phi(\|w'\|_\infty) < \|\gamma\|_{L^1}.$$

We define ψ as in (2.3), Ψ as in (2.4) and, for all $\lambda > 0$, $\mathcal{I}_{\lambda,\mu} : H_0^1(0,T) \to \mathbb{R}$ by setting

$$\mathcal{I}_{\lambda,\mu}(v) = \int_0^T \Psi(v')\,dt - \lambda \int_0^T F(t,v)\,dt - \mu \int_0^T G(t,v)\,dt.$$

$\mathcal{I}_{\lambda,\mu}$ is C^1 and weakly lower semicontinuous; moreover, it is coercive and bounded from below. In particular $\mathcal{I}_{\lambda,\mu}$ satisfies the Palais-Smale condition. Consequently, for each $\lambda > 0$ there exists $u_1 \in H_0^1(0,T)$ such that

$$\mathcal{I}_{\lambda,\mu}(u_1) = \min_{v \in H_0^1(0,T)} \mathcal{I}_{\lambda,\mu}(v).$$

Observe that $u_1 \in W^{2,1}(0,T)$ and is a solution of the problem

$$\begin{cases} -(\psi(u'))' = \lambda f(t,u) + \mu g(t,u) & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases} \tag{2.25}$$

Note also that, by (2.24), if $\lambda \in]0,1[$ we have

$$\mathcal{I}_{\lambda,\mu}(u_1) \le \mathcal{I}_{\lambda,\mu}(w) < -c_f < 0. \tag{2.26}$$

Using the fact that $\lambda f(t,s) + \mu g(t,s) \ge 0$ for a.e. $t \in [0,T]$ and every $s \le 0$, and arguing as in the proof of Theorem 2.1, we see that any solution u of (2.25) satisfies $u \ge 0$. Therefore u_1 is a positive solution of (2.25).

As in the proof of Theorem 2.3 the second solution will be found by using the mountain pass theorem. We take $\varepsilon > 0$ such that

$$\tfrac{1}{2} - \mu\varepsilon(\tfrac{T}{\pi})^2 > 0.$$

By assumption (h_{12}) there exists r such that $0 < r < \|w\|_{H_0^1}$ and

$$G(t,s) \le \varepsilon s^2$$

for a.e. $t \in [0,T]$ and every $s \in [0,r]$. Take now $v \in H_0^1(0,T)$, with $0 < \|v\|_{H_0^1} \le r/c_\infty$ and $c_\infty > 1$ defined as in (2.21). We have, using also (2.5),

$$\begin{aligned} \int_0^T \Psi(v')\,dt - \mu \int_0^T G(t,v)\,dt &\ge \tfrac{1}{2} \int_0^T |v'|^2\,dt - \mu\varepsilon \int_0^T |v|^2\,dt \\ &\ge \|v\|_{H_0^1}^2 (\tfrac{1}{2} - \mu\varepsilon(\tfrac{T}{\pi})^2) > 0. \end{aligned} \tag{2.27}$$

Take a constant $\lambda(\mu) \in]0,1[$ such that

$$\frac{r^2}{c_\infty^2}(\tfrac{1}{2} - \mu\varepsilon(\tfrac{T}{\pi})^2) - \lambda(\mu)c_f > 0$$

and pick any $\lambda \in]0,\lambda(\mu)[$. By (2.27) we have

$$\mathcal{I}_{\lambda,\mu}(v) > 0$$

for all $v \in H^1_0(0,T)$ such that $\|v\|_{H^1_0} = r/c_\infty$. Since also (2.26) holds, by the mountain pass theorem we conclude that the functional $\mathcal{I}_{\lambda,\mu}$ has a critical point u_2, with $\mathcal{I}_{\lambda,\mu}(u_2) > 0$. Therefore u_2 is a positive solution of (2.25). Since $\mathcal{I}_{\lambda,\mu}(u_1) < 0$ we have $u_1 \neq u_2$.

