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On Variational Inequalities Driven by Elliptic Operators Not in Divergence Form

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Abstract

In this paper we study semilinear variational inequalities driven by an elliptic operator not in divergence form modeled by

$$\begin{cases} \langle Au, v-u\rangle \geq \displaystyle\int_\Omega |u(x)|^{s-1}u(x)(v(x)-u(x))\,dx & \text{for any } v \in H^1_0(\Omega),\ v \leq \psi \\ u \in H^1_0(\Omega),\ u \leq \psi\,, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$, with smooth boundary, A is the elliptic operator, not in divergence form, given by

$$Au = -\sum_{i,j=1}^N D_i\left(a_{ij}(x)D_j\,u\right) + \sum_{i=1}^N a_i(x)D_iu + a_0(x)u\,.$$

Here a_{ij}, a_i, $i, j = 1, \ldots, N$, and a_0 satisfy suitable regularity conditions, while $1 < s < 4/(N-2)$ and the obstacle ψ is a function sufficiently smooth. Even if this problem is not variational in nature, we will prove the existence of non-trivial non-negative solutions for it, performing a variational approach combined with a penalization technique. This kind of approach seems to be new for problems of this type. We also prove a $C^{1,\alpha}$–regularity result for the solutions of our problem.

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1 Introduction

1.1 Semilinear elliptic variational inequalities

Variational inequalities play an important role in the mathematical context both for their academic interest and for their many applications to various problems arising from physics, optimal control, financial mathematics, statistics and economics, such as, for instance, the obstacle problem, the fluid filtration problem in porous media, the elastic-plastic torsion problem, the stopping time problem, the real options approach to investments, only to cite few of them. In particular semilinear variational inequalities are widely studied by many authors (see, e.g., [3, 9, 10, 14, 15, 16, 17, 18, 19, 22, 23, 27, 31] and references therein) using different approaches, like variational and topological methods, sub and super-solutions, fixed point theorems, penalization techniques, a priori estimates on the solutions, approximation and regularization arguments. In this paper we study the following semilinear elliptic variational inequality:

$$\begin{cases} \langle Au, v-u\rangle \geq \displaystyle\int_\Omega f(x,u(x))(v(x)-u(x))\,dx \text{ for any } v\in H^1_0(\Omega),\ v\leq\psi \\ u\in H^1_0(\Omega),\ u\leq\psi\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N\geq 3$, with smooth boundary, A is the elliptic operator, not in divergence form, given by

$$Au=-\sum_{i,j=1}^N D_i\Big(a_{ij}(x)D_j\,u\Big)+\sum_{i=1}^N a_i(x)D_iu+a_0(x)u\,.$$

Here $a_{ij}:\overline{\Omega}\to\mathbb{R}$ are functions of class $C(\overline{\Omega})$ such that $a_{ij}=a_{ji}$, $i,j=1,\dots,N$, and

$$\lambda|\xi|^2\leq\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j\leq\Lambda|\xi|^2 \tag{1.2}$$

for any $x\in\Omega$ and $\xi\in\mathbb{R}^N$, for some positive constants λ and Λ, while

$$\begin{gathered} a_i,a_0:\Omega\to\mathbb{R},\ i=1,\dots,N\,, \\ \text{are bounded measurable functions with } a_0\geq 0 \text{ a.e. in } \Omega\,. \end{gathered} \tag{1.3}$$

In the following we denote by a the vector-valued function $a=(a_i)_{i=1,\dots,N}$ and by

$$\|a\|_\infty=\max_{i=1,\dots,N}\|a_i\|_\infty\,.$$

The obstacle function ψ is such that

$$\psi\in H^1_0(\Omega)\ \text{ with }\ \psi\geq 0\ \text{ a.e. in }\ \Omega\,; \tag{1.4}$$

$$\psi\in H^2(\Omega)\cap L^\infty(\Omega)\quad\text{and}\quad D_i\Big(a_{ij}(\cdot)D_j\,\psi\Big)\in L^\infty(\Omega)\ \ i,j=1,\dots,N\,. \tag{1.5}$$

Of course, as a model for ψ we can take any non-negative function in $C^2_0(\overline{\Omega})$. While the nonlinear term $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a function satisfying the following conditions:

$$f \text{ is locally Lipschitz continuous in } \overline{\Omega}\times\mathbb{R}\,,\text{ uniformly with respect to } x\in\overline{\Omega}; \tag{1.6}$$

$$\begin{gathered}\text{there exist } c_1 > 0 \text{ and } 1 < s < 4/(N-2), \text{ such that} \\ |f(x,t)| \le c_1(1+|t|^s) \text{ in } \overline{\Omega}\times\mathbb{R}\,;\end{gathered} \tag{1.7}$$

$$\lim_{|t|\to 0}\frac{f(x,t)}{|t|} = 0 \text{ uniformly in } x \in \overline{\Omega}\,; \tag{1.8}$$

$$\begin{gathered}\text{there exists } \mu > 2 \text{ such that for any } x \in \overline{\Omega}, t \in \mathbb{R}\setminus\{0\} \\ 0 < \mu F(x,t) \le t f(x,t)\,,\end{gathered} \tag{1.9}$$

where the function F is the primitive of f with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau\,. \tag{1.10}$$

Note that f grows subcritically, since $4/(N-2) < 2^* = 2N/(N-2)$ when $N \ge 3$. In the sequel we will denote by L_R, $R > 0$, the best Lipschitz constant of f, i.e.

$$L_R = \sup\left\{\frac{|f(x,t_1)-f(x,t_2)|}{|t_1-t_2|}\,,\ x\in\overline{\Omega}\,, t_i\in\mathbb{R}\,, |t_i|\le R\,, i=1,2, t_1\ne t_2\right\}. \tag{1.11}$$

A model for f is given by the function

$$f(x,t) = b(x)|t|^{s-1}t\,g(t)\,,$$

with $g \in Lip_{\mathrm{loc}}(\mathbb{R})\cap L^\infty(\mathbb{R})$, $g > 0$ in $\mathbb{R}$, while $b \in Lip_{\mathrm{loc}}(\overline{\Omega})$, $b > 0$ in $\overline{\Omega}$ and s is given in assumption (1.7).

1.2 Main results of the paper

Due to the presence of a lower order term containing Du in the elliptic operator A, problem (1.1) has not a variational structure, in the sense that it is not the Euler–Lagrange equation of some functional. Despite that, using the techniques introduced in [6, 11, 12] for studying nonlinear PDEs without a variational structure, here we will study the variational inequality (1.1) via critical points theorems. The idea consists of freezing the gradient Du appearing in the lower order term of A and of defining a new elliptic operator A_w. Then, we associate with the variational inequality (1.1) another variational inequality driven by the elliptic operator A_w: the advantage of considering this new problem is that it has a variational nature, so it can be studied through classical critical points theorems. Precisely, we associate with the variational inequality driven by A_w a penalized equation (i.e. an equation containing a penalization term which gives the information that the solution stays below the obstacle) which is variational in nature. In order to get a solution for it, along this paper we will use the Mountain Pass Theorem, see [2], thanks to our assumptions on the nonlinearity f.

Finally, in order to come back to the original problem, we will perform an iteration scheme, using the techniques of [6, 11, 12]. The penalization method we will use was first introduced by Bensoussan and Lions in [4], while the iterative technique we will perform was firstly used in [6] by Defigueiredo, Girardi and Matzeu (see also [11, 12, 19, 20, 21, 26]) in order to study a semilinear equation governed by the Laplacian operator $-\Delta$, when the nonlinear term depends also on the gradient of the solution, i.e. when it is of the form $f = f(\cdot, u, Du)$. In the present paper we will adapt all these techniques in order to study a variational inequality driven by an operator not in divergence form. Following this strategy we will prove the existence and regularity result stated here below:

Theorem 1.1 *Assume conditions* (1.2)–(1.9) *hold true*[1]. *Moreover suppose that there exists* $e \in H_0^1(\Omega)$ *such that*[2]

$$\|e\| > 4H/\lambda \tag{1.12}$$

[1] When $N = 3$ we also need to assume that $s < 2$ in (1.7). See the end of Subection 4.2 for more details.

[2] Note that e depends only on a_{ij}, $i, j = 1,\ldots,N$, a_0, c_1, c_2, c_3, s, μ and Ω. See Subsection 2.3 for the details.

$$\left(\frac{\Lambda}{2}+\frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 < c_2\|e\|_\mu^\mu - c_3|\Omega| - H\,\|e\| \tag{1.13}$$

$$0 \le e \le \psi \;\; a.e. \; in\, \Omega\,. \tag{1.14}$$

Then, there exist two positive constants R and C depending only on a_{ij}, $i, j = 1, 2, 3$, a_0, c_1, s, ψ, N and Ω, such that, if

$$\|a\|_\infty < C \tag{1.15}$$

and

$$L_R < (\lambda\lambda_1)/2\,, \tag{1.16}$$

then problem (1.1) *admits a non-trivial non-negative solution u belonging to $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in$* $(0,1)$.

