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On the Existence and Breaking Symmetry of the Ground State Solution of Hardy Sobolev Type Equations with Weighted p-LAPLACIAN

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Abstract

We study the existence of ground state solution, cylindrically symmetric solution, breaking cylindrical symmetry of ground state solution of the following Hardy-Sobolev type equation with weighted p-Laplacian: $-div(|y|^a|\nabla u|^{p-2}\nabla u) - \lambda|y|^{a-p}|u|^{p-2}u = |y|^{-b_q}|u|^{q-2}u \ in \ \mathbb{R}^N$ where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $b_q = N - \frac{q}{p}(N - p + a)$, $a > p - k$ and $\lambda < (\frac{k-p+a}{p})^p$.

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1 Introduction

In this article we study the quasilinear singular elliptic equation involving partial weight, more precisely let us consider the following problem

$$\begin{aligned} -div(|y|^a|\nabla u|^{p-2}\nabla u) - \lambda|y|^{a-p}|u|^{p-2}u = |y|^{-b_q}|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N \\ \int_{\mathbb{R}^N} |y|^a|\nabla u|^p < \infty \end{aligned} \tag{1.1}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $1 \le k < N$, $p < q \le p^* = \frac{Np}{N-p}$, $1 < p < N$, $a > p - k$, $b_q = N - \frac{q}{p}(N + a - p)$, $\lambda < \left(\frac{k+a-p}{p}\right)^p$.

A large number of papers deal with (1.1) and with similar variational problems. We quote for example [2], [4], [5], [6], [8] - [17], [19] - [21], [23] - [25] and the references therein. In particular in [10], [13], [16], [19] analysis of breaking symmetry of ground state solutions were studied.

When $a = 0$ then $\operatorname{div}(|y|^a|\nabla u|^{p-2}\nabla u) = \Delta_p u$ is the standard p-Laplacian operator. If $p < N, q = p^*, \lambda = 0$ then $b_{p^*} = 0$ and the equation (1.1) reduces to

$$-\Delta_p u = u^{p^*-1} \text{ in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} |\nabla u|^p dx < \infty \tag{1.2}$$

which is well studied in the famous paper by Aubin [1] and Talenti [23]. It was shown that the solution of (1.2) is achieved in $D^{1,p}(\mathbb{R}^N)$ by a radially symmetric function [see [23]]. Up to our knowledge all the available results in the literature are done in the case $p = 2$ or $k = N$ or $\lambda = 0$ or $a = 0$. In the case $k = N$, Catrina and Wang have proved existence for $p = 2, q \le 2^*$. Also in the case $k = N$ Musina has proved existence, break of symmetry of ground state solution in [19].

There exists a constant $\lambda_{p,a} > 0$ (see Lemma 2.1) such that

$$\lambda_{p,a} \int_{\mathbb{R}^N} |y|^{a-p}|u|^p dx < \int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx \tag{1.3}$$

for all $u \in C_0^\infty\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right)$. Inequality (1.3) was proved by Caffarelli, Kohn, Nirenberg in [7] for spherical weights. General proof for the cylindrical case follows from [3] with suitable modification of the arguments. We have sketched the proof in Lemma 2.1. Using (1.3) we can define the reflexive Banach space $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ as the completion of $C_0^\infty\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right)$ with respect to the norm $\left(\int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx\right)^{\frac{1}{p}}$ and we also obtain $\left(\int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p) dx\right)^{\frac{1}{p}}$ is an equivalent norm to $\left(\int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx\right)^{\frac{1}{p}}$ when $\lambda < \lambda_{p,a}$. Now let us consider the following Hardy-Sobolev type inequality:

$$C\left(\int_{\mathbb{R}^N} |y|^{-b_q}|u|^q\right)^{\frac{p}{q}} \le \int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx \tag{1.4}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. This inequality was proved by Maz'ya in Section 2.1.6 in [18] for $k < N$ and by Caffarelli, Kohn, Nirenberg in [7] for spherical weight.

Now using (1.3) and (1.4) for $\lambda < \left(\frac{k+a-p}{p}\right)^p$,

$$S_p(a, \lambda, q) = \inf_{u \in D^{1,p}(\mathbb{R}^N, |y|^a dx), u \not\equiv 0} \frac{\int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} |y|^{a-p}|u|^p dx}{\left(\int_{\mathbb{R}^N} |y|^{-b_q}|u|^q dx\right)^{\frac{p}{q}}} \tag{1.5}$$

is positive. Define

$$A(u) = \int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)\, dx, \quad B(u) = \left(\int_{\mathbb{R}^N} |y|^{-b_q}|u|^q dx\right)^{\frac{p}{q}}.$$

Thus

$$S_p(a, \lambda, q) = \inf_{u \in D^{1,p}(\mathbb{R}^N, |y|^a dx), u \not\equiv 0} \frac{A(u)}{B(u)}. \tag{1.6}$$

It is easy to see that

$$\begin{aligned} S_p(a, 0, p) &= \lambda_{p,a} \\ S_p(0, 0, p^*) &= S_p \end{aligned}$$

where

$$S_p = \inf_{u \in D^{1,p}(\mathbb{R}^N,),u \not\equiv 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*}\right)^{\frac{p}{p^*}}}$$

the best Sobolev constant which is achieved by $u(x) = \left(\frac{1}{1+|x|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}$, see [1] for details.

