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The Sturm-Liouville Hierarchy of Evolution Equations II

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Abstract

In a previous paper [15] we introduced the Sturm-Liouville (SL) hierarchy of evolution equations. This hierarchy includes the Korteveg-de Vries (K-dV) and the Camassa-Holm (CH) hierarchies. We also defined and discussed in detail the algebro-geometric solutions of the SL-hierarchy. In this paper, we broaden the class of algebro-geometric solutions in a substantial way. Namely, we define and discuss solutions of the SL-hierarchy lying in an isospectral class of the Sturm-Liouville problem $-(p\varphi')' + q\varphi = \lambda y\varphi$, which is determined by data related to a Riemann surface of "infinite genus".

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1 Introduction

In a previous paper [15] we introduced the "Sturm-Liouville hierarchy" (briefly SL-hierarchy) of nonlinear evolution equations. This hierarchy includes the Korteweg-de Vries and the Camassa-Holm hierarchies in a natural sense. We also discussed certain special solutions -the so called algebro-geometric solutions- of our SL-hierarchy. These solutions are obtained by making reference to a compact hyperelliptic Riemann surface.

The point of this paper is to extend the notion of algebro-geometric solution of the SL-hierarchy to certain situations in which the underlying Riemann surface has infinitely many gaps which cluster at infinity. The approach we take is to depart from the results of [15], in which the SL-hierarchy was defined and solved in the case of a nonsingular (finite genus and compact) Riemann surface. (Actually, we will refine here the definition of *Sturm-Liouville hierarchy* of evolution equations, in the sense that each equation in the hierarchy consists of a system made of two evolution equations.)

The approach we take is that of considering the limits of some appropriate quantities which are determined using the structure of a sequence of nonsingular Riemann surface. These quantities are basically differential equations for *poles* as will be explained. We show that a limiting family of infinitely many differential equations for the poles exists and admits solutions. These solutions are then shown to give rise to solutions of the SL-hierarchy.

In a bit more detail, the SL-hierarchy is based on the Sturm-Liouville spectral problem

$$-(p\varphi')' + q\varphi = \lambda y\varphi.$$

The starting point in its construction is a so-called zero-curvature equation

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A, B] = 0, \tag{1.1}$$

whose coefficient matrices A, B will be defined below. Here $[A, B] = AB - BA$ is the commutator of A and B. We observe that the coefficients of B are determined by three quantities T, U, V, which are functions of $t, x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and p, q, y. A *solution* of the SL-hierarchy is in the first place a triple T, U, V such that the relation (1.1) holds. One uses the structure of the zero-curvature equation (1.1) to determine differential relations for p, q, y.

In [15], we introduced a Riemann surface into the picture, via a condition which is adjoined to the relation (1.1). As stated above, this surface is compact and has finite genus. In the present paper, we allow the Riemann surface to have infinitely many singularities, and view it as an appropriate limit of finite genus surfaces. In this way, we obtain the limiting family of infinitely many differential equations for the pole motion. We will write down explicit formulas for the solutions of the Sturm-Liouville hierarchy of evolution equations. These solutions are expressed via the *pole motion* and give rise to a curve in an *isospectral set* of Sturm-Liouville potentials $a = (p, q, y)$.

There is a good amount of literature concerning hierarchies of evolution equations. We mention [4, 18, 16, 22, 24] for the K-dV equation, [1, 8, 23] for the Camassa-Holm hierarchy, and [9] and [20] for other types of evolution equations. The present paper essentially extends the known results in two ways: first of all we introduce hierarchies of evolution equations which generalize both the K-dV and the Camassa-Holm hierarchies. Our hierarchies contain two coupled nonlinear evolution equations, one for the function $q(t, x)$ and the other for the function $y(t, x)$. Moreover, the presence of the function $p(t, x)$ allows us to introduce some hierarchies which can be used to study the so called “small dispersion limits” of certain PDEs (see [15] for examples on this topic). Second, the family of initial conditions (or, better, the family of potentials) for which we will be able to solve the hierarchy is larger than those considered in most of the literature, and is strictly larger than those used in approaches which are similar to ours: it includes periodic potentials, algebro-geometric potentials, the potentials introduced by Levitan [16], and other families of reflectionless potentials. However, we are not yet able to treat the cases when the potential (which gives rise to the initial condition) has spectrum with a Cantor part; indeed we require that the so-called spectral gaps cluster at ∞ and satisfy some decay conditions in order for certain infinite products to converge. We do believe that it is possible to enlarge the family of initial conditions to other potentials as well by using methods similar to those introduced in the present paper.

The paper is organized as follows: in Section 2 we review some results concerning the so-called algebro-geometric Sturm-Liouville potentials. We will put in evidence the dynamical approach to the study of these potentials, and will state some useful results concerning the link between the concept of exponential dichotomy and the spectrum of the Sturm-Liouville operator. The material discussed in Section 2 can be found in [6, 12]. In Section 3 we briefly introduce the SL-hierarchy in the case of algebro-geometric Sturm-Liouville potentials. We refer the reader to [15] for a detailed discussion of the hierarchy and of its solutions.

Section 4 contains the results concerning the selection of the initial data for which we will solve the SL-hierarchy. We will also use some facts proved in [15], and discuss a characterization of the algebro-geometric Sturm-Liouville potentials by means of a so-called *stationary zero-curvature equation.*

In the final Section 5 we introduce and solve the SL-hierarchy, when the initial data are chosen in accordance with the results in Section 4. We will give explicit formulas for the solutions of the hierarchy in terms of the zeros of the Green's functions associated to a time-dependent family of Sturm-Liouville operators. We will not discuss here the nature of the motion of these poles (the so-called pole motion) in detail, but we plan to discuss it more deeply in a future publication.

We finish this Introduction by introducing some notation which we will use throughout all the paper. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, we will use both the symbols f_t, f_x and $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$ to denote the partial derivatives of f. The symbol D will denote the operator of differentiation with respect to $x \in \mathbb{R}$, and $\mathbb{M}(n, \mathbb{C})$ will denote the set of $n \times n$ matrices with complex coefficients.

2 Preliminaries

This section contains basic concepts concerning the spectral theory of the Sturm-Liouville operator. The facts discussed here can be found in ([6, 12, 14], see also [13]).

Let us denote by $\mathcal{E}_2 = \{b = (p, \mathcal{M}) : \mathbb{R} \to \mathbb{R}^2 \mid b \in C^1(\mathbb{R})$, b is uniformly continuous and $p(x) \geq \delta$, $\delta \leq \mathcal{M}(x) \leq \Delta$ for every $x \in \mathbb{R}\}$. Moreover, set $\mathcal{E}_3 = \{a = (p, q, y) : \mathbb{R} \to \mathbb{R}^3 \mid a$ is uniformly continuous and bounded, $p \in C^1(\mathbb{R})$, and $p(x) \geq \delta$, $\delta \leq y(x) \leq \Delta$ for every $x \in \mathbb{R}\}$. The sets $\mathcal{E}_2$ and $\mathcal{E}_3$ are equipped with the standard topology on uniform convergence on compact subsets of $\mathbb{R}$.

For $a \in \mathcal{E}_3$, let us consider the Sturm-Liouville operator

$$L_a : \mathcal{D} \to L^2(\mathbb{R}, ydx) : \varphi \mapsto \frac{1}{y}(-DpD + q)\varphi,$$

where D is the operator of differentiation with respect to $x \in \mathbb{R}$. We will often refer to an element $a \in \mathcal{E}_3$ as a *potential.* The domain $\mathcal{D}$ of L_a is given by $\mathcal{D} = \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \in L^2(\mathbb{R}, ydx)$, φ is absolutely continuous and $D^2\varphi \in L^2(\mathbb{R}, ydx)\}$. It is well-known that L_a admits a self-adjoint extension to all $L^2(\mathbb{R}, ydx)$ which we denote again by L_a. Let Σ_a be the spectrum of L_a. It is a fact that $\Sigma_a \subset \mathbb{R}$ is bounded below and unbounded above: moreover, the set $R_a = \mathbb{R} \setminus \Sigma_a$ is (at most) a countable union of disjoint open (possibly unbounded) intervals.

The family of operators we discuss in these lines contains the Schrödinger operator (obtained by setting $a = (1, q, 1)$) and the so-called *acoustic operator* (obtained by setting $a = (1, 1, y)$).

Beyond the intrinsic relevance of the spectral theory of the Sturm-Liouville operator L_a, there is a fundamental connection between this operator and a family of nonlinear evolution equations. This connection (whose prototypical example was discovered in the '60s in [7] for the case of the Schrödinger operator and the K-dV equation) is the topic of this paper, and has encouraged the study of various classes of Sturm-Liouville operators. We will delay the description of the connection between the Sturm-Liouville operator and families of evolution equations to the next sections: now we discuss certain results concerning the spectral theory of the so-called *algebro-geometric Sturm-Liouville potentials*. A dynamical approach has turned out to be very useful in studying the above operators. Let $a \in \mathcal{E}_3$ and let L_a be the associated Sturm-Liouville operator. Together with L_a one can define the eigenvalue equation

$$E_a(\varphi, \lambda) = -DpD\varphi + q\varphi = \lambda y\varphi$$

where $\lambda \in \mathbb{C}$. This equation can be expressed in matrix form as

$$X' = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix} X,$$

where $X = \begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}$. We will often write $a(x,\lambda) = \begin{pmatrix} 0 & 1/p(x) \\ q(x) - \lambda y(x) & 0 \end{pmatrix}$, whenever $a(x) = (p(x), q(x), y(x)) \in \mathcal{E}_3$.

Define a function $A : \mathcal{E}_3 \to \mathbb{M}(2,\mathbb{C}) : a \mapsto a(0,\lambda)$. We denote by $\{\tau_s\}_{s\in\mathbb{R}}$ the translation flow on $\mathcal{E}_3$, i.e. if $a(\cdot) \in \mathcal{E}_3$, then $\tau_s(a) = a(s+\cdot) \in \mathcal{E}_3$. We apply a construction of Bebutov type to a fixed element $a_0 \in \mathcal{E}_3$, as follows. Pick $a_0 \in \mathcal{E}_3$, and consider the set $\mathcal{A} = cls\ Hull(a_0) = cls\{\tau_s(a_0) \mid s \in \mathbb{R}\}$. Then $\mathcal{A}$ is a compact, τ_s-invariant subset of $\mathcal{E}_3$, in the sense that $\tau_s(\mathcal{A}) = \mathcal{A}$, for every $s \in \mathbb{R}$. The Bebutov construction allows us to use dynamical methods to study the operator L_a: indeed, it makes now sense to speak of the family of linear systems

$$\begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}' = A(\tau_x(a))\begin{pmatrix} \varphi \\ p\varphi' \end{pmatrix}, \qquad a \in \mathcal{A}. \tag{2.1}$$

One important consequence of the Bebutov construction is that there is a subset $\mathcal{A}_1$ of $\mathcal{A}$ which has considerable properties of recurrence. For example, let μ be an ergodic measure on $\mathcal{A}$ (hence μ is a τ_s-invariant Borel probability measure which has the additional property of being indecomposable: if $B \subset \mathcal{A}$ is a Borel set and $\mu(\tau_s(B)\Delta B) = 0$ for every $s \in \mathbb{R}$, then either $\mu(B) = 0$ or $\mu(B) = 1$). It is a standard fact that such a measure exists [21]. If necessary, let us redefine $\mathcal{A}$ to be the topological support of μ. Then there is a set $\mathcal{A}_1 \subset \mathcal{A}$ such that each $a \in \mathcal{A}_1$ is both positively and negatively Poisson recurrent: there exist real sequences $(t_n) \to \infty$ and $(s_n) \to -\infty$ such that $\lim_{n\to\infty} \tau_{t_n}(a) = a$ and $\lim_{n\to\infty} \tau_{s_n}(a) = a$.

A fundamental tool in the study of the family (2.1) is the concept of *exponential dichotomy*. Let $\Phi_a(x)$ be the fundamental matrix solution of the family (2.1).

Definition 2.1 *The family* (2.1) *is said to have an exponential dichotomy over $\mathcal{A}$ if there are positive constants η, ρ, together with a continuous, projection valued function $P : \mathcal{A} \to \mathbb{M}_2(\mathbb{C})$ such that the following estimates holds:*

(i) $|\Phi_a(x)P(a)\Phi_a(s)^{-1}| \le \eta e^{-\rho(x-s)}, \qquad x \ge s,$

(ii) $|\Phi_a(x)(I - P(a))\Phi_a(s)^{-1}| \le \eta e^{\rho(x-s)}, \qquad x \le s.$

One has the following fundamental result (see [10])

Theorem 2.1 *Let $\mathcal{A}$ be obtained by a Bebutov type construction as above. Consider the family* (2.1). *If $a \in \mathcal{A}$ has dense orbit, then the spectrum Σ_a of the operator L_a equals the set*

$$\Sigma_{ed} := \{\lambda \in \mathbb{C} \mid \textit{the family } (2.1) \textit{ does } \textbf{not} \textit{ admit an exponential dichotomy over } \mathcal{A}\}.$$

It is known that, if $\Im\lambda \neq 0$, then the family (2.1) admits an exponential dichotomy over $\mathcal{A}$, hence $\Sigma_a \subset \mathbb{R}$. Another consequence of the above theorem is that if $a \in \mathcal{E}_3$ and $\mathcal{A} = cls\ Hull(a)$ then the spectrum of L_a and that of all the operators $L_{\tau_x(a)}$ coincide, i.e., $\Sigma_a = \Sigma_{\tau_x(a)} = \Sigma_{ed}$ for every $x \in \mathbb{R}$ [6].

Now, let $a \in \mathcal{E}_3$ and let us fix the Dirichlet boundary condition $\varphi(0) = 0$ at $x = 0$. There are well-defined unbounded self-adjoint operators $L_a^\pm$ which are defined in $L^2(\mathbb{R}^\pm, ydx)$ and which are determined by the formula

$$L_a(\varphi) = \frac{1}{y}[-(p\varphi')' + q\varphi]$$

and the Dirichlet condition in $x = 0$. If $\Im\lambda \neq 0$, we define the Weyl m-functions $m_\pm(a,\lambda)$ to be those complex numbers which parametrize $\ker P(a)$ and $\operatorname{Im} P(a)$, as follows:

$$\operatorname{Im} P(a) = \operatorname{Span}\begin{pmatrix} 1 \\ m_+(a,\lambda) \end{pmatrix} \qquad \ker P(a) = \operatorname{Span}\begin{pmatrix} 1 \\ m_-(a,\lambda) \end{pmatrix}.$$

Note that, since $a \in \mathcal{A}$ and $\det \Phi_a(x) = 1$ for every $x \in \mathbb{R}$, both $\ker P(a)$ and $\operatorname{Im} P(a)$ are complex lines in $\mathbb{C}^2$.