The constant L_R mentioned in Theorem 1.1 is defined in formula (1.11), while the explicit formulas for R and C will be given in Subsection 2.3 (see formulas (1.26) and (1.27), respectively). The constants c_2, c_3 and H appearing in (1.12) and (1.13) will be given in the forthcoming Lemma 2.2 and in (1.23). Note that assumption (1.13) is satisfied if $\|e\| >> 1$. Indeed, since span$\{e\}$ is a finite dimensional subspace of $H^1_0(\Omega)$, it is easily seen that for any $v \in$ span$\{e\}$

$$\kappa\|v\| \le \|v\|_\mu \le K\|v\|$$

for some κ and K positive. Hence,

$$\begin{aligned}&\left(\frac{\Lambda}{2}+\frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 - c_2\|e\|_\mu^\mu + c_3|\Omega| + H\,\|e\|\\ &\le \left(\frac{\Lambda}{2}+\frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 - c_2\kappa^\mu\|e\|^\mu + c_3|\Omega| + H\|e\| < 0\,,\end{aligned}$$

if $\|e\| >> 1$, being $\mu > 2$ by assumption (1.9). Moreover (1.14) is not a restrictive condition on the obstacle function ψ. Indeed, in [27, Proposition 4.1] the authors prove that (1.13) is a *natural* assumption to have non-trivial non-negative solutions for (1.1) since, if ψ is 'small', the only non-negative solution of problem (1.1) is the trivial one. For further comments on hypotheses (1.12)–(1.14) we refer to [10, 18]. In the sequel the existence result will be obtained, as we said before, using critical points theory and an iterative technique, while for the regularity of the solutions we will use the Lewy-Stampacchia estimates for variational inequalities (see [13]) and also the regularity theory for elliptic equations (see, for instance, [5, 8, 30]). This paper is organized as follows. In Section 2 we introduce the notations used along the paper and we give some preliminary estimates on the nonlinearity. Finally, we define the constants appearing in the main Theorem 1.1. In Section 3 we illustrate the strategy we will follow in order to get our existence and regularity result: it is based on a penalization method combined with variational methods and an iterative procedure. Section 4 is devoted to the study of the penalized equation associated with problem (1.1) (here we will apply the Mountain Pass Theorem), while in Section 5 we perform the iteration scheme and we conclude the proof of the main result of the paper.

2 Preliminaries

2.1 Notations

Throughout the paper we denote by $H^1_0(\Omega)$ the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |Du|^2 dx\right)^{1/2} \tag{1.17}$$

and by $L^q(\Omega)$, with $q \in [1, \infty)$, the usual Lebesgue space with the norm defined as

$$\|u\|_q = \left(\int_\Omega |u|^q dx\right)^{1/q}.$$

Note that, taking into account these definitions, for any $u \in H_0^1(\Omega)$ the following inequality holds true[3]

$$\sum_{i=1}^N \|D_i u\|_2 \leq \sqrt{N}\left(\sum_{i=1}^N \|D_i u\|_2^2\right)^{1/2} = \sqrt{N}\|u\|, \tag{1.18}$$

where $D_i u$ is the i–th component of the gradient Du. This will be useful in the sequel. Moreover, $C^{1,\alpha}(\overline{\Omega})$ is equipped with the usual norm $\|\cdot\|_{1,\alpha}$, while $C_R^{1,\alpha}(\overline{\Omega})$ will be the following set

$$C_R^{1,\alpha}(\overline{\Omega}) = \left\{u \in C^{1,\alpha}(\overline{\Omega}) : \|u\|_{1,\alpha} \leq R\right\}$$

with $\alpha \in (0,1)$ and $R > 0$. Finally, in the sequel λ_1 will be the first eigenvalue of the Laplacian operator $-\Delta$ in Ω with homogeneous Dirichlet boundary data, that is

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |Du|^2 dx}{\displaystyle\int_\Omega |u|^2 dx},$$

while S_q will be the best constant in the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q \leq 2^*$.

2.2 Some useful estimates

Here, from the structural assumptions on f we derive some bounds from above and below for the nonlinear term and its primitive, which will be useful along the paper. This part is quite standard and does not take into account the fact that the problem is not variational in nature: the reader familiar with these estimates may go directly to Subsection 2.3.

Lemma 2.1 *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying conditions* (1.6)–(1.8). *Then, for any $\delta > 0$ there exists $\eta(\delta) > 0$ such that for any $x \in \overline{\Omega}$ and for any $t \in \mathbb{R}$*

$$|f(x,t)| \leq 2\delta|t| + (s+1)\eta(\delta)|t|^s \tag{1.19}$$

and so, as a consequence,

$$|F(x,t)| \leq \delta\,|t|^2 + \eta(\delta)\,|t|^{s+1}, \tag{1.20}$$

where F is defined as in (1.10).

Lemma 2.2 *Let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.6) *and* (1.9). *Then, there exist two positive constants c_2 and c_3 such that for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$*

$$F(x,t) \geq c_2|t|^\mu - c_3. \tag{1.21}$$

For the proof of Lemmas 2.1 and 2.2 we refer, respectively, to [28, Lemma 3] and [29, Lemma 4] (similar estimates are also in [25, 30]).

[3] As a consequence of the Cauchy–Schwarz inequality, it holds true that $\left(\sum_{i=1}^N \beta_i\right)^2 \leq N\sum_{i=1}^N \beta_i^2$ for $\beta_i \geq 0$, $i = 1, \ldots, N$.

2.3 Definition of the constants appearing in Theorem 1.1

In the sequel we will denote by $\bar{A}$ the elliptic operator defined as follows

$$\bar{A}u := -\sum_{i,j=1}^{N} D_i\Big(a_{ij}(x)D_j\, u\Big)\,. \tag{1.22}$$

Let $e \in H_0^1(\Omega)$ be the function given in Theorem 1.1 . Let us define the constants

$$H = \frac{\lambda}{4}\left(\frac{\lambda}{8}\cdot\frac{1}{\eta(\lambda\lambda_1/8)S_{s+1}^{s+1}}\right)^{1/(s-1)} \tag{1.23}$$

$$T = \|e\| \tag{1.24}$$

and

$$\sigma = \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)T^2 + HT \tag{1.25}$$

where λ, Λ are as in (1.2) and η is given in (1.19). Note that H depends only on a_{ij}, $i, j = 1, \ldots, N$, η, s and Ω and therefore only on a_{ij}, $i, j = 1, \ldots, N$, c_1, s and Ω, while T depends only on e and, therefore, only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, c_2, c_3, s, μ and Ω . Hence also σ depends only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, c_2, c_3, s, μ and Ω . Finally, let $\varepsilon' = 1 - \frac{s(N-2)}{4} \in (0, 1)$ (thanks to the choice of s, cfr. assumption (1.7)) and let us define the following constants

$$\tilde{C} = k^{(N-2)/N}\,[N/(N-2)]^{(N-2)/2}\ ,$$

with k suitable positive constant depending on ε' (see the proof of [24, Theorem 2.4] for more details),

$$K_1 = \tilde{C}^{N/2\varepsilon'}\left[(c_1 + \|\bar{A}\psi\|_\infty + \|a_0\|_\infty)(|\Omega| + S_{2^*}^{s}\|\psi\|^s)|\Omega| + \frac{\|\psi\|}{\sqrt{\lambda_1}}\right],$$

$$K_2 = \tilde{C}^{N/2\varepsilon'}|\Omega|(|\Omega| + S_{2^*}^{s}\|\psi\|^s)\,,$$

$$\hat{C} = K_1 + 1\,,$$

and

$$\overline{C} = c_1|\Omega|\Big(1 + \hat{C}^s\Big) + |\Omega|\,\|\bar{A}\psi\|_\infty + |\Omega|\hat{C}\|a_0\|_\infty\,,$$

all depending only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, s, ψ, N and Ω . Furthermore, let C_{Mor} be the embedding constant in the Morrey Theorem and C_{CZ} be the constant in the Calderón-Zygmund Theorem applied in $L^q(\Omega)$, $q \in [1, +\infty)$. It is well known that the constant C_{Mor} depends only on s and Ω, while C_{CZ} depends only on a_{ij}, $i, j = 1, \ldots, N$, q and Ω. Now we can define the constants R and C appearing in the Theorem 1.1. We put

$$R = 2C_{\mathrm{Mor}}C_{\mathrm{CZ}}\overline{C} \tag{1.26}$$

and

$$C = \min\left\{\frac{\sqrt{\lambda_1}H}{\sqrt{N\,|\Omega|}\,R}\,, \frac{1}{2C_{\mathrm{Mor}}C_{\mathrm{CZ}}|\Omega|}\,, \frac{1}{K_2 R}\,, \frac{\lambda\sqrt{\lambda_1}}{2}\right\}. \tag{1.27}$$

Both R and C depend only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, s, ψ, N and Ω .