In this article we focus on existence of ground state solutions, existence of symmetric solutions and break of symmetry of ground state solutions. In particular, for existence of ground state solutions we prove

Theorem 1.1 *Let p, q, a, b_q satisfy the same condition as in* (1.1). *Then if*

1. $q < p^*$ *or*
2. $q = p^*$ *and* $S_p(a, \lambda, p^*) < S_p$

$S_p(a, \lambda, q)$ *is achieved.*

The proof of this theorem closely follows the proof of Theorem 0.1 in [15] which deals with the case $\lambda = 0$. It will be shown that the existence of ground state solutions assures the non uniqueness of solution to the problem (1.1) for $\lambda << 0$. The next a natural question: is there any sufficient condition that implies $S_p(a, \lambda, p^*) < S_p$? It is shown in Section 3 that $S(a, \lambda, p^*) \leq S_p$. For the strict inequality we have proved the following partial result

Theorem 1.2 *(i) If $a = 0$ then $S_p(0, \lambda, p^*) < S_p$ if and only if $0 < \lambda < \lambda_{p,0}$.*

(ii) If $p - k < a < 0$ then there exists $\lambda^* < \lambda_{p,a} \frac{N-1}{N-p} \frac{pa}{N+a}$ such that whenever, $\lambda^* < \lambda < \lambda_{p,a}$ then $S_p(a, \lambda, p^*) < S_p$.

This result is in the spirit of a result in [19]. Therefore when a is negative we have obtained a range of λ for which $S(a, \lambda, p^*)$ is achieved. But this result does not guarantee the existence of solution for the case $\lambda = 0$. However from the Theorem 0.2 in [15] we obtain $S(a, 0, p^*)$ is achieved when $p-k < a < 0$. The main tools which are used to prove Theorem 1.1 are Ekland's variational principle, Rellich's compactness theorem and suitable rescaling arguments. The crucial step is Proposition 3.1. Thanks to Proposition 3.1 we can find a weakly convergent minimizing sequence $\{u_n\}$ whose L^q norms are bounded away from 0 on a compact subset of $\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right)$. In this way we exclude concentration at 0 and vanishing (see Section 3 for definition) and thus we overcome the lack of compactness produced by dilation in $\mathbb{R}^N$ and translation in $\mathbb{R}^{N-k}$.

In Section 5 we wonder whether the problem (1.1) has a cylindrically symmetric solution $u(x) = u(|y|, z)$ or does break of symmetry occur for certain values of λ. We prove

Theorem 1.3 *Let $k \geq 2,\ 1 < p < q$ and $a > p - k$. Then equation* (1.1) *has a nonnegative cylindrically symmetric solution u i.e $u(x) = u(|y|, z)$ for any $\lambda < \lambda_{p,a}$.*

We also prove that when $\lambda << 0$ then the ground state solution can not be cylindrically symmetric more precisely

Theorem 1.4 *Let $k \geq 2,\ 2 \leq p < q,\ a > p - k,\ q \leq p^*$. Then there exists $\lambda_* = \lambda(a, p, q, k, n)$ such that for any $\lambda < \lambda_*$ no minimizer of $S_p(a, \lambda, q)$ is cylindrically symmetric.*

Therefore when $2 \le p < q < p^*$, $\lambda << 0$ the problem (1.1) has at least two distinct solutions.This paper is organized in 5 sections. Section 2 is devoted to preliminaries. In Section 3 we have pointed out the non compactness of the minimization problem [see Section 3 for details] and proved a Proposition regarding that. In Section 4 we have proved Theorem 1.1 and Theorem 1.2. In Section 5 we have proved Theorem 1.3 and Theorem 1.4.

2 Preliminaries

In this section we recall some well known inequalities. We start with the following version of Hardy's inequality

Lemma 2.1 *For $1 < p < N$ and $a > p - k$ we have*

$$\lambda_{p,a} \int_{\mathbb{R}^N} |y|^{a-p} |u|^p dx < \int_{\mathbb{R}^N} |y|^a |\nabla u|^p dx \tag{2.7}$$

for all $u \in C_0^\infty(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ with $\lambda_{p,a} = \left(\frac{k+a-p}{p}\right)^p$.

The idea of the proof follows from [3] with suitable modification. For the convenience of the reader we will sketch the proof below. To prove this lemma we need the following lemma.

Lemma 2.2

- *Let $p \ge 2$. Then there exists a constant $C(p) > 0$ s.t*

$$|f - g|^p_{L^p(\mathbb{R}^N)} - |f|^p_{L^p(\mathbb{R}^N)} \ge C|g|^p_{L^p(\mathbb{R}^N)} - p \int_{\mathbb{R}^N} |f|^{p-2} f g \tag{2.8}$$

  for all $f, g \in L^p(\mathbb{R}^N)$.

- *For $1 < p < 2$, there exists a constant $C(p) > 0$ s.t*

$$|f - g|^p_{L^p(\mathbb{R}^N)} - |f|^p_{L^p(\mathbb{R}^N)} \ge C \int_{\mathbb{R}^N} \frac{|g|^2}{(|f| + |g|)^{2-p}} - p \int_{\mathbb{R}^N} |f|^{p-2} f g \tag{2.9}$$

  for all $f, g \in L^p(\mathbb{R}^N)$.