Next, let $a = (p, q, y) \in \mathcal{E}_3$ be a Sturm-Liouville potential. Consider the (unbounded, self-adjoint) operator $L_a = \frac{1}{y}[-DpD + q]$ on $L^2(\mathbb{R}, ydx)$. We will define the Green's function for the operator L_a. As is well known, the Green's function $\mathcal{G}_a(x, s, \lambda)$ is the kernel of the resolvent operator $(L_a - \lambda I)^{-1}$ acting on $L^2(\mathbb{R}, ydx)$ ($\Im\lambda \neq 0$). This means that, if one considers the nonhomogeneous equation $-(p\psi')' + q\psi = \lambda y\psi + yf$, where $f \in L^2(\mathbb{R}, ydx)$ and $\Im\lambda \neq 0$, one has

$$\psi(x) = \int_{\mathbb{R}} \mathcal{G}(x, s, \lambda) f(s) ds.$$

There is an interesting relation between the Weyl m-functions $m_\pm(a, \lambda)$ and the *diagonal Green's function* $\mathcal{G}_a(\lambda) := \mathcal{G}_a(0, 0, \lambda)$, namely, one has

$$\mathcal{G}_a(\lambda) = \frac{y(0)}{m_-(a, \lambda) - m_+(a, \lambda)} \qquad \Im\lambda \neq 0.$$

We now make use of the Bebutov flow. Let $a \in \mathcal{E}_3$, and consider the values $\tau_x(a)$ when $x \in \mathbb{R}$. We already noticed that the spectrum of the translated operators $L_{\tau_x(a)}$ equals that of the original operator L_a, i.e., $\Sigma_a = \Sigma_{\tau_x(a)}$ for every $x \in \mathbb{R}$. When $x \in \mathbb{R}$, let us define $m_\pm(x, \lambda) := m_\pm(\tau_x(a), \lambda)$, and $\mathcal{G}_a(x, \lambda) := \mathcal{G}_{\tau_x(a)}(\lambda) = \dfrac{y(x)}{m_-(x, \lambda) - m_+(x, \lambda)}$ ($\Im\lambda \neq 0$). Some observations concerning the dynamical definition of the Weyl m-functions imply that $\mathcal{G}_a(x, \lambda) = \mathcal{G}_{\tau_x(a)}(0, 0, \lambda) = \mathcal{G}_a(x, x, \lambda)$ whenever $x \in \mathbb{R}$ and $\Im\lambda \neq 0$. One can show that for every $x \in \mathbb{R}$, the non-tangential limit

$$\mathcal{G}_a(x, \eta) = \lim_{\varepsilon \to 0} \mathcal{G}_a(x, \eta + i\varepsilon)$$

exists for a.a. $\eta \in \mathbb{R}$ (see [5]). Let us also note that the Weyl m-functions satisfy the Riccati equation

$$m' + \frac{1}{p} m^2 = q - \lambda y, \qquad \Im\lambda \neq 0. \tag{2.2}$$

It is well-known that the behavior of the function $\mathcal{G}(x, \lambda)$ is fundamental for the study of the properties of the spectrum of the Sturm-Liouville operator. In fact, one gives "names" to potentials $\in \mathcal{E}_3$ according to how their *diagonal Green's function* behaves. In this paper, we will consider two particular families of potentials which lie in $\mathcal{E}_3$, namely *algebro-geometric potentials* and *reflectionless potentials.*

Definition 2.2 *A potential $a \in \mathcal{E}_3$ is called algebro-geometric if the following properties are satisfied:*

(i) the spectrum Σ_a of the operator L_a is a finite union of disjoint closed intervals, plus an halfline:

$$\Sigma_a = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2n}, \infty).$$

(ii) for every $x \in \mathbb{R}$, $\Re\mathcal{G}_a(x, \eta) = 0$ for a.a. $\eta \in \Sigma_a$.

A potential $a \in \mathcal{E}_3$ is called reflectionless if:

(i) the spectrum Σ_a of the operator L_a has locally positive Lebesgue measure, i.e., if $\lambda \in \Sigma_a$ and $I \subset \mathbb{R}$ is an open interval which contains λ, then $I \cap \Sigma_a$ has positive Lebesgue measure;

(ii) for every $x \in \mathbb{R}$, $\Re\mathcal{G}_a(x, \eta) = 0$ for a.a. $\eta \in \Sigma_a$.

Algebro-geometric Sturm-Liouville potentials were introduced in [12] (see also [6] and [13]): they are an extension of the well-known algebro-geometric potentials Schrödinger potentials [4]. Arguing as in [6, 12] one can prove that for an algebro-geometric potential condition (ii) actually holds *for every* $\eta \in \Sigma_a$.

Reflectionless Sturm-Liouville potentials have been introduced in [14], as an extension of the homologous potentials in the Schrödinger case, as described in [3]. See also [24] for more information.

Condition (ii) in the above definition has some fundamental consequences. First of all, if $a \in \mathcal{E}_3$ is reflectionless, every potential $\tau_x(a)$ is reflectionless as well for every $x \in \mathbb{R}$ (and, in particular, if a is algebro-geometric, then $\tau_x(a)$ is algebro-geometric as well for every $x \in \mathbb{R}$). Second, it can be proved (see [6, 12, 14]) that both $m_+(x,\cdot)$ and $m_-(x,\cdot)$ extend holomorphically through every open inteval which lies in the spectrum Σ_a. Denoting by $h_\pm(x,\cdot)$ these extensions, one obtains

$$h_+(x,\lambda) = \begin{cases} m_+(x,\lambda), & \Im\lambda > 0 \\ m_-(x,\lambda), & \Im\lambda < 0 \end{cases} \qquad \text{and} \qquad h_-(x,\lambda) = \begin{cases} m_-(x,\lambda), & \Im\lambda > 0 \\ m_+(x,\lambda), & \Im\lambda < 0 \end{cases}$$

Some other considerations (see [6, 12]) allow to conclude that, if $a \in \mathcal{E}_3$ is an algebro-geometric potential, then (1) the spectrum Σ_a of the operator L_a has no isolated eigenvalues; (2) the Weyl m-functions $m_\pm(x,\cdot)$ extend meromorphically through $\mathbb{R} \setminus \Sigma_a$ for every $x \in \mathbb{R}$. In particular, if $a \in \mathcal{E}_3$ is a fixed algebro-geometric potential, then there exists exactly one point $P_i(a)$ in each interval $I_i = [\lambda_{2i-1}, \lambda_{2i}]$ ($i = 1,\ldots,g$) such that either $|m_+(P_i(a) + i\varepsilon)|$ or $|m_-(P_i(a) + i\varepsilon)|$ is singular as $\varepsilon \to 0$, and this singularity is a simple pole. These points $P_i(a)$ ($i = 1,\ldots,g$) correspond to the isolated eigenvalues of the half-line restricted operators $L_a^\pm : L^2(\mathbb{R}^\pm, y dx) \to L^2(\mathbb{R}^\pm, y dx)$ with initial condition $\varphi(0) = 0$. Clearly, if $P_i(a)$ is a pole for m_+, then it is an isolated eigenvalue of L_a^+ and vice-v ersa.

Let $a \in \mathcal{E}_3$ be algebro-geometric. We now bring in to the scene the Bebutov flow, to obtain *moving eigenvalues* $P_i(x) := P(\tau_x(a))$. These eigenvalues lie in the spectral gaps I_i for every $x \in \mathbb{R}$.

We further introduce a Riemann surface which is fundamental to the purpose of describing the motion of the eigenvalues $P_1(x),\ldots,P_g(x)$. Let $\mathcal{R}$ be the Riemann surface of the algebraic relation

$$w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1)\ldots(\lambda - \lambda_{2g}).$$

Then $\mathcal{R}$ is a torus having exactly g holes corresponding to the spectral gaps $I_i = [\lambda_{2i-1}, \lambda_{2i}]$ ($i = 1,\ldots,g$). Denote with π the standard projection of $\mathcal{R}$ to the Riemann sphere $\hat{\mathbb{C}}$. Then π is 2-1 except at the points $\lambda_0, \lambda_1, \ldots, \lambda_{2g}$ and ∞, where it is 1-1. We will call the points $\lambda_0, \lambda_1, \lambda_{2g}, \infty$ the ramification points of $\mathcal{R}$. Denote points in $\mathcal{R}$ by P. If $\lambda \in \hat{\mathbb{C}}$ is not a ramification point, then $\pi^{-1}(\lambda) = \{P^+, P^-\}$. Roughly speaking, P^+ corresponds to the positive square root of $-(\lambda - \lambda_0)(\lambda - \lambda_1)\ldots(\lambda - \lambda_{2g})$, while P^- to the negative square root of $-(\lambda - \lambda_0)(\lambda - \lambda_1)\ldots(\lambda - \lambda_{2g})$. In fact, one can define a function $k(P)$ on $\mathcal{R}$ first by defining $k(0^+)$ (resp. $k(0^-)$) to be the positive (resp. negative) square root of $\lambda_0\lambda_1\ldots\lambda_{2g}$, then using analytic continuation. One obtains a function $k(P)$ which is single valued on $\mathcal{R}$ and such that $k(P)$ has as values the positive and negative square roots of w^2. In particular, if λ is not a ramification point, and if $\pi^{-1}(\lambda) = \{P^+, P^-\}$, then $k(P^+)$ (resp. $k(P^-)$) is the positive (resp. negative) square root of w^2.

Now, let us define $c_i = \pi^{-1}(I_i)$ ($i = 1,\ldots,g$). The curves c_i are circles and correspond to the "holes" of $\mathcal{R}$. A convenient way of describing this correspondence is that of using a polar coordinate, as follows. Let $I_i = [\lambda_{2i-1}, \lambda_{2i}]$, and let $c_i = \pi^{-1}(I_i)$. If $P_i \in c_i$, the value $\pi(P_i) \in I_i$ can be written as

$$\pi(P_i) = (\lambda_{2i-1} - \lambda_{2i})\sin^2\frac{\theta_i}{2} + \lambda_{2i},$$

where θ_i is an appropriate value $\theta_i \in [0, 2\pi]$: if $\theta_i \in (0,\pi)$, then we have that $k(P_i)$ is positive, while if $\theta_i \in (\pi, 2\pi)$, then $k(P_i)$ is negative (note that $k(P)$ is real when $P \in c_i$). From now on, we will often commit the abuse of notation of denoting by P_i both points in c_i and their projections in I_i $(i = 1, \dots, g)$.

Let $a \in \mathcal{E}_3$ be algebro-geometric with spectrum $\Sigma_a = [\lambda_0, \lambda_1] \cup \dots \cup [\lambda_{2g}, \infty)$. Let us look at the poles $P_1(x), \dots, P_g(x)$ of $m_\pm(x, \lambda)$. The motion of the points $P_1(x), \dots, P_g(x)$ takes place in $I_1 \times \dots \times I_g$, hence in $c_1 \times \dots \times c_g$ via the correspondence $\pi(c_i) = I_i$. Each point $P_i(x) \in I_i$ can be written as

$$P_i(x) = (\lambda_{2i-1} - \lambda_{2i}) \sin^2 \frac{\theta_i(x)}{2} + \lambda_{2i},$$

which defines the angular coordinate $\theta_i(x)$ of the point $P_i(x)$ $(i = 1, \dots, g)$.

Now, we review some reasoning made in [12, 6]: first, letting $a = (p, q, y) \in \mathcal{E}_3$ be an algebro-geometric potential, one can define a single meromorphic function $M(P)$ on $\mathcal{R}$ by setting $M(0^+) = m_+(0)$ and $M(0^-) = m_-(0)$, and then by using analytic continuation along curves in $\mathcal{R}$. In this way, one has $M(P) = m_+(P)$ and $M \circ \sigma(P) = m_-(P)$ on $\mathcal{R}$. Next, we expand M near ∞ to obtain

$$M(P) = m_+(P) = \frac{Q(\lambda) + \sqrt{py}k(P)}{H(\lambda)}, \quad M \circ \sigma(P) = m_-(P) = \frac{Q(\lambda) - \sqrt{py}k(P)}{H(\lambda)},$$

where $\lambda = \pi(P)$, $H(\lambda) = \prod_{i=1}^{g} (\lambda - \pi(P_i))$, and $Q(\lambda)$ is a polynomial of degree g in $\lambda \in \mathbb{C}$. Here we adopt the convention that $\sqrt{py} > 0$ whenever g is even, while $\sqrt{py} < 0$ when g is odd. To determine $Q(\lambda)$, we argue as follows: in view of the definition of $m_-(P)$ on $\mathcal{R}$, one has that m_- is finite at the p oints $P_1(x), \dots, P_g(x)$, hence we must have $Q(P_i(x)) = \sqrt{p(x)y(x)}k(P_i(x))$ $(i = 1, \dots, g)$. Next, we note also that, in view of (2.2), $Q(P_i(x)) = \frac{p(x)}{2} H_x(P_i(x))$ for every $i = 1, \dots, g$. Moreover, one can show (see [6, 12]) that $\frac{Q(\lambda)}{\lambda^g} \to \frac{-(p(x)y(x))_x}{4y(x)}$ as $\lambda \to \infty$ on $\mathcal{R}$, hence we can write

$$Q(\lambda) = \frac{p(x)}{2} \sqrt{p(x)y(x)} \left(\frac{1}{\sqrt{p(x)y(x)}} H(\lambda) \right)_x .$$

Some computations now show that we can relate the triple $a = (p, q, y) \in \mathcal{E}_3$ to the motion of the points P_i as follows: let $\mathcal{M}(x) = m_-(x, 0) - m_+(x, 0)$. Then one obtains

$$P_{i,x}(x) = \frac{(-1)^g \mathcal{M}(x) k(P_i(x)) \prod_{i=1}^{g} P_i(x)}{p(x) k(0^+) \prod_{j \neq i} (P_i(x) - P_j(x))}. \tag{2.3}$$

The system (2.3) is a system of g ODE's. It is intended to take place in $\mathbb{R}$, hence the value $k(P_i)$ must be given a sign according to the "position" of the point P_i on $\mathcal{R}$. This may create a little bit of confusion, but we can easily avoid that by using polar coordinates $\theta_i(x)$. In fact, one can write down the derivative of $\theta_i(x)$ instead of that of $P_i(x)$, then use the expression which defines $\theta_i(x)$ to recover $P_i(x)$. In this way, one has

$$\begin{aligned} \theta_{i,x}(x) = (-1)^g \frac{\mathcal{M}(x)}{p(x)} \frac{\sqrt{(P_j(x) - \lambda_0)}}{\sqrt{\lambda_0}} \times \\ \times \prod_{j=1}^{g} \frac{P_j(x)}{\sqrt{\lambda_{2j-1}\lambda_{2j}}} \prod_{j \neq i} \frac{\sqrt{(\lambda_{2j-1} - P_i(x))(\lambda_{2j} - P_j(x))}}{P_j(x) - P_i(x)}. \end{aligned}$$

Now, once we have determined the motion of the points $P_i(x)$ $(i = 1, \dots, g)$, we use the Taylor coefficients of the expansion of M at ∞ to obtain

$$y(x) = \frac{\mathcal{M}^2(x)}{4p(x)} \frac{\prod_{i=1}^{g} [P_i(x)]^2}{k^2(0^+)}, \tag{2.4}$$

$$q(x) = y(x)\left(\lambda_0 + \sum_{i=1}^{g}(\lambda_{2i} + \lambda_{2i-1} - 2P_i(x))\right) + g(x) + \frac{g^2(x)}{p(x)}, \tag{2.5}$$

where $g(x) = -\dfrac{(p(x)y(x))_x}{4y(x)}$.