3 Strategy for proving Theorem 1.1

Problem (1.1) is not variational in nature, due to the presence of Du in the lower order term of A. Despite that, we will treat it via variational techniques. Precisely, we associate, in a suitable way, with our problem a semilinear elliptic equation and we essentially adapt to our case the idea introduced in [6] which consists of 'freezing' the gradient of u in the lower order terms of the equation. In this way we have to manage a PDE which can be studied using the classical critical points theorems (in this work we will use the Mountain Pass Theorem by Ambrosetti and Rabinowitz [2]). Precisely, the strategy for proving our main theorem can be summarized as follows: let $\bar{\alpha} \in (0, 1)$ be fixed and let R be as in (1.26) and C as in (1.27). Now we proceed by steps:

- problem (1.1) has no variational nature, as we said before. Hence, in order to make this problem variational, it is enough to 'freeze' the gradient Du appearing in the lower order term of A. For this in the sequel we will fix w in $H_0^1(\Omega) \cap C_R^{1,\bar{\alpha}}(\overline{\Omega})$ and we will consider the following variational inequality

$$\begin{cases} \langle A_w u_w, v - u_w \rangle \geq \displaystyle\int_\Omega f(x, u_w(x))(v(x) - u_w(x))\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } v \in H_0^1(\Omega),\ v \leq \psi \\ u_w \in H_0^1(\Omega),\ u_w \leq \psi\,. \end{cases} \tag{1.28}$$

Here A_w is the elliptic operator in divergence form defined as follows

$$A_w u = -\sum_{i,j=1}^N D_i\left(a_{ij}(x) D_j\, u\right) + \sum_{i=1}^N a_i(x) D_i w + a_0(x) u\,; \tag{1.29}$$

- in order to study (1.28) we will perform the classical techniques of critical points theory combined with a penalization method. For this, first of all we will associate with the variational inequality (1.28) the following penalized equation

$$\begin{cases} \langle A_w u_w^\varepsilon, v \rangle + \dfrac{1}{\varepsilon} \displaystyle\int_\Omega (u_w^\varepsilon - \psi)^+(x) v(x)\, dx = \int_\Omega f(x, u_w^\varepsilon(x)) v(x)\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } v \in H_0^1(\Omega) \\ u_w^\varepsilon \in H_0^1(\Omega), \end{cases} \tag{1.30}$$

where $\varepsilon > 0$ is the penalization parameter, while

$$\frac{1}{\varepsilon} \int_\Omega (u_\varepsilon - \psi)^+(x) v(x)\, dx$$

represents the penalization term which contains the information that u_w^ε stays below the obstacle (otherwise the penalization term blows up as $\varepsilon \to 0$). Problem (1.30) is the Euler–Lagrange equation of the functional

$$I_w^\varepsilon : H_0^1(\Omega) \to \mathbb{R}$$

defined as

$$\begin{aligned} I_w^\varepsilon(u) = &\frac{1}{2} \sum_{i,j=1}^N \int_\Omega a_{ij}(x) D_i u(x) D_j u(x)\, dx + \frac{1}{2} \int_\Omega a_0(x) u^2(x)\, dx \\ &+ \frac{1}{2\varepsilon} \int_\Omega \left((u - \psi)^+\right)^2(x)\, dx - \int_\Omega F(x, u(x))\, dx \\ &+ \sum_{i=1}^N \int_\Omega a_i(x) u(x) D_i w(x)\, dx. \end{aligned}$$

Note that I_w^ε is well–defined in $H_0^1(\Omega)$ and it is Fréchet differentiable in $H_0^1(\Omega)$, thanks to the Sobolev embedding theorems and (1.7). Also critical points of I_w^ε are weak solutions of (1.30). In order to get a non-trivial non-negative weak solution u_w^ε of (1.30), in the sequel we will apply the Mountain Pass Theorem to the functional I_w^ε ;

- now, we have to come back to problem (1.28). For this we will suitably estimate the $H_0^1(\Omega)$–norm of u_w^ε. Thanks to these estimates we will show that u_w^ε weakly converges to some $u_w \in H_0^1(\Omega)$ as $\varepsilon \to 0$. Moreover, we will prove that such a u_w is a non-trivial non-negative solution of the variational inequality (1.28) ;
- finally, in order to come back to the original problem (1.1) we need some regularity properties on u_w (i.e. $u_w \in C^{1,\alpha}$ for any $\alpha \in (0,1)$) and a suitable estimate on the $C^{1,\alpha}$–norm of u_w. By these ingredients we will construct a non-trivial non-negative solution u of problem (1.1) through an iterative technique. Precisely, we will fix $w = u_0$ in $H_0^1(\Omega) \cap C_R^{1,\bar\alpha}(\overline{\Omega})$ and we will consider the following sequence of semilinear variational inequality

$$\begin{cases} \langle A_n u_n, v - u_n\rangle \geq \displaystyle\int_\Omega f(x, u_n(x))(v(x) - u_n(x))\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } v \in H_0^1(\Omega),\ v \leq \psi \\ u_n \in H_0^1(\Omega),\ u_n \leq \psi\,, \end{cases} \tag{1.31}$$

where

$$A_n u = A_{u_{n-1}} u\,, \quad n \in \mathbb{N}\,.$$

For any fixed n, equation (1.31) admits a non-trivial non-negative solution u_n . We will show that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_0^1(\Omega)$ and so, as a consequence of this, it converges to some $u \in H_0^1(\Omega)$. Finally, we will prove that such a u is a non-trivial non-negative solution of problem (1.1) which has good regularity properties and this ends to proof of Theorem 1.1 .

4 Critical points of the functional I_w^ε

This section is devoted to the study of problem (1.30) . In the sequel $\bar\alpha \in (0,1)$, w in $H_0^1(\Omega)\cap C_R^{1,\bar\alpha}(\overline{\Omega})$ and $\varepsilon > 0$ are fixed. First of all, thanks to the Mountain Pass Theorem we will prove that the functional I_w^ε admits a non-trivial non-negative critical point $u_w^\varepsilon \in H_0^1(\Omega)$. Later, we will give some estimates on u_w^ε . We would like to remark that in this section (as well as in the next Subsection 5.1) we only need that $w \in H_0^1(\Omega)$ and $\|w\| \leq R\,|\Omega|^{1/2}$, while the $C^{1,\bar\alpha}$–regularity will be used from Subsection 5.2 on. Of course, if $w \in C_R^{1,\bar\alpha}(\overline{\Omega})$ then

$$\|w\| = \left(\int_\Omega |Dw(x)|^2\,dx\right)^{1/2} \leq R\,|\Omega|^{1/2}\,.$$

This estimate will be useful in the sequel.

4.1 Existence of a non-trivial non-negative critical point u_w^ε for I_w^ε

Thanks to our assumptions on the nonlinear term f, in order to get a non-trivial critical point for I_w^ε, we will apply the Mountain Pass Theorem. For this we have to prove that I_w^ε has a suitable geometric structure and satisfies a compactness condition. First of all, let us study the geometry of I_w^ε.

Claim 1 *There exist $\rho > 0$ and $\beta > 0$ such that for any $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \rho$ it results that $I_w^\varepsilon(u) \geq \beta$.*

Proof. The non-negativity of a_0 and of the penalization term, Hőlder inequality, (1.2) , (1.18) , (1.20) , the choice of w and the definition of λ_1 yield

$$
\begin{aligned}
I_w^\varepsilon(u) &\geq \frac{\lambda}{2}\int_\Omega |Du(x)|^2 dx - \delta\int_\Omega |u(x)|^2 dx - \eta(\delta)\int_\Omega |u(x)|^{s+1} dx - \frac{\sqrt{N}\|a\|_\infty}{\sqrt{\lambda_1}}\|w\|\|u\| \\
&\geq \left(\frac{\lambda}{4} - \frac{\delta}{\lambda_1}\right)\|u\|^2 - \eta(\delta)\|u\|_{s+1}^{s+1} + \frac{\lambda}{4}\|u\|^2 - \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\|u\| \\
&= \left[\left(\frac{\lambda}{4} - \frac{\delta}{\lambda_1}\right) - \eta(\delta)S_{s+1}^{\,s+1}\|u\|^{s-1}\right]\|u\|^2 + \left[\frac{\lambda}{4}\|u\| - \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\right]\|u\|,
\end{aligned}
$$

for any $\delta > 0$ and for some positive constant $\eta(\delta)$. Choosing $\delta = \frac{\lambda\lambda_1}{8}$ we obtain

$$
I_w^\varepsilon(u) \geq \left(\frac{\lambda}{8} - \eta(\lambda\lambda_1/8)S_{s+1}^{\,s+1}\|u\|^{s-1}\right)\|u\|^2 + \left[\frac{\lambda}{4}\|u\| - \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\right]\|u\| .
$$

Now, let us take $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. Since (1.15) and (1.27) hold true, we can choose $\rho > 0$ such that $\frac{\lambda}{8} > \eta(\lambda\lambda_1/8)S_{s+1}^{\,s+1}\rho^{s-1}$ and $\frac{\lambda}{4}\rho > \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty$; so that we get

$$
I_w^\varepsilon(u) \geq \beta\,,
$$

for some positive β depending only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, s and Ω .

Claim 2 *There exist $\tilde{R} > \rho$ and $e \in H_0^1(\Omega)$ such that $\|e\| \geq \tilde{R}$ and $I_w^\varepsilon(e) < 0$, where ρ is given in Claim 1.*

Proof. Let e be as in Theorem 1.1 and let $\tilde{R} = 4H/\lambda$. By the choice of ρ in Claim 1, it is easily seen that

$$
\rho < \left(\frac{\lambda}{8}\cdot\frac{1}{\eta(\lambda\lambda_1/8)S_{s+1}^{\,s+1}}\right)^{1/(s-1)} = \frac{4}{\lambda}H = \tilde{R}\,.
$$

Since $e \leq \psi$ a.e. in Ω (see (1.14)), we have

$$
\begin{aligned}
I_w^\varepsilon(e) = \frac{1}{2}\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_ie(x)D_je(x)\,dx + \frac{1}{2}\int_\Omega a_0(x)e^2(x)\,dx - \int_\Omega F(x, e(x))\,dx \\
+ \sum_{i=1}^N \int_\Omega a_i(x)e(x)D_iw(x)\,dx\,,
\end{aligned}
$$

so that, by (1.2) , (1.18), (1.21) and the properties of w, we deduce that

$$
I_w^\varepsilon(e) \leq \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 - c_2\|e\|_\mu^\mu + c_3|\Omega| + \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\,\|e\|\,,
$$

which, by (1.15), (1.27) and, finally, (1.13), yields

$$
I_w^\varepsilon(e) \leq \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 - c_2\|e\|_\mu^\mu + c_3|\Omega| + H\,\|e\| < 0\,.
$$

