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Dual Spaces of Weighted Multi-Parameter Hardy Spaces Associated with the Zygmund Dilation *

Xiaolong Han, Guozhen Lu,† Yayuan Xiao

Department of Mathematics

Wayne State University, Detroit, Michigan 48202

e-mail: xlhan@math.wayne.edu, gzlu@math.wayne.edu, and xiao@math.wayne.edu

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Abstract

In this paper, we apply the discrete Littlewood-Paley-Stein analysis to prove the duality theorem of weighted multi-parameter Hardy spaces associated with Zygmund dilations, i.e., $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$ for $0 < p \leq 1$. Our theorems extend the $H^p_{\mathcal{Z}} - CMO^p_{\mathcal{Z}}$ duality theorems established in [13] (see also [12]) for non-weighted multi-parameter Hardy spaces associated with the Zygmund dilation.

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1 Introduction and statements of main results

The celebrated $H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ duality theorem was proved by C. Fefferman and E. M. Stein [5, 9] in one-parameter case. In multi-parameter setting, S-Y. A. Chang and R. Fefferman [2, 3] proved that the dual space of the product $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the product $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, using the bi-Hilbert transform. We also refer the reader to Ferguson and Lacey [10] for another characterization of the product BMO spaces.

Among the multi-parameter structures, the Zygmund dilation is perhaps the simplest one after pure product space dilations. (See R. Fefferman's survey [7].) Recently, Y. Han and the second author [11, 12, 13] developed a unified approach to multi-parameter Hardy space theory by using the discrete multi-parameter Littlewood-Paley-Stein analysis, and the $H^p_{\mathcal{Z}} - CMO^p_{\mathcal{Z}}$ duality theorem in [13] is one of their main theorems, where $H^p_{\mathcal{Z}}$ is the multi-parameter Hardy space associated with the Zygmund dilations and $CMO^p_{\mathcal{Z}}$ is the Carleson measure spaces associated with the Zygmund dilations.

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†Corresponding Author: Guozhen Lu at gzlu@math.wayne.edu.

The main result in this paper is to characterize the dual spaces of the weighted multi-parameter Hardy spaces associated with the Zygmund dilations, that is, $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$ for all $0 < p \le 1$ and $\omega \in A_\infty(\mathcal{Z})$. Such Carleson measure spaces $CMO^p_{\mathcal{Z}}(\omega)$ play the same role as the John-Nirenberg BMO spaces in the duality $H^1(\mathbb{R}^n) - BMO(\mathbb{R}^n)$ in the one-parameter setting when $p = 1$. In the pure product setting with arbitrary k parameters, the dual spaces of weighted Hardy spaces $H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ for the weight function $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ have been identified in [17].

Let us first establish the preliminaries for the Zygmund dilations and recall the related background briefly. In $\mathbb{R}^3$, the Zygmund dilation is given by $\rho_{s,t}(x, y, z) = (sx, ty, stz)$ for $s, t > 0$, and the maximal operator associated to the Zygmund dilations is defined by

$$\mathcal{M}_{\mathcal{Z}} f(x, y, z) = \sup_{\substack{(x,y,z)\in Q\\ Q\in\mathcal{R}_{\mathcal{Z}}}} \frac{1}{|Q|} \int_Q |f|, \tag{1.1}$$

where $\mathcal{R}_{\mathcal{Z}}$ is the class of rectangles whose sides are parallel to the axes and have side lengths of the form s, t, and st. As a special case of Córdoba's solution [1] of Zygmund's conjecture, the operator $\mathcal{M}_{\mathcal{Z}}$ is bounded from the Orlicz space $L\log^+ L(Q_1)$ to weak $L^1(Q_1)$. (Q_1 is the unit cube in $\mathbb{R}^3$.) The weighted L^p boundedness of $\mathcal{M}_{\mathcal{Z}}$ for $1 < p < \infty$ was proved by R. Fefferman [6]. (See also [8].) Various generalizations can be found in [16].

Ricci and Stein [19] introduced the singular integral operator $T_{\mathcal{Z}}$ as $T_{\mathcal{Z}} = K_{\mathcal{Z}} * f$ and proved its L^p $(1 < p < \infty)$ boundedness, where

$$K_{\mathcal{Z}}(x, y, z) = \sum_{j,k} 2^{-2(j+k)} \phi_{j,k}\left(\frac{x}{2^j}, \frac{y}{2^k}, \frac{z}{2^{j+k}}\right),$$

and the functions $\phi_{j,k}$ on $\mathbb{R}^3$ satisfy

$$\int_{\mathbb{R}^2} \phi_{j,k}(x, y, z)dxdy = \int_{\mathbb{R}^2} \phi_{j,k}(x, y, z)dydz = \int_{\mathbb{R}^2} \phi_{j,k}(x, y, z)dzdx = 0.$$

R. Fefferman and Pipher in [FP] further showed that $T_{\mathcal{Z}}$ is bounded in weighted L^p_w spaces for $1 < p < \infty$ when the weights w satisfy an analogous condition of Muckenhoupt associated to the Zygmund dilation (see below). In fact, they proved that if $K_{\mathcal{Z}}$ is the kernel of the Ricci-Stein operator $T_{\mathcal{Z}}$ satisfying the above cancelation condition, then $K_{\mathcal{Z}}$ can be decomposed into $K_{\mathcal{Z}} = K^{(1)}_{\mathcal{Z}} + K^{(2)}_{\mathcal{Z}}$ such that

$$\int_{\mathbb{R}} K^{(1)}_{\mathcal{Z}}(x, y, z)dx = 0, \int_{\mathbb{R}^2} K^{(1)}_{\mathcal{Z}}(x, y, z)dydz = 0$$

$$\int_{\mathbb{R}} K^{(2)}_{\mathcal{Z}}(x, y, z)dy = 0, \int_{\mathbb{R}^2} K^{(2)}_{\mathcal{Z}}(x, y, z)dxdz = 0.$$

Subsequently, they proved that each of the operators with the kernels $K^{(1)}_{\mathcal{Z}}$ and $K^{(2)}_{\mathcal{Z}}$ are bounded on L^p_w for $1 < p < \infty$.

The authors in [13] proved that both the convolution and non-convolution type Ricci-Stein operators are bounded on $H^p_{\mathcal{Z}}$ and $BMO_{\mathcal{Z}}$[1]. In [15], the boundedness of Ricci-Stein singular integrals on weighted multi-parameter Hardy spaces $H^p_{\mathcal{Z}}(w)$ when $w \in A_\infty(\mathcal{Z})$ was established. It is interesting to note that we only require $w \in A_\infty(\mathcal{Z})$ which is much weaker than the usual requirement $w \in A_1$ for boundedness of singular integral operators on weighted Hardy spaces. Using the A_∞ weight to consider the boundedness of singular integrals on weighted multi-parameter Hardy spaces seems to be first used in [4]. (See also [18] for the case of more parameters.)

[1]The multi-parameter Hardy space associated with the Zygmund dilations $H^p_{\mathcal{Z}}$ is defined in the following content, see [13] for more information, where one can also find a nice historical note in the introductory section.

Let us denote by $\mathcal{S}(\mathbb{R}^n)$ as the space of Schwartz functions in $\mathbb{R}^n$. The test function defined on $\mathbb{R}^3$ is given by

$$\psi(x,y,z) = \psi^{(1)}(x)\psi^{(2)}(y,z),$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ satisfy

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1 \text{ for all } \xi_1 \in \mathbb{R}\backslash\{0\},$$

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi_2, 2^{-k}\xi_3)|^2 = 1 \text{ for all } (\xi_2,\xi_3) \in \mathbb{R}^2\backslash\{(0,0)\},$$

and the moment conditions

$$\int_{\mathbb{R}} x^\alpha \psi^{(1)}(x)dx = \int_{\mathbb{R}^2} y^\beta z^\gamma \psi^{(2)}(y,z)dydz = 0$$

for all integers $\alpha, \beta, \gamma \geq 0$. By taking Fourier transform, it is easy to see the continuous version of Calderón's identity

$$f(x,y,z) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y,z), \tag{1.2}$$

where

$$\psi_{j,k}(x,y,z) = 2^{2(j+k)}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y, 2^{j+k} z), \tag{1.3}$$

and the series converges in $L^2(\mathbb{R}^3)$.

Now we define the Littlewood-Paley-Stein square function of f associated with the Zygmund dilation,

$$g_{\mathcal{Z}}(f)(x,y,z) = \left\{ \sum_{j,k} |\psi_{j,k} * f(x,y,z)|^2 \right\}^{1/2}. \tag{1.4}$$

From Ricci and Stein's L^p boundedness of the operator $T_{\mathcal{Z}}$, together with the L^2 convergence of Calderón's identity, one can obtain the L^p estimate of $g_{\mathcal{Z}}$ as $\|g_{\mathcal{Z}}(f)\|_{L^p} \approx \|f\|_{L^p}$ for $1 < p < \infty$. Precisely, there exist two constants $C_1, C_2 > 0$ independent of f such that

$$C_1\|f\|_{L^p} \leq \|g_{\mathcal{Z}}(f)\|_{L^p} \leq C_2\|f\|_{L^p}. \tag{1.5}$$

To apply this Littlewood-Paley-Stein analysis to Hardy spaces and weighted Hardy spaces, we need to introduce a proper test function space.

Definition 1.1 *A Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^3$ is said to be a product test function on $\mathbb{R}\times\mathbb{R}^2$ if $f \in \mathcal{S}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}} x^\alpha f(x,y,z)dx = \int_{\mathbb{R}^2} y^\beta z^\gamma f(x,y,z)dydz = 0$$

for all nonnegative integers α, β, and γ.