The motion of the points $P_1(x), \ldots P_g(x)$ can be better understood by using instruments of algebraic geometry, and in particular by using the Abel map and a generalized Jacobi variety. Moreover, the potential $a = (p, q, y) \in \mathcal{E}_3$ can be expressed by means of a Generalized Riemann Theta function: but this is another story, and we address the reader to [12, 13, 6] for further details on this topic.

Another important fact concerning algebro-geometric potentials $a = (p, q, y) \in \mathcal{E}_3$ is that the above process can be inverted, as follows: let $\Lambda_0 = \{\lambda_0 < \lambda_1 < \ldots < \lambda_{2g}\}$ be a finite set of $2g$ positive real numbers, let $\mathcal{R}$ be the Riemann surface defined as above, let us set $I_j = [\lambda_{2j-1}, \lambda_{2j}]$ $(j = 1, \ldots, g)$, let us pick points $P_1(0) \in c_1, \ldots, P_g(0) \in c_g$ $(\pi(c_j) = I_j$, for every $j = 1, \ldots, g)$, and let us fix a function $b = (p, \mathcal{M}) \in \mathcal{E}_2$. Next, let the points $P_1(x), \ldots, P_g(x)$ be determined as to satisfy the system (2.3), with initial condition $(P_1(0), \ldots, P_g(0))$. Then define $y(x)$ and $q(x)$ as in (2.4) and (2.5) respectively. It turns out [12, 6] that the triple $a = (p, q, y) \in \mathcal{E}_3$ is an algebro-geometric potential whose spectrum equals the set $\Sigma = [\lambda_0, \lambda_1] \cup \ldots [\lambda_{2g}, \infty)$.

3 The g-th order Sturm-Liouville hierarchy of evolution equation

In this Section we review some results of [15], in which a Sturm-Liouville hierarchy of evolution equations was introduced and solved when the initial data is a given algebro-geometric Sturm-Liouville potential. The procedure which we follow for determining the solution of the Sturm-Liouville hierarchy consists of several steps, which we summarize as follows:

1. Let us fix a set $\Lambda_0 = \{\lambda_0 < \lambda_1 < \ldots < \lambda_{2g}\} \subset \mathbb{R}$ where $\lambda_i > 0$ for every $i = 0, \ldots, 2g$. Let $b = (p, \mathcal{M}) \in \mathcal{E}_2$, and let us determine an algebro-geometric Sturm-Liouville potential $a = (p, q, y) \in \mathcal{E}_3$ via the procedure described in the previous section.

2. Let

$$U(x, \lambda) = \frac{2(-1)^{g+1} p(x) k(0^+)}{\mathcal{M}(x) \prod_{i=1}^{g} P_i(x)} \prod_{i=1}^{g} (\lambda - P_i(x)).$$

Choose numbers $k, r \in \{0, 1, \ldots, g-1\}$ such that $k \le r$, and define $T(x, \lambda)$ and $V(x, \lambda)$ in such a way that

$$T(x, \lambda) = \frac{\lambda^{-k} p(x)}{2} \left(\frac{U(x, \lambda)}{p(x)}\right)_x$$

and

$$T_x(x, \lambda) + \frac{\lambda^{-k}}{p(x)}(q(x) - \lambda y(x))(V(x, \lambda) - U(x, \lambda)) = 0.$$

3. Let

$$B_g = \begin{pmatrix} -T & \lambda^{-k} U/p \\ \lambda^{-k}(q - \lambda y)V & T \end{pmatrix}.$$

Then one can show that, if $A = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix}$, then the so-called stationary zero-curvature condition holds, i.e.,

$$-B_{g,x} + [A, B_g] = 0$$

where $[A, B_g] = AB_g - B_gA$ is the commutator of A and B_g. Moreover, one easily checks that

$$\frac{d}{dx}\det B_g = 0.$$

In view of the above relation, one makes the following Ansatz:

$$\frac{p^2}{4}\left[\left(\frac{U}{p}\right)_x\right]^2 + \frac{1}{p}(q - \lambda y)UV = k^2(\lambda) = -\prod_{i=0}^{2g}(\lambda - \lambda_i). \tag{3.1}$$

4. Let $a = (p, q, y) \in \mathcal{E}_3$ be an algebro-geometric potential corresponding to the pair $b = (p, \mathcal{M}) \in \mathcal{E}_2$ and the set $\Lambda_0 = \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$ as in point 1. Let us keep the numbers $k \leq r \in \{0, 1, \ldots, g - 1\}$ fixed. We introduce a parameter $t \in \mathbb{R}$ in all the functions involved. So one introduces a one-parameter family of Sturm-Liouville potentials $a(t, x) = (p(t, x), q(t, x), y(t, x))$, together with matrices

$$A(t, x, \lambda) = \begin{pmatrix} 0 & 1/p(t, x) \\ q(t, x) - \lambda y(t, x) & 0 \end{pmatrix}$$

$$B_g(t, x, \lambda) = \begin{pmatrix} -T(t, x, \lambda) & \lambda^{-k}\dfrac{U(t, x, \lambda)}{p(t, x)} \\ \lambda^{-k}(q(t, x) - \lambda y(t, x))V(t, x, \lambda) & T(t, x, \lambda) \end{pmatrix}$$

$$B_r(t, x, \lambda) = \begin{pmatrix} -T_r(t, x, \lambda) & \lambda^{-k}\dfrac{U_r(t, x, \lambda)}{p(t, x)} \\ \lambda^{-k}(q(t, x) - \lambda y(t, x))V_r(t, x, \lambda) & T_r(t, x, \lambda) \end{pmatrix}$$

where $U(t, x, \lambda)$ is a polynomial of degree g in λ and whose coefficients depend on t and x, $T(t, x, \lambda)$ and $V(t, x, \lambda)$ are defined as in point (2), $U_r(t, x, \lambda)$ is a polynomial of degree r in λ whose coefficients depend on t and x, and $T_r(t, x, \lambda)$ and $V(t, x, \lambda)$ are defined by

$$\left(\frac{1}{p}\right)_t - \lambda^{-k}\left(\frac{U_r}{p}\right)_x + \frac{2}{p}T_r = 0,$$

$$T_{r,x} + \frac{\lambda^{-k}}{p}(q - \lambda y)(V_r - U_r) = 0.$$

We ask if $U(t, x, \lambda)$ and $U_r(t, x, \lambda)$ can be chosen in such a way that

$$\begin{cases} -B_{g,x} + [A, B_g] = 0 \\ A_t - B_{r,x} + [A, B_r] = 0 \\ \dfrac{d}{dx}\det B_g = 0, \qquad \text{and (3.1) holds} \end{cases} \tag{3.2}$$

for all $(t, x) \in \mathbb{R}^2$ and all $\lambda \neq 0$. One can show that the validity of (3.2) implies that each $a(t, x)$ is algebro-geometric, i.e., the flow determined by equations (3.2) is isospectral: in particular each $a(t, x)$ is algebro-geometric and its spectrum equals the set $\Sigma = [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$. See [15], Section 3 and Section 4.

5. The answer to question in point (4) is positive. In fact, one can define the coefficients of the polynomial $U_r(t, x, \lambda)$ beginning from the relation

$$U_{r,x} = \frac{\lambda^{-k}}{p}\left[\frac{\mathcal{M}_t}{\mathcal{M}} + p\left(\frac{1}{p}\right)_t\right] - \frac{\mathcal{M}_x}{\mathcal{M}}U_r + \sum_{i=1}^{g}\left[\frac{\lambda^k}{P_i^k}U_r - U_r(P_i)\right]\frac{\lambda P_{i,x}}{P_i(\lambda - P_i)}.$$

It turns out that the coefficients of U_r can be determined via a recursion relation. It also turns out that the points $P_i(t,x)$ satisfy the system

$$\begin{cases} P_{i,x} = \dfrac{(-1)^g k(P_i)\mathcal{M}\prod_{j=1}^g P_j}{pk(0^+)\prod_{j\neq i}(P_i - P_j)} \\ P_{i,t} = \dfrac{U_r(P_i)}{P_i^k} P_{i,x} \end{cases}.$$

We will explain this construction in more detail and provide some examples in the next sections.

In Section 5, we will see how exactly this construction produces a pair of nonlinear evolution equations which we will call the r-th order Sturm-Liouville evolution equations.

4 The stationary infinite-order Sturm-Liouville hierarchy

This section provides a procedure to construct the initial conditions for which our hierarchy of evolution equations will be solved. The procedure we will use here is based on a so called algebro-geometric approximation. Such a technique has been shown to be successful in the study of the K-dV hierarchy of evolution equations (see [24]).

But let's try to be methodical, and let us assume we are given a sequence of positive real numbers $\Lambda_0 = \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \ldots\} \subset \mathbb{R}$ such that $\lambda_0 < \lambda_1 < \ldots \lambda_{2g} < \ldots$. Let us set $I_k = [\lambda_{2k-1}, \lambda_{2k}]$. Let $d_k = \lambda_{2k} - \lambda_{2k-1}$ be the length of the interval I_k. Further, set $h_{jk} = dist(I_j, I_k)$ and $h_{0k} = \lambda_{2k-1} - \lambda_0$.

We make the following additional assumptions on the sequence Λ_0:

Hypotheses 4.1 *The sequence Λ_0 satisfies:*

(i) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$;

(ii) $\displaystyle\sum_{k=1}^{\infty} d_k < \infty$;

(iii) $\displaystyle\sup_{k\in\mathbb{N}} d_k h_{0k} < \infty$;

(iv) $\displaystyle\sup_{j\in\mathbb{N}} \sum_{k\neq j} \frac{\sqrt{d_k}}{h_{jk}} := D_1 < \infty$

Note that the infinite products

$$\prod_{j=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_j}\right) \quad \text{and} \quad \prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{z_j}\right), \ (z_j \in I_j)$$

converge. Below we will introduce an entire function $U(x,\lambda)$ in the λ-complex plane having as zeros exactly one point in each interval I_j, namely $s_j(x) \in I_j$. To be clearer, for every $x \in \mathbb{R}$, we choose points $s_1(x) \in I_1, \ldots, s_g(x) \in I_g, \ldots$, then construct an entire function $U(x,\lambda)$ of the complex variable λ whose zeros are exactly the points $s_1(x), \ldots, s_g(x), \ldots$. The points $s_j(x)$ depend on the parameter $x \in \mathbb{R}$. We make the a-priori assumption that $s_j \in C^\infty(\mathbb{R})$. It will be clear later on that it suffices to assume $s_j \in C^1(\mathbb{R})$.

Let $k \geq 0$ be any positive integer, and define

$$T(x,\lambda) = \frac{p}{2\lambda^k}\left(\frac{U}{p}\right)_x. \tag{4.1}$$

Further, we choose $V(x,\lambda)$ as to satisfy

$$T_x(x,\lambda) + \frac{\lambda^{-k}}{p(x)}(q(x) - \lambda y(x))(V(x,\lambda) - U(x,\lambda)) = 0. \tag{4.2}$$

Here, the functions p, q, y are the coordinates of a triple $a \in \mathcal{E}_3$, i.e., $a = (p, q, y) \in \mathcal{E}_3$.

We look for a map $x \mapsto \{s_1(x), \dots, s_g(x), \dots\}$ and an element $a = (p, q, y) \in \mathcal{E}_3$ for which the following stationary zero-curvature relation holds

$$-B_x + [A, B] = 0, \tag{4.3}$$

where

$$A = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -T & \dfrac{U}{p\lambda^k} \\ \dfrac{1}{\lambda^k}(q-\lambda)V & T \end{pmatrix}.$$

We will prove that, under an additional condition concerning the structure of B, the only possibility for (4.3) to be valid is that a is a reflectionless potential which is the limit on the compact subsets of $\mathbb{R}$ of a sequence of algebro geometric potentials $\{a_n\}$. Further, the zeros $s_j(x)$ of U will be the zeros of the diagonal Green's function $\mathcal{G}(x,\lambda)$ of L_a .

First, pick a pair $(p, \mathcal{M}) \in \mathcal{E}_2$. We set $\tilde{U} = U/p$ as before, and write

$$\tilde{U}(x,\lambda) = \frac{-2}{\mathcal{M}(x)} \prod_{j=0}^{\infty} \left(1 - \frac{\lambda}{s_j(x)}\right). \tag{4.4}$$

Since $\tilde{U}(x,\lambda)$ is an entire function, the Taylor series

$$\tilde{U}(x,\lambda) = \sum_{i=0}^{\infty} \tilde{u}_i(x)\lambda^i$$

converges for all $\lambda \in \mathbb{C}$.