For a proof see Lemma 3.1 in [3]

Proof of Lemma 2.1. for $u \in C_0^\infty(\mathbb{R}^N \setminus \{y = 0\})$ define

$$I(u) = \int |y|^a |\nabla u|^p - \left(\frac{k - p + a}{p}\right)^p \int |y|^{a-p} |u|^p.$$

Now set $v(x) = |y|^H u(x)$ where $H = \frac{k-p+a}{p} > 0$. Then by direct calculation we obtain

$$|y|^a \left(|\nabla u|^p - H^p \frac{|u|^p}{|y|^p}\right) = |y|^{-k} (|Hv\nabla(|y|) - |y|\nabla v|^p - |Hv|^p)$$

and therefore

$$I[u] = \int_{\mathbb{R}^N} |y|^{-k} \left(|Hv\nabla(|y|) - |y|\nabla v\right|^p - |Hv|^p\right) dx.$$

Now replacing f and g by $|y|^{\frac{-k}{p}} Hv\nabla(|y|)$ and $|y|^{\frac{-k}{p}}|y|\nabla v$ respectively in Lemma 2.2 we obtain for $p \geq 2$,

$$I[u] \geq c\int_{\mathbb{R}^N} |y|^{-k}||y|\nabla v|^p dx - pH^{p-1}\int_{\mathbb{R}^N} |y|^{1-k}(|v|^{p-2}v\nabla(|y|))\cdot\nabla v dx.$$

Now applying integration by parts to $\int_{\mathbb{R}^N} |y|^{1-k}\left(|v|^{p-2}v\nabla(|y|)\right)\cdot\nabla v$ we obtain

$$-p\int_{\mathbb{R}^N} |y|^{1-k}\left(|v|^{p-2}v\nabla(|y|)\right)\cdot\nabla v dx = \int_{\mathbb{R}^N} |y|^{-k}|v|^p\left(|y|\Delta(|y|) + 1 - k\right) dx = 0.$$

Therefore we get

$$I[u] \geq c\int_{\mathbb{R}^N} |y|^{p-k}|\nabla v|^p dx > 0$$

as $\int_{\mathbb{R}^N} |y|^{p-k}|\nabla v|^p dx = 0$ implies v =constant which means $u = 0$.

Similarly for $1 < p < 2$ from Lemma 2.2 we obtain

$$I[u] \geq c\int_{\mathbb{R}^N} \frac{|y|^{2-k}|\nabla v|^2}{(|Hv| + |y||\nabla v|)^{2-p}} dx > 0.$$

Hence for $1 < p < N$ we obtain

$$\lambda_{p,a}\int_{\mathbb{R}^N} |y|^{a-p}|u|^p dx < \int_{\mathbb{R}^N} |y|^a|\nabla u|^p dx.$$

□

3 Non-compactness

In this section we will observe the lack of compactness phenomenon by studying the minimization problem (1.5).

If $S_p(a,\lambda,q)$ is attained by $u \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ then for $R_n > 0$ and $z_n \in \mathbb{R}^{N-k}$ define $u_n(x) = R_n^{\frac{N+a-p}{p}} u(R_n y, R_n z + z_n)$. Now by simple computation it follows that $A(u_n) = A(u)$ and $B(u_n) = B(u)$ implies $S_p(a,\lambda,q)$ is also achieved by $u_n \forall n \geq 1$. When $z_n = 0$, it is easy to check that $|y|^a|\nabla u_n|^p \to 0$ in $L^1\{|x| > R\}$ if $R_n \to \infty$ and which immediately implies u_n "concentrates at 0". Similarly we can exhibit "vanishing" if we allow $R_n \to 0$, i.e. $|y|^a|\nabla u_n|^p \to 0$ in $L^1_{loc}(\mathbb{R}^N)$. In case $q = p^*$, we have additional type of lack of compactness by the group of translation in the z variable as it is already pointed out in the introduction of [15] as a consequence of Proposition 1.3 in [15] .

For $\varepsilon > 0$, choose $u_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that

$$S_p \leq \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx}{\left(\int_{\mathbb{R}^N} |u_\varepsilon|^{p^*}\right)^{\frac{p}{p^*}}} < S_p + \varepsilon.$$

Now fix $y_0 \neq 0$ and define $u_n^\varepsilon(x) = R_n^{\frac{N-p}{p}} u_\varepsilon(R_n(x - x_0))$, where $x_0 = (y_0, z_0)$. Thus

$$\begin{aligned}
\lim_{R_n\to\infty} & \frac{A(u_n^\varepsilon)}{\left(\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}} |u_n^\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}} \\
= & \lim_{R_n\to\infty} \frac{\int_{\mathbb{R}^N} \left(|\frac{y}{R_n} + y_0|^a |\nabla u_\varepsilon|^p - \lambda |\frac{y}{R_n} + y_0|^{a-p} R_n^{-p} |u_\varepsilon|^p\right) dx}{\left(\int_{\mathbb{R}^N} |\frac{y}{R_n} + y_0|^{\frac{Na}{N-p}} |u_\varepsilon|^{p^*}\right)^{\frac{p}{p^*}}} \\
= & \lim_{R_n\to\infty} \frac{\int_{\mathbb{R}^N} \left(|\nabla u_\varepsilon|^p - \frac{\lambda}{|R_n y_0|^p} |u_\varepsilon|^p\right) dx}{\left(\int_{\mathbb{R}^N} |u_\varepsilon|^{p^*}\right)^{\frac{p}{p^*}}} \\
= & \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*}\right)^{\frac{p}{p^*}}}
\end{aligned}$$

Hence $S_p(a, \lambda, p^*) \leq \lim_{\varepsilon\to 0} \lim_{R_n\to\infty} \frac{A(u_n^\varepsilon)}{\left(\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}} |u_n^\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}} = S_p$. i.e. $S_p(a, \lambda, p^*) \leq S_p$.