If f is a product test function on $\mathbb{R}\times\mathbb{R}^2$, we denote $f \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and the norm of f is defined by the norm of Schwartz test function. We denote the dual of $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ by $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, so the Littlewood-Paley-Stein square function $g_{\mathcal{Z}}$ can be defined for all distributions in $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$. Thus for $0 < p < \infty$, the multi-parameter Hardy space associated with Zygmund dilations can be defined as

$$H^p_{\mathcal{Z}}(\mathbb{R}^3) = \{f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))' : g_{\mathcal{Z}}(f) \in L^p(\mathbb{R}^3)\},$$

and $H^p_{\mathcal{Z}}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$ follows immediately from (1.5) above. See [12] and [13] for the thorough study of such $H^p_{\mathcal{Z}}$ spaces including the duality theory and boundedness of convolution and non-convolution operators associated to the Zgymund dilations.

Given $1 < p < \infty$, a nonnegative function ω on $\mathbb{R}^3$ is said to belong to $A_p(\mathcal{Z})$ if

$$\sup_{Q\in R_{\mathcal{Z}}}\left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\right)^{p-1} = \|\omega\|_{A_p(\mathcal{Z})} < \infty.$$

When $p = 1$, $\omega \in A_1(\mathcal{Z})$ if there exists $C > 0$ such that $\mathcal{M}_{\mathcal{Z}}(\omega)(x) \le C\omega(x)$ for almost every $x \in \mathbb{R}^3$. Finally, we define

$$A_\infty(\mathcal{Z}) = \bigcup_{1\le p<\infty} A_p(\mathcal{Z}).$$

Notice that if $\omega \in A_\infty(\mathcal{Z})$, then $\omega \in A_{q_\omega}(\mathcal{Z})$, where $q_\omega = \inf\{q : \omega \in A_q(\mathcal{Z})\}$. Now let us introduce the two spaces that we study in this paper.

Definition 1.2 ($H^p_{\mathcal{Z}}(\omega)$) *Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, the multi-parameter Hardy space associated with the Zygmund dilation is defined as $H^p_{\mathcal{Z}}(\omega) = \{f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))' : g_{\mathcal{Z}}(f) \in L^p_\omega)\}$. If $f \in H^p_{\mathcal{Z}}(\omega)$, the norm of f is defined by $\|f\|_{H^p_{\mathcal{Z}}(\omega)} = \|g_{\mathcal{Z}}(f)\|_{L^p_\omega}$.*

Definition 1.3 ($CMO^p_{\mathcal{Z}}(\omega)$) *Let $0 < p \le 1$, $\omega \in A_\infty(\mathcal{Z})$ and $\psi_{j,k}$ be the same as in (1.3), we say that $f \in CMO^p_{\mathcal{Z}}(\omega)$ if $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$ with the finite norm $\|f\|_{CMO^p_{\mathcal{Z}}(\omega)}$ defined by*

$$\sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}|\psi_{j,k}*f(x_I,y_J,z_R)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}},$$

where the supremum is taken over all open sets Ω in $\mathbb{R}^3$ with $\omega(\Omega) < \infty$, I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N, and x_I, y_J, and z_R are any fixed points in I, J and R, respectively.

Remark 1.1 In Definitions 1.2 and 1.3 above, the definitions of $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$ involve $\psi_{j,k}$, to show these definitions are well defined, we need to prove that they are independent of the choice of functions $\psi_{j,k}$. Precisely, we use sup-inf comparison principle of first kind as Theorem 2.2 in Section 2 to show that $H^p_{\mathcal{Z}}(\omega)$ is well-defined and the following sup-inf comparison principle of second kind, which is one of our major theorems, to prove that $CMO^p_{\mathcal{Z}}(\omega)$ is well-defined.

Theorem 1.1 (Sup-inf comparison principle of second kind) *Let $0 < p \le 1$ and $\omega \in A_\infty(\mathcal{Z})$. Suppose $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and $\psi_{j,k}, \phi_{j,k}$ are defined as in* (1.3). *Then for $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$,*

$$\sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}\sup_{u\in I,v\in J,w\in R}|\psi_{j,k}*f(u,v,w)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}$$

$$\approx \sup_{\Omega}\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}\inf_{u\in I,v\in J,w\in R}|\phi_{j,k}*f(u,v,w)|^2\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}.$$

where Ω are all open sets in $\mathbb{R}^3$ with $\omega(\Omega) < \infty$, I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N.

Then we can prove that the space $CMO^p_{\mathcal{Z}}(\omega)$ is exactly the dual space of $H^p_{\mathcal{Z}}(\omega)$ for $0 < p \le 1$. More precisely,

Theorem 1.2 ($H^p_{\mathcal{Z}}(\omega) - CMO^p_{\mathcal{Z}}(\omega)$) *Let $0 < p \le 1$ and $\omega \in A_\infty(\mathcal{Z})$. Then $(H^p_{\mathcal{Z}}(\omega))^* = CMO^p_{\mathcal{Z}}(\omega)$, namely the dual space of $H^p_{\mathcal{Z}}(\omega)$ is $CMO^p_{\mathcal{Z}}(\omega)$. More precisely, if $g \in CMO^p_{\mathcal{Z}}(\omega)$, the map ℓ_g given by $\ell_g(f) =< f, g >$, defined initially for $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$, extends to a continuous linear functional on $H^p_{\mathcal{Z}}(\omega)$ with $\|\ell_g\| \approx \|g\|_{CMO^p_{\mathcal{Z}}(\omega)}$. Conversely, for every $\ell \in (H^p_{\mathcal{Z}}(\omega))^*$ there exists some $g \in CMO^p_{\mathcal{Z}}(\omega)$ so that $\ell = \ell_g$. In particular, we denote $CMO^1_{\mathcal{Z}}(\omega)$ by $BMO_{\mathcal{Z}}(\omega)$ and then $(H^1_{\mathcal{Z}}(\omega))^* = BMO_{\mathcal{Z}}(\omega)$.*

Finally, combining the boundedness of Ricci-Stein operators $T_{\mathcal{Z}}$ on weighted Hardy space $H^1_{\mathcal{Z}}(\omega)$ derived in [15] and the above duality theorem, we can deduce immediately the following boundedness theorem of $T_{\mathcal{Z}}$ on $BMO_{\mathcal{Z}}(\omega)$.

Theorem 1.3 *Assume $\omega \in A_\infty(\mathcal{Z})$. Then the Ricci-Stein operator $T_{\mathcal{Z}}$ is bounded on weighted space $BMO_{\mathcal{Z}}(\omega)$.*

The rest of our paper will be organized as follows. In Section 2, we collect several known results on the discrete Calderón's identity and sup-inf comparison principle of first kind which are used to prove that $H^p_{\mathcal{Z}}(\omega)$ is well-defined. In Section 3, we show the well-definedness of $CMO^p_{\mathcal{Z}}(\omega)$ using sup-inf comparison of second kind and almost orthogonality estimate. Section 4 is devoted to proving the duality theorem, namely, Theorem 1.2.

We shall point out in the end of the introduction that the main tool in this paper, the discrete multi-parameter Littlewood-Paley-Stein analysis, is a relatively unified theory with a whole scheme, some theorems and lemmas we use here originate from the work [13]. An interested reader should consult the papers [11, 12, 13, 14] and related works mentioned therein. (See also [4], [18] and [17] where some nice application of the discrete Littlewood-Paley-Stein analysis was given in weighted setting.)

2 Discrete Calderón's identity, sup-inf comparison principle of first kind

To show the definition of $H^p_{\mathcal{Z}}(\omega)$ is independent of the choice of functions $\psi_{j,k}$ and thus well defined in Definition 1.2, we need to recall some results associated with the Zygmund dilation. First, we require the discrete version of Calderón's identity.

Theorem 2.1 (Discrete Calderón's identity) *Suppose that $\psi_{j,k}$ are the same as in (1.3). Then*

$$f(x, y, z) = \sum_{j,k} \sum_{I,J,R} |I||J||R| \widetilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R) \tag{2.6}$$

converges in $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and its dual space $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$, where I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large integer N, x_I, y_J, z_R are any fixed points in I, J, R, respectively, and $\widetilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R) \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$.

The complete proof of Theorem 2.1 is contained in [13], for the reader's convenience, we provide a sketch of the proof here. An observation shows that the continuous version of Calderón's identity (1.2) converges in the norm of $\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$ and in the dual space $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$. The explicit expression of $\widetilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)$ can also be found in [12, 13].

The discrete Calderón's identity enables us to derive the following weighted version sup-inf comparison principle of first kind, whose proof is included in [15]. It is an extension of the non-weighted one first derived in [12, 13].

Theorem 2.2 (Sup-inf comparison principle of first kind) *Let $0 < p < \infty$ and $\omega \in A_\infty(\mathcal{Z})$, suppose $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and $\psi_{j,k}, \phi_{j,k}$ are defined as in* (1.3). *Then for $f \in (\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$,*

$$
\begin{aligned}
&\left\| \left\{ \sum_{j,k} \sum_{I,J,R} \sup_{u\in I, v\in J, w\in R} |\psi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^p_\omega} \\
\approx\ &\left\| \left\{ \sum_{j,k} \sum_{I,J,R} \inf_{u\in I, v\in J, w\in R} |\phi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right\}^{\frac{1}{2}} \right\|_{L^p_\omega},
\end{aligned}
\tag{2.7}
$$

where I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N, χ_I, χ_J, and χ_R are indicator functions of I, J, and R, respectively.