In a way which is similar to that of the finite-gap stationary case, the relation (4.3) gives the recursions

$$\begin{cases} -2\mathcal{D}_{py}\tilde{u}_{j-1} = DpDpD\tilde{u}_j - 2\mathcal{D}_{pq}\tilde{u}_j, \quad j = 1, 2, \dots \\ DpDpD\tilde{u}_0 - \mathcal{D}_{pq}\tilde{u}_0 = 0. \end{cases} \tag{4.5}$$

The system (4.5) allows to determine all the coefficients $\tilde{u}_j(x)$ of the Taylor expansion of $\tilde{U}$, beginning with $\tilde{u}_0$.

We use the notation

$$k_\infty(\lambda) = \sqrt{\prod_{i=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_i}\right)}.$$

Our aim is to characterize the triples $a = (p, q, y) \in \mathcal{E}_3$ for which a relation of the type (4.3) holds.

The relation (3.1) retains validity in this case as well, hence we have the additional relation $\frac{d}{dx}\det B = 0$. Motivated by what we did above, we make the Ansatz:

$$p^2\frac{\tilde{U}_x^2}{4} + \frac{1}{p}(q - \lambda y)UV = const = \prod_{i=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_i}\right). \tag{4.6}$$

Computing (4.6) at a zero $s_j(x)$ we obtain

$$p^2\frac{\tilde{U}_x(s_j)^2}{4} = k_\infty^2(s_j).$$

Moreover,

$$\tilde{U}_x(s_j) = \frac{2}{\mathcal{M}(x)}\prod_{k\neq j}\left(1 - \frac{s_j}{s_k}\right)s_{j,x},$$

hence

$$\begin{aligned} s_{j,x}(x) = \pm &\frac{\mathcal{M}(x)\sqrt{(s_j(x) - \lambda_0)(\lambda_{2j} - s_j(x))(s_j(x) - \lambda_{2j-1})}}{p(x)\sqrt{\lambda_0}}\times \\ &\times\prod_{k=1}^{\infty}\frac{s_k(x)}{\sqrt{\lambda_{2k-1}\lambda_{2k}}}\prod_{l\neq j}\frac{\sqrt{(\lambda_{2l-1} - s_j(x))(\lambda_{2l} - s_j(x))}}{s_l(x) - s_j(x)}. \end{aligned} \tag{4.7}$$

At this point, formula (4.7) is only formal. Indeed, we must show that the right-hand side of (4.7) is meaningful, and that each equation in (4.7) admits a solution. Note that (4.7) is a system of infinitely many linear differential equations.

Moreover, there is an ambiguity which follows from the sign $\pm$ in (4.7). As discussed earlier, to determine if $s_j(x)$ is either increasing or decreasing, one can argue as follows: let us fix an index $j \in \mathbb{N}$, and let us consider the circle c_j obtained by taking two copies of the interval $I_j = [\lambda_{2j-1}, \lambda_{2j}]$ and glueing them together by identifying their endpoints. The motion of the point $s_j(x)$ is then transferred into c_j by observing that each time the point $s_j(x)$ crosses an endpoint of I_j its derivative changes sign and $s_j(x)$ jumps to the other copy of I_j. Under this point of view, one can introduce an *angular coordinate* for each point $s_j(x)$ in such a way that

$$s_j(x) = \lambda_{2j-1} + d_j\sin^2\frac{\theta_j(x)}{2}, \qquad j \in \mathbb{N}. \tag{4.8}$$

In this way, one has

$$s_{j,x}(x) = \frac{1}{2}d_j\theta_{j,x}(x)\sin\theta_j(x), \qquad j \in \mathbb{N}, \tag{4.9}$$

hence $\theta_{j,x}(x) = \dfrac{2s_{j,x}(x)}{d_j\sin\theta_j(x)}$ for every $x \in \mathbb{R}$ for which $\sin\theta_j(x) \neq 0$.

Now, we use (4.7) to obtain

$$\begin{aligned} \theta_{j,x}(x) = \pm &\frac{|\sin\theta_j(x)|}{\sin\theta_j(x)}\frac{\mathcal{M}(x)}{p(x)}\frac{\sqrt{(s_j(x) - \lambda_0)}}{\sqrt{\lambda_0}}\times \\ &\times\prod_{k=1}^{\infty}\frac{s_k(x)}{\sqrt{\lambda_{2k-1}\lambda_{2k}}}\prod_{l\neq j}\frac{\sqrt{(\lambda_{2l-1} - s_j(x))(\lambda_{2l} - s_j(x))}}{s_l(x) - s_j(x)}. \end{aligned} \tag{4.10}$$

The definition of the coordinate $\theta_j(x)$ implies that $\pm\dfrac{|\sin\theta_j(x)|}{\sin\theta_j(x)} = 1$, hence the system (4.10) is more suitable to study the motion of the points $s_j(x)$. Moreover, a simple observation shows that $\theta_{j,x}(x)$

is well defined also at those points where $\sin\theta_j(x) = 0$, hence (4.10) makes sense for every $x \in \mathbb{R}$. Another important consequence of the choice of the angular coordinate is that one can (formally) differentiate each equation in (4.10) with respect to x, while equations in (4.7) have singularities at those values for which $s_j(x)$ reaches an endpoint of I_j.

We now introduce a space which will be useful in the next lines. Let $\mathcal{I} = I_1 \times I_2 \times \cdots \times I_g \times \ldots$. Denote points in $\mathcal{I}$ by $[X]$. So if $[X] \in \mathcal{I}$, then $[X]$ is a sequence $(X_1, X_2, \ldots, X_g, \ldots)$ such that $X_i \in I_i$ for every $i \in \mathbb{N}$. We introduce the following metric on $\mathcal{I}$:

$$\|[X] - [Y]\|_{\mathcal{I}} := \sup_{k\in\mathbb{N}} \frac{|X_k - Y_k|}{\sqrt{d_k}}.$$

Note that $\|\cdot\|_{\mathcal{I}}$ is well defined since $|X_k - Y_k| \leq d_k$ for every $k \in \mathbb{N}$.

Alternatively, one can choose to work with the angular coordinates. In this case, the suitable space is given by $C = [0, 2\pi]^{\aleph_0}$ and the metric becomes

$$\|[\theta] - [\rho]\|_C = \sup_{k\in\mathbb{N}} \sqrt{d_k}|\theta_k - \rho_k|.$$

Now, let us introduce the functions $F_j : \mathcal{I} \to \mathbb{R}$, as follows

$$F_j([X]) = \frac{\prod_{k\in\mathbb{N}} X_k}{k_\infty(0)} \frac{k_\infty(X_j)}{\prod_{k\neq j}(X_k - X_j)}.$$

We are ready to prove the following result.

Proposition 4.1 *Let the sequence Λ_0 satisfy Hypotheses 4.1. Then the maps F_j are well defined, for every $j \in \mathbb{N}$.*

Proof. We will show that the infinite products appearing in $F_j([X])$ are bounded. In particular, we will give an uniform bound with respect to $[X] \in \mathcal{I}$ for each F_j. The computation we are going to make here is similar to one made in [14]. We have

$$\sup_{[X]\in\mathcal{I}} |F_j([X])| \leq \prod_{k\in\mathbb{N}} \left(1 + \frac{d_k}{h_{0k}}\right) d_j \frac{\sqrt{h_{0j} + d_j}}{\sqrt{\lambda_0}} \prod_{l\neq j} \left(1 + \frac{d_l}{h_{jl}}\right).$$

Hence the functions F_j are well defined for every $j \in \mathbb{N}$.

The inequality in the above proof shows that

$$|F_j([X])| \leq C d_j \sqrt{d_j + h_{0j}} \leq C C_1 \sqrt{d_j},$$

hence

$$\sup_{j\in\mathbb{N}} \frac{|F_j([X])|}{\sqrt{d_j}} \leq C C_1, \qquad \sup_{j\in\mathbb{N}} |F_j([X])| < C_2,$$

uniformly with respect to $[X] \in \mathcal{I}$, where $C_1 = \sup_{j\in\mathbb{N}} (d_j + \sqrt{d_j h_{0j}})$.

Note that

$$s_{j,x}(x) = \pm \frac{\mathcal{M}(x)}{p(x)} F_j([s(x)]), \qquad [s(x)] = (s_1(x), s_2(x), \ldots, s_n(x), \ldots) \tag{4.11}$$

Moreover

$$\theta_{j,x}(x) = \frac{\mathcal{M}(x)}{p(x)} G_j([s(x)]), \tag{4.12}$$

where

$$G_j([X]) = \frac{\sqrt{(X_j - \lambda_0)}}{\sqrt{\lambda_0}} \prod_{k=1}^{\infty} \frac{X_k}{\sqrt{\lambda_{2k-1}\lambda_{2k}}} \prod_{l \neq j} \frac{\sqrt{(\lambda_{2l-1} - X_j)(\lambda_{2l} - X_j)}}{X_l - X_j}.$$

The functions G_j are defined on $\mathcal{I}$: however, one can think at those functions as defined on C, via the correspondence we described above: $X_j = X_j(\theta_j) = \lambda_{2j-1} + d_j \sin^2 \frac{\theta_j}{2}$. One obtains a sequence $[X[\theta]] = (X_1(\theta_1), \dots, X_n(\theta_n), \dots)$, and functions $G_j : C \to \mathbb{R} : [\theta] \mapsto G_j([X([\theta]])$. Using the estimates in Proposition 4.1 we obtain

$$|G_j([s(x)]) \leq C \sqrt{d_j + h_{0j}}$$

and

$$|F_j([s(x)])| \leq d_j |G_j([s(x)])|.$$

Observe that by using (4.10) instead of (4.7) we earn regularity but we lose uniform boundedness.

The following estimates are similar to those proved in [14].

Proposition 4.2 *Let $j, k \in \mathbb{N}$. Then*

$$\left| \frac{\partial G_j}{\partial X_k}([X]) \right| \leq \sup_{[X] \in \mathcal{I}} \left| G_j([X]) \left(\frac{1}{X_k} + \frac{1}{X_j - X_k} \right) \right|, \qquad j \neq k \in \mathbb{N} \tag{4.13}$$

$$\begin{aligned} \left| \frac{\partial G_j}{\partial X_j}([X]) \right| \leq \sup_{[X] \in \mathcal{I}} \left| G_j([X]) \right| \Bigg| \frac{1}{X_j} + \frac{1}{2(X_j - \lambda_0)} + \\ + \frac{1}{2} \sum_{j \neq s \in \mathbb{N}} \left(\frac{1}{(X_j - \lambda_{2s})} + \frac{1}{(X_j - \lambda_{2s-1})} - \frac{2}{(X_j - X_s)} \right) \Bigg|. \end{aligned} \tag{4.14}$$

Note that equations (4.13) and (4.14) imply that

$$\sum_{k \in \mathbb{N}} \sup_{[X] \in \mathcal{I}} \left| \sqrt{d_k} \frac{\partial G_j}{\partial X_k}([X]) \right| \leq C(2D_1 + D_1^2) \sqrt{d_j + h_{0j}}.$$

We have the instruments to prove uniqueness of the solution of the system (4.7), when the initial data $(s_1(0), s_2(0), \dots, s_g(0), \dots) \in \mathcal{I}$ is given.

Theorem 4.2 *Let Λ_0 satisfy Hypotheses 4.1. Choose a point* $[\tilde{s}(0)] := (\tilde{s}_1(0), \tilde{s}_2(0), \dots, \tilde{s}_g(0), \dots) \in \mathcal{I}$, *and let* $(p, \mathcal{M}) \in \mathcal{E}_2$. *Assume that a solution* $[s(x)] := (s_1(x), s_2(x), \dots, s_g(x), \dots)$ *of the system* (4.7) *exists for every* $x \in \mathbb{R}$, *that* $[s(x)] \in \mathcal{I}$ *for every* $x \in \mathbb{R}$ *and that* $[s(0)] = [\tilde{s}(0)]$. *Then the solution* $[s(x)]$ *is unique.*

Proof. The proof repeats the same arguments as in ([14], Theorem 4.5). We observe that if a solution of then it has to satisfy $[s(x)] \in \mathcal{I}$ for every $x \in \mathbb{R}$. We will only sketch the proof of uniqueness, which is based on a Gronwall-type inequality. Assume that there are two solutions $[q(x)]$ and $[s(x)]$ of (4.7) which satisfy the same initial condition $[s(0)]$. Then

$$\begin{aligned} \frac{|s_{j,x}(x) - q_{j,x}(x)|}{\sqrt{d_j}} &\leq \left| \frac{\mathcal{M}(t)}{p(t)} \right| \frac{d_j(|G_j([s(x)]) - G_j([q(x)])|)}{\sqrt{d_j}} \leq \\ &\leq \frac{\Delta}{\delta} \sum_{k \in \mathbb{N}} \sqrt{d_k} \sqrt{d_j} \sup_{[X] \in \mathcal{I}} \left| \frac{\partial G_r([X])}{\partial X_k} \right| \left\| [s(x)] - [q(x)] \right\|_{\mathcal{I}} \end{aligned}$$

Using the estimates for the derivatives of G_r in the previous Proposition, we obtain

$$\sum_{k\in\mathbb{N}} \sqrt{d_j}\,\sqrt{d_k} \sup_{[X]\in\mathcal{I}} \left|\frac{\partial G_j}{\partial X_k}([X])\right| \le CC_1(2D_1 + D_1^2),$$

hence

$$\sup_{j\in\mathbb{N}} \frac{|s_{j,x}(x) - q_{j,x}(x)|}{\sqrt{d_j}} \le \tilde{C}\|[s(x)] - [q(x)]\|_{\mathcal{I}},$$

where $\tilde{C} = \dfrac{\Delta}{\delta}CC_1(2D_1 + D_1^2)$.

Next, consider the map $F : x \mapsto \left\|[s(x)] - [q(x)]\right\|_{\mathcal{I}}$. Then $F(x)$ is continuous, and the above estimate implies that

$$F(x) \le \tilde{C}\int_0^x F(s)ds.$$

This means that $F(x) \equiv 0$ by standard arguments. The theorem is proved.