This concludes the proof of the Claim 2 . Roughly speaking, Claims 1 and 2 say that the functional I_w^ε has the geometrical structure required by the Mountain Pass Theorem. It is standard to prove that I_w^ε verifies the Palais–Smale condition. For the details we refer to [19, 21] . Then, by the Mountain Pass Theorem, the functional I_w^ε has a non-trivial critical point u_w^ε such that

$$
I_w^\varepsilon(u_w^\varepsilon) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_w^\varepsilon(\gamma(t)) \geq \beta > 0\,, \tag{1.32}
$$

where $\Gamma = \{\gamma \in C([0,1];\mathbb{R}) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. Hence, u_w^ε is a non-trivial solution of equation (1.30). Here β is the positive constant given in Claim 1. In order to get a non-negative solution for (1.30) it is convenient to modify the functional I_w^ε as follows

$$\tilde{I}_w^\varepsilon(u) = \frac{1}{2}\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_iu(x)D_ju(x)\,dx + \frac{1}{2}\int_\Omega a_0(x)u^2(x)\,dx + \frac{1}{2\varepsilon}\int_\Omega \Big((u-\psi)^+\Big)^2(x)\,dx - \int_\Omega \tilde{F}(x,u(x))\,dx\,,$$

where

$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,\tau)\,d\tau$$

and

$$\tilde{f}(x,t) = \begin{cases} f(x,t) - \sum_{i=1}^N a_i(x)D_iw(x) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0\,. \end{cases}$$

Of course $\tilde{I}_w^\varepsilon$ is well–defined and Fréchet differentiable in $H_0^1(\Omega)$ and, moreover, it satisfies all the assumptions of the Mountain Pass Theorem. Hence it has a non-trivial critical point $\tilde{u}_w^\varepsilon$. Let us show that $\tilde{u}_w^\varepsilon$ is a non-trivial non-negative critical point of I_w^ε. Since $\tilde{u}_w^\varepsilon$ is a critical point of $\tilde{I}_w^\varepsilon$ we have that for any $v \in H_0^1(\Omega)$

$$\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i\tilde{u}_w^\varepsilon(x)D_jv(x)\,dx + \int_\Omega a_0(x)\tilde{u}_w^\varepsilon(x)v(x)\,dx + \frac{1}{\varepsilon}\int_\Omega (\tilde{u}_w^\varepsilon - \psi)^+(x)v(x)\,dx = \int_\Omega \tilde{f}(x,\tilde{u}_w^\varepsilon(x))v(x)\,dx\,,$$

so that, taking $v = (\tilde{u}_w^\varepsilon)^-$ (i.e. the negative part of $\tilde{u}_w^\varepsilon$) we get

$$\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i\tilde{u}_w^\varepsilon(x)D_j(\tilde{u}_w^\varepsilon)^-(x)\,dx + \int_\Omega a_0(x)\tilde{u}_w^\varepsilon(x)(\tilde{u}_w^\varepsilon)^-(x)\,dx + \frac{1}{\varepsilon}\int_\Omega (\tilde{u}_w^\varepsilon - \psi)^+(x)(\tilde{u}_w^\varepsilon)^-(x)\,dx = \int_\Omega \tilde{f}(x,\tilde{u}_w^\varepsilon(x))(\tilde{u}_w^\varepsilon)^-(x)\,dx\,,$$

which, taking into account (1.2), the non-negativity of a_0, the definition of $\tilde{f}$ and the fact that $\psi \geq 0$ a.e. in Ω, gives

$$\lambda\|(\tilde{u}_w^\varepsilon)^-\| \leq \sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i(\tilde{u}_w^\varepsilon)^-(x)D_j(\tilde{u}_w^\varepsilon)^-(x)\,dx + \int_\Omega a_0(x)\big((\tilde{u}_w^\varepsilon)^-\big)^2(x)\,dx = 0\,.$$

Hence $(\tilde{u}_w^\varepsilon)^- = 0$ a.e. in Ω, that is $\tilde{u}_w^\varepsilon \geq 0$ a.e. in Ω. Finally, since $\tilde{u}_w^\varepsilon$ is non-negative in Ω and it is a critical point of $\tilde{I}_w^\varepsilon$, it is easily seen that it is also a critical point for the original functional I_w^ε, thanks to the definition of $\tilde{f}$. In this way we have shown that problem (1.30) admits a non-trivial non-negative solution which, in the sequel, we will denote by u_w^ε.

4.2 Estimates on u_w^ε

Now, we have to come back to the variational inequality (1.28). For this we need some estimates, uniformly with respect to ε, on the $H_0^1(\Omega)$–norm of u_w^ε. First of all, note that the Mountain Pass characterization of the critical level of I_w^ε gives

$$I_w^\varepsilon(u_w^\varepsilon) \leq \max_{t\in[0,1]} I_w^\varepsilon(\gamma(t)) \;\text{ for all } \gamma \in \Gamma.$$

Taking $\gamma(t) = te$, where $t \in [0,1]$, by (1.2), (1.9), the properties of e (note that $te \le e \le \psi$ a.e. in Ω) and of w, (1.15) and (1.27) we get

$$
\begin{aligned}
I^\varepsilon_w(u^\varepsilon_w) &\le \max_{t\in[0,1]} \left\{ \frac{\Lambda t^2}{2}\|e\|^2 + \frac{t^2}{2\lambda_1}\|a_0\|_\infty\|e\|^2 - \int_\Omega F(x,te(x))dx \right. \\
&\qquad\qquad \left. + \frac{\sqrt{N}tR\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\|e\| \right\} \\
&\le \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)\|e\|^2 + \frac{\sqrt{N}R\,|\Omega|^{1/2}}{\sqrt{\lambda_1}}\|a\|_\infty\|e\| \\
&< \left(\frac{\Lambda}{2} + \frac{\|a_0\|_\infty}{2\lambda_1}\right)T^2 + HT = \sigma\,,
\end{aligned}
\tag{1.33}
$$

where σ is the constant defined in (1.25). Note that σ does not depend on ε. At this point we can prove the following result:

Proposition 4.1 *Let $\psi \in H^1_0(\Omega)$ and $s < 2$ in* (1.7). *Then, there exists a positive constant $\tilde\kappa$ independent of ε, such that $\|u^\varepsilon_w\| \le \tilde\kappa$ for any $\varepsilon > 0$.*

Proof. We argue by contradiction and we suppose that, up to a subsequence (still denoted by u^ε_w),

$$\|u^\varepsilon_w\| \longrightarrow \infty \quad \text{as} \quad \varepsilon \to 0\,. \tag{1.34}$$

As a consequence of this and of (1.2) we have that

$$\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle := \sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_iu^\varepsilon_w(x)D_ju^\varepsilon_w(x)\,dx \to +\infty \;\text{ as }\; \varepsilon \to 0 \tag{1.35}$$

and also

$$\left\|\frac{u^\varepsilon_w}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle^{1/2}}\right\| = \frac{\|u^\varepsilon_w\|}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle^{1/2}} \le \frac{1}{\sqrt{\lambda}}\,,$$

so that $\frac{u^\varepsilon_w}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle^{1/2}}$ is bounded, uniformly with respect to ε, in $H^1_0(\Omega)$. Hence, there exists $\tilde u \in H^1_0(\Omega)$ such that[4]

$$\frac{u^\varepsilon_w}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle^{1/2}} \to \tilde u \quad \text{weakly in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \to 0. \tag{1.36}$$

Since u^ε_w is a solution of problem (1.30), taking $v = \dfrac{u^\varepsilon_w}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle}$ as a test function in (1.30), passing to the limit and taking into account (1.35), we obtain that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\left(\int_\Omega \frac{f(x,u^\varepsilon_w(x))u^\varepsilon_w(x)}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle}\,dx \;-\frac{1}{\varepsilon}\int_\Omega \frac{(u^\varepsilon_w-\psi)^+(x)u^\varepsilon_w(x)}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle}\,dx\right) \\
= 1 + \int_\Omega a_0(x)\tilde u^2(x)\,dx\,.
\end{aligned}
\tag{1.37}
$$

By (1.33) and (1.35) we also deduce that $I^\varepsilon_w(u^\varepsilon_w)/\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle \to 0$ as $\varepsilon \to 0$. Hence,

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\left(\int_\Omega \frac{F(x,u^\varepsilon_w(x))}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle}\,dx - \frac{1}{2\varepsilon}\int_\Omega \frac{\left((u^\varepsilon_w-\psi)^+(x)\right)^2}{\langle \tilde A u^\varepsilon_w, u^\varepsilon_w\rangle}\,dx\right) \\
= \frac{1}{2} + \frac{1}{2}\int_\Omega a_0(x)\tilde u^2(x)\,dx\,.
\end{aligned}
\tag{1.38}
$$

[4] Of course $\tilde u$ depends on w, but, for the sake of simplicity, we omit it in the notation.