From this sup-inf comparison principle, we introduce the discrete Littlewood-Paley-Stein square function

$$
g^d_{\mathcal{Z}}(f)(x,y,z) = \left\{ \sum_{j,k} \sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)|^2 \chi_I(x)\chi_J(y)\chi_R(z) \right\}^{\frac{1}{2}}, \tag{2.8}
$$

from which we can conclude the $H^p_{\mathcal{Z}}(\omega)$ norm of f can be characterized using a discrete form

$$
\|f\|_{H^p_{\mathcal{Z}}(\omega)} \approx \|g^d_{\mathcal{Z}}(f)\|_{L^p_\omega}.
$$

Thus, we conclude that $H^p_{\mathcal{Z}}(\omega)$ is well-defined by Theorem 2.2.

3 Proof of Theorem 1.1

The purpose of this section is to get the sup-inf comparison principle of second kind, i.e., Theorem 1.1, to ensure that the space $CMO^p_{\mathcal{Z}}(\omega)$ in Definition 1.3 is well-defined. First, we recall an "almost orthogonality lemma", and refer the reader to [13] for its detailed proof.

Lemma 3.1 (Almost orthogonality estimate) *If $\psi, \phi \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, define*

$$
\psi_{t,s}(x,y,z) = t^{-2}s^{-2}\psi\left(\frac{x}{t}, \frac{y}{s}, \frac{z}{ts}\right)
$$

and $\phi_{t',s'}$ is defined similarly. Then, for any positive integers L and M, there exists a constant $C = C(L,M) > 0$ such that

$$
\begin{aligned}
&|\psi_{t,s} * \phi_{t',s'}(x,y,z)| \\
\le\ & C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^L \left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^L \frac{(t\vee t')^M}{(t\vee t' + |x|)^{M+1}} \frac{(s\vee s')^M}{t^*(s\vee s' + |y| + \frac{|z|}{t^*})^{M+2}},
\end{aligned}
\tag{3.9}
$$

where $t^* = t$ if $s > s'$, $t^* = t'$ if $s \le s'$, $t \wedge s = \min(t,s)$, and $t \vee s = \max(t,s)$.

Together with the discrete Calderón's identity and some geometric properties of multi-parameter rectangles, Theorem 1.1 can be proved by a delicate study of the Zygmund rectangles. Its proof adapts ideas from the proof of the nonweighted version given in [13]. We carefully incorporate the weight function ω into the argument. For completeness, we include a detailed proof here.

Proof. [Proof of Theorem 1.1] For simplicity, we denote $f_{j,k} = f_Q$, where $Q = I\times J\times R \subseteq \mathbb{R}^3$, I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N. While x_I, y_J, and z_R are any fixed points in I, J, and R, respectively. Then, we can rewrite the discrete Calderón identity (2.6) on $(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$ as

$$f(x,y,z)=\sum_{j',k'}\sum_{Q'=I'\times J'\times R'}|I'||J'||R'|\widetilde{\phi}_{Q'}(x,y,z,x_{I'},y_{J'},z_{R'})(\phi_{Q'}*f)(x_{I'},y_{J'},z_{R'}).$$

Thus, for all $(x,y,z)\in Q$,

$$\psi_Q*f(x,y,z)=\sum_{j',k'}\sum_{Q'=I'\times J'\times R'}|Q'|\psi_Q*\widetilde{\phi}_{Q'}(x,y,z,x_{I'},y_{J'},z_{R'})(\phi_{Q'}*f)(x_{I'},y_{J'},z_{R'}),$$

where I', J', and R' are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I')=2^{-j'-N}$, $\ell(J')=2^{-k'-N}$, and $\ell(R')=2^{-j'-k'-2N}$ for a fixed large positive integer N. While $x_{I'}$, $y_{J'}$, and $z_{R'}$ are any fixed points in I', J', and R', respectively.

From the almost orthogonality estimates (3.9) in Lemma 3.1, by choosing $t=2^{-j}$, $t'=2^{-j'}$, $s=2^{-k}$, and $s'=2^{-k'}$,

$$\begin{aligned}&|\psi_Q*f(x,y,z)|^2 \qquad (3.10)\\ \le\;& C\sum_{Q'=I'\times J'\times R'}|Q'|\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^L\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^L\frac{(|I|\vee|I'|)^M}{(|I|\vee|I'|+d(I,I'))^{M+1}}\\ &\times\frac{(|J|\vee|J'|)^M}{t^*(|J|\vee|J'|+d(J,J')+\frac{d(R,R')}{t^*})^{M+2}}|\phi_{Q'}*f(x_{I'},y_{J'},z_{R'})|^2,\end{aligned}$$

where $d(I,I')$ denotes the distance between the two intervals I and I', $|Q'|=|I'||J'||R'|$, $t^*=|I|$ when $|J|\ge|J'|$, and $t^*=|I'|$ when $|J|<|J'|$, the constant C depends only on M, L, and functions ψ and ϕ. Write

$$P_Q=\sup_{x\in I,y\in J,z\in R}|\psi_Q*f(x,y,z)|^2,$$

and

$$F_Q=\inf_{x\in I,y\in J,z\in R}|\phi_Q*f(x,y,z)|^2.$$

Since $x_{I'}$, $y_{J'}$, and $z_{R'}$ in (3.10) are arbitrary in I', J' and R', we have

$$\sum_{Q\subseteq\Omega}\frac{|I\times J\times R|^2}{\omega(I\times J\times R)}P_Q\le C\sum_{Q\subseteq Q'\subseteq\Omega}\widetilde{r}(Q,Q')P(Q,Q')\frac{|I'\times J'\times R'|^2}{\omega(I'\times J'\times R')}F_{Q'}, \qquad (3.11)$$

where

$$\widetilde{r}(Q,Q')=\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L-2}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L-2}\left(\frac{|R|}{|R'|}\wedge\frac{|R'|}{|R|}\right)^{-2}\frac{\omega(I'\times J'\times R')}{\omega(I\times J\times R)},$$

and

$$\begin{aligned}P(Q,Q')\;=\;&\frac{(|I|\vee|I'|)^{M+1}}{(|I|\vee|I'|+d(I,I'))^{M+1}}\frac{(|J|\vee|J'|)^{M+1}}{(|J|\vee|J'|+d(J,J')+\frac{d(R,R')}{t^*})^{M+1}}\\ &\times\frac{|R|\vee|R'|}{t^*(|J|\vee|J'|)+t^*d(J,J')+d(R,R')}.\end{aligned}$$

Since $\omega\in A_\infty(\mathcal{Z})$ and $Q\subseteq Q'$, there exists q_ω and $1\le q_\omega<\infty$ such that

$$\frac{\omega(I'\times J'\times R')}{\omega(I\times J\times R)}\le C\left(\frac{|I'\times J'\times R'|}{|I\times J\times R|}\right)^{q_\omega}.$$

Thus,

$$\widetilde{r}(Q,Q') \le r(Q,Q'),$$

where

$$r(Q,Q') = \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^{L-q_\omega-2} \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^{L-q_\omega-2} \left(\frac{|R|}{|R'|} \wedge \frac{|R'|}{|R|}\right)^{-q_\omega-2}.$$

We thus have

$$\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subseteq\Omega} P_Q \frac{|Q|^2}{\omega(Q)} \le C \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subseteq Q'\subseteq\Omega} r(Q,Q')P(Q,Q')\cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}. \tag{3.12}$$

To show Theorem 1.1, we only need to estimate the right hand side of (3.12). That is, to prove that it can be controlled by

$$\sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q'\subseteq\Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.$$

For $i,l \ge 0$, set

$$\Omega^{i,l} = \bigcup_{Q=I\times J\times R\subseteq\Omega} 3(2^i I \times 2^l J \times 2^{i+l} R).$$

Then, write

$$B_{0,0} = \{Q' = I'\times J'\times R' : 3Q' \cap \Omega^{0,0} \ne \emptyset\},$$

and for $i,l \ge 1$,

$$B_{i,0} = \{Q' = I'\times J'\times R' : 3(2^i I'\times J'\times 2^i R') \cap \Omega^{i,0} \ne \emptyset, 3(2^{i-1} I'\times J'\times 2^{i-1} R') \cap \Omega^{i,0} = \emptyset\},$$

$$B_{0,l} = \{Q' = I'\times J'\times R' : 3(I'\times 2^l J'\times 2^l R') \cap \Omega^{0,l} \ne \emptyset, 3(I'\times 2^{l-1} J'\times 2^{l-1} R') \cap \Omega^{0,l} = \emptyset\},$$

$$B_{i,l} = \{Q' = I'\times J'\times R' : 3(2^i I'\times 2^l J'\times 2^{i+l} R') \cap \Omega^{i,l} \ne \emptyset,\ 3(2^{i-1} I'\times 2^{l-1} J'\times 2^{i+l-2} R') \cap \Omega^{i,l} = \emptyset\}.$$

Note that since $\bigcup_{Q'} = \bigcup_{i,l\ge0} \bigcup_{Q'\in B_{i,l}}$, the right hand of (3.12) can be bounded by

$$\begin{aligned} &\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{Q\subseteq\Omega} \left[\sum_{Q'\in B_{0,0}} + \sum_{i\ge1}\sum_{Q'\in B_{i,0}} + \sum_{l\ge1}\sum_{Q'\in B_{0,l}} + \sum_{i,l\ge1}\sum_{Q'\in B_{i,l}}\right] \\ &\times r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')} \\ \triangleq\ & I + II + III + IV. \end{aligned} \tag{3.13}$$