Now we shift our attention to the question of the existence of a solution of the system (4.7). We will give a proof which is based on a so-called *algebro-geometric approximation*. This method has been successfully applied in [14] to construct reflectionless Sturm-Liouville potentials with prescribed properties. We repeat here the ideas of [14]. Fix $n \in \mathbb{N}$. Let us consider the set $\Lambda_n = \{\lambda_0, \lambda_1, \ldots, \lambda_{2n}\} \subset \Lambda_0$, where Λ_0 satisfies Hypotheses (4.1). Let $\Sigma_n = [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$. Further, let us fix a point $[P(0)] = (P_1(0), \ldots, P_n(0), \ldots) \in \mathcal{I}$, and let us consider the first n components of $[P(0)]$, which we denote by $\mathcal{P}^n(0) := (P_1(0), \ldots, P_n(0))$. Let us fix a pair $(p, \mathcal{M}_n) \in \mathcal{E}_2$, and let $a_n = (p, q_n, y_n) \in \mathcal{E}_3$ be the algebro geometric Sturm-Liouville potentials obtained as in Section 2. Then Σ_n is the spectrum of the operator $L_n := L_{a_n}$, and $\mathcal{G}_n(x, \eta) := \mathcal{G}_{a_n}(x, \eta) = 0$ for a.a. $\eta \in \mathbb{R}$ and for all $x \in \mathbb{R}$. Recall that the (simple) zeros of $\mathcal{G}_n(x, \eta)$, namely $P_1^{(n)}(x), \ldots P_n^{(n)}(x)$, satisfy the system

$$P_{j,x}^{(n)}(x) = (-1)^n \frac{\mathcal{M}_n(x)\prod_{i=1}^n P_i^{(n)}(x)k_n(P_j^{(n)}(x))}{p(x)k_n(0^+)\prod_{k\neq j}(P_j^{(n)}(x) - P_k^{(n)}(x))} = \frac{\mathcal{M}_n(x)}{p(x)}F_j^{(n)}(x). \tag{4.15}$$

as in Section 2.

Again, one obtains a more accurate description of the motion of the points $P_j^{(n)}(x)$ by using angular coordinates, as follows:

$$P_j^{(n)}(x) = \lambda_{2j-1} + d_j \sin^2\frac{\theta_j^{(n)}(x)}{2}, \qquad j = 1, \ldots, n, \quad n \in \mathbb{N}. \tag{4.16}$$

Next, we already stated in Section 3 that there exists a matrix

$$B_n = \begin{pmatrix} -T_n & \dfrac{U_n}{p\lambda^k} \\ \dfrac{1}{\lambda^k}(q_n - \lambda y_n)V_n & T_n \end{pmatrix}$$

such that, if

$$A_n = \begin{pmatrix} 0 & \dfrac{1}{p} \\ q_n - \lambda y_n & 0 \end{pmatrix},$$

then

$$-B_{n,x} + [A_n, B_n] = 0.$$

Here $U_n = \sum_{i=0}^{n} u_j^{(n)}(x)\lambda^j$, T_n and V_n are defined via the relations

$$T_n = \frac{p}{2\lambda^k}\left(\frac{U_n}{p}\right)_x, \tag{4.17}$$

and

$$T_{n,x}(x,\lambda) + \frac{\lambda^{-k}}{p(x)}(q(x) - \lambda y(x))(V_n(x,\lambda) - U_n(x,\lambda)) = 0. \tag{4.18}$$

Moreover, one has that (3.1) holds, i.e., $\frac{d}{dx}\det B_n = 0$, which translates to

$$p^2\frac{\tilde{U}_{n,x}^2}{4} + \frac{1}{p}(q_n - \lambda y_n)U_n V_n = \prod_{i=0}^{2n}\left(1 - \frac{\lambda}{\lambda_i}\right), \tag{4.19}$$

where

$$\tilde{U}_n(x,\lambda) = \frac{-2}{\mathcal{M}_n(x)}\prod_{i=1}^{n}\left(1 - \frac{\lambda}{P_i^{(n)}(x)}\right)$$

(for the details, see again [15]). The validity of the stationary zero-curvature condition $-B_{n,x} + [A_n, B_n] = 0$ (or, *equivalently*, the fact that a_n is an algebro-geometric potential) implies that

$$y_n(x) = \frac{1}{p(x)(u_n^{(n)}(x))^2 k_n^2(0^+)} = \frac{\mathcal{M}_n^2(x)}{4p(x)}\frac{\prod_{i=1}^{n}\left[P_i^{(n)}(x)\right]^2}{k_n^2(0^+)}, \tag{4.20}$$

for every $n \in \mathbb{N}$. Moreover, one has

$$q_n(x) = y_n(x)\left(\lambda_0 + \sum_{i=1}^{n}\left(\lambda_{2i} + \lambda_{2i-1} - 2P_i^{(n)}(x)\right)\right) + g_{n,x}(x) + \frac{g_n^2(x)}{p(x)}, \tag{4.21}$$

where $g_n(x) = -\frac{(p(x)y_n(x))_x}{4y_n(x)}$.

We have made all these preparatory remarks with the goal of solving the problem of the existence of a solution of the system (4.7). As anticipated before, we will prove the existence of a solution of (4.7) by approximation.

For this, we first fix a sequence $\{(p, \mathcal{M}_n)\} \subset \mathcal{E}_2$ such that both $\mathcal{M}_n$ and $\mathcal{M}_{n,x}$ converge to $\mathcal{M}$ and $\mathcal{M}_x$ uniformly on compact subsets of $\mathbb{R}$. Next, compare (4.15) and (4.7); it is clear that the right-hand side of (4.15) converges pointwise to the right-hand side of (4.7). Here we take the point of view that for each $n \geq 1$, $I_1 \times \cdots \times I_n$ is embedded in $\mathcal{I}$ via the map $(P_1, \ldots, P_n) \mapsto (P_1, \ldots, P_n, P_{n+1}(0), \ldots)$.

Let us consider the sequence $\{P_j^{(n)}(x)\}_n$. We will apply the Ascoli-Arzelà Theorem to this sequence. For, we first prove that $\{P_j^{(n)}(x)\}_n$ is equibounded on all $\mathbb{R}$. This is a very easy consequence of Hypotheses 4.1: we have

$$|P_j^{(n)}(x)| \leq h_{0j} + d_j$$

for every $x \in \mathbb{R}$. Next, we prove equicontinuity of $\{P_j^{(n)}(x)\}_n$ by showing that the sequence of the derivatives $\{P_{j,x}^{(n)}(x)\}_n$ is equibounded on $\mathbb{R}$. We have

$$\begin{aligned}|P_{j,x}^{(n)}(x)| &\leq \left|\frac{\mathcal{M}_n(x)}{p(x)\sqrt{\lambda_0}}\right| \prod_{i=1}^{n} \frac{P_i^{(n)}(x)}{\sqrt{\lambda_{2i-1}\lambda_{2i}}} \left|\frac{k_n(P_j^{(n)}(x))}{\prod_{k\neq j}(P_j^{(n)}(x) - P_k^{(n)}(x))}\right| \leq \\ &\leq \frac{\Delta}{\delta\sqrt{\lambda_0}} d_j \sqrt{h_{0j} + d_j} \prod_{k=1}^{\infty}\left(1 + \frac{d_k}{h_{0k}}\right) \prod_{l\neq j}^{\infty}\left(1 + \frac{d_l}{h_{jl}}\right) \leq \\ &\leq \frac{\Delta}{\delta} C_2.\end{aligned}$$

By the Ascoli-Arzelà Theorem, there exists a subsequence $\{P_j^{(n_k)}(x)\}_k \subset \{P_j^{(n)}(x)\}_n$ which converges uniformly on compact subsets of $\mathbb{R}$ to a function $P_j : \mathbb{R} \to I_j$ ($j \in \mathbb{N}$).

We want to show that $P_j(x) = s_j(x)$ for every $j \in \mathbb{N}$, where $s_j(x)$ is expressed by (4.7). To do this, we prove the following

Proposition 4.3 *For every $j \in \mathbb{N}$, the sequence $\{P_{j,x}^{(n)}(x)\}_n$ admits a subsequence which converges uniformly on compact subsets of $\mathbb{R}$.*

Proof. The proof is the same as that in [14]. We repeat here the main ideas.

It suffices to show that the sequence $\{F_j^{(n)}(x)\}_n$ is equicontinuous and equibounded on all $\mathbb{R}$. That $\{F_j^{(n)}(x)\}_n$ is equibounded follows from the estimate

$$|F_j^{(n)}(x)| \leq C_2.$$

To prove equicontinuity, we use the angular coordinates $\theta_j^{(n)}(x) \in [0, 2\pi]$. Indeed, if the sequence $\{\theta_{j,x}^{(n)}(x)\}_n$ is equicontinuous, the relation (4.16) shows that the sequence $\{P_{j,x}^{(n)}(x)\}_n$ is equicontinuous as well. The system (4.15) translates to

$$\begin{aligned}\theta_{j,x}^{(n)}(x) &= (-1)^n \frac{\mathcal{M}_n(x)}{p(x)} \frac{\prod_{k=1}^{n} P_k^{(n)}(x)}{k_n(0^+)} \sqrt{P_j^{(n)}(x) - \lambda_0} \times \\ &\times \prod_{l\neq j,\, l=1}^{n} \frac{\sqrt{(P_j^{(n)}(x) - \lambda_{2l-1})(P_j^{(n)}(x) - \lambda_{2l})}}{P_j^{(n)}(x) - P_l^{(n)}(x)} := \\ &:= \frac{\mathcal{M}_n(x)}{p(x)} G_j^{(n)}(x).\end{aligned}$$

Again, note the ambiguity deriving from the sign of $k_n(P_j^{(n)}(x))$ in (4.15) disappears when considering the angular coordinates. We will prove that the sequence $\{\theta_{j,xx}^{(n)}(x)\}_n$ is equibounded, which implies that $\{\theta_{j,x}^{(n)}(x)\}_n$ is equicontinuous. It suffices to show that the sequence $\{G_{j,x}^{(n)}(x)\}_n$ is equibounded, because

$$|\theta_{j,xx}^{(n)}(x)| \leq \left|\left(\frac{\mathcal{M}_n(x)}{p(x)}\right)_x\right| C\sqrt{d_j + h_{0j}} + \frac{\Delta}{\delta}|G_{j,x}^{(n)}(x)|.$$

We have

$$|G_{j,x}^{(n)}(x)| \leq \sum_{k=1}^{n} \sup_{[X]\in\mathcal{I}} \left|\frac{\partial G_j^{(n)}}{\partial X_k}(x)\right| \left|P_{j,x}^{(n)}(x)\right| \leq \frac{\Delta}{\delta} \sum_{k=1}^{n} \sup_{[X]\in\mathcal{I}} \left|\frac{\partial G_j^{(n)}}{\partial X_k}(x)\right| |F_j^{(n)}(x)|.$$

We use estimates analogous to those in (4.13) and (4.14), to obtain

$$|G_{j,x}^{(n)}(x)| \leq \frac{\Delta}{\delta} 2C^2 C_1 (2D_1 + D_1^2) \sqrt{d_j + h_{0j}} = C(j).$$

This shows that

$$|\theta^{(n)}_{j,xx}(x)| \le C_2(j),$$

and that

$$|P^{(n)}_{j,xx}(x)| \le C_3(j).$$

The constants $C_1(j), C_2(j)$ and $C_3(j)$ depend on j, and in particular one has

$$C_1, C_2 \sim d_j \sqrt{d_j + h_{0j}}, \qquad C_3 \sim d_j(d_j + h_{0j}).$$

The proof now follows by applying the Ascoli-Arzelà Theorem.

A further elementary argument of calculus allows us to state the following result.

Theorem 4.3 *Let Λ_0 be a sequence which satisfies Hypotheses 4.1. Let $[s(0)] \in \mathcal{I}$ be a fixed point. Then there exists a unique solution $[s(x)] : \mathbb{R} \to \mathcal{I}$ of the system (4.7). Moreover, if $[s(x)] = (s_1(x), \dots, s_n(x), \dots)$ we have*

$$s_j(x) = \lim_{n\to\infty} P^{(n)}_j(x)$$

and

$$s_{j,x}(x) = \lim_{n\to\infty} P^{(n)}_{j,x}(x)$$

uniformly on compact subsets of $\mathbb{R}$.

Let us consider the sequence Λ_0 which satisfies Hypotheses 4.1. We prove the following result

Proposition 4.4 *Let Λ_0 be given in such a way that Hypotheses 4.1 holds. For every $n \in \mathbb{N}$, let $(p, \mathcal{M}_n) \in \mathcal{E}_2$ be such that $(p, \mathcal{M}_n) \to (p, \mathcal{M}) \in \mathcal{E}_2$ and $\mathcal{M}_{n,x} \to \mathcal{M}_x$ and $\mathcal{M}_{n,xx} \to \mathcal{M}_{xx}$ uniformly on compact subsets of $\mathbb{R}$. Define the triple $a_n = (p, q_n, y_n)$, where*

$$y_n = \frac{\mathcal{M}_n^2(x)}{4p(x)} \frac{\prod_{j=1}^n \left[P^{(n)}_j(x)\right]^2}{k_n^2(0^+)},$$

$$q_n(x) = y_n(x)\left(\lambda_0 + \sum_{i=1}^n \left(\lambda_{2i} + \lambda_{2i-1} - 2P^{(n)}_i(x)\right)\right) + g_{n,x}(x) + \frac{g_n^2(x)}{p(x)},$$

where $g_n(x) = -\dfrac{(p(x)y_n(x))_x}{4y_n(x)}$ and the points $P^{(n)}_j(x)$ are defined by (4.16). Then

$$\lim_{n\to\infty} y_n(x) = y(x) = \frac{\mathcal{M}^2(x)}{4p(x)} \frac{\prod_{j=1}^\infty s_j^2(x)}{k_\infty^2(0)}$$

and

$$\lim_{n\to\infty} q_n(x) = q(x) = y(x)\left(\lambda_0 + \sum_{i=1}^\infty \left(\lambda_{2i} + \lambda_{2i-1} - 2s_j(x)\right)\right) + g_x(x) + \frac{g^2(x)}{p(x)},$$

uniformly on compact subsets of $\mathbb{R}$. Here $g(x) = -\dfrac{(p(x)y(x))_x}{4y(x)}$.