Combining (1.37) and (1.38) with $(F4)$ we get

$$\limsup_{\varepsilon\to 0}\Big[(\mu-2)\int_\Omega \frac{F(x,u_w^\varepsilon(x))}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx - \frac{1}{\varepsilon}\int_\Omega \frac{(u_w^\varepsilon-\psi)^+(x)\psi(x)}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx\Big] \le 0. \tag{1.39}$$

Taking ψ as a test function in (1.30) (this choice is admissible since $\psi \in H_0^1(\Omega)$ by assumption) we have

$$\begin{aligned}\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_iu_w^\varepsilon(x)D_j\psi(x)\,dx &+ \int_\Omega a_0(x)u_w^\varepsilon(x)\psi(x)\,dx \\ &+ \frac{1}{\varepsilon}\int_\Omega (u_w^\varepsilon-\psi)^+(x)\psi(x)\,dx \\ &= \int_\Omega f(x,u_w^\varepsilon(x))\psi(x)\,dx - \sum_{i=1}^N\int_\Omega a_i(x)\psi(x)D_iw(x)\,dx\,.\end{aligned} \tag{1.40}$$

Note that, by (1.19), (1.35), (1.36) and the dominated convergence theorem we have

$$\begin{aligned}\Big|\int_\Omega \frac{f(x,u_w^\varepsilon(x))\psi(x)}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx\Big| &\le 2\delta\int_\Omega \frac{|u_w^\varepsilon(x)||\psi(x)|}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx \\ &+ (s+1)\eta(\delta)\int_\Omega \frac{|u_w^\varepsilon(x)|^s}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle^{s/2}}\cdot\frac{|\psi(x)|}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle^{1-s/2}}\,dx \to 0,\end{aligned}$$

as ε goes to zero, since $s<2$. Hence

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{f(x,u_w^\varepsilon(x))\psi(x)}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx = 0\,.$$

Thus, dividing (1.40) by $\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle$ and passing to the limit as ε goes to zero we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \frac{(u_w^\varepsilon-\psi)^+(x)\psi(x)}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx = 0\,,$$

so that, (1.9) and (1.39) yield

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{F(x,u_w^\varepsilon(x))}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx = 0\,. \tag{1.41}$$

Then, (1.38) and the non-negativity of a_0 a.e. in Ω (see (1.3)) give

$$0 \le \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_\Omega \frac{\left((u_w^\varepsilon-\psi)^+\right)^2(x)}{\langle \tilde{A}u_w^\varepsilon, u_w^\varepsilon\rangle}\,dx = -\frac{1}{2}\left(1+\int_\Omega a_0(x)\tilde{u}^2(x)\,dx\right) < 0\,,$$

which is a contradiction. Hence, (1.34) can not occur. This ends the proof of Proposition 4.1 .

We note that in the proof of Proposition 4.1 we need the assumption $s<2$ in order to get a contradiction. Recalling the choice of s (see assumption (1.7)) this condition is restrictive only when $N=3$, since for $N>3$ is automatically satisfied, as, in this case, $4/(N-2)\le 2$.

5 An iterative scheme

In this section we will come back to the variational inequality (1.28) . First of all, we will prove that (1.28) admits a non-trivial non-negative solution and later we will discuss the regularity of such a solution. This last fact will be used in order to construct a non-trivial solution of the original problem (1.1) .

5.1 A solution u_w for the variational inequality driven by A_w

This subsection is devoted to the existence of a solution for problem (1.28). Thanks to Proposition 4.1 we can show the following result:

Proposition 5.1 *Let ψ be as in* (1.4) *and $s < 2$ in* (1.7). *Then the variational inequality* (1.28) *admits a non-negative solution $u_w \in H_0^1(\Omega)$. Moreover, there exists a positive constant $\tilde{\beta}$, depending only on a_{ij}, $i, j = 1, \ldots, N$, a_0, c_1, s and Ω, such that u_w satisfies the following inequality*

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\, dx + \int_\Omega a_0(x) u_w^2(x)\, dx \\
&\quad - 2\int_\Omega F(x, u_w(x))\, dx + 2\sum_{i=1}^{N} \int_\Omega a_i(x) D_i w(x) u_w(x)\, dx \\
&\quad - \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j (u_w - \psi)^+(x)\, dx \\
&\quad - \int_\Omega a_0(x) u_w(x)(u_w - \psi)^+(x)\, dx + \int_\Omega f(x, u_w(x))(u_w - \psi)^+(x)\, dx \\
&\quad - \sum_{i=1}^{N} \int_\Omega a_i(x) D_i w(x)(u_w - \psi)^+(x)\, dx \\
&\quad \geq \tilde{\beta}\,.
\end{aligned} \tag{1.42}$$

As a consequence, $u_w \not\equiv 0$ in Ω.

Proof. By Proposition 4.1 it is easy to see that there exists $u_w \in H_0^1(\Omega)$ such that

$$u_w^\varepsilon \to u_w \quad \text{weakly in} \quad H_0^1(\Omega) \quad \text{as } \varepsilon \to 0. \tag{1.43}$$

By (1.33), (1.2), the non-negativity of a_0, (1.20), the choice of w and Sobolev embedding theorem it follows that

$$\begin{aligned}
\frac{1}{2\varepsilon}\|(u_w^\varepsilon - \psi)^+\|_2^2 &\leq \sigma - \frac{\lambda}{2}\|u_w^\varepsilon\|^2 + \int_\Omega F(x, u_w^\varepsilon(x))\, dx - \sum_{i=1}^{N} \int_\Omega a_i(x) u_w^\varepsilon(x) D_i w(x) \\
&\leq \sigma + \delta \int_\Omega |u_w^\varepsilon(x)|^2 dx + \eta(\delta) \int_\Omega |u_w^\varepsilon(x)|^{s+1}\, dx + \sqrt{N} R\, |\Omega|^{1/2} \|a\|_\infty \|u_w^\varepsilon\|_2 \\
&\leq \sigma + \frac{\delta}{\lambda_1}\|u_w^\varepsilon\|^2 + \eta(\delta) S_{s+1} \|u_w^\varepsilon\|^{s+1} + \frac{\sqrt{N} R\, |\Omega|^{1/2}}{\sqrt{\lambda_1}} \|a\|_\infty \|u_w^\varepsilon\| \leq K\,,
\end{aligned}$$

where $K := \sigma + \frac{\delta}{\lambda_1}\tilde{\kappa}^2 + \eta(\delta) S_{s+1} \tilde{\kappa}^{s+1} + \frac{\sqrt{N} R\, |\Omega|^{1/2}}{\sqrt{\lambda_1}} \|a\|_\infty \tilde{\kappa}$, thanks to Proposition 4.1 . Note that K does not depend on ε, since $\tilde{\kappa}$ does not. Thus

$$\|(u_w^\varepsilon - \psi)^+\|_2 \leq 2K\sqrt{\varepsilon}$$

for any $\varepsilon > 0$. As a consequence

$$(u_w^\varepsilon - \psi)^+ \to 0 \;\text{ in }\; L^2(\Omega)$$

as $\varepsilon \to 0$. By (1.43) we get $(u_w^\varepsilon - \psi)^+ \to (u_w - \psi)^+$ in $L^2(\Omega)$ as $\varepsilon \to 0$, so that we deduce that $(u_w - \psi)^+ = 0$ a.e. in Ω, that is

$$u_w \leq \psi \quad \text{a.e. in } \Omega\,. \tag{1.44}$$

Now we claim that u_w^ε strongly converges to u_w in $H_0^1(\Omega)$ as $\varepsilon \to 0$. First of all, note that by (1.43) we get

$$Du_w^\varepsilon \to Du_w \quad \text{weakly in} \quad (L^2(\Omega))^N \tag{1.45}$$

$$u_w^\varepsilon \to u_w \quad \text{in} \quad L^q(\Omega) \tag{1.46}$$

$$u_w^\varepsilon \to u_w \quad \text{a.e. in} \quad \Omega \tag{1.47}$$

as $\varepsilon \to 0$ and there exists $h \in L^q(\Omega)$ such that

$$|u_w^\varepsilon(x)| \le h(x) \quad \text{a.e. in} \quad \Omega \tag{1.48}$$

for any $q \in [1, 2^*)$. As a consequence of (1.6), (1.7), (1.47), (1.48) and the dominated convergence theorem we have that

$$\int_\Omega f(x, u_w^\varepsilon(x))u_w^\varepsilon(x)\,dx \to \int_\Omega f(x, u_w(x))u_w(x)\,dx \tag{1.49}$$

and

$$\int_\Omega f(x, u_w^\varepsilon(x))u_w(x)\,dx \to \int_\Omega f(x, u_w(x))u_w(x)\,dx \tag{1.50}$$

as $\varepsilon \to 0$. Taking u_w^ε as a test function in (1.30) and taking into account (1.29) we have

$$\begin{aligned}\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i u_w^\varepsilon(x)D_j u_w^\varepsilon(x)\,dx + \int_\Omega a_0(x)(u_w^\varepsilon)^2(x)\,dx + \frac{1}{\varepsilon}\int_\Omega (u_w^\varepsilon - \psi)^+(x)u_w^\varepsilon(x)\,dx \\ = \int_\Omega f(x, u_w^\varepsilon(x))u_w^\varepsilon(x)\,dx - \sum_{i=1}^N \int_\Omega a_i(x)u_w^\varepsilon(x)D_i w(x)\,dx,\end{aligned}$$

while, taking u_w as a test function in (1.30) we obtain

$$\begin{aligned}\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_i u_w^\varepsilon(x)D_j u_w(x)\,dx + \int_\Omega a_0(x)u_w^\varepsilon(x)u_w(x)\,dx \\ + \frac{1}{\varepsilon}\int_\Omega (u_w^\varepsilon - \psi)^+(x)u_w(x)\,dx \\ = \int_\Omega f(x, u_w^\varepsilon(x))u_w(x)\,dx - \sum_{i=1}^N \int_\Omega a_i(x)u_w(x)D_i w(x)\,dx\,.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ in the latter two relations and taking into account (1.45), (1.46), (1.49) and (1.50) we have