The proof of the following Proposition has been adopted from [15] (Proposition 1.3) which deals with $\lambda = 0$.

Proposition 3.1 *Let u_n be a bounded sequence in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ and satisfy*

$$-div(|y|^a |\nabla u_n|^{p-2} \nabla u_n) - \lambda |y|^{a-p} |u_n|^{p-2} u_n = |y|^{-b_q} |u_n|^{q-2} u_n + f_n \tag{3.10}$$

where $f_n \to 0$ in $(D^{1,p}(\mathbb{R}^N, |y|^a dx))'$ which is a dual space of $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ and a, p, b_q, q satisfy the assumption in (1.1). *Then upto a subsequence either $u_n \to 0$ strongly in $L^q(\mathbb{R}^N, |y|^{-b_q} dx)$ or there exists $R_n \in (0, \infty)$, $z_n \in \mathbb{R}^{N-k}$ such that* $\lim_{n\to\infty} \int_K |y|^{-b_q} |\tilde{u_n}|^q dx > 0$ *where $K = \{(y, z) \in \mathbb{R}^N : \frac{1}{2} < |y| < 1, |z| < 1\}$ and $\tilde{u_n}(y, z) = R_n^{\frac{N-p+a}{p}} u_n(R_n y, R_n z + z_n)$.*

Before starting the proof let us recall a lemma from [15]:

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ is a bounded domain. Then $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ is compactly embedded in $L^p(\Omega, |y|^a dx)$.*

For the proof of this lemma see Lemma 1.1 in [15].

Proof of Proposition 3.1. Since u_n is a bounded sequence in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$, there exists $u \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ such that $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ and $L^q(\mathbb{R}^N, |y|^{-b_q} dx)$. If $u \neq 0$, then we have $\int_{\mathbb{R}^N} |y|^{-b_q} |u|^q dx > 0$. Therefore upto rescaling we have $\int_K |y|^{-b_q} |u|^q dx > 0$. Now using lower semi-continuity of the norm in L^q we have $\int_K |y|^{-b_q} |u_n|^q dx > 0$. Therefore assume $u = 0$. We can also assume that $\int_{\mathbb{R}^N} |y|^{-b_q} |u|^q dx > 0$, otherwise by the equivalence of norm $A(u_n)^{\frac{1}{p}}$ with $(\int_{\mathbb{R}^N} |y|^a |u_n|^p dx)^{\frac{1}{p}}$ implies $u_n \to 0$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. Therefore we choose $\delta > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |y|^{-b_q} |u|^q dx > \delta^{\frac{q}{q-p}}$$

and $\delta < S_p(a, \lambda, q)$. Let $Q_n(r)$ denotes the concentration function

$$Q_n(r) = \sup_{x=(0,z)\in\mathbb{R}^N} \int_{B(x,r)} |y|^{-b_q}|u_n|^q dx$$

which is a continuous function. Thus we can choose $R_n > 0$ and $x_n = (y_n, z_n) \in \mathbb{R}^N$ such that

$$Q_n(R_n) = \delta^{\frac{q}{q-p}} = \int_{B(x_n,R_n)} |y|^{-b_q}|u_n|^q dx.$$

Now define $\tilde{u_n}(x) = R_n^{\frac{N-p+a}{p}} u_n(R_n y, R_n z + z_n)$. It is easy that $\|u_n\|_{D^{1,p}(\mathbb{R}^N,|y|^a dx)} = \|\tilde{u_n}\|_{D^{1,p}(\mathbb{R}^N,|y|^a dx)}$. So there exists $\tilde{u} \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ such that $\tilde{u_n} \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. By the change of variable we also have

$$\sup_{x=(0,z)} \int_{B(x,1)} |y|^{-b_q}|\tilde{u_n}|^q dx = \int_{B(0,1)} |y|^{-b_q}|\tilde{u_n}|^q dx = \delta^{\frac{q}{q-p}}$$

and

$$-div(|y|^a|\nabla\tilde{u_n}|^{p-2}\nabla\tilde{u_n}) - \lambda|y|^{a-p}|\tilde{u_n}|^{p-2}\tilde{u_n} = |y|^{-b_q}|\tilde{u_n}|^{q-2}\tilde{u_n} + \tilde{f_n}$$

where $\tilde{f_n} \to 0$ in $(D^{1,p}(\mathbb{R}^N, |y|^a dx))'$. As before if $\tilde{u} \neq 0$, we are done. Therefore assume $\tilde{u} = 0$. Now let us choose a finite number of points $\eta_1, \ldots, \eta_t \in \mathbb{R}^{N-k}$ such that $\bar{B}_1^{N-k}(0) \subset \cup_{i=1}^t B_{\frac{1}{2}}^{N-k}(\eta_i)$. Now choose $\{\psi\}_{i=1}^t \in C_0^\infty(B_1^{N-k}(\eta_i))$ such that $\psi \equiv 1$ in $B_{\frac{1}{2}}^{N-k}(\eta_i)$ and $\phi \in C_0^\infty(B_1^k(0))$ such that $\phi \equiv 1$ in $B_{\frac{1}{2}}^k(0)$, $0 \le \psi_i, \phi \le 1$. Take $\tilde{u_n}(\psi_i\phi)^p \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ as a test function and thus we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |y|^a|\nabla\tilde{u_n}|^{p-2}\nabla\tilde{u_n}\nabla\left(\tilde{u_n}(\phi\psi_i)^p\right) dx \quad &- \quad \lambda \int_{\mathbb{R}^N} |y|^{a-p}|\tilde{u_n}\phi\psi|^p dx \\
&= \quad \int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}|^{q-p}|\tilde{u_n}\phi\psi|^p dx \\
&+ \quad o(1).
\end{aligned}$$