Here we only show the estimate of I, then the estimates for the other three terms can follow similarly. Notice that if $Q' \in B_{0,0}$, then $3Q' \cap \Omega^{0,0} \ne \emptyset$. Let

$$\mathcal{F}_h^{0,0} = \{Q' \in B_{0,0} : |3Q' \cap \Omega^{0,0}| \ge \frac{1}{2^h}|3Q'|\},$$

$$\mathcal{D}_h^{0,0} = \mathcal{F}_h^{0,0} \backslash \mathcal{F}_{h-1}^{0,0},$$

and

$$\Omega_h^{0,0} = \bigcup_{Q'\in D_h^{0,0}} Q',$$

where $h \ge 0$ and $\mathcal{F}_{-1}^{0,0} = \emptyset$. Without loss of generality we may assume that for any open set $\Omega \subset \mathbb{R}^3$,

$$\sum_{Q=I\times J\times R\subseteq\Omega} \frac{|I\times J\times R|^2}{\omega(I\times J\times R)} F_Q \le C\omega(\Omega)^{\frac{2}{p}-1}. \tag{3.14}$$

Since $\bigcup_{h\geq 0} \mathcal{D}_h^{0,0} = B_{0,0}$, we have

$$I \leq \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} \sum_{Q\subset\Omega} r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}. \tag{3.15}$$

To estimate (3.15), for each $Q' \in B_{0,0}$ and $i', l', v' \geq 1$, we decompose $\{Q \subset \Omega\}$ into 8 pieces as follows:

$$\begin{aligned}
A_{0,0,0}(Q') &= \{Q \subset \Omega : d(I,I') \leq |I|\vee|I'|,\ d(J,J') \leq |J|\vee|J'|, d(R,R') \leq |R|\vee|R'|\},\\
A_{i',0,0}(Q') &= \{Q \subset \Omega : 2^{i'-1}(|I|\vee|I'|) < d(I,I') \leq 2^{i'}(|I|\vee|I'|), d(J,J') \leq |J|\vee|J'|,\\
&\qquad d(R,R') \leq |R|\vee|R'|\},\\
A_{0,l',0}(Q') &= \{Q \subset \Omega : d(I,I') \leq |I|\vee|I'|,\ 2^{l'-1}(|J|\vee|J'|) < d(J,J') \leq 2^{l'}(|J|\vee|J'|),\\
&\qquad d(R,R') \leq |R|\vee|R'|\},\\
A_{0,0,v'}(Q') &= \{Q \subset \Omega : d(I,I') \leq |I|\vee|I'|,\ d(J,J') \leq |J|\vee|J'|,\\
&\qquad 2^{v'-1}(|R|\vee|R'|) < d(R,R') \leq 2^{v'}(|R|\vee|R'|)\},\\
A_{i',l',0}(Q') &= \{Q \subset \Omega : 2^{i'-1}(|I|\vee|I'|) < d(I,I') \leq 2^{i'}(|I|\vee|I'|),\\
&\qquad 2^{l'-1}(|J|\vee|J'|) < d(J,J') \leq 2^{l'}(|J|\vee|J'|),\ d(R,R') \leq |R|\vee|R'|\},\\
A_{i',0,v'}(Q') &= \{Q \subset \Omega : 2^{i'-1}(|I|\vee|I'|) < d(I,I') \leq 2^{i'}(|I|\vee|I'|), d(J,J') \leq |J|\vee|J'|,\\
&\qquad 2^{v'-1}(|R|\vee|R'|) < d(R,R') \leq 2^{v'}(|R|\vee|R'|)\},\\
A_{0,l',v'}(Q') &= \{Q \subset \Omega : d(I,I') \leq |I|\vee|I'|,\ 2^{l'-1}(|J|\vee|J'|) < d(J,J') \leq 2^{l'}(|J|\vee|J'|),\\
&\qquad 2^{v'-1}(|R|\vee|R'|) < d(R,R') \leq 2^{v'}(|R|\vee|R'|)\},\\
A_{i',l',v'}(Q') &= \{Q \subset \Omega : 2^{i'-1}(|I|\vee|I'|) < d(I,I') \leq 2^{i'}(|I|\vee|I'|),\\
&\qquad 2^{l'-1}(|J|\vee|J'|) < d(J,J') \leq 2^{l'}(|J|\vee|J'|),\\
&\qquad 2^{v'-1}(|R|\vee|R'|) < d(R,R') \leq 2^{v'}(|R|\vee|R'|)\}.
\end{aligned}$$

Then, (3.15) becomes

$$\begin{aligned}
I &\leq \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} \sum_{Q\subset\Omega} r(Q,Q')P(Q,Q')\cdot m_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&= \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} \Big(\sum_{Q\in A_{0,0,0}(Q')} + \sum_{i'\geq 1} \sum_{Q\in A_{i',0,0}(Q')}\\
&\quad + \sum_{l'\geq 1} \sum_{Q\in A_{0,l',0}(Q')} + \sum_{v'\geq 1} \sum_{Q\in A_{0,0,v'}(Q')} + \sum_{i',l'\geq 1} \sum_{Q\in A_{i',l',0}(Q')}\\
&\quad + \sum_{i',v'\geq 1} \sum_{Q\in A_{i',0,v'}(Q')} + \sum_{l',v'\geq 1} \sum_{Q\in A_{0,l',v'}(Q')} + \sum_{i',l',v'\geq 1} \sum_{Q\in A_{i',l',v'}(Q')} \Big)\\
&\quad \cdot r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}\\
&\triangleq I_1 + \cdots + I_8.
\end{aligned}$$

In the following proof, we will give the estimates for I_1 and I_4 separately and the estimates for I_2, I_3, I_5, I_6, I_7, and I_8 can be handled similarly.

(i). To estimate

$$I_1 = \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q'\in\mathcal{D}_h^{0,0}} \sum_{Q\in A_{0,0,0}(Q')} r(Q,Q')P(Q,Q')\cdot F_{Q'}\frac{|Q'|^2}{\omega(Q')}, \tag{3.16}$$

we divide $\{Q \in A_{0,0,0}(Q')\}$ into 6 cases, and note that $3Q \cap 3Q' \neq \emptyset$ for $Q \in A_{0,0,0}(Q')$.
Case 1. $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \leq |R|$.

We will use a similar idea of analyzing geometric properties of the intervals (such analysis is similar to what was used in [3] in a less complicated situation). Since

$$\frac{|I|}{|3I'|}|3Q'| = |I| \times |3J'| \times |3R'| \leq 9|3Q \cap 3Q'| \leq |3Q' \cap \Omega^{0,0}| < \frac{9}{2^{h-1}}|3Q'|,$$

then $|I| \leq 2^{-h+5}|I'|$ and thus $|I'| \sim 2^{h-5+n}|I|$ for some $n \geq 0$. Moreover, for each given such n, the number of such I's is no more than $5 \cdot 2^{h-5+n}$. As for J, we have $|J| \sim 2^m|J'|$ for some $m \geq 0$ and for each m, the number of such J's is no more than 5. Since $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|R| \sim 2^{-(h-5+n)}2^m|R'|$. Note that $|R| \geq |R'|$, thus $m > h - 5 + n$. Furthermore, for each fixed n and m, the number of such R's is no more than 5. Thus,

$$\begin{aligned}
&\sum_{\text{Case 1}} r(Q,Q')P(Q,Q')\\
\leq\ & C\sum_{\text{Case 1}} \left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2}\left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\frac{|R|}{|I||J|}\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0} 2^{(5-h-n)(L-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{(h-5+n-m)(-q_\omega-2)}\cdot 2^n\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0} 2^{-hL}\cdot 2^{5L}\cdot 2^{-m(L-2q_\omega-4)}\cdot 2^{-4n}\\
\leq\ & C2^{-hL}.
\end{aligned}$$

Case 2: $|I'| \geq |I|$, $|J'| \leq |J|$, and $|R'| \geq |R|$.

Since

$$\frac{|I||R|}{|3I'||3R'|}|3Q'| = |I| \times |3J'| \times |R| \leq 3|3Q' \cap 3Q| \leq 3|3Q' \cap \Omega^{0,0}| < \frac{3}{2^{h-1}}|3Q'|,$$

then $|I'||R'| \sim 2^{h-5+n}|I||R|$. As for J, $|J| \sim 2^m|J'|$. So for each m, the number of such J's is no more than 5. Noting that $|R| = |I| \times |J|$ and $|R'| = |I'| \times |J'|$, we have $|I'||I'||J'| \sim 2^{h-5+n}|I||I||J|$, which yields that $|I'|^2 \sim 2^{h-5+n+m}|I|^2$, that is, $|I'| \sim 2^{(h-5+n+m)/2}|I|$. Hence for each n and m, the number of such I's is less than $5 \cdot 2^{(h+m+n)/2}$. Since we can obtain that $|R'| \sim 2^{(h-5+n-m)/2}|R|$, and $|R'| \geq |R|$, we have $m \leq h - 5 + n$. For each fixed n and m, the number of such R's is less than $5 \cdot 2^{(h-5+n-m)/2}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case 2}} r(Q,Q')P(Q,Q')\\
\leq\ & C\sum_{\text{Case 2}} \left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2}\left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R|}{|R'|}\right)^{-q_\omega-2}\frac{|R'|}{|I||J|}\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0} 2^{\frac{1}{2}(5-h-n-m)(L-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{\frac{1}{2}(5-h-n+m)(-q_\omega-2)}2^{h+n}\\
=\ & \sum_{n\geq 0}\sum_{m\geq 0} 2^{-h(\frac{L}{2}-q_\omega-3)}\cdot 2^{\frac{5}{2}L-5q_\omega-10}\cdot 2^{-n(\frac{L}{2}-q_\omega-3)}\cdot 2^{-m(\frac{3}{2}L-2q_\omega-4)}\\
\leq\ & C2^{-h(\frac{L}{2}-q_\omega-3)}.
\end{aligned}$$

Case 3: $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \leq |R|$.