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w^\varepsilon(x) D_j u_w^\varepsilon(x)\,dx &= -\int_\Omega a_0(x) u_w^2(x)\,dx \\
&\quad - \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (x) u_w^\varepsilon(x)\,dx \\
&\quad + \int_\Omega f(x, u_w(x)) u_w(x)\,dx - \sum_{i=1}^{N} \int_\Omega a_i(x) u_w(x) D_i w(x)\,dx \\
&= \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\,dx + \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (x) u_w(x)\,dx \\
&\quad - \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (x) u_w^\varepsilon(x)\,dx \\
&= \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\,dx + \limsup_{\varepsilon\to 0} \Bigg(\frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (x) u_w(x)\,dx \\
&\quad - \frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (x) u_w^\varepsilon(x)\,dx \Bigg) \\
&= \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\,dx + \limsup_{\varepsilon\to 0} \Bigg(\frac{1}{\varepsilon} \int_\Omega (u_w^\varepsilon - \psi)^+ (u_w(x) - \psi(x))dx \\
&\quad - \frac{1}{\varepsilon} \int_\Omega \left[(u_w^\varepsilon - \psi)^+(x)\right]^2 dx \Bigg) \\
&\le \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\,dx \\
&\le \liminf_{\varepsilon\to 0} \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w^\varepsilon(x) D_j u_w^\varepsilon(x)\,dx,
\end{aligned}
$$

since $u_w \le \psi$ a.e. in Ω (see (1.44)) and thanks to the weak l.s.c. of the norm. As a consequence we get

$$
\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w^\varepsilon(x) D_j u_w^\varepsilon(x)\,dx \to \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_w(x) D_j u_w(x)\,dx\,,
$$

so

$$
\|u_w^\varepsilon\| \to \|u_w\|
$$

as $\varepsilon \to 0$. Here we have used the fact that, by (1.2) the function

$$
H_0^1(\Omega) \ni u \mapsto \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u(x) D_j u(x)\,dx
$$

defines a norm on $H_0^1(\Omega)$ equivalent to the usual one given in (1.17). Since $H_0^1(\Omega)$ is a separable Hilbert space, we get

$$
u_w^\varepsilon \to u_w \quad \text{in} \quad H_0^1(\Omega) \tag{1.51}
$$

and the claim is proved. Now we are ready to show that u_w is a solution of (1.28). Since u_w^ε is a solution of (1.30), taking $v - u_w^\varepsilon$, with $v \in H_0^1(\Omega)$ and $v \le \psi$ a.e. in Ω, as a test function in (1.30) we obtain that

$$\begin{aligned}\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}(x)D_iu_w^{\varepsilon}(x)D_j(v-u_w^{\varepsilon})(x)\,dx+\int_{\Omega}a_0(x)u_w^{\varepsilon}(x)(v(x)-u_w^{\varepsilon}(x))\,dx\\ +\frac{1}{\varepsilon}\int_{\Omega}(u_w^{\varepsilon}-\psi)^{+}(x)(v(x)-u_w^{\varepsilon}(x))\,dx\\ =\int_{\Omega}f(x,u_w^{\varepsilon}(x))(v(x)-u_w^{\varepsilon}(x))\,dx\\ -\sum_{i=1}^{N}\int_{\Omega}a_i(x)D_iw(x)(v(x)-u_w^{\varepsilon}(x))\,dx\,.\end{aligned}\tag{1.52}$$

The choice of v yields

$$\frac{1}{\varepsilon}\int_{\Omega}(u_w^{\varepsilon}-\psi)^{+}(x)(v(x)-u_w^{\varepsilon}(x))\,dx\le\frac{1}{\varepsilon}\int_{\Omega}(u_w^{\varepsilon}-\psi)^{+}(x)(\psi(x)-u_w^{\varepsilon}(x))\,dx\le 0\,.$$

By this fact, passing to the limit in (1.52) and taking into account (1.46)–(1.48), (1.51) and the dominated convergence theorem, we get

$$\begin{aligned}\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}(x)D_iu_w(x)D_j(v-u_w)(x)\,dx+\int_{\Omega}a_0(x)u_w(x)(v(x)-u_w(x))\,dx\\ \ge\int_{\Omega}f(x,u_w(x))(v(x)-u_w(x))\,dx-\sum_{i=1}^{N}\int_{\Omega}a_i(x)D_iw(x)(v(x)-u_w(x))\,dx,\end{aligned}$$

that is, by (1.29)

$$\begin{cases}\langle A_wu_w,v-u_w\rangle\ge\displaystyle\int_{\Omega}f(x,u_w(x))(v(x)-u_w(x))\,dx & \text{for any } v\in H_0^1(\Omega),\ v\le\psi\\ u_w\in H_0^1(\Omega),\ u_w\le\psi\,. & \end{cases}$$

Thus u_w is a solution of the variational inequality (1.28). Of course, u_w is non-negative, since u_w^{ε} does and (1.47) holds true. Finally, we have to show that (1.42) is valid. At this purpose, let us take $(u_w^{\varepsilon}-\psi)^{+}$ as a test function in (1.30), we have

$$\begin{aligned}\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}(x)D_iu_w^{\varepsilon}(x)D_j(u_w^{\varepsilon}-\psi)^{+}(x)\,dx+\int_{\Omega}a_0(x)u_w^{\varepsilon}(x)(u_w^{\varepsilon}-\psi)^{+}(x)\,dx\\ +\frac{1}{\varepsilon}\int_{\Omega}\left((u_w^{\varepsilon}-\psi)^{+}\right)^2(x)\,dx\\ =\int_{\Omega}f(x,u_w^{\varepsilon}(x))(u_w^{\varepsilon}-\psi)^{+}(x)\,dx-\sum_{i=1}^{N}\int_{\Omega}a_i(x)D_iw(x)(u_w^{\varepsilon}-\psi)^{+}(x)\,dx\,.\end{aligned}$$

As a consequence of this and of (1.32) we have

$$\begin{aligned}2I_w^{\varepsilon}(u_w^{\varepsilon})-\sum_{i,j=1}^{N}\int_{\Omega}a_{ij}(x)D_iu_w^{\varepsilon}(x)D_j(u_w^{\varepsilon}-\psi)^{+}(x)\,dx\\ -\int_{\Omega}a_0(x)u_w^{\varepsilon}(x)(u_w^{\varepsilon}-\psi)^{+}(x)\,dx-\frac{1}{\varepsilon}\int_{\Omega}\left((u_w^{\varepsilon}-\psi)^{+}\right)^2(x)\,dx\\ +\int_{\Omega}f(x,u_w^{\varepsilon}(x))(u_w^{\varepsilon}-\psi)^{+}(x)\,dx-\sum_{i=1}^{N}\int_{\Omega}a_i(x)D_iw(x)(u_w^{\varepsilon}-\psi)^{+}(x)\,dx\\ \ge 2\beta>0\,,\end{aligned}$$

that is, writing explicitly $I_w^\varepsilon(u_w^\varepsilon)$,

$$\begin{aligned}
&\sum_{i,j=1}^N \int_\Omega a_{ij}(x) D_i u_w^\varepsilon(x) D_j u_w^\varepsilon(x)\,dx + \int_\Omega a_0(x)(u_w^\varepsilon)^2(x)\,dx - 2\int_\Omega F(x,u_w^\varepsilon(x))\,dx \\
&\quad + 2\sum_{i=1}^N \int_\Omega a_i(x) D_i w(x) u_w^\varepsilon(x)\,dx - \sum_{i,j=1}^N \int_\Omega a_{ij}(x) D_i u_w^\varepsilon(x) D_j (u_w^\varepsilon - \psi)^+(x)\,dx \\
&\quad - \int_\Omega a_0(x) u_w^\varepsilon(x)(u_w^\varepsilon - \psi)^+(x)\,dx + \int_\Omega f(x,u_w^\varepsilon(x))(u_w^\varepsilon - \psi)^+(x)\,dx \\
&\quad - \sum_{i=1}^N \int_\Omega a_i(x) D_i w(x)(u_w^\varepsilon - \psi)^+(x)\,dx \\
&\geq 2\beta > 0\,,
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and taking into account (1.51) and the assumptions on f (in particular (1.6) and (1.7) in order to apply again the dominated convergence theorem) we get (1.42) with $\tilde\beta = 2\beta$. As a consequence of (1.42) we deduce that $u_w \not\equiv 0$ in Ω. Indeed, otherwise by (1.42) and using also the fact that $\psi \geq 0$ a.e. in Ω (see (1.4)), we would get a contradiction. This ends the proof of Proposition 5.1 .