Now doing the direct calculation and using Lemma 3.1 we can simplify the left hand side(LHS) as

$$\begin{aligned}
\text{LHS} &= \int_{\mathbb{R}^N} |y|^a|\nabla(\tilde{u_n}\phi\psi)|^p dx - \lambda \int_{\mathbb{R}^N} |y|^{a-p}|\tilde{\phi}\psi|^p dx + o(1) \\
&\ge S_p(a, \lambda, q)\left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx\right)^{\frac{p}{q}} + o(1)
\end{aligned}$$

And using Hólder inequality right hand side (RHS) can be estimated as

$$\begin{aligned}
\text{RHS} &\le \left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx\right)^{\frac{p}{q}} \left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}|^q dx\right)^{\frac{q-p}{q}} \\
&\le \delta\left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx\right)^{\frac{p}{q}}
\end{aligned}$$

Hence we have

$$S_p(a, \lambda, q)\left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx\right)^{\frac{p}{q}} \le \delta\left(\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx\right)^{\frac{p}{q}} + o(1).$$

Now since $\delta < S_p(a, \lambda, q)$, we have $\int_{\mathbb{R}^N} |y|^{-b_q}|\tilde{u_n}\phi\psi|^q dx = o(1)$. Therefore

$$\int_{B^k_{\frac{1}{2}}(0)\times B^{N-k}_1(0)} |y|^{-b_q}|\tilde{u_n}|^q dx \le \sum_{i=1}^{t} \int_{B^k_{\frac{1}{2}}(0)\times B^{N-k}_{\frac{1}{2}}(\eta_i)} |y|^{-b_q}|\tilde{u_n}|^q dx = o(1).$$

Comparing the above estimate with the fact that

$$\int_{B(0,1)} |y|^{-b_q}|\tilde{u_n}|^q dx = \delta^{\frac{q}{q-p}} > 0,$$

we obtain

$$\int_{\{\frac{1}{2}<|y|<1\}\times B^{N-k}_1(0)} |y|^{-b_q}|\tilde{u_n}|^q dx > 0. \qquad \square$$

4 Existence of ground state solution

If $S_p(a, \lambda, q)$ is achieved by a function $u \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ then by using the standard arguments [22], we obtain u is a nonnegative weak solution of (1.1) upto a multiplicative constant and this solution is called ground state solution. Theorem 1.1 deals with the condition which assures $S_p(a, \lambda, q)$ to be achieved.

Proof of Theorem 1.1. Let $\{u_n\}$ be a minimizing sequence of $S_p(a, \lambda, q)$. Then applying Ekland's variational principle we can assume

$$\begin{aligned}\int_{\mathbb{R}^N} (|y|^a|\nabla u_n|^p - \lambda|y|^{a-p}|u_n|^p)\, dx &= S_p(a, \lambda, q)^{\frac{q}{q-p}} + o(1)\\ &= \int_{\mathbb{R}^N} |y|^{-b_q}|u_n|^q dx + o(1), \end{aligned}\tag{4.11}$$

$$-div(|y|^a|\nabla u_n|^{p-2}\nabla u_n) - \lambda|y|^{a-p}|u_n|^{p-2}u_n = |y|^{-b_q}|u_n|^{q-2}u_n + f_n \tag{4.12}$$

where $f_n \to 0$ in $(D^{1,p}(\mathbb{R}^N, |y|^a dx))'$. Now applying Hardy's inequality (2.7) we obtain

$$\int_{\mathbb{R}^N} (|y|^a|\nabla u_n|^p - \lambda|y|^{a-p}|u_n|^p)dx > \left(1 - \frac{\lambda}{\lambda_{p,a}}\right)\int_{\mathbb{R}^N} |y|^a|\nabla u_n|^p dx$$

which immediately implies $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ by (4.11), since $\lambda < \lambda_{p,a}$. Therefore there exists a $u \in D^{1,p}(\mathbb{R}^N, |y|^a dx)$ such that $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. We already know from (4.11) that $u_n \not\to 0$ in $L^q(\mathbb{R}^N, |y|^{-b_q} dx)$. Therefore upto rescaling from Proposition (3.1) we obtain

$$\int_K |y|^{-b_q}|u_n|^q dx > 0 \tag{4.13}$$

where $K = \{(y, z) \in \mathbb{R}^N : \frac{1}{2} < |y| < 1, |z| < 1\}$.