Finally, we observe that there exists a local minimum point u_3 of $\mathcal{I}_{\lambda,\mu}$, with $\|u_3\|_{H^1_0} < r/c_\infty$. To verify that $u_3 \neq 0$ we argue as in the proof of Theorem 2.1. Consider a function $\zeta \in H^1_0(0,T)$, a constant $K > 0$ and a strictly decreasing sequence $(c_n)_n$ as in (2.8), with the further property, which follows from (h_{16}), that

$$G(t, c_n\zeta(t)) \geq -Kc_n^2\zeta(t)^2$$

for a.e. $t \in [a,b]$ and all n. Then we compute, using also (2.5),

$$\begin{aligned}\mathcal{I}_{\lambda,\mu}(c_n\zeta) &= \int_0^T \Psi(c_n\zeta')\,dt - \lambda\int_0^T F(t, c_n\zeta)\,dt - \mu\int_0^T G(t, c_n\zeta)\,dt\\ &\leq c_n^2\Big(\tfrac{1}{2}\sigma\|\zeta'\|^2_{L^2} - \lambda\int_c^d \frac{F(t,c_n)}{c_n^2}\,dt + (\lambda+\mu)K\|\zeta\|^2_{L^2}\Big).\end{aligned}$$

Hence, we conclude that $\mathcal{I}_{\lambda,\mu}(u_3) \leq \mathcal{I}_{\lambda,\mu}(c_n\zeta) < 0$ for large n and, in particular, $u_3 \neq 0$. Observe that, by (2.27), $-c_f < \mathcal{I}_{\lambda,\mu}(u_3)$. Since, by (2.26), $\mathcal{I}_{\lambda,\mu}(u_1) < -c_f$, we conclude that $u_1 \neq u_3$. Therefore u_1, u_2 and u_3 are positive solutions of (2.25) and, by the Claim in Step 1 of Theorem 2.1, of (2.23) as well.

Example 2.3 Take $p \in \,]0,1[$ and $q \in \,]1,+\infty[$. Let $m, n : [0,T] \to \mathbb{R}$ be continuous functions, with $m^+ > 0$ and $n^+ > 0$. Then Theorem 2.4 yields the existence of a constant $\mu^* > 0$ and of a function $\lambda : \,]\mu^*, +\infty[\, \to \mathbb{R}^+_0 \cup \{+\infty\}$ such that, for all $\mu > \mu^*$ and all $\lambda \in \,]0, \lambda(\mu)[$, the problem

$$\begin{cases}-\big(u'/\sqrt{1-u'^2}\big)' = \lambda m(t)u^p + \mu n(t)u^q & \text{in }]0,T[,\\ u(0) = u(T) = 0\end{cases}$$

has at least three positive solutions.

2.4 Existence of infinitely many positive solutions

The case of an oscillatory potential

Theorem 2.5 *Assume* (h_1), (h_2), (h_3), (h_4) *and*

(h_{17}) *there exists a continuous function* $l : [0,T/2] \to \mathbb{R}$ *such that* $f(t,s) \leq l(s)$, *for a.e.* $t \in [0,T]$ *and every* $s \in [0,T/2]$, *and*

$$\liminf_{s\to 0^+} \frac{L(s)}{s^2} = 0,$$

where $L(s) = \int_0^s l(\xi)\,d\xi$.

Then there exists a sequence $(u_k)_k$ *of positive solutions of* (1.1) *such that*

$$\lim_{k\to+\infty} \|u_k\|_\infty = 0.$$

Proof. As in the proof of Theorem 2.1 we replace f with a function, we still denote by f, which satisfies all the assumptions of the theorem, agrees with the original function in $[0,T]\times[0,T/2]$, vanishes outside the rectangle $[0,T]\times[-T,T]$, and is such that $f(t,s) \geq 0$ for a.e. $t \in [0,T]$ and every $s \leq 0$.

Suppose first that there exists a strictly decreasing sequence $(R_k)_k$ such that $\lim_{k\to+\infty} R_k = 0$ and $l(R_k) \leq 0$. In this case we set, for each k, $\beta_k = R_k$.