The spectrum Σ_a of the operator $L_a := \dfrac{1}{y}(-DpD+q)$ equals the set $\bigcap_{n=1}^\infty \Sigma_{a_n}$. Moreover $m_\pm(a_n, \lambda) \to m_\pm(a, \lambda)$ uniformly on compact subsets of $\mathbb{C}^+$, which implies that $a = (p, q, y)$ is reflectionless.

Proof. Actually, we only have to prove that y_n (resp. q_n) converges to y (resp. q) uniformly on compact subsets of $\mathbb{R}$. The other statements of the proposition follow from this convergence by the results in [14]. We will repeat briefly here some on the main steps for completeness.

We will use the familiar method which consists on applying the Ascoli-Arzelà theorem. The sequence $\{y_n\}_n$ is equibonded on all $\mathbb{R}$, because

$$|y_n(x)| \leq \frac{\Delta^2}{4\delta^2\lambda_0}\prod_{j=1}^{n}\left(1+\frac{d_j}{h_{0j}}\right)^2 = \frac{\Delta^2}{4\delta^2\lambda_0}\tilde{D}_1^2 < \infty,$$

where

$$\tilde{D}_1 = \prod_{j=1}^{\infty}\left(1+\frac{d_j}{h_{0j}}\right).$$

Moreover

$$|y_{n,x}(x)| \leq |\sqrt{y_n(x)}| \left|\left(\frac{\mathcal{M}_n(x)}{\sqrt{p(x)}}\frac{\prod_{j=1}^n P_j^{(n)}(x)}{k_n(0^+)}\right)_x\right|.$$

Now, to show that the sequence $\{y_n(x)\}_n$ is equibounded it suffices to show that

$$\left(\frac{\prod_{j=1}^n P_j^{(n)}(x)}{k_n(0^+)}\right)_x$$

is equibounded. This is easy, because

$$\left|\left(\frac{\prod_{j=1}^n P_j^{(n)}(x)}{k_n(0^+)}\right)_x\right| \leq \frac{\Delta}{\delta\sqrt{\lambda_0}}CC_1\tilde{D}_1\left(\sum_{j=1}^{\infty}\frac{\sqrt{d_j}}{h_{0j}}\right) \leq \frac{\Delta}{\delta\sqrt{\lambda_0}}CC_1\tilde{D}_1 D_1 < \infty.$$

Now, since y_n converges to y uniformly on compact subsets of $\mathbb{R}$, using the Riccati equation (2.2) and the assumptions concerning the functions $\mathcal{M}_n$, we have

$$q_n = -\frac{1}{2}\left(\frac{p\mathcal{M}_{n,x}}{\mathcal{M}_n}\right)_x + \frac{1}{4p}\left[\mathcal{M}_n^2 + \left(\frac{p\mathcal{M}_{n,x}}{\mathcal{M}_n}\right)^2\right] \to$$
$$\to -\frac{1}{2}\left(\frac{p\mathcal{M}_x}{\mathcal{M}}\right)_x + \frac{1}{4p}\left[\mathcal{M}^2 + \left(\frac{p\mathcal{M}_x}{\mathcal{M}}\right)^2\right] = q,$$

uniformly on compact subsets of $\mathbb{R}$. We have proved that it can be assumed that the sequences $\{q_n\}_n$ and $\{y_n\}_n$ converge to q and y respectively, uniformly on compact subsets of $\mathbb{R}$.

Now, the second part of the proof goes as follows: one uses some results in [11] and the decreasing-disc construction of the Weyl m-functions to show that $m_\pm(a_n,\lambda)$ converge to $m_\pm(a,\lambda)$ uniformly on compact subsets of $\mathbb{C}^+$. Hence one has that $\mathcal{G}(a_n,\lambda)$ converges to $\mathcal{G}(a,\lambda)$ uniformly on compact subsets of $\mathbb{C}^+$. Next we consider the functions $h_n(\lambda) = \ln\mathcal{G}(a_n,\lambda)$ and $h(\lambda) = \ln\mathcal{G}(a,\lambda)$. They are holomorphic in $\mathbb{C}^+$, and their imaginary parts $\Im h_n(\lambda)$ and $\Im h(\lambda)$ are bounded between 0 and π. Let us consider a compact interval $I \subset \mathbb{R}$, and let D be a semidisc contained in $\mathbb{C}^+$ whose diameter is exactly I. It is known that $\Im h_n$ and $\Im h$ have nontangential limits defined in the boundary ∂D of the semidisc D. Since h_n converges to h uniformly on compact subsets of $\mathbb{C}^+$, we see that $\Im h_n$ converges to $\Im h$ weakly in $L^2(\partial D)$, and hence weakly in $L^2(I)$ (see the proof of Lemma 7.4 in [17]). Now, let us consider a Borel set $B \subset \mathbb{R}$ which is contained in the spectrum of L_{a_n} for all $n \in \mathbb{N}$. By assumption, we have $\Re\mathcal{G}(a_n,\lambda) = 0$ for a.a. $\lambda \in B\cap I$, indeed *for all* $\lambda \in B\cap I$, because each a_n is algebro-geometric. This implies that $\Im h_n = \pi/2$ for all $\lambda \in B\cap I$, and hence $\Im h = \pi/2$ for a.a. $\lambda \in B\cap I$. But this tells us exactly that $\Re\mathcal{G}(a,\lambda) = 0$ for a.a. $\lambda \in B\cap I$. The above arguments can be repeated for every potential $\tau_x(a_n)$, hence one obtains that $\Re\mathcal{G}(a,x,\lambda) = 0$ for a.a. λ lying in B, if B is a Borel set which is contained in the spectrum of $L_{\tau_x(a_n)}$ for every $n \in \mathbb{N}$.

Now, $\Sigma_{a_{n+1}} \subset \Sigma_{a_n}$ for all $n \in \mathbb{N}$. Let us set $\Sigma_* = \bigcap_{n\in\mathbb{N}} \Sigma_{a_n}$. From what we have said, for each $x \in \mathbb{R}$, $\Re G(a,x,\lambda) = 0$ for a.a. $\lambda \in \Sigma_*$. So to prove that a is reflectionless, it is sufficient to prove that $\Sigma_* = \Sigma_a$.

First, one can show that $\Im\mathcal{G}(a,\lambda + i0) > 0$ for a.a. $\lambda \in \Sigma_*$. This implies that $\Sigma_* \subset \Sigma_a$. Moreover, if $\lambda \notin \Sigma_*$, and λ is a real number, then there exists an interval $J = [\lambda - \delta, \lambda + \delta]$ such that $J \cap \Sigma_{a_n} = \emptyset$ for all sufficiently large $n \in \mathbb{N}$. The spectral measure e_n associated to a_n converges in the weak-$*$ sense to the spectral measure e associated to a. Now, $\int_J de_n = 0$ for sufficiently large $n \in \mathbb{N}$, and hence $\int_J de = 0$. This means that $J \cap \Sigma_a = \emptyset$, and so $\Sigma_a \subset \Sigma_*$. The proof is complete.

At this point, we introduce some notation which will be useful in the following lines.

Definition 4.1 *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n \ldots\}$ be any sequence of positive real numbers such that $\lambda_{2i-1} < \lambda_{2i}$ and $(\lambda_{2i-1}, \lambda_{2i}) \cap \Lambda = \emptyset$ for every $i \in \mathbb{N}$ (the sequence Λ need not have cluster points). Let*

$$C(\Lambda) := (-\infty, \lambda_0) \bigcup_{i\in\mathbb{N}} (\lambda_{2i-1}, \lambda_{2i}),$$

and

$$\Sigma = \mathbb{R} \setminus C(\Lambda).$$

Assume further that Σ has locally positive Lebesgue measure. We define a set $\mathcal{R}_\Lambda \subset \mathcal{E}_3$ as follows: $\mathcal{R}_\Lambda = \{a = (p,q,y) \in \mathcal{E}_3 \mid a$ is a reflectionless Sturm-Liouville potential and the spectrum of the full-line operator L_a equals $\Sigma\}$.

We have proved that, given a sequence Λ_0 satisfying Hypotheses 4.1, an element $[s(0)] \in \mathcal{I}$, and two fixed positive functions $p, \mathcal{M} \in \mathcal{E}_2$ as in Proposition 4.4, it is possible to construct a sequence $\{a_n\}_n = \{(p, q_n, y_n)\}_n \subset \mathcal{E}_3$ consisting of algebro-geometric Sturm-Liouville potentials which converges to a triple $a = (p,q,y) \in \mathcal{E}_3$ uniformly on compact subsets of $\mathbb{R}$. The triple $a \in \mathcal{E}_3$ obtained via the approximation described above has the fundamental property of being a reflectionless Sturm-Liouville potential lying in $\mathcal{R}_{\Lambda_0}$.

Now, let Λ_0 be a given sequence which satisfies Hypotheses 4.1 and let $a = (p,q,y) \in \mathcal{R}_{\Lambda_0}$ be obtained via the algebro-geometric approximation described in the above lines. Let

$$A = \begin{pmatrix} 0 & 1/p(x) \\ q(x) - \lambda y(x) & 0 \end{pmatrix},$$

$U(x,\lambda) = p(x)\tilde{U}(x,\lambda)$, $\tilde{U}(x,\lambda)$ defined as in (4.3) and $V(x,\lambda)$ and $T(x,\lambda)$ as in (4.2) and (4.1) respectively. We want to show that the zero-curvature relation $-B_x + [A,B] = 0$ is satisfied. For this, one starts again with algebro-geometric potentials. So, let $a_n = (p, q_n, y_n)$ be the algebro-geometric potentials which approximate $a = (p,q,y)$. To each potential a_n one can associate the matrix

$$A_n = \begin{pmatrix} 0 & 1/p(x) \\ q_n(x) - \lambda y_n(x) & 0 \end{pmatrix}.$$

Clearly, one has

$$\lim_{n\to\infty} A_n = A,$$

uniformly on compact subsets of $\mathbb{R}$, and for every $\lambda \in \mathbb{C}$. As explained in Section 3, there exists a polynomial

$$U_n(x,\lambda) = \sum_{j=0}^{n} u_j^{(n)}(x)\lambda^j = \frac{-2p(x)}{\mathcal{M}_n(x)} \prod_{j=1}^{n} \left(1 - \frac{\lambda}{P_j^{(n)}(x)}\right),$$

such that the n-tuple $(P_1^{(n)}(x), \dots, P_n^{(n)}(x))$ is the solution of the system (4.15) with initial data $(s_1(0), \dots, s_n(0))$. If $T_n(x, \lambda)$ and $V_n(x, \lambda)$ are defined as in (4.17) and (4.18) respectively, and if we define

$$B_n(x,\lambda) = \begin{pmatrix} -T_n & \dfrac{U_n}{p\lambda^k} \\ \dfrac{1}{\lambda^k}(q_n - \lambda y_n)V_n & T_n \end{pmatrix},$$

then the stationary zero curvature equation of order n is satisfied, i.e.,

$$-B_{n,x}(x,\lambda) + [A_n(x,\lambda), B_n(x,\lambda)] = 0,$$

together with (4.19).

Let us think of $B_n(x,\lambda)$ as a function $B_n : \mathbb{R} \times \mathbb{C} \to \mathbb{M}(2,\mathbb{C})$, and equip the space $\mathbb{M}(2,\mathbb{C})$ with the usual norm.

Next, let us fix $x \in \mathbb{R}$, and consider the maps $\lambda \mapsto U_n(x,\lambda)$. We claim that the family $\{U_n(x,\cdot)\}$ is normal in the complex plane $\mathbb{C}$, hence it converges uniformly on compact subsets of $\mathbb{C}$ to a function $U(x,\cdot)$ which clearly must coincide with that function $U(x,\cdot)$ obtained by multiplying by $p(x)$ the function $\tilde{U}(x,\cdot)$ defined as in (4.4). To show this, it suffices to prove that the family $\{U_n(x,\lambda)\}$ is uniformly bounded on compact subsets of $\mathbb{C}$, and in fact if $|\lambda| \le k$,

$$|U_n(x,\lambda)| \le \frac{2\Delta}{\delta} \prod_{i=1}^{\infty} \left(1 + \frac{k}{\lambda_{2i-1}}\right), \tag{4.22}$$

and the claim follows since the infinite product in (4.22) converges. Note that (4.22) implies that the maps $x \mapsto U_n(x,\lambda)$ are uniformly bounded on all $\mathbb{R}$, for every fixed $\lambda \in \mathbb{C}$. A simple application of the Cauchy Theorem shows that for every $x \in \mathbb{R}$, the coefficients $u_j^{(n)}(x)$ of $U_n(x,\lambda)$ converge to the coefficients $u_j(x)$ of the Taylor expansion of $U(x,\lambda)$ $(j = 1, 2, \dots)$.

We now prove that $U_n(x,\lambda) \to U(x,\lambda)$ uniformly for x in compact subsets of $\mathbb{R}$, whenever λ lies in a compact subset $K \subset \mathbb{C}$. Let $K = \{\lambda \in \mathbb{C} \mid |\lambda| \le k\}$. Let us estimate the values $|U_{n,x}(x,\lambda)|$ when $\lambda \in K$. We have, simplifying the notation,

$$\begin{aligned} |U_{n,x}| &\le \left|\frac{p_x}{p} - \frac{\mathcal{M}_{n,x}}{\mathcal{M}_n}\right| |U_n| + \left|\frac{2p}{\mathcal{M}_n}\right| \left(\sum_{j=1}^{n} \frac{|\lambda P_{j,x}^{(n)}|}{(P_j^{(n)})^2} \prod_{l \ne j} \left(1 + \frac{|\lambda|}{P_l^{(n)}}\right)\right) \le \\ &\le \frac{2\Delta}{\delta} \prod_{i=1}^{\infty} \left(1 + \frac{k}{\lambda_{2i-1}}\right) \left(\alpha + \frac{\Delta}{\delta} k C C_1 \left(\sum_{j \in \mathbb{N}} \frac{\sqrt{d_j}}{h_{0j}^2}\right)\right) < \infty, \end{aligned} \tag{4.23}$$

where $\alpha = \sup_{x \in \mathbb{R}} \left|\dfrac{p_x}{p} - \dfrac{\mathcal{M}_{n,x}}{\mathcal{M}_n}\right|$.