This can be handled in a way similar to that of Case 1, and we have

$$\sum_{\text{Case 3}} r(Q,Q')P(Q,Q') \leq C2^{-hL}.$$

Case 4: $|I'| \leq |I|$, $|J'| \geq |J|$, and $|R'| \geq |R|$.

This can be handled in a similar way to that of Case 2, and we have

$$\sum_{\text{Case } 4} r(Q,Q')P(Q,Q') \leq C2^{-h(\frac{L}{2}-q_\omega-2)}.$$

Case 5: $|I'| \geq |I|$, $|J'| \geq |J|$, and thus $|R'| \geq |R|$.

Since

$$|I| \times |J| \times |R| \leq |3Q' \cap 3Q| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then $|Q'| \sim 2^{h-1+n}|Q|$ for some $n \geq 0$. Note that for each n, the number of such Q's is less than $(2^n)^3 = 2^{3n}$. More precisely, we have $(|I'||J'|)^2 \sim 2^{h-1+n}(|I||J|)^2$. Thus,

$$\begin{aligned}
&\sum_{\text{Case } 5} r(Q,Q')P(Q,Q') \\
\leq\ & C\sum_{\text{Case } 5}\left(\frac{|I||J|}{|I'||J'|}\right)^{L-q_\omega-2}\left(\frac{|R|}{|R'|}\right)^{-q_\omega-2}\frac{|R'|}{|I'||J'|} \\
\leq\ & C\sum_{n\geq 0} 2^{-\frac{h-1+n}{2}(L-q_\omega-2)} \cdot 2^{-\frac{h-1+n}{2}(-q_\omega-2)} \cdot 2^{3n} \\
=\ & C\sum_{n\geq 0} 2^{-h(\frac{L}{2}-q_\omega-2)} \cdot 2^{\frac{L}{2}-q_\omega-2} \cdot 2^{-n(\frac{L}{2}-q_\omega-5)} \\
\leq\ & C2^{-h(\frac{L}{2}-q_\omega-2)}.
\end{aligned}$$

Case 6: $|I'| \leq |I|$, $|J'| \leq |J|$, and thus $|R'| \leq |R|$.

Since

$$|I'| \times |J'| \times |R'| \leq |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then we can see that in this case, h must be less than 3. From $|I'| \leq |I|$, we have $|I| \sim 2^n|I'|$ for some $n \geq 0$ and for each given such n, the number of such I's is less than 5. Similarly, from $|J'| \leq |J|$, we have $|J| \sim 2^m|J'|$ and for each m, the number of such J's is less than 5. Hence we have $|R| \sim 2^{n+m}|R'|$, and for each n and m, the number of such R's is less than 5. Thus,

$$\begin{aligned}
&\sum_{\text{Case } 6} r(Q,Q')P(Q,Q') \\
\leq\ & C\sum_{\text{Case } 6}\left(\frac{|I'||J'|}{|I||J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\frac{|R|}{|I||J|} \\
\leq\ & C\sum_{n\geq 0}\sum_{n\geq 0}(2^{-n-m})^{L-q_\omega-2} \cdot (2^{-n-m})^{-q_\omega-2} \cdot 2^{-mL} \\
=\ & C\sum_{n\geq 0}\sum_{n\geq 0} 2^{-n(L-2q_\omega-4)} \cdot 2^{-m(2L-2q_\omega-4)} \\
\leq\ & C.
\end{aligned}$$

Before we combine these 6 cases above, observe that since $|\Omega_h^{0,0}| \leq Ch2^h|\Omega^{0,0}|$, $|\Omega^{0,0}| \leq C|\Omega|$, and $\omega \in A_\infty(\mathcal{Z})$, which is a doubling measure, together with (3.14), we have

$$\begin{aligned}
&\sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}\omega(\Omega_h^{0,0})^{\frac{2}{p}-1} \\
\leq\ & \sum_h 2^{-h(\frac{L}{2}-q_\omega-3)}\omega(Ch^2 2^h\Omega^{0,0})^{\frac{2}{p}-1} \\
\leq\ & C\omega(\Omega^{0,0})^{\frac{2}{p}-1} \\
\leq\ & C\omega(\Omega)^{\frac{2}{p}-1}.
\end{aligned}$$

Thus, combining the above 6 cases, I_1 in (3.16) can be estimated as:

$$\begin{aligned} I_1 \le\ & \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \left(\sum_{\text{Case 1}} + \cdots + \sum_{\text{Case 5}} \right) r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ & + \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{\text{Case 6}} r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ \le\ & C \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} 2^{-h(\frac{L}{2}-q_\omega-3)} \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ & + \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{3} \sum_{Q' \in \mathcal{D}_h^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ \le\ & C \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h 2^{-h(\frac{L}{2}-q_\omega-3)} \omega(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega_h^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ & + \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{3} \omega(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\omega(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega_h^{0,0}} F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ \le\ & C \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h 2^{-h(\frac{L}{2}-q_\omega-3)} (h^2 2^h)^{\frac{2}{p}-1} \omega(\Omega)^{\frac{2}{p}-1} \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ & + \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{3} (h^2 2^h)^{\frac{2}{p}-1} \omega(\Omega)^{\frac{2}{p}-1} \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')} \\ \le\ & C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}, \end{aligned}$$

where we choose L large enough, and the estimate of I_1 is finished. Next we move our attention to the estimate of I_4.

(**iv**) To estimate

$$I_4 = \frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_h \sum_{Q' \in \mathcal{D}_h^{0,0}} \sum_{v' \ge 1} \sum_{Q \in A_{0,0,v'}(Q')} r(Q,Q')P(Q,Q') \cdot F_{Q'} \frac{|Q'|^2}{\omega(Q')}, \tag{3.17}$$

similar to what we did in (**i**), we divide $\{Q \in A_{0,0,v'}(Q')\}$ into 6 cases for each $v' \ge 1$.

Case 1: $|I'| \ge |I|$, $|J'| \le |J|$, and $|R'| \le |R|$.

Note that in this case, $3(I' \times J' \times R') \cap 3(I \times J \times 2^{v'} R) \ne \emptyset$. Since

$$\frac{|I|}{|3I'|} |3Q'| = |I| \times |3J'| \times |3R'| \le |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}} |3Q'|,$$

then $|I'| \sim 2^{h-1+n}|I|$ for some $n \ge 0$, and for each n, the number of such I's is no more than $5 \cdot 2^n$. As for J, $|J| \sim 2^m |J'|$. And for each m, the number of such J's is no more than 5. Note that $2^{v'-1}|R| < d(R,R') \le 2^{v'}|R|$, which yields that $3R' \cap 3 \cdot 2^{v'} R \ne \emptyset$. Moreover, from $|R| \sim 2^{-(h-1+n)} 2^m |R'|$ and $|R| \ge |R'|$, we have $m > h-1+n$ and for each fixed v', n and m, the number of such R's is less than $5 \cdot 2^{v'}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case }1} r(Q,Q')P(Q,Q')\\
\leq\ & C\sum_{\text{Case }1} C\sum_{\text{Case }1}\left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2}\left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\\
&\frac{(|J|)^{M+1}}{\left(|J|+\frac{2^{v'-1}|R|}{|I|}\right)^{M+1}}\frac{|R|}{|I||J|+2^{v'-1}|R|}\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0}2^{-(h-1+n)(L-q_\omega-2)}\cdot 2^{(h-1+n-m)(-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{n}\cdot 2^{-v'(M+2)}\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0}2^{-hL}\cdot 2^{-m(L-2q_\omega-4)}\cdot 2^{-n(L-1)}\cdot 2^{L}\cdot 2^{-v'(M+2)}\\
\leq\ & C2^{-hL}2^{-v'(M+2)}.
\end{aligned}$$

Case 2: $|I'|\geq|I|$, $|J'|\leq|J|$, and $|R'|\geq|R|$.

Note that in this case, $3(I'\times J'\times 2^{v'}R')\cap 3(I\times J\times R)\neq\emptyset$. Since

$$\frac{|I||R|}{|3I'||3R'|}|3Q'|=|I|\times|3J'|\times|R|\leq|3Q'\cap\Omega^{0,0}|<\frac{1}{2^{h-1}}|3Q'|,$$

then $|I'||R'|\sim 2^{h-1+n}|I||R|$. As for J, $|J|\sim 2^m|J'|$. So for each m, the number of such J's is no more than 5. Noting that $|R|=|I|\times|J|$ and $|R'|=|I'|\times|J'|$, we have $|I'||I'||J'|\sim 2^{h-1+n}|I||I||J|$, which yields that $|I'|^2\sim 2^{h-1+n+m}|I|^2$, that is, $|I'|\sim 2^{(h-1+n+m)/2}|I|$. Hence for each n and m, the number of such I's is less than $5\cdot 2^{(h+m+n)/2}$. Also we can obtain that $|R'|\sim 2^{(h-1+n-m)/2}|R|$. Since $|R'|\geq|R|$, we have $m\leq h-1+n$. Moreover, note that $2^{v'-1}|R'|<d(R,R')\leq 2^{v'}|R'|$, which yields that $3\cdot 2^{v'}R'\cap 3R\neq\emptyset$. For each fixed v', n and m, the number of such R's is less than $5\cdot 2^{(h+n-m)/2}2^{v'}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case }2} r(Q,Q')P(Q,Q')\\
\leq\ & C\sum_{\text{Case }1}\left(\frac{|I|}{|I'|}\right)^{L-q_\omega-2}\left(\frac{|J'|}{|J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\\
&\cdot\frac{(|J|)^{M+1}}{\left(|J|+\frac{2^{v'-1}|R'|}{|I|}\right)^{M+1}}\frac{|R'|}{|I||J|+2^{v'-1}|R'|}\\
\leq\ & C\sum_{n\geq 0}\sum_{m\geq 0}2^{-\frac{h-1+n}{2}(L-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{-\frac{h-1+n-m}{2}(-q_\omega-2)}\\
&\cdot 2^{h+n}2^{v'}2^{-v'(M+2)}\\
=\ & C\sum_{n\geq 0}\sum_{m\geq 0}2^{-h(\frac{L}{2}-q_\omega-1)}\cdot 2^{-m(\frac{3L}{2}-q_\omega-4)}\cdot 2^{-n(\frac{L}{2}-q_\omega-1)}\cdot 2^{\frac{L}{2}-q}\\
\leq\ & C2^{-h(\frac{L}{2}-q_\omega-1)}2^{-v'(M+1)}.
\end{aligned}$$

Case 3: $|I'|\leq|I|$, $|J'|\geq|J|$, and $|R'|\leq|R|$.