Now we claim that $u \neq 0$. When $q < p^*$, then by Rellich's theorem we obtain

$$0 < \lim_{n\to\infty} \int_K |y|^{-b_q}|u_n|^q dx = \int_K |y|^{-b_q}|u|^q dx$$

which immediately implies $u \neq 0$ for $q < p^*$. When $q = p^*$ we will prove this fact by contradiction. So assume $u = 0$. Now let us choose $\phi \in C_0^\infty(\mathbb{R}^k)$ such that $\phi \equiv 0$ in $|y| < \frac{1}{4}$, $\phi \equiv 1$ in $\frac{1}{2} < |y| < 1$,

$0 \le \phi \le 1$ and $\psi \in C_0^\infty(\mathbb{R}^{N-k})$ such that $\psi \equiv 1$ in $|z| \le 1$, $0 \le \psi \le 1$. Thus $\phi\psi = 1$ in K. Now let us consider $u_n(\phi\psi)^p$ as a test function. Therefore a computation as in Proposition (3.1) leads to

$$\begin{aligned}\int_{\mathbb{R}^N} (|y|^a|\nabla(u_n\phi\psi)|^p - \lambda|y|^{a-p}|u_n\phi\psi|^p)\,dx \quad &\le \quad \int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}}|u_n|^{p^*-p}|u_n\phi\psi|^p dx \\ &+ \quad o(1).\end{aligned}$$

Applying H'older inequality in RHS as we have done in Proposition (3.1) we obtain

$$\text{RHS} \le S_p(a,\lambda,p^*)\left(\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}}|u_n\phi\psi|^q dx\right)^{\frac{p}{p^*}} + o(1).$$

And we have

$$\nabla(|y|^{\frac{a}{p}}u_n\phi\psi) = |y|^{\frac{a}{p}}\nabla(u_n\phi\psi) + \nabla(|y|^{\frac{a}{p}})u_n\phi\psi.$$

Since $\phi\psi$ has compact support in $\mathbb{R}^N$ and vanishes near $\{y = 0\}$, applying Rellich's theorem we obtain $\nabla(|y|^{\frac{a}{p}})u_n\phi\psi \to 0$ in $L^p(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |y|^{a-p}|u_n\phi\psi|^p dx \to 0.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla(|y|^{\frac{a}{p}}u_n\phi\psi)|^p dx = \int_{\mathbb{R}^N} ||y|^{\frac{a}{p}}\nabla(u_n\phi\psi)|^p dx + o(1).$$

Thus we have

$$\int_{\mathbb{R}^N} |\nabla(|y|^{\frac{a}{p}}u_n\phi\psi)|^p dx \le S_p(a,\lambda,p^*)\left(\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}}|u_n\phi\psi|^{p^*} dx\right)^{\frac{p}{p^*}} + o(1).$$

Hence applying Sobolev inequality we obtain

$$S_p\left(\int_{\mathbb{R}^N} |(|y|^{\frac{a}{p}}u_n\phi\psi)|^{p^*} dx\right)^{\frac{p}{p^*}} \le S_p(a,\lambda,p^*)\left(\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}}|u_n\phi\psi|^{p^*} dx\right)^{\frac{p}{p^*}} + o(1).$$

Now since $S_p(a,\lambda,p^*) < S_p$, we obtain $\int_{\mathbb{R}^N} |y|^{\frac{Na}{N-p}}|u_n\phi\psi|^{p^*} dx = o(1)$.
i.e. $\int_K |y|^{\frac{Na}{N-p}}|u_n|^{p^*} dx = o(1)$ which is a contraction to (4.13). Hence $u \ne 0$.

Next we claim $u_n \to u$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. To prove the claim, notice that since $u_n \rightharpoonup u \ne 0$, u solves (1.1). Thus

$$\begin{aligned}\int_{\mathbb{R}^N} \quad (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)dx &= \int_{\mathbb{R}^N} |y|^{-b_q}|u|^q dx \\ \le \quad & S_p(a,\lambda,q)^{-\frac{q}{p}}\left(\int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)\,dx\right)^{\frac{q}{p}}.\end{aligned}$$

Since $u \ne 0$, we can conclude $\int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)\,dx \ge S_p(a,\lambda,q)^{\frac{q}{q-p}}$. By the lower semi continuity of the norm in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$ we have

$$\int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)\,dx \le S_p(a,\lambda,q)^{\frac{q}{q-p}}.$$

Hence $\int_{\mathbb{R}^N} (|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p)\,dx = S_p(a,\lambda,q)^{\frac{q}{q-p}}$ i.e. $S_p(a,\lambda,q)$ is achieved by u and $u_n \to u$ in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. □

Proof of Theorem 1.2.

(i) We have already seen that $S_p(a,\lambda,p^*) \le S_p$. Now

$$S_p(0,\lambda,p^*) = \inf_{u\in D^{1,p}(\mathbb{R}^N),u\not\equiv 0} \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^p - \lambda|\frac{u}{y}|^p\right)dx}{\left(\int_{\mathbb{R}^N}|u|^{p^*}\right)^{\frac{p}{p^*}}}.$$

Thus $\lambda \le 0 \Longleftrightarrow S_p(0,\lambda,p^*) \ge S_p$. i.e. for $\lambda \le 0$ we have $S_p(0,\lambda,p^*) = S_p$.

Let us consider u as the Aubin-Talenti function i.e, $u = \left(\frac{1}{1+|x|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}$. We know that $u \in D^{1,p}(\mathbb{R}^N)$ and S_p is achieved by u. Now if $\lambda > 0$ then

$$\begin{aligned} S_p(0,\lambda,p^*) &\le \frac{\int_{\mathbb{R}^N}(|\nabla u|^p - \lambda|\frac{u}{y}|^p)dx}{\left(\int_{\mathbb{R}^N}|u|^{p^*}\right)^{\frac{p}{p^*}}} \\ &= S_p - \frac{\lambda\int_{\mathbb{R}^N}|\frac{u}{y}|^p dx}{\left(\int_{\mathbb{R}^N}|u|^{p^*}\right)^{\frac{p}{p^*}}} \\ &< S_p. \end{aligned}$$

Hence $S_p(0,\lambda,p^*)$ is achieved if and only if $0 < \lambda < \lambda_{p,0}$.