It follows from the above estimates that the derivatives $U_{n,x}$ are uniformly bounded whenever λ lies in a compact subset of $\mathbb{C}$, hence we can find a subsequence in $\{U_n(x,\lambda)\}$ which converges to $U(x,\lambda)$ uniformly on compact subsets of $\mathbb{R}$, whenever λ lies in a compact subset of $\mathbb{C}$.

We claim that, for every $\lambda \in \mathbb{C}$, $B_n(x,\lambda)$ is bounded uniformly on compact subsets of $\mathbb{R}$, and converges to $B(x,\lambda)$ pointwise in x. To prove this, we use the zero-curvature relations $-B_{n,x} + [A_n, B_n] = 0$ and (4.19). Fix $\lambda \in \mathbb{C}$. It is clear now that the maps $x \mapsto A_n(x,\lambda)$ converge to $A(x,\lambda)$ uniformly on compact subsets of $\mathbb{R}$. Moreover, let us examine the map $x \mapsto B_n(x,\lambda)$. We will show that all the entries of B_n are uniformly bounded on compact subsets of $\mathbb{R}$. We observed in the above lines that the functions $U_n(x,\lambda)$ are uniformly bounded on each compact subset of $\mathbb{R}$ and converge to $U(x,\lambda)$ uniformly on compact subsets of $\mathbb{R}$. Moreover, the functions $U_{n,x}(x,.\lambda)$ are uniformly bounded on each compact subset of $\mathbb{R}$ in view of (4.23). Recall that $T_n(x,\lambda)$ and $V_n(x,\lambda)$ are defined via the relations (4.17) and (4.18) respectively, hence $T_n(x,\lambda)$ is uniformly bounded on each compact subset of $\mathbb{R}$. Next, we use (4.19) to see that

$$(q_n - \lambda y_n)V_n = \left(k_n^2(\lambda) - \frac{p^2\tilde{U}_{n,x}^2}{4}\right)\frac{p}{U_n}$$

for every $\lambda \neq s_j(x)$. This shows that the functions $(q_n(x) - \lambda y_n(x))V_n(x,\lambda)$ are uniformly bounded in each compact subset of $\mathbb{R}$ (if $\lambda = s_j(x)$ for some $j \in \mathbb{N}$, then after some cancellations, one obtains in the right-hand side of the above expression a bounded quantity as well). We conclude that the maps $x \mapsto B_n(x,\lambda)$ are uniformly bounded on each compact subset of $\mathbb{R}$. Clearly $B_n(x,\lambda) \to B(x,\lambda)$ pointwise in x, for every $\lambda \in \mathbb{C}$.

Integrating the stationary zero-curvature relation with respect to x we have

$$B_n(x,\lambda) - B_n(0,\lambda) = \int_0^x [A_n(s,\lambda), B_n(s,\lambda)]ds.$$

In view of the above observations, we can apply the dominated convergence theorem to see that

$$B(x,\lambda) - B(0,\lambda) = \int_0^x [A(s,\lambda), B(s,\lambda)]ds.$$

In conclusion,

$$B_x(x,\lambda) = [A(x,\lambda), B(x,\lambda)]$$

holds, together with (4.6).

We summarize the results so far obtained by using the notation we have introduced in Definition 4.1. We take a sequence $\Lambda_0 = \{\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots\}$ satisfying Hypotheses 4.1, then pick a pair $(p, \mathcal{M}) \in \mathcal{E}_2$ as in Proposition 4.4. We assume that the stationary zero-curvature equation

$$-B_x + [A,B] = 0$$

holds, together with the condition

$$\lambda^k \det B = \prod_{i=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_j}\right).$$

Then set $U(x,\lambda) = \sum_{j=0}^{\infty} u_j(x)\lambda^j$, $T(x,\lambda)$ as in (4.1) and $V(x,\lambda)$ as in (4.2). We write $U(x,\lambda)$ as an infinite product, as follows:

$$U(x,\lambda) = \frac{-2p(x)}{\mathcal{M}(x)}\prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{s_j(x)}\right).$$

The zero-curvature equation determines the system (4.7) for the motion of the zeros $s_j(x)$ of $U(x,\lambda)$, with a fixed initial condition $[s(0)] \in \mathcal{I}$. The system (4.7) admits a unique solution $[s(x)] \in \mathcal{I}$ which we find by an algebro-geometric approximation. It turns out that the triple $a = (p,q,y) \in \mathcal{E}_3$ obtained by such an algebro-geometric approximation is a reflectionless Sturm-Liouville potential and that the spectrum Σ_a of the operator L_a is given by

$$\Sigma_a = \bigcup_{i=1}^{\infty}[\lambda_{2i-2}, \lambda_{2i-1}],$$

i.e., $a \in \mathcal{R}_{\Lambda_0}$.

Moreover, arguing exactly as in [15], we can go backwards, i.e., if we take a reflectionless Sturm-Liouville potential $a = (p, q, y) \in \mathcal{R}_{\Lambda_0}$, and if we set $A(x,\lambda) = \begin{pmatrix} 0 & 1/p(x) \\ q(x) - \lambda y(x) & 0 \end{pmatrix}$, then there exists a matrix $B(x,\lambda)$ of the type described above such that

$$-B_x + [A, B] = 0, \qquad \lambda^k \det B = \prod_{j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right).$$

In conclusion, we can state the following

Theorem 4.4 *A triple $a(x) = (p(x), q(x), y(x))$ lies in $\mathcal{R}_{\Lambda_0}$ if and only if there exists a matrix $B(x,\lambda) = \begin{pmatrix} -T & \lambda^{-k} U/p \\ \lambda^{-k}(q - \lambda y)V & T \end{pmatrix}$, where $U(x,\lambda)$ is an entire function with zeros $s_j(x)$ ($j \in \mathbb{N}$) satisfying* (4.7)*, $T(x,\lambda)$ and $V(x,\lambda)$ are defined as in* (4.1) *and* (4.2) *respectively, such that*

$$-B_x(x,\lambda) + [A(x,\lambda), B(x,\lambda)] = 0, \qquad \lambda^k \det B(x,\lambda) = \prod_{j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right)$$

are valid.

5 The time-dependent hierarchy

The aim of this section is to introduce and solve a time-dependent hierarchy of evolution equations. The solution of this hierarchy will be given with initial data a reflectionless potential $a(x) \in \mathcal{R}_{\Lambda_0}$.

Let us fix two numbers $r \geq k \geq 0$ and introduce a time dependency in our discussion, as follows: let us consider matrices

$$A(t,x,\lambda) = \begin{pmatrix} 0 & 1/p(t,x) \\ q(t,x) - \lambda y(t,x) & 0 \end{pmatrix},$$

$$B(t,x,\lambda) = \begin{pmatrix} -T(t,x,\lambda) & \lambda^{-k} \dfrac{U(t,x,\lambda)}{p(t,x)} \\ \lambda^{-k}(q(t,x) - \lambda y(t,x))V(t,x,\lambda) & T(t,x,\lambda) \end{pmatrix}$$

and

$$B_r(t,x,\lambda) = \begin{pmatrix} -T_r(t,x,\lambda) & \lambda^{-k} \dfrac{U_r(t,x,\lambda)}{p(t,x)} \\ \lambda^{-k}(q(t,x) - \lambda y(t,x))V_r(t,x,\lambda) & T_r(t,x,\lambda) \end{pmatrix}.$$

Here the matrix $B(t,x,\lambda)$ is analogous to that in the stationary case, but now the coefficients depend on t as well. The matrix B_r has entries which we will define shortly.

We pose the following basic problem: do there exist matrices A, B and B_r as above in such a way that

$$\begin{cases} A_t(t,x,\lambda) - B_{r,x}(t,x,\lambda) + [A(t,x,\lambda), B_r(t,x,\lambda)] = 0 \\ -B_x(t,x,\lambda) + [A(t,x,\lambda), B(t,x,\lambda)] = 0 \end{cases} \tag{5.1}$$

and the Ansatz

$$\lambda^k \det B(t,x,\lambda) = \prod_{i=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_i}\right) \tag{5.2}$$

hold for every $t, x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$?

Before analysing in detail the problem we have just introduced, we make a fundamental observation. The second equation in (5.1) and (5.2) are equivalent to saying that the map $t \mapsto a(t,x) = (p(t,x), q(t,x), y(t,x))$ is a curve in the set $\mathcal{R}_{\Lambda_0}$.

The first equation in (5.1) gives

$$\begin{cases} T_{r,x} + \dfrac{\lambda^{-k}}{p}(q - \lambda y)(U_r - V_r) = 0 \\ \\ \left(\dfrac{1}{p}\right)_t - \lambda^{-k}\left(\dfrac{U_r}{p}\right)_x + \dfrac{2}{p}T_r = 0, \end{cases} \tag{5.3}$$

and, setting $\tilde{U}_r = U_r/p$,

$$\begin{aligned} 2\lambda^k(q-\lambda y)_t - \frac{p_t}{p}(q-\lambda y) + \left(p\left(\frac{p_t}{p}\right)_x\right)_x &= 2(p(q-\lambda y))_x\tilde{U}_r + \\ &+ 4p(q-\lambda y)\tilde{U}_{r,x} - (p(p\tilde{U}_{r,x})_x)_x. \end{aligned} \tag{5.4}$$

The relations (5.3) define T_r and V_r once U_r is known, hence we have to look for formulas to determine U_r and the triple $a(t,x) = (p(t,x), q(t,x), y(t,x))$. For the moment, assume that $\tilde{U}_r$ is a polynomial of degree r in $\lambda \in \mathbb{C}$ and with coefficients which depend on t and x, i.e.,

$$\tilde{U}_r(t,x,\lambda) = \sum_{j=0}^{r} f_j(t,x)\lambda^j.$$

Then, as we explained a few lines ago, T_r is defined by the second equation in (5.3) and V_r by the first equation on (5.3). The problem then lies in proving that $\tilde{U}_r$ can be taken to be a polynomial of degree r in λ. If we agree in taking $\tilde{U}_r$ as above, then the equation (5.4) provides $r+2$ relations: it will turn out (see the observations after the relations (5.11)) that one coefficient of the polynomial $\tilde{U}_r$ is obtained once the pair $(p(t,x), \mathcal{M}(t,x)) \in \mathcal{E}_2$ is fixed, so r of those relations are used to determine the other coefficients $f_j(t,x)$ recursively, while the remaining 2 relations provide compatibility conditions for the triple $a(t,x)$. These conditions translate into 2 evolution equations (one for the function $q(t,x)$ and the other for the function $y(t,x)$) which we call the r-th order Sturm-Liouville evolution equations (briefly SL-equations). In a little bit more detail, we will call the *SL-equations* those equations which arise from the relation (5.4) when we compute the coefficients of degree $k+1$ and k. In particular, these coefficients give rise to two equations of the form (setting $f_{-1} = f_{r+1} = 0$)

$$\begin{cases} q_t = \mathcal{Q}(t,x,f_k,f_{k-1},q,q_x,q_{xx},\ldots,y,y_x,y_{xx},\ldots,p,p_x,p_{tx},\ldots) \\ y_t = \mathcal{Y}(t,x,f_{k+1},f_k,q,q_x,q_{xx},\ldots,y,y_x,y_{xx},\ldots,p,p_x,p_{tx},\ldots). \end{cases}$$

When r varies over $\mathbb{N}$, we have the Sturm-Liouville hierarchy of evolution equations (briefly, SL-hierarchy).

Let us consider some simple examples. Assume $k = 0$, and fix $p(t,x) = y(t,x) = 1$. Then $\tilde{U}_r = U_r$ and (5.4) reads

$$2q_t = 2q_x U_r + 4(q-\lambda)U_{r,x} - U_{r,xxx}.$$

This is the standard K-dV hierarchy ([4]). For $r = 1$, set $U_1(t,x,\lambda) = f_1(t,x)\lambda + f_0(t,x)$. Then

$$\begin{cases} f_{1,x}(t,x) = 0; \\ 2q_x(t,x)f_1(t,x) - 4f_{0,x}(t,x) = 0; \\ 2q_t(t,x) = 2q_x(t,x)f_0(t,x) + 4q(t,x)f_{0,x}(t,x) - f_{0,xxx}(t,x). \end{cases}$$

If $f_1(t,x) = c_1$, we obtain $c_1 q_x(t,x) = 2 f_{0,x}(t,x)$, which implies $f_0(t,x) = \frac{c_1}{2} q(t,x) + c_2$. Hence the last relation in the system above gives us

$$q_t(t,x) = \frac{3}{2} c_1 q(t,x) q_x(t,x) - \frac{c_1}{4} q_{xxx}(t,x) + c_2 q_x(t,x)$$

which is a generalized version of the classical K-dV equation. If $c_1 = 1$ and $c_2 = 0$, we obtain the classical K-dV equation, i.e.,

$$q_t(t,x) = \frac{3}{2} q(t,x) q_x(t,x) - \frac{1}{4} q_{xxx}(t,x).$$

As another example, let us assume that $k = 1$ and let $p(t,x) = q(t,x) = 1$ be fixed. Then (5.4) translates to

$$2\lambda^2 y_t(t,x) = 2\lambda y_x(t,x) U_r(t,x,\lambda) - 4(1 - \lambda y(t,x)) U_{r,x}(t,x,\lambda) + U_{r,xxx}(t,x,\lambda).$$

This is a version of the Camassa-Holm hierarchy (another one can be obtained by setting $k = r$ as in [8]). If $r = 1$, a possible solution is given by

$$\begin{cases} f_0 = c_1; \\ c_1 y(t,x) + c_2 = 2 f_1(t,x) - \frac{1}{2} f_{1,xx}(t,x); \\ y_t(t,x) = y_x(t,x) f_1(t,x) + 2 y(t,x) f_{1,x}(t,x). \end{cases}$$

This system is a generalized version of the Camassa-Holm equation. The classical Camassa-Holm equation is obtained by setting $c_1 = 1$ and $c_2 = 0$ (see [2]).

Note that the constants in these constructions can be chosen at will. There are other interesting examples which can be given (see [15] for more information).

In the examples above just one coordinate of the triple $a(t,x)$ is unknown, and in fact only one evolution equation appears. Instead, we will prove the following result.