This can be handled similarly as Case 1, and we have

$$\sum_{\text{Case }3} r(Q,Q')P(Q,Q')\leq C2^{-hL}2^{-v'(M+2)}.$$

Case 4: $|I'|\leq|I|$, $|J'|\geq|J|$, and $|R'|\geq|R|$.

This can be handled similarly as Case 2, and we have

$$\sum_{\text{Case 4}} r(Q,Q')P(Q,Q') \le C2^{-h(\frac{L}{2}-q_\omega-1)}2^{-\nu'(M+1)}.$$

Case 5: $|I'| \ge |I|$, $|J'| \ge |J|$, and thus $|R'| \ge |R|$.

Since

$$|I| \times |J| \times |R| \le |3Q' \cap \Omega^{0,0}| < \frac{1}{2^{h-1}}|3Q'|,$$

then $|Q'| \sim 2^{h-1+n}|Q|$. And for each ν' and n, the number of such Q's is less than $2^{\nu'}(2^n)^3$. More precisely, we have $(|I'||J'|)^2 \sim 2^{h-1+n}(|I||J|)^2$. Thus,

$$\begin{aligned}
&\sum_{\text{Case 5}} r(Q,Q')P(Q,Q')\\
\le\ & C\sum_{\text{Case 5}}\left(\frac{|I||J|}{|I'||J'|}\right)^{L-q_\omega-2}\left(\frac{|R|}{|R'|}\right)^{-q_\omega-2}\\
&\cdot\frac{(|J'|)^{M+1}}{(|J'|+\frac{2^{\nu'-1}|R'|}{|I'|})^{M+1}}\frac{|R'|}{|I'||J'|+2^{\nu'-1}|R'|}\\
\le\ & C\sum_{n\ge 0} 2^{\frac{-(h-1+n)}{2(L-q_\omega-2)}}\cdot 2^{\frac{-(h-1+n)}{2(-q_\omega-2)}}\cdot 2^{\nu'}2^{3n}2^{-\nu'(M+2)}\\
\le\ & C2^{-h(\frac{L}{2}-q_\omega-2)}\cdot 2^{-n(\frac{L}{2}-q_\omega-5)}\cdot 2^{\frac{L}{2}-q_\omega-2}\cdot 2^{-\nu'(M+1)}\\
\le\ & C2^{-h\frac{L}{2-q_\omega-2}}2^{-\nu'(M+1)}.
\end{aligned}$$

Case 6: $|I'| \le |I|$, $|J'| \le |J|$, and thus $|R'| \le |R|$.

Since

$$|I'| \times |J'| \times |R'| \le |3Q' \cap \Omega^{0,0,}| < \frac{1}{2^{h-1}}|3Q'|,$$

then we can see that in this case, h must be less than 3. And from $|I'| \le |I|$, we have $|I| \sim 2^n|I'|$ and for each n, the number of such I is less than 5. Similarly, from $|J'| \le |J|$, we have $|J| \sim 2^m|J'|$ and for each m, the number of such J is less than 5. Hence we have $|R| \sim 2^{n+m}|R'|$, and for each ν', n, m, the number of such R is less than $5 \cdot 2^{\nu'}$. Thus,

$$\begin{aligned}
&\sum_{\text{Case 6}} r(Q,Q')P(Q,Q')\\
\le\ & C\sum_{\text{Case 6}}\left(\frac{|I'||J'|}{|I||J|}\right)^{L-q_\omega-2}\left(\frac{|R'|}{|R|}\right)^{-q_\omega-2}\\
&\cdot\frac{(|J|)^{M+1}}{(|J|+\frac{2^{\nu'-1}|R|}{|I|})^{M+1}}\frac{|R|}{|I||J|+2^{\nu'-1}|R|}\\
\le\ & C\sum_{n\ge 0}\sum_{m\ge 0} 2^{-n(L-q_\omega-2)}\cdot 2^{-m(L-q_\omega-2)}\cdot 2^{(-n-m)(-q_\omega-2)}\cdot 2^{\nu'}2^{-\nu'(M+2)}\\
\le\ & C2^{-\nu'(M+1)}.
\end{aligned}$$

Thus, combining the above 6 cases, by choosing L and M large enough, I_4 in (3.17) becomes

$$I_4 \le C\sup_{\Omega^*}\frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}}\sum_{Q'\subseteq\Omega^*} F_{Q'}\frac{|Q'|^2}{\omega(Q')}.$$

Using the same techniques, we are able to control the other 8 integrates for I, therefore give the estimate for I, that is,

$$I \leq C \sup_{\Omega^*} \frac{1}{\omega(\Omega^*)^{\frac{2}{p}-1}} \sum_{Q' \subseteq \Omega^*} F_{Q'} \frac{|Q'|^2}{\omega(Q')}.$$

Similarly, II, III, IV in (3.13) can be bounded by the right hand side of the above inequality. Hence the proof of the theorem is complete.

Finally we show that $CMO^p_{\mathcal{Z}}(\omega)$ is well defined as a corollary of sup-inf comparison principle of second kind, Theorem 1.1.

Corollary 3.1 *The definition of $CMO^p_{\mathcal{Z}}(\omega)$ in Definition 1.3 is independent of the choice of $\psi_{j,k}$, therefore it is well-defined.*

4 Proof of Theorem 1.2

In this section, we prove the $(H^p_{\mathcal{Z}}(\omega))^* - CMO^p_{\mathcal{Z}}(\omega)$ duality theorem, i.e., Theorem 1.2, and we need to introduce the following two sequence spaces.

Definition 4.1 **($s^p_{\mathcal{Z}}(\omega)$ and $c^p_{\mathcal{Z}}(\omega)$)** *Let $\omega \in A_\infty(\mathcal{Z})$, $j,k \in \mathbb{Z}$, and I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$, and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N. The sequence $s = \{s_{I\times J\times R}\}$ is said to be in the sequence space $s^p_{\mathcal{Z}}(\omega)$ if*

$$\|s\|_{s^p_{\mathcal{Z}}(\omega)} = \|\left\{\sum_{j,k} \sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}}\|_{L^p_\omega} < \infty, \tag{4.18}$$

and the sequence $t = \{t_{I\times J\times R}\}$ is said to be in the sequence space $c^p_{\mathcal{Z}}(\omega)$ if

$$\|t\|_{c^p_{\mathcal{Z}}(\omega)} = \sup_{\Omega} \left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I\times J\times R\subseteq\Omega} |t_{I\times J\times R}|^2 \frac{|I\times J\times R|}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}} < \infty, \tag{4.19}$$

for all open sets Ω in $\mathbb{R}^3$ with $\omega(\Omega) < \infty$, and $I \times J \times R$ run over all dyadic cubes with side-lengths defined above.

We derive the following duality theorem for these sequence spaces.

Theorem 4.1 **($s^p_{\mathcal{Z}}(\omega) - c^p_{\mathcal{Z}}(\omega)$)** *$(s^p_{\mathcal{Z}}(\omega))^* = c^p_{\mathcal{Z}}(\omega)$, precisely, let $\omega \in A_\infty(\mathcal{Z})$ and $0 < p \leq 1$, the map which maps $s = \{s_{I\times J\times R}\}$ to $< s, t >= \sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R}$ defines a continuous linear functional on $s^p_{\mathcal{Z}}(\omega)$ with operator norm $\|t\|_{(s^p_{\mathcal{Z}}(\omega))^*} \approx \|t\|_{c^p_{\mathcal{Z}}(\omega)}$, and every $\ell \in (s^p_{\mathcal{Z}}(\omega))^*$ is of this form for some $t \in c^p_{\mathcal{Z}}(\omega)$.*

Proof. [Proof of Theorem 4.1] First we show that $c^p_{\mathcal{Z}}(\omega) \subseteq (s^p_{\mathcal{Z}}(\omega))^*$. Suppose $t = \{t_{I\times J\times R}\} \in c^p_{\mathcal{Z}}(\omega)$ and $s = \{s_{I\times J\times R}\} \in s^p_{\mathcal{Z}}(\omega)$, set

$$h(x,y,z) = \left\{\sum_{j,k} \sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}},$$

which means $\|s\|_{s^p_{\mathcal{Z}}(\omega)} = \|h\|_{L^p_\omega}$. Then we write $\Omega_i = \{(x,y,z) : h(x,y,z) > 2^i\}$, and

$$B_i = \{I\times J\times R : \omega(I\times J\times R\cap\Omega_i) > \frac{1}{2}\omega(I\times J\times R), \omega(I\times J\times R\cap\Omega_{i+1}) \leq \frac{1}{2}\omega(I\times J\times R)\}.$$

Thus,

$$\sum_{j,k}\sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R}=\sum_{i}\sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}.$$