(ii) Define $\lambda_0 = \lambda_{p,a}\frac{N-1}{N-p}\frac{pa}{N+a} < 0$. Now consider u as in (i), i.e. the Aubin-Talenti function. Therefore

$$\begin{aligned} \int_{\mathbb{R}^N}(|y|^a|\nabla u|^p - \lambda_0|y|^{a-p}|u|^p)dx &< \left(1-\frac{\lambda_0}{\lambda_{p,a}}\right)\int_{\mathbb{R}^N}|y|^a|\nabla u|^p dx \\ &= \left(1-\frac{(N-1)pa}{(N-p)(N+a)}\right)\int_{\mathbb{R}^N}|y|^a|\nabla u|^p dx. \end{aligned}$$

Hence

$$S_p(a,\lambda_0,p^*) < \frac{N}{N-p}\,\frac{N-p-a(p-1)}{N+a}\,\frac{\int_{\mathbb{R}^N}|y|^a|\nabla u|^p dx}{\left(\int_{\mathbb{R}^N}|y|^{\frac{Na}{N-p}}|u|^{p^*}dx\right)^{\frac{p}{p^*}}}.$$

And the RHS of the above expression is less than equal to S_p by Theorem 0.2 in [15]. Hence we have $S_p(a,\lambda_0,p^*) < S_p$. Now as the map $\lambda \longmapsto S_p(a,\lambda,p^*)$ is continuous and non increasing, we have $S_p(a,\lambda_0-\varepsilon,p^*) < S_p$ for ε is small enough. We can also conclude the theorem using $\lambda \longmapsto S_p(a,\lambda,p^*)$ non increasing function. □

5 Cylindrically symmetric solution and break of symmetry

Define $D^{1,p}_{cyln}(\mathbb{R}^N,|y|^a dx)$ to be the subspace of $D^{1,p}(\mathbb{R}^N,|y|^a dx)$ such that

$$D^{1,p}_{cyln}(\mathbb{R}^N,|y|^a dx) = \{u \in D^{1,p}(\mathbb{R}^N,|y|^a dx) : u(x) = u(|y|,z)\}.$$

Proof of Theorem 1.3. Let us take $u \in C_0^\infty\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}\right)$ and $u(x) = u(|y|, z)$. Then

$$\frac{\displaystyle\int_{\mathbb{R}^N}\left(|y|^a|\nabla u|^p dx - \lambda\int_{\mathbb{R}^N}|y|^{a-p}|u|^p\right)dx}{\left(\displaystyle\int_{\mathbb{R}^N}|y|^{-b_q}|u|^q dx\right)^{\frac{p}{q}}}$$

$$= w_k^{\frac{q-p}{q}}\frac{\displaystyle\int_{\mathbb{R}^{N-k}}\int_0^\infty\left(r^{a+k-1}(u_r^2+|\nabla_z u|^2)^{\frac{p}{2}} - \lambda r^{a-p+k-1}|u|^p\right)drdz}{\left(\displaystyle\int_{\mathbb{R}^{N-k}}\int_0^\infty r^{-(b_q-k+1)}|u|^q drdz\right)^{\frac{p}{q}}}.$$

We can conclude RHS of the above expression is positive by applying Hardy's inequality (2.7) and Hardy-Sobolev type inequality (1.4) with $k = 1$, i.e N is replaced by $N - k + 1$ and a is replaced by $a + k - 1$ which is strictly greater than $p - 1$. And correspondingly we see b_q will be replaced by $N - k + 1 - \frac{q}{p}(N - k + 1 - p + a + k - 1)$ which is equal to $b_q - k + 1$.
Therefore we can define

$$S_{p,cyln}(a,\lambda,q) = \inf_{u\in D^{1,p}_{cyln}(\mathbb{R}^N,|y|^a dx),u\not\equiv 0}\frac{\displaystyle\int_{\mathbb{R}^N}\left(|y|^a|\nabla u|^p dx - \lambda\int_{\mathbb{R}^N}|y|^{a-p}|u|^p\right)dx}{\left(\displaystyle\int_{\mathbb{R}^N}|y|^{-b_q}|u|^q dx\right)^{\frac{p}{q}}} \tag{5.14}$$

which is positive. We claim that $S_{p,cyln}(a, \lambda, q)$ is achieved in $D^{1,p}_{cyln}(\mathbb{R}^N, |y|^a dx)$ since because of the reduction of dimension in cylindrically symmetric case, problem (1.1) with critical exponent $q = p^*$ reduces to the subcritical problem in R^{N-k+1}. Therefore we can conclude the theorem in the same way as the proof of Theorem 1.1, only we have to take the cut off functions to be cylindrically symmetric i.e $\phi(y, z) = \phi(|y|, z)$. □

It is clear from the definition of $S_{p,cyln}(a, \lambda, q)$ and $S_p(a, \lambda, q)$ that $S_p(a, \lambda, q) \leq S_{p,cyln}(a, \lambda, q)$. Now notice that if $S_p(a, \lambda, q) < S_{p,cyln}(a, \lambda, q)$ then u can not be cylindrically symmetric i.e. break of symmetry occurs if $S_p(a, \lambda, q)$ is achieved by u.