Suppose next that there exists $r > 0$ such that $l(s) > 0$ for every $s \in]0, r]$. By (h_{17}) we can find a strictly decreasing sequence $(R_k)_k$ such that $\lim_{k\to+\infty} R_k = 0$ and

$$\lim_{k\to+\infty} \frac{L(R_k)}{R_k^2} = 0. \tag{2.28}$$

We set, for $s \geq 0$,

$$\kappa(s) = \frac{1+s}{\sqrt{2+s}}.$$

For each $R \in]0, r]$ we define

$$T(R) = \int_0^R \frac{\kappa(L(R) - L(s))}{\sqrt{L(R) - L(s)}}\, ds = \int_0^1 \frac{R\kappa(L(R) - L(Rs))}{\sqrt{L(R) - L(Rs)}}\, ds.$$

This function is called the time-map and is such that the problem

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = l(u) & \text{in }]0, T(R)[, \\ u(0) = R,\ u'(0) = 0, \end{cases} \tag{2.29}$$

has a positive solution satisfying $u(T(R)) = 0$. We easily have, for each $R \in]0, r]$,

$$T(R) \geq \frac{R}{\sqrt{2L(R)}}.$$

Hence, by (2.28) we obtain

$$\lim_{k\to+\infty} T(R_k) = +\infty.$$

Consequently, there exists a sequence $(\beta_k)_k$ of solutions of the equation in (2.29), which are strictly positive on $[0, T]$ and such that

$$\lim_{k\to+\infty} \beta_k(t) = 0 \tag{2.30}$$

uniformly in $[0, T]$. We define ψ as in (2.3), Ψ as in (2.4) and $\mathcal{I}$ as in (2.7).

In both cases, for each k, β_k is an upper solution of the problem

$$\begin{cases} -(\psi(u'))' = f(t, u) & \text{in }]0, T[, \\ u(0) = u(T) = 0 \end{cases} \tag{2.31}$$

and, by (h_4), $\alpha = 0$ is a lower solution of (2.31). Applying [9, Lemma 2.1] yields the existence of a solution u_k of (2.31) such that $\alpha \leq u_k \leq \beta_k$ and

$$\mathcal{I}(u_k) = \min_{\substack{v\in H_0^1(0,T)\\ \alpha\leq v\leq\beta_k}} \mathcal{I}(v).$$

To verify that $u_k \neq 0$ we argue as in the proof of Theorem 2.1. Consider a function $\zeta \in H_0^1(0, T)$, a constant $K > 0$ and a strictly decreasing sequence $(c_n)_n$ as in (2.8). Then we easily compute, using also (2.5),

$$\begin{aligned} \mathcal{I}(c_n\zeta) &= \int_0^T \Psi(c_n\zeta')\, dt - \int_0^T F(t, c_n\zeta)\, dt \\ &\leq c_n^2\Big(\tfrac{1}{2}\sigma\|\zeta'\|_{L^2}^2 - \int_c^d \frac{F(t, c_n)}{c_n^2}\, dt + K\|\zeta\|_{L^2}^2\Big). \end{aligned}$$

Hence we get $\mathcal{I}(u_k) \leq \mathcal{I}(c_n\zeta) < 0$, for large n, and we conclude that $u_k \neq 0$. Therefore u_k is a positive solution of (2.31). Hence, by the Claim in Step 1, it is a positive solution of (1.1). As (2.30) holds uniformly on $[0, T]$, possibly passing to a subsequence, we may assume that $\beta_{k+1} < \beta_k$ and $\max \beta_{k+1} < \max u_k$ for all k. Hence we obtain $u_{k+1} \neq u_i$ for each $i \leq k$.

Example 2.4 Take $p \in]0,1[$ and $q \in]1,+\infty[$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(s) = (s^{p+1} + s^{q+1}) + (s^{p+1} - s^{q+1}) \sin(\ln |s|)$$

if $s \neq 0$ and $F(0) = 0$. Set $f = F'$. Then Theorem 2.5 yields the existence of infinitely many positive solutions of the problem

$$\begin{cases} -\Big(u' / \sqrt{1 - u'^2}\Big)' = f(u) & \text{in }]0,T[, \\ u(0) = u(T) = 0. \end{cases}$$

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