Note that $0<p\leq 1$, by Cauchy-Schwartz's inequality,

$$\begin{aligned}
&|\sum_{i}\sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}|\\
\leq\ & \sum_i\left(\sum_{I\times J\times R\in B_i}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac12}\left(\sum_{I\times J\times R\in B_i}|\bar{t}_{I\times J\times R}|^2\frac{|I\times J\times R|}{\omega(I\times J\times R)}\right)^{\frac12}\\
\leq\ & \left\{\sum_i\left(\sum_{I\times J\times R\in B_i}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac p2}\left(\sum_{I\times J\times R\in B_i}|t_{I\times J\times R}|^2\frac{|I\times J\times R|}{\omega(I\times J\times R)}\right)^{\frac p2}\right\}^{\frac1p}\\
\leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)}\left\{\sum_i\omega(\Omega_i)^{1-\frac p2}\left(\sum_{I\times J\times R\in B_i}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac p2}\right\}^{\frac1p}.
\end{aligned}$$

Where the last inequality follows from the fact that if $I\times J\times R\in B_i$, then there exists $0<\theta<1$ such that

$$I\times J\times R\subseteq\{(x,y,z):\mathcal{M}_{\mathcal{Z}}(\chi_{\Omega_i})(x,y,z)>\theta\}\triangleq\widetilde{\Omega}_i,$$

together with $\omega(\widetilde{\Omega}_i)\leq C\omega(\Omega_i)$, imply

$$\left(\sum_{I\times J\times R\in B_i}|t_{I\times J\times R}|^2\frac{|I\times J\times R|}{\omega(I\times J\times R)}\right)^{\frac12}\leq C\|t\|_{c^p_{\mathcal{Z}}(\omega)}\omega(\Omega_i)^{\frac1p-\frac12}.$$

We claim for now

$$\sum_{I\times J\times R\in B_i}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\leq C2^{2i}\omega(\Omega_i). \tag{4.20}$$

Assume this claim for the moment, then

$$\begin{aligned}
&|\sum_{i}\sum_{I\times J\times R\in B_i} s_{I\times J\times R}\bar{t}_{I\times J\times R}|\\
\leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)}(\sum_i 2^{ip}\omega(\Omega_i))^{\frac1p}\\
\leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)}\|h\|_{L^p_\omega}\\
\leq\ & C\|t\|_{c^p_{\mathcal{Z}}(\omega)}\|s\|_{s^p_{\mathcal{Z}}(\omega)},
\end{aligned}$$

therefore $c^p_{\mathcal{Z}}(\omega)\subseteq(s^p_{\mathcal{Z}}(\omega))^*$. To show the claim (4.20), it is sufficient to prove

$$\sum_{I\times J\times R\in B_i}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\leq C\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}h^2(x,y,z)\omega dxdydz$$

because

$$\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}h^2(x,y,z)\omega dxdydz\leq 2^{2(i+1)}\omega(\widetilde{\Omega}_i)\leq C2^{2i}\omega(\Omega_i).$$

However,

$$\begin{aligned}
&\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} h^2(x,y,z)\omega dxdydz \\
= &\int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} \sum_{I\times J\times R} |s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\omega dxdydz \\
\geq &\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega((\widetilde{\Omega}_i\backslash\Omega_{i+1})\cap(I\times J\times R))}{\omega(I\times J\times R)}\frac{\omega(I\times J\times R)}{|I\times J\times R|} \\
\geq &\frac{1}{2}\sum_{I\times J\times R\in B_i} |s_{I\times J\times R}|^2 \frac{\omega(I\times J\times R)}{|I\times J\times R|},
\end{aligned}$$

since for $I\times J\times R\in B_i$,

$$\omega((\widetilde{\Omega}_i\cap I\times J\times R)) > \frac{1}{2}\omega(I\times J\times R),$$

and

$$\omega((\Omega_{i+1}\cap I\times J\times R)) \leq \frac{1}{2}\omega(I\times J\times R).$$

Then $I\times J\times R\in\widetilde{\Omega}_i$, hence

$$\omega((\widetilde{\Omega}_i\backslash\Omega_{i+1})\cap(I\times J\times R)) > \frac{1}{2}\omega(I\times J\times R).$$

The claim is verified.

Next we shall prove that $(s^p_{\mathcal{Z}}(\omega))^*\subseteq c^p_{\mathcal{Z}}(\omega)$. Let $\ell\in(s^p_{\mathcal{Z}}(\omega))^*$, then there exists some $t=\{t_{I\times J\times R}\}$ such that $\forall s=\{s_{I\times J\times R}\}\in s^p_{\mathcal{Z}}(\omega)$,

$$\ell(s)=\sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R}.$$

For an open set Ω in $\mathbb{R}^3$ with $\omega(\Omega)<\infty$, define

$$\|s\|_{s^p_{\mathcal{Z},\Omega}(\omega)}=\left\{\int_\Omega\left(\sum_{I\times J\times R\subseteq\Omega}|s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right)^{\frac{p}{2}}\omega dxdydz\right\}^{\frac{1}{p}},$$

and

$$\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}=\left(\sum_{I\times J\times R\subseteq\Omega}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac{1}{2}}.$$

Then, by Hölder's inequality,

$$\begin{aligned}
&\|s\|_{s^p_{\mathcal{Z},\Omega}(\omega)} \\
= &\left\{\int_\Omega\left(\sum_{I\times J\times R\subseteq\Omega}|s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right)^{\frac{p}{2}}\omega dxdydz\right\}^{\frac{1}{p}} \\
\leq &\,\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\left\{\int_\Omega\sum_{I\times J\times R\subseteq\Omega}|s_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\omega dxdydz\right\}^{\frac{1}{2}} \\
= &\,\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{I\times J\times R\subseteq\Omega}|s_{I\times J\times R}|^2\frac{\omega(I\times J\times R)}{|I\times J\times R|}\right)^{\frac{1}{2}} \\
= &\,\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}.
\end{aligned}$$

Thus, we compute

$$
\begin{aligned}
&\left\{\frac{1}{\omega(\Omega)^{\frac{2}{p}-1}}\sum_{j,k}\sum_{I\times J\times R\subseteq\Omega}|t_{I\times J\times R}|^2\frac{|I\times J\times R|}{\omega(I\times J\times R)}\right\}^{\frac{1}{2}}\\
=&\ \frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}}\sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\le 1}\Big|\sum_{I\times J\times R\subseteq\Omega}s_{I\times J\times R}\bar{t}_{I\times J\times R}\Big|\\
\le&\ \frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}}\sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\le 1}\|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}\|s_{I\times J\times R}\|_{s^p_{\mathcal{Z},\Omega}(\omega)}\\
=&\ \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}\sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\le 1}\frac{1}{\omega(\Omega)^{\frac{1}{p}-\frac{1}{2}}}\|s_{I\times J\times R}\|_{s^p_{\mathcal{Z},\Omega}(\omega)}\\
=&\ \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}\sup_{\|s\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\le 1}\|s_{I\times J\times R}\|_{\ell^2_{\mathcal{Z},\Omega}(\omega)}\\
\le&\ \|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}
\end{aligned}
$$

for all Ω. Therefore, by taking the supremum, $t\in c^p_{\mathcal{Z}}(\omega)$ and $\|t\|_{c^p_{\mathcal{Z}}(\omega)}\le\|t\|_{(s^p_{\mathcal{Z}}(\omega))^*}$, which implies $(s^p_{\mathcal{Z}}(\omega))^*\subseteq c^p_{\mathcal{Z}}(\omega)$, and thus the proof of Theorem 4.1 is complete.

In order to pass the duality theory from sequence spaces to $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$, we need the following lemmas.

Lemma 4.1 *Let $j,k,j',k'\in\mathbb{Z}$ and I,J,R,I',J',R' be dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I)=2^{-j-N}$, $\ell(J)=2^{-k-N}$, $\ell(R)=2^{-j-k-2N}$, $\ell(I')=2^{-j'-N}$, $\ell(J')=2^{-k'-N}$, and $\ell(R')=2^{-j'-k'-2N}$ for a fixed large positive integer N, $\{a_{I',J',R'}\}$ be any given sequence, $x_{I'}$, $y_{J'}$, and $z_{R'}$ be any points in I', J' and R'. Then for any $u,u^*\in I$, $v,v^*\in J$, $w,w^*\in R$ we have*

$$
\begin{aligned}
&\sum_{I',J',R'}\frac{2^{-(j\wedge j')M_1}2^{-(k\wedge k')M_2}|I'||J'||R'|}{(2^{-(j\wedge j')}+|u-x_{I'}|)^{1+M_1}2^{-j^*}(2^{-(k\wedge k')}+|v-y_{J'}|+\frac{|w-z_{R'}|}{2^{-j^*}})^{2+M_2}}|a_{I',J',R'}|\\
\le&\ C2^{4N(\frac{1}{r}-1)}2^{\tau}\left\{\mathcal{M}_{\mathcal{Z}}\Big(\sum_{I',J',R'}|a_{I',J',R'}|^r\chi_{I'}\chi_{J'}\chi_{R'}\Big)(u^*,v^*,w^*)\right\}^{1/r},
\end{aligned}
$$

where $j^*=j$ if $k<k'$, and $j^*=j'$ if $k\ge k'$. $\mathcal{M}_{\mathcal{Z}}$ is the maximal operator associated with Zygmund dilations defined in (1.1), and $\max\{\frac{2}{1+M_1},\frac{2}{2+M_2}\}<r\le 1$. The summation is taken for all I',J',R' with the fixed side-length defined above and τ is defined as follows,

$$
\tau=\begin{cases}(\frac{2}{r}-2)(j'+k'-j-k) & \text{if } j<j' \text{ and } k<k',\\ (\frac{2}{r}-1)(j'-j) & \text{if } j<j' \text{ and } k\ge k',\\ j-j'+(\frac{2}{r}-2)(k'-k) & \text{if } j\ge j' \text{ and } k<k',\\ 0 & \text{if } j\ge j' \text{ and } k\ge k'.\end{cases}
$$

The detailed proof of Lemma 4.1 can be found in [13].