Proof of Theorem 1.4. Let us recall the definition of $A(u)$ and $B(u)$ from Section 3. We also define $Q(u) = \frac{A(u)}{B(u)}$. We will prove this theorem by contradiction. Assume u is cylindrically symmetric local minimum of Q in $D^{1,p}(\mathbb{R}^N, |y|^a dx)$. Since Q is homogeneous we can assume $B(u) = 1$. Thus we have

$$Q'(u).v = 0,\quad Q''(u)[v, v] \geq 0\ \ \forall v \in D^{1,p}(\mathbb{R}^N, |y|^a dx).$$

By computing the derivative we get from above that

$$A'(u).v = Q(u)B'(u).v \ \text{ and }\ A''(u)[v, v] \geq Q(u)B''(u)[v, v]. \tag{5.15}$$

From the definition of $B(u)$ we can compute

$$B'(u).v = p\left(\int_{\mathbb{R}^N}|y|^{-b_q}|u|^q\right)^{\frac{p}{q}-1}\int_{\mathbb{R}^N}|y|^{-b_q}|u|^{q-2}uv = p\int_{\mathbb{R}^N}|y|^{-b_q}|u|^{q-2}uv.$$

And similarly

$$B''(u)[v, v] = p\left(\frac{p-q}{q}\left(\int_{\mathbb{R}^N}|y|^{-b_q}|u|^{q-2}uvdx\right)^2 + (q-1)\int_{\mathbb{R}^N}|y|^{-b_q}|u|^{q-2}v^2dx\right),$$

$$A''(u)[v,v] = p(p-1)\left(\int_{\mathbb{R}^N}\left(|y|^a|\nabla u|^{p-2}|\nabla v|^2 - \lambda|y|^{a-p}|u|^{p-2}v^2\right)dx\right).$$

Now let us choose the test function $v = u\phi_1$ where $\phi_1 \in H^1(S^{k-1})$ is an eigen function of the Laplace-Beltrami operator on S^{k-1} corresponding to the smallest positive eigen value, that is

$$\frac{1}{|S^{k-1}|}\int_{S^{k-1}}\phi_1^2 = 1, \quad \int_{S^{k-1}}\phi_1 = 0, \quad -\Delta_\sigma\phi_1 = (k-1)\phi_1.$$

Therefore using this test function we obtain $B''(u)[v,v] = p(q-1)$. Now note that,

$$|\nabla v|^2 = \phi_1^2|\nabla u|^2 + u^2|\nabla\phi_1|^2 = \phi_1^2|\nabla u|^2 + u^2|y|^{-2}|\nabla_\sigma\phi_1|^2.$$

Since $p \geq 2$, we have

$$\begin{aligned}
A''(u)[v,v] &= p(p-1)(\int_{\mathbb{R}^N}(|y|^a|\nabla u|^p| + (k-1)|y|^{a-2}|\nabla u|^{p-2}|u|^2 \\
&- \lambda|y|^{a-p}|u|^{p-2}v^2)dx) \\
&\leq p(p-1)n(u) \\
&+ p(k-1)(p-1)\left(\int_{\mathbb{R}^N}|y|^a|\nabla u|^p\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}|y|^{a-p}|u|^p\right)^{\frac{2}{p}}.
\end{aligned}$$

Hence from (5.15) and using the relation $B(u) = 1$, that is $Q(u) = A(u)$,we have

$$\begin{aligned}
p(q-1)A(u) &\leq p(p-1)A(u) \\
&+ p(k-1)(p-1)\left(\int_{\mathbb{R}^N}|y|^a|\nabla u|^p\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}|y|^{a-p}|u|^p\right)^{\frac{2}{p}} \\
A(u) &= \int_{\mathbb{R}^N}|y|^a|\nabla u|^p - \lambda|y|^{a-p}|u|^p dx \\
&\leq \frac{(p-1)(k-1)}{q-p}\left(\int_{\mathbb{R}^N}|y|^a|\nabla u|^p\right)^{\frac{p-2}{p}}. \\
&\left(\int_{\mathbb{R}^N}|y|^{a-p}|u|^p\right)^{\frac{2}{p}}
\end{aligned} \tag{5.16}$$

Now set $t > 0$ such that $t^p = \dfrac{\int_{\mathbb{R}^N}|y|^a|\nabla u|^p}{\int_{\mathbb{R}^N}|y|^{a-p}|u|^p}$. Also set $\alpha = \frac{k-p+a}{p}$. Then $t > \alpha$. Also set $\gamma = \frac{(p-1)(k-1)}{q-p}$.

Hence from (5.16) we obtain $\lambda \geq t^p - \gamma t^{p-2}$. Now using elementary calculus we can calculate infimum of $t \longmapsto t^p - \gamma t^{p-2}$ on the set $\{t > \alpha\}$. Therefore we obtain

$$\begin{aligned}
\lambda &> \alpha^p - \gamma\alpha^{p-2} \text{ if } p = 2 \text{ or } a \geq p - k + (\gamma p(p-2))^{\frac{1}{2}} \\
\lambda &\geq -2(\frac{\gamma}{p})^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}} \text{ otherwise}
\end{aligned}$$

Therefore if we choose λ such that

$$\begin{aligned}
\lambda &\leq \alpha^p - \gamma\alpha^{p-2} \text{ if } p = 2 \text{ or } a \geq p - k + (\gamma p(p-2))^{\frac{1}{2}} \\
\lambda &< -2(\frac{\gamma}{p})^{\frac{p}{2}}(p-2)^{\frac{p-2}{2}} \text{ otherwise}
\end{aligned}$$

then no minimizer of $S_p(a,\lambda,q)$ is cylindrically symmetric that is $S_p(a,\lambda,q) < S_{p,cyln}(a,\lambda,q)$. $\square$

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