5.2 Regularity of u_w

In this subsection we will discuss the regularity of the solution u_w found in Subsection 5.1. Precisely, we will show that for any $\alpha \in (0,1)$ the function $u_w \in C^{1,\alpha}(\overline{\Omega})$ and will also derive a suitable estimate on its $C^{1,\alpha}$–norm. This regularity result will be used in order to construct a non-trivial solution of the original problem (1.1) . In the sequel we will adapt to the variational inequality (1.1) the techniques used in [21, 26], where PDEs not in divergence form where considered. Here the main difficulty is related to the fact that we have to manage variational inequalities and not simply an equation. We will essentially argue as in [19], where we considered a variational inequality driven by an elliptic operator in divergence form and with a nonlinearity depending also on the gradient of the solution. The main tools for getting our regularity result are the Lewy-Stampacchia estimates for solutions of variational inequalities (see [13]) and the regularity theory for elliptic PDEs (see, for instance, [5, 8, 30]). As we said before, from now on it will be important the fact that $w \in C^{1,\bar\alpha}(\overline{\Omega})$. Also, note that, until now, we have used only the assumption (1.4) on the obstacle ψ. The regularity condition (1.5) will be used in this subsection in order to prove the following regularity result:

Proposition 5.2 *Assume the function ψ satisfies* (1.4) *and* (1.5) *and let u_w be the function given in Proposition 5.1 . Then $u_w \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. Moreover, the following relations hold true*

$$\textit{if}\ \ \|w\|_{1,\bar\alpha} \leq R\ \ \textit{then}\quad \|u_w\|_{1,\alpha} < R\,. \tag{1.53}$$

$$\|u_w\| \leq R|\Omega|^{1/2}\,. \tag{1.54}$$

Proof. In the following let $\bar A$ the operator defined in (1.22) and let us denote by $\bar f$ the nonlinear term given by

$$\bar f(x,u) := f(x,u) - \sum_{i=1}^N a_i(x) D_i w\,. \tag{1.55}$$

With this notation, since u_w is a solution of the variational inequality (1.28), it satisfies the following variational inequality

$$\begin{cases} \langle \bar{A}u_w, v - u_w\rangle + \displaystyle\int_\Omega a_0(x)u_w(x)(v(x) - u_w(x))\,dx \geq \int_\Omega \bar{f}(x, u_w(x))(v(x) - u_w(x))\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } v \in H_0^1(\Omega),\ v \leq \psi \\ u_w \in H_0^1(\Omega),\ u_w \leq \psi\,. \end{cases}$$

Hence, the Lewy-Stampacchia estimates for variational inequality yield

$$\inf\left\{\bar{f}(\cdot, u_w), \bar{A}\psi\right\} \leq \bar{A}u_w + a_0(\cdot)u_w \leq \bar{f}(\cdot, u_w).$$

Taking into account that Ω is bounded and $s > 1$, the Sobolev embedding theorem implies that $u_w \in L^{2^*}(\Omega) \cap L^{2^*/s}(\Omega)$. Moreover, by (1.7) and the fact that $w \in C_R^{1,\bar{\alpha}}(\overline{\Omega})$ it easily follows that $\bar{f}(\cdot, u_w) \in L^{2^*/s}(\Omega)$. Furthermore, by assumption (1.5) we get that the following inequalities

$$\inf\left\{\bar{f}(\cdot, u_w), \bar{A}\psi\right\} - a_0(\cdot)u_w \leq \bar{A}u_w \leq \bar{f}(\cdot, u_w) - a_0(\cdot)u_w$$

hold true in $L^{2^*/s}(\Omega)$. Then $u_w \in H_0^1(\Omega)$ is a weak solution of the problem

$$\begin{cases} \bar{A}u_w = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.56}$$

where $g = h - a_0(\cdot)u_w$ in Ω for some $h \in L^{2^*/s}(\Omega)$ such that

$$\inf\left\{\bar{f}(\cdot, u_w), \bar{A}\psi\right\} \leq h \leq \bar{f}(\cdot, u_w)\,.$$

It is easily seen that h satisfies the following inequality

$$|h(x)| \leq |\bar{f}(x, u_w(x))| + |\bar{A}\psi(x)| \quad \text{a.e. in} \quad \Omega\,,$$

so that g verifies a.e. $x \in \Omega$

$$\begin{aligned} |g(x)| &\leq |h(x)| + |a_0(x)u_w(x)| \\ &\leq |\bar{f}(x, u_w(x))| + |\bar{A}\psi(x)| + |a_0(x)u_w(x)| \\ &\leq c_1(1 + |u_w(x)|^s) + \|a\|_\infty R + \|\bar{A}\psi\|_\infty + \|a_0\|_\infty |u_w(x)|\,, \end{aligned} \tag{1.57}$$

thanks to assumptions (1.5), (1.7) and the regularity of a, a_0, w and ψ. Since $0 \leq u_w \leq \psi$ a.e. in Ω, by (1.5) we get that $u_w \in L^\infty(\Omega)$ (alternatively, by standard arguments, we can show that u_w is a classical solution of (1.56) (see, e.g., [1, 8])). Thus, by (1.57) the function $g \in L^\infty(\Omega)$ and so $g \in L^q(\Omega)$ for all $q \in [1, +\infty)$ and

$$\|g\|_q \leq c_1|\Omega|\left(1 + \|u_w\|_\infty^s\right) + |\Omega|\,\|a\|_\infty R + |\Omega|\,\|\bar{A}\psi\|_\infty + |\Omega|\,\|a_0\|_\infty \|u_w\|_\infty\,. \tag{1.58}$$

Now, let us estimate the L^∞–norm of u_w. First of all note that $|g(x)| = \tilde{g}(x)\,(1 + |u_w(x)|)$, where

$$\begin{aligned} \tilde{g}(x) &= \frac{|g(x)|}{1 + |u_w(x)|} \\ &\leq \frac{c_1(1 + |u_w(x)|^s) + \|a\|_\infty R + \|\bar{A}\psi\|_\infty\| + \|a_0\|_\infty |u_w(x)|}{1 + |u_w(x)|} \\ &\leq \left(c_1 + \|a\|_\infty R + \|\bar{A}\psi\|_\infty\| + \|a_0\|_\infty\right)\left(1 + |u_w(x)|^s\right). \end{aligned}$$

As a consequence of this and of the fact that Ω is bounded, $\tilde{g} \in L^{2^*/s}(\Omega)$ and

$$\|\tilde{g}\|_{2^*/s} \leq \Big(c_1 + \|a\|_\infty R + \|\bar{A}\psi\|_\infty\| + \|a_0\|_\infty\Big)\Big(|\Omega| + \|u_w\|_{2^*}^s\Big). \tag{1.59}$$

Then, using [24, Theorem 2.4] with $\varepsilon' = 1 - \frac{s(N-2)}{4} \in (0,1)$ (thanks to the choice of s), we get the following estimate

$$|u_w(x)| \leq \tilde{C}^{N/2\varepsilon'}\Big(\|u_w\|_2 + |\Omega|\,\|\tilde{g}\|_{2^*/s}\Big) \quad \text{in} \quad \Omega\,, \tag{1.60}$$

where $\tilde{C} = k^{(N-2)/N}\,[N/(N-2)]^{(N-2)/2}$ with k suitable positive constant depending on ε' (see [24, Theorem 2.4] for more details). Thus, by (1.59) and (1.60), the Sobolev embeddings theorems and the fact that $0 \leq u_w \leq \psi$, we have

$$\|u_w\|_\infty \leq K_1 + K_2\|a\|_\infty R\,, \tag{1.61}$$

where $K_1 = \tilde{C}^{N/2\varepsilon'}\left[(c_1 + \|\bar{A}\psi\|_\infty\| + \|a_0\|_\infty)(|\Omega| + S_{2^*}^{\,s}\|\psi\|^s)|\Omega| + \frac{\|\psi\|}{\sqrt{\lambda_1}}\right]$ and $K_2 = \tilde{C}^{N/2\varepsilon'}|\Omega|(|\Omega| + S_{2^*}^{\,s}\|\psi\|^s)$ depend only on a_{ij}, $i,j = 1,\ldots,N$, a_0, c_1, s, ψ, N and Ω. Then, by (1.15) and (1.27) we get

$$\|u_w\|_\infty \leq K_1 + 1 =: \hat{C}\,, \tag{1.62}$$

where $\hat{C}$ depends only on a_{ij}, $i,j = 1,\ldots,N$, a_0, c_1, s, ψ, N and Ω. Hence, by (1.58) and (1.62), we get

$$\begin{aligned}\|g\|_q &\leq c_1|\Omega|\Big(1 + \hat{C}^s\Big) + |\Omega|\,\|a\|_\infty R + |\Omega|\,\|\bar{A}\psi\|_\infty + |\Omega|\hat{C}\|a_0\|_\infty \\ &\leq \overline{C} + |\Omega|\,\|a\|_\infty R\,,\end{aligned} \tag{1.63}$$

where $\overline{C} = c_1|\Omega|\Big(1 + \hat{C}^s\Big) + |\Omega|\,\|\bar{A}\psi\|_\infty + |\Omega|\hat{C}\|a_0\|_\infty$ depends only on a_{ij}, $i,j = 1,\ldots,N$, a_0, c_1, s, ψ, N and Ω. Hence $\overline{C}$ is independent of w and R. Being $a_{ij} \in C(\overline{\Omega})$, $i,j = 1,\ldots,N$, by the Caldéron-Zygmund Theorem (see, for instance [8, Lemma 9.17]) we also have that $u_w \in H^{2,q}(\Omega)$ and

$$\|u_w\|_{2,q} \leq C_{\rm CZ}\|g\|_q\,, \tag{1.64}$$

where $C_{\rm CZ}$ is a positive constant depending only on Ω, q and the coefficients a_{ij}, $i,j = 1,\ldots,N$. Taking $q > N$, by Morrey Theorem (see, for instance [7, Section 5.6, Theorem 5]), we easily deduce that $u_w \in C^{1,\alpha}(\overline{\Omega})$, for any $\alpha \in (0,1)$, and

$$\|u_w\|_{1,\alpha} \leq C_{\rm Mor}\|u_w\|_{2,q}\,, \tag{1.65}$$

where $C_{\rm Mor}$ is a positive constant depending only on α and Ω. Now, in order to conclude the proof of Proposition 5.2 it remains to prove that (1.53) and (1.54) hold true. For this, by (1.63)–(1.65) we get

$$\|u_w\|_{1,\alpha} \leq C_{\rm Mor}\,C_{\rm CZ}\Big[\overline{C} + |\Omega|\,\|a\|_\infty R\Big]\,, \tag{1.66}$$

and so, by (1.15), (1.26) and (1.27) we have

$$\|u_w\|_{1,\alpha} \leq R/2 + C_{\rm Mor}\,C_{\rm CZ}|\Omega|\,\|a\|_\infty R < R/2 + R/2 = R\,.$$

In conclusion we have shown (1.53). Finally, note that, since $u_w \in C_R^{1,\alpha}(\overline{\Omega})$, then

$$|Du_w(x)| \leq R \quad \text{in} \quad \Omega\,.$$

Thus,

$$\|u_w\| = \left(\int_\Omega |Du_w(x)|^2\,dx\right)^{1/2} \leq R\,|\Omega|^{1/2}\,.$$

Hence, Proposition 5.2 is completely proved.