Lemma 4.2 *Let $\omega\in A_\infty(\mathcal{Z})$, $j,k\in\mathbb{Z}$, $\psi_{j,k}$ be the same as in (1.3) and I, J, and R are dyadic intervals in $\mathbb{R}$ with interval-length $\ell(I)=2^{-j-N}$, $\ell(J)=2^{-k-N}$, and $\ell(R)=2^{-j-k-2N}$ for a fixed large positive integer N. For any $f\in(\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3))'$, we define a mapping S by*

$$
S(f)=\left\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}|R|^{\frac{1}{2}}\psi_{j,k}*f(x_I,y_J,z_R)\right\}.
$$

For any sequence $s = \{s_{I\times J\times R}\}$, *we define a mapping* T *by*

$$T(s) = \sum_{j,k}\sum_{I\times J\times R} s_{I\times J\times R}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}|R|^{\frac{1}{2}}\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R),$$

where $\widetilde{\psi}_{j,k}$ *are same as in the discrete Calderón's identity in Theorem 2.1. Then,* S *is bounded from* $H^p_{\mathcal{Z}}(\omega)$ *to* $s^p_{\mathcal{Z}}(\omega)$*, and from* $CMO^p_{\mathcal{Z}}(\omega)$ *to* $c^p_{\mathcal{Z}}(\omega)$*. While* T *is bounded from* $s^p_{\mathcal{Z}}(\omega)$ *to* $H^p_{\mathcal{Z}}(\omega)$*, and from* $c^p_{\mathcal{Z}}(\omega)$ *to* $CMO^p_{\mathcal{Z}}(\omega)$*. Moreover,* $T\circ S$ *is the identity map on both* $H^p_{\mathcal{Z}}(\omega)$ *and* $CMO^p_{\mathcal{Z}}(\omega)$*.*

Proof. [Proof of Lemma 4.2] If $f \in H^p_{\mathcal{Z}}(\omega)$, then by the definition of $H^p_{\mathcal{Z}}(\omega)$ in Definition 1.2 together with discrete Littlewood-Paley-Stein square function (2.8),

$$\begin{aligned}
&\|h(f)\|_{s^p_{\mathcal{Z}}(\omega)}\\
= \;& \left\|\left\{\sum_{j,k}\sum_{I\times J\times R}|h(f)_{I\times J\times R}|^2|I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}}\right\|_{L^p_\omega}\\
= \;& \left\|\left\{\sum_{j,k}\sum_{I,J,R}|(\psi_{j,k}*f)(x_I,y_J,z_R)|^2\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}}\right\|_{L^p_\omega}\\
\le \;& C\|f\|_{H^p_{\mathcal{Z}}(\omega)}.
\end{aligned}$$

Similarly, by the aid of sup-inf comparison principle of second kind in Theorem 1.1, we can show S is bounded from $CMO^p_{\mathcal{Z}}(\omega)$ to $c^p_{\mathcal{Z}}(\omega)$.

To show T is bounded from $s^p_{\mathcal{Z}}(\omega)$ to $H^p_{\mathcal{Z}}(\omega)$, by using almost orthogonality estimate in Lemma 3.1, we get

$$\begin{aligned}
&\sum_{j',k'}\sum_{I',J',R'}|\psi_{j',k'}*T(s)(x_{I'},y_{J'},z_{R'})|^2\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
= \;& \sum_{j',k'}\sum_{I',J',R'}|\psi_{j',k'}*(\sum_{j,k}\sum_{I\times J\times R}s_{I\times J\times R}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}|R|^{\frac{1}{2}}\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R))\\
& (x_{I'},y_{J'},z_{R'})|^2\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
\le \;& \sum_{j',k'}\sum_{I',J',R'}\sum_{j,k}\sum_{I,J,R}2^{-|j-j'|L}2^{-|k-k'|L}|I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}}|R|^{-\frac{1}{2}}\frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'}+|x_{I'}-x_I|)^{M+1}}\\
& \times\frac{2^{-(k\wedge k')M}}{2^{-j^*}(2^{-k\wedge k'}+|y_{J'}-y_J|+2^{j^*}|z_{R'}-z_R|)^{M+2}}|s_{I\times J\times R}|\chi_{I'}(x)\chi_{J'}(y)\chi_{R'}(z)\\
\le \;& C\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}2^{\tau}\left\{\mathcal{M}_{\mathcal{Z}}(\sum_{I,J,R}|I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}}|R|^{-\frac{1}{2}}|s_{I\times J\times R}|\chi_I\chi_J\chi_R)^r\right\}^{\frac{1}{r}},
\end{aligned}$$

in which we applied Lemma 4.1 to get the last inequality, and use the weighted inequalities for vector-valued maximal operator associated with Zygmund dilations (See [15]), we will be able to derive that

$$\begin{aligned}
&\|T(s)\|_{H^p_{\mathcal{Z}}(\omega)} \\
= \;& \|\left\{\sum_{j',k'}\sum_{I',J',R'} |(\psi_{j',k'} * T(s))(x'_I, y'_J, z'_R)|^2 \chi'_I(x)\chi'_J(y)\chi'_R(z)\right\}^{\frac{1}{2}} \|_{L^p_\omega} \\
\le \;& C\|\sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{\tau} \left\{\mathcal{M}_{\mathcal{Z}}(\sum_{I,J,R} |I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}}|R|^{-\frac{1}{2}}|s_{I\times J\times R}|\chi_I\chi_J\chi_R)^r\right\}^{\frac{1}{r}} \|_{L^p_\omega} \\
\le \;& C\|\left\{\sum_{j,k}\sum_{I\times J\times R} |s_{I\times J\times R}|^2 |I|^{-1}|J|^{-1}|R|^{-1}\chi_I(x)\chi_J(y)\chi_R(z)\right\}^{\frac{1}{2}} \|_{L^p_\omega} \\
= \;& C\|s\|_{s^p_{\mathcal{Z}}(\omega)}.
\end{aligned}$$

Similarly, we can prove T is bounded from $c^p_{\mathcal{Z}}(\omega)$ to $CMO^p_{\mathcal{Z}}(\omega)$, and it is evident that $T \circ S$ is the identity map on $H^p_{\mathcal{Z}}(\omega)$ and $CMO^p_{\mathcal{Z}}(\omega)$.

Combining Theorem 4.1 and Lemma 4.2, we are able to prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] First, if $g \in CMO^p_{\mathcal{Z}}(\omega)$, the map ℓ_g is given by $\ell_g(f) =< f, g >$ for $f \in \mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$

$$\begin{aligned}
&\ell_g(f) = | < f, g > | \\
= \;& | < \sum_{j,k}\sum_{I,J,R} |I||J||R|\widetilde{\psi}_{j,k}(x, y, z, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R), g > | \\
= \;& | < S(f), S(g) > | \\
\le \;& \|S(f)\|_{s^p_{\mathcal{Z}}(\omega)}\|S(g)\|_{c^p_{\mathcal{Z}}(\omega)} \\
\le \;& C\|f\|_{H^p_{\mathcal{Z}}(\omega)}\|g\|_{CMO^p_{\mathcal{Z}}(\omega)}.
\end{aligned}$$

Since $S_{\mathcal{Z}}(\mathbb{R}^3)$ is dense in $H^p_{\mathcal{Z}}(\omega)$ (See [15]), Hahn-Banach Theorem implies that the map $\ell_g =< f, g >$ can be extended to a continuous linear functional on $H^p_{\mathcal{Z}}(\omega)$, and $\|\ell_g\| \le C\|g\|_{CMO^p_{\mathcal{Z}}(\omega)}$.

Conversely, for every $\ell \in (H^p_{\mathcal{Z}}(\omega))^*$, consider $\ell_T = \ell \circ T$ defined on $s^p_{\mathcal{Z}}(\omega)$, and for every $s \in s^p_{\mathcal{Z}}(\omega)$,

$$|\ell_T(s)| = |\ell(T(s))| \le \|\ell\|\|T(s)\|_{H^p_{\mathcal{Z}}(\omega)} \le C\|\ell\|\|s\|_{s^p_{\mathcal{Z}}(\omega)},$$

which implies $\ell_T \in (s^p_{\mathcal{Z}}(\omega))^*$, then by Theorem 4.1, there exists $t = \{t_{I\times J\times R}\} \in c^p_{\mathcal{Z}}(\omega)$ such that

$$\ell_T(s) =< s, t >= \sum_{I\times J\times R} s_{I\times J\times R}\bar{t}_{I\times J\times R},$$

for all $s \in s^p_{\mathcal{Z}}(\omega)$, and

$$\|t\|_{c^p_{\mathcal{Z}}(\omega)} \approx \|\ell_T\| \le C\|\ell\|.$$

From Lemma 4.2, $T \circ S$ is the identity map on $H^p_{\mathcal{Z}}(\omega)$, thus $\ell = \ell \circ T \circ S = \ell_T \circ S$, and

$$\ell(f) = \ell_T(S(f)) =< S(f), t >=< f, g >,$$

in which $g = T(t)$. This shows $\ell = \ell_g$ for $g \in CMO^p_{\mathcal{Z}}(\omega)$, and $\|g\|_{CMO^p_{\mathcal{Z}}(\omega)} \le C\|t\|_{c^p_{\mathcal{Z}}(\omega)} \le C\|\ell\|$, which completes the proof of Theorem 1.2.

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