Note that both (1.53) and (1.54) say that the $C^{1,\alpha}$–norm and the $H_0^1(\Omega)$–norm of u_w can be controlled independently of w. In particular, on both these norms of u_w we have the same control as in the analogous ones of w. This will be crucial in the next subsection in order to perform the iterative technique and to get the existence of a non-trivial solution for the variational inequality (1.1).

5.3 End of the proof of Theorem 1.1

Now we are ready to prove Theorem 1.1 . For this we will use an iterative technique, introduced in [6] for studying elliptic PDEs with gradient dependent nonlinearities through variational methods. Let us fix $\bar{\alpha} \in (0,1)$ and $u_0 \in C^{1,\bar{\alpha}}_R(\overline{\Omega}) \cap H^1_0(\Omega)$ and consider the variational inequality (1.31) . By Propositions 5.1 and 5.2 every problem (1.31) admits a non-trivial non-negative solution $u_n \in H^1_0(\Omega)$ such that $\|u_n\| \le R\,|\Omega|^{1/2}$, $u_n \in C^{1,\alpha}_R(\overline{\Omega})$ for any $\alpha \in (0,1)$, that is $u_n \in C^{1,\alpha}(\overline{\Omega})$ and $\|u_n\|_{1,\alpha} \le R$ for any $n \in \mathbb{N}$. In particular we have that

$$|u_n(x)| \le R \quad \text{in } \Omega \quad \text{for any } n \in \mathbb{N}. \tag{1.67}$$

Moreover, for any $v \in H^1_0(\Omega)$ with $v \le \psi$ a.e. in Ω we have that

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_n(x) D_j(v(x) - u_n(x))\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega a_i(x) D_i u_{n-1}(x)(v(x) - u_n(x))\, dx \\
&\quad + \int_\Omega a_0(x) u_n(x)(v(x) - u_n(x))\, dx \ge \int_\Omega f(x, u_n(x))(v(x) - u_n(x)) dx
\end{aligned} \tag{1.68}$$

and

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u_{n+1}(x) D_j(v - u_{n+1})(x)\, dx \\
&\qquad + \sum_{i=1}^{N} \int_\Omega a_i(x) D_i u_n(x)(v(x) - u_{n+1}(x))\, dx \\
&\qquad + \int_\Omega a_0(x) u_{n+1}(x)(v(x) - u_{n+1}(x))\, dx \\
&\ge \int_\Omega f(x, u_{n+1}(x))(v(x) - u_{n+1}(x))\, dx\,.
\end{aligned} \tag{1.69}$$

Taking $v = u_{n+1}$ as a test function in (1.68) and $v = u_n$ in (1.69), adding (1.68) and (1.69) we get

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i(u_{n+1} - u_n)(x) D_j(u_{n+1} - u_n)(x)\, dx \\
&\qquad + \sum_{i=1}^{N} \int_\Omega a_i(x)(D_i u_n(x) - D_i u_{n-1}(x))(u_{n+1}(x) - u_n(x))\, dx \\
&\qquad + \int_\Omega a_0(x)(u_{n+1}(x) - u_n(x))^2\, dx \\
&\le \int_\Omega (f(x, u_{n+1}(x)) - f(x, u_n(x)))\,(u_{n+1}(x) - u_n(x)) dx\,,
\end{aligned}$$

so that, by (1.2) and the non-negativity of a_0 , yields

$$
\begin{aligned}
\lambda \|u_{n+1}-u_n\|^2 &\le \int_\Omega (f(x,u_{n+1}(x))-f(x,u_n(x)))\,(u_{n+1}(x)-u_n(x))dx \\
&\qquad -\sum_{i=1}^N \int_\Omega a_i(x)(D_iu_n(x)-D_iu_{n-1}(x))(u_{n+1}(x)-u_n(x))\,dx \\
&\le L_R \int_\Omega |u_{n+1}(x)-u_n(x)|^2 dx \\
&\qquad + \|a\|_\infty \int_\Omega |Du_n(x)-Du_{n-1}(x)|(u_{n+1}(x)-u_n(x))dx \\
&\le \frac{L_R}{\lambda_1}\|u_{n+1}-u_n\|^2 + \frac{\|a\|_\infty}{\sqrt{\lambda_1}}\|u_n-u_{n-1}\|\|u_{n+1}-u_n\|,
\end{aligned}
$$

for any $n \in \mathbb{N}$, being (1.6), (1.11) and (1.67) valid. As a consequence we get

$$
\lambda \|u_{n+1}-u_n\| \le \frac{L_R}{\lambda_1}\|u_{n+1}-u_n\| + \frac{\|a\|_\infty}{\sqrt{\lambda_1}}\|u_n-u_{n-1}\|\,,
$$

and so, by (1.15) and (1.27),

$$
\lambda\|u_{n+1}-u_n\| < \frac{L_R}{\lambda_1}\|u_{n+1}-u_n\| + \frac{\lambda}{2}\|u_n-u_{n-1}\|\,,
$$

that is

$$
\|u_{n+1}-u_n\| < \hat{\kappa}\|u_n-u_{n-1}\| \quad \forall\ n \in \mathbb{N},
$$

where $\hat{\kappa} = \frac{\lambda}{2}\cdot\left(\lambda - \frac{L_R}{\lambda_1}\right)^{-1} \in (0,1)$ thanks to assumption (1.16). Thus, u_n is a Cauchy sequence in $H_0^1(\Omega)$ and so

$$
u_n \to u \quad \text{in } H_0^1(\Omega) \text{ as } n \to \infty \tag{1.70}
$$

for some $u \in H_0^1(\Omega)$. Passing to the limit in (1.68) as $n \to \infty$ it is easy to see that

$$
\langle Au, v-u\rangle \ge \int_\Omega f(x,u(x))(v(x)-u(x))\,dx \qquad \text{for any } v \in H_0^1(\Omega),\ v \le \psi\,.
$$

Moreover, since $0 \le u_n \le \psi$ a.e. in Ω for any $n \in \mathbb{N}$, we get that $0 \le u \le \psi$ a.e. in Ω. Ultimately, u is a solution of (1.1). Now, it remains to show that $u \not\equiv 0$ in Ω. For this, note that by Proposition 5.1 the function u_n satisfies (1.42) with $w = u_{n-1}$, that is

$$
\begin{aligned}
&\sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_iu_n(x)D_ju_n(x)\,dx + \int_\Omega a_0(x)u_n^2(x)\,dx \\
&\quad - 2\int_\Omega F(x,u_n(x))\,dx + 2\sum_{i=1}^N \int_\Omega a_i(x)D_iu_{n-1}(x)u_n(x)\,dx \\
&\quad - \sum_{i,j=1}^N \int_\Omega a_{ij}(x)D_iu_n(x)D_j(u_n-\psi)^+(x)\,dx - \int_\Omega a_0(x)u_n(x)(u_n-\psi)^+(x)\,dx \\
&\quad + \int_\Omega f(x,u_n(x))(u_n-\psi)^+(x)\,dx - \sum_{i=1}^N \int_\Omega a_i(x)D_iu_{n-1}(x)(u_n-\psi)^+(x)\,dx \\
&\quad \ge \tilde{\beta} > 0\,.
\end{aligned}
$$

Thus, passing to the limit as $n \to \infty$ and taking into account (1.70) we get

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u(x) D_j u(x)\, dx + \int_\Omega a_0(x) u^2(x)\, dx \\
&\quad - 2 \int_\Omega F(x, u(x))\, dx + 2 \sum_{i=1}^{N} \int_\Omega a_i(x) D_i u(x) u(x)\, dx \\
&\quad - \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) D_i u(x) D_j (u-\psi)^+(x)\, dx - \int_\Omega a_0(x) u(x) (u-\psi)^+(x)\, dx \\
&\quad + \int_\Omega f(x, u(x)) (u-\psi)^+(x)\, dx - \sum_{i=1}^{N} \int_\Omega a_i(x) D_i u(x) (u-\psi)^+(x)\, dx \\
&\quad \geq \tilde{\beta} > 0\,,
\end{aligned}$$

which yields that $u \not\equiv 0$ in Ω. Finally, since $u_n \in C_R^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$ and any $n \in \mathbb{N}$, the sequences $(u_n)_n$ and $(Du_n)_n$ are equicontinuous and equibounded in $\overline{\Omega}$. The Ascoli-Arzelà theorem implies that $u_n \to u$ and $Du_n \to Du$ uniformly in $\overline{\Omega}$ as $n \to \infty$, so that $u \in C^1(\overline{\Omega})$. By the same arguments we also deduce that $u \in C_R^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$.

Note that, for the arguments we used here to prove Theorem 1.1, it is crucial to show that the sequence u_n is a Cauchy sequence in $H_0^1(\Omega)$. Indeed, this implies that the whole sequence u_n (and not only a subsequence) strongly converges in $H_0^1(\Omega)$.

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