Theorem 5.1 *Let $\Lambda_0 = \{\lambda_0, \lambda_1, \dots\}$ be a sequence satisfying Hypotheses 4.1. Choose a family $\{p(t,x), \mathcal{M}(t,x)\}_{t\in\mathbb{R}} \subset \mathcal{E}_2$.*

Then there exist functions $q(t,x)$ and $y(t,x)$ together with a polynomial $U_r(t,x,\lambda)$ such that the family $\{p(t,x), q(t,x), y(t,x)\}_{t\in\mathbb{R}}$ lies in $\mathcal{R}_{\Lambda_0}$ and the zero-curvature relations (5.1) *and* (5.2) *are valid. Moreover, the functions $q(t,x)$ and $y(t,x)$ satisfy the r-th order SL-equations. They are expressed by the formulas*

$$y(t,x) = \frac{\mathcal{M}^2(t,x)}{4 p(t,x) \lambda_0} \prod_{i=1}^{\infty} \frac{s_i(t,x)}{\lambda_{2-1} \lambda_{2i}}$$

$$q(t,x) = y(t,x) \left(\lambda_0 \sum_{i=1}^{\infty} (\lambda_{2i} + \lambda_{2i-1} - 2 s_i(t,x)) \right) + g_x(t,x) + \frac{g^2(t,x))}{p(t,x)},$$

where $g(t,x) = -\frac{(p(t,x) y(t,x))_x}{4 y(t,x)}$, and $s_i(t,x)$ satisfies the system (5.9) *below ($i \in \mathbb{N}$).*

To prove this result, we first recall and deepen some facts which we have stated in Section 4.

If $\Lambda_0 = \{\lambda_0, \lambda_1, \dots, \lambda_{2n}\}$ is a finite set, then Theorem 5.1 is proved in [15]. In fact, the following procedure can be used to solve the problem.

1. The Weyl m-functions $m_\pm$ now depend on t as well. We find that they satisfy the Riccati equations

$$M_x + \frac{1}{p} M^2 = q - \lambda y,$$

$$M_t = 2T_r M - \lambda^{-k}\frac{U_r}{p}M^2 + \lambda^{-k}(q - \lambda y)V_r.$$

Since $m_\pm$ now depend on t as well, we will have $\mathcal{M}(t,x) = m_-(t,x,0) - m_+(t,x,0)$.

2. If the relations (5.1) and (5.2) are valid, then we have

$$\tilde{U}_t(t,x,\lambda) = \frac{2}{P(t,x)}\left[T(t,x,\lambda)U_r(t,x,\lambda) - T_r(t,x,\lambda)U(t,x,\lambda)\right], \tag{5.5}$$

$$T_t(t,x,\lambda) = \frac{q(t,x) - \lambda y(t,x)}{\lambda^k p(t,x)}\left[V_r(t,x,\lambda)U(t,x,\lambda) - U_r(t,x,\lambda)V(t,x,\lambda)\right], \tag{5.6}$$

$$\begin{aligned}&[(q(t,x) - \lambda y(t,x))V(t,x,\lambda)]_t = \\ &= 2(q(t,x) - \lambda y(t,x)))[T_r(t,x,\lambda)V(t,x,\lambda) - V_r(t,x,\lambda)T(t,x,\lambda)],\end{aligned} \tag{5.7}$$

or, equivalently,

$$-B_t + [B_r, B] = 0. \tag{5.8}$$

3. Equation (5.5) implies that, if $s_1(t,x),\dots,s_n(t,x)$ are the zeros of the polynomial $U(t,x)$, then

$$\begin{cases} s_{i,x} = \dfrac{(-1)^n k(s_i(t,x))\mathcal{M}(t,x)\prod_{l=1}^n s_l(t,x)}{p(t,x)k(0^+)\prod_{j\neq i}[s_i(t,x) - s_j(t,x)]}, \\ \\ s_{i,t}(t,x) = \dfrac{U_r(t,x,s_i(t,x))}{s_i^k(t,x)}s_{i,x}(t,x). \end{cases} \tag{5.9}$$

4. The map $t \mapsto a(t,x) = (p(t,x), q(t,x), y(t,x))$ is a curve in $\mathcal{R}_{\Lambda_0}$ consisting of algebro-geometric Sturm-Liouville potentials .

5. The polynomial $U_r(t,x,\lambda)$ is constructed as to satisfy (5.5) for every $t,x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. If

$$\tilde{U}_r(t,x,\lambda) = \frac{U_r(t,x,\lambda)}{p(t,x)} = \sum_{j=0}^r f_j(t,x)\lambda^j,$$

then the coefficients $f_j(t,x)$ can be determined recursively by the relation

$$\begin{aligned}\tilde{U}_{r,x}(\lambda) = \frac{\lambda^k}{p}\left[\frac{\mathcal{M}_t}{\mathcal{M}} + p\left(\frac{1}{p}\right)_t\right] - \frac{\mathcal{M}_x}{\mathcal{M}}\tilde{U}_r(\lambda) + \\ + \sum_{i=0}^n\left[\frac{\lambda^k}{s_i^k}\tilde{U}_r(s_i) - \tilde{U}_r(\lambda)\right]\frac{\lambda s_{i,x}}{s_i(\lambda - s_i)},\end{aligned} \tag{5.10}$$

where we omitted to write explicitly the dependence of the functions with respect to t and x.

6. We introduce the following notation: if $\mathcal{S}_n(t,x) = (s_1(t,x),\dots,s_n(t,x))$,

$$\varsigma_i(\mathcal{S}_n(t,x)) = (-1)^i\sum_{\ell\in\Lambda_i} s_{\ell_1}(t,x)s_{\ell_2}(t,x)\dots s_{\ell_i}(t,x), \qquad 1 \le i \le g,$$

where

$$\Lambda_i = \{\ell \in \mathbb{N}^i \mid 1 \le \ell_1 < \ell_2 < \dots < \ell_i \le n\},$$

and

$$\sigma_i^{(j)}(\mathcal{S}_n(t,x)) = (-1)^i \sum_{\ell\in\Lambda_i^{(j)}} s_{\ell_1}(t,x)s_{\ell_2}(t,x)\dots s_{\ell_i}(t,x), \qquad 1\le i\le g-1,$$

where

$$\Lambda_i^{(j)} = \{\ell\in\mathbb{N}^i \mid 1\le \ell_1<\ell_2<\dots<\ell_i\le n, \quad \ell_k\ne j\}.$$

In other words, ς_i and $\sigma_i^{(j)}$ are symmetric functions of $(s_1(t,x),\dots,s_n(t,x))$. Moreover, we denote by $\mathcal{S}_n^{-1}(t,x)$ the n-tuple

$$\mathcal{S}_n^{-1}(t,x) = \left(\frac{1}{s_1(t,x)},\dots,\frac{1}{s_n(t,x)}\right),$$

and by

$$H(t,x) = \mathcal{M}(t,x)\prod_{i=1}^n s_i(t,x).$$

After some computations, one obtains the following formulas for the coefficients $f_1(t,x)$, … , $f_n(t,x)$:

$$\begin{aligned}
f_j(t,x) &= \frac{1}{\mathcal{M}(t,x)}\left(\sum_{i=j}^r (-1)^i c_i\varsigma_{n-i+j}(\mathcal{S}_n^{-1}(t,x))\right), \qquad j=k+1,\dots,r\\
f_k(t,x) &= \frac{1}{\mathcal{M}(t,x)}\left(\sum_{i=0}^k c_i\varsigma_{k-i}(\mathcal{S}_n^{-1}(t,x))+\right.\\
&\qquad \left.+\int_0^x\left(\frac{\mathcal{M}_t(t,s)}{p(t,s)}-\frac{\mathcal{M}(t,s)p_t(t,s)}{p^2(t,s)}\right)ds\right)\\
f_j(t,x) &= \frac{1}{\mathcal{M}(t,x)}\left(\sum_{i=0}^j c_i\varsigma_{j-i}(\mathcal{S}_n^{-1}(t,x))\right), \qquad j=0,\dots,k-1
\end{aligned} \tag{5.11}$$

Note that if $k\ne 0$, then $f_0(t,x) = \dfrac{c_0}{\mathcal{M}(t,x)}$, while if $k=0$ then

$$f_0(t,x) = \frac{1}{\mathcal{M}(t,x)}\left(c_0+\int_0^x\left(\frac{\mathcal{M}_t(t,s)}{p(t,s)}-\frac{\mathcal{M}(t,s)p_t(t,s)}{p^2(t,s)}\right)ds\right).$$

In both cases, one has that the coefficient $f_0(t,x)$ depends only on the choice of the functions $p(t,x), \mathcal{M}(t,x)$ (recall the statement we made about the recursion for the coefficients $f_j(t,x)$ some lines after equation (5.4)).

7. Conversely, if we define $\tilde{U}_r(t,x,\lambda)$ as above, the system (5.9) admits a unique solution when the initial condition $(s_1(0,0),\dots,s_n(0,0))$ is given. We then discover that the relations (5.5)–(5.7) are satisfied and that the zero-curvature system (5.1) is satisfied once we define $T_r(t,x,\lambda)$ and $V_r(t,x,\lambda)$ via the relations (5.3).

We use these results which are valid when Λ_0 is a finite set to show that the procedure can be extended to the case when Λ_0 is an infinite discrete set which satisfies Hypotheses 4.1. The method is again based on the algebro-geometric approximation. We will show that, if we fix a set Λ_0 which satisfies Hypotheses 4.1, and if the initial condition is furnished by a triple $a(0,x) = (p(0,x), q(0,x), y(0,x))$ which is obtained via the algebro-geometric approximation we have described in the previous section, then there exists a curve $t\mapsto a(t,x) = (p(t,x),q(t,x),y(t,x))\in\mathcal{R}_{\Lambda_0}$ for which the zero-curvature system (5.1) and the Ansatz (5.2) are satisfied (for $k\le r$ fixed a priori), and this is equivalent to finding a solution of the r-th equation of the Sturm-Liouville hierarchy.

We are obliged to make the notation slightly heavier. So, let us fix numbers $k\le r<n\in\mathbb{N}$. Corresponding to those numbers there is a solution of the Sturm-Liouville hierarchy (5.1). We mean

that there exist matrices $B_n(t,x,\lambda)$, $B_r^{(n)}(t,x,\lambda)$ and $A_n(t,x,\lambda)$ such that (5.1) and (5.2) are valid. Here $U_n(t,x,\lambda)$ is as in Section 4, but has entries which depend on t as well,

$$A_n(t,x,\lambda)=\begin{pmatrix} 0 & 1/p(t,x) \\ q_n(t,x)-\lambda y_n(t,x) & 0 \end{pmatrix},$$

while $B_r^{(n)}(t,x,\lambda)$ is defined as

$$B_r^{(n)}(t,x,\lambda)=\begin{pmatrix} -T_r^{(n)}(t,x,\lambda) & \lambda^{-k}\dfrac{U_r^{(n)}(t,x,\lambda)}{p(t,x)} \\ \lambda^{-k}(q_n(t,x)-\lambda y_n(t,x))V_r^{(n)}(t,x,\lambda) & T_r^{(n)}(t,x,\lambda) \end{pmatrix}.$$

The index n which appears indicates that the construction of the solution of the Sturm-Liouville hierarchy is carried out by choosing the set $\Lambda_0^{(n)}=\{\lambda_0,\lambda_1,\ldots,\lambda_{2n}\}\subset\Lambda_0$. This produces a curve $t\mapsto a_n(t,x)=(p(t,x),q_nt,x),y_n(t,x))\in\mathcal{R}_{\Lambda_0^{(n)}}$.

We immediately observe that for every fixed $t\in\mathbb{R}$ and $\lambda\in\mathbb{C}$, the matrices $A_n(t,x,\lambda)$ and $B_n(t,x,\lambda)$ are uniformly bounded for x in compact subsets of $\mathbb{R}$ and converge to matrices

$$A(t,x,\lambda)=\begin{pmatrix} 0 & 1/p(t,x) \\ q(t,x)-\lambda y(t,x) & 0 \end{pmatrix}$$

and

$$B(t,x,\lambda)=\begin{pmatrix} -T(t,x,\lambda) & \lambda^{-k}\dfrac{U(t,x,\lambda)}{p(t,x)} \\ \lambda^{-k}(q(t,x)-\lambda y(t,x))V(t,x,\lambda) & T(t,x,\lambda) \end{pmatrix}$$

(see the previous section).

We have to study the behaviour of the matrix $B_r^{(n)}(t,x,\lambda)$. As is clear from the arguments we used the previous section, we only need to study the behaviour of the polynomials $U_r^{(n)}(t,x,\lambda)$, because the other entries of $B_r^{(n)}(t,x,\lambda)$ will inherit the desired properties by means of the relation (5.3).

Now, we have

$$U_r^{(n)}(t,x,\lambda)=\sum_{i=0}^{r}f_j^{(n)}(t,x)\lambda^j,$$

and the coefficients $f_j^{(n)}(t,x)$ satisfy the relations (5.11), where now indices n appear, i.e.,

$$\begin{aligned}
f_j^{(n)}(t,x)&=\frac{1}{\mathcal{M}_n(t,x)}\left(\sum_{i=j}^{r}(-1)^i c_i\varsigma_{n-i+j}(\mathcal{S}_n^{-1}(t,x))\right),\ j=k+1,\ldots,r\\
f_k^{(n)}(t,x)&=\frac{1}{\mathcal{M}_n(t,x)}\left(\sum_{i=0}^{k}c_i\varsigma_{k-i}(\mathcal{S}_n^{-1}(t,x))+\right.\\
&\qquad\left.+\int_0^x\left(\frac{\mathcal{M}_{n,t}(t,s)}{p(t,s)}-\frac{\mathcal{M}_n(t,s)p_t(t,s)}{p^2(t,s)}\right)ds\right)\\
f_j^{(n)}(t,x)&=\frac{1}{\mathcal{M}_n(t,x)}\left(\sum_{i=0}^{j}c_i\varsigma_{j-i}(\mathcal{S}_n^{-1}(t,x))\right),\ j=0,\ldots,k-1
\end{aligned}\tag{5.12}$$

and

$$\mathcal{M}_n(t,x)=\frac{-2k_n(0^+)}{\tilde{U}_n(t,x,0)}.$$

We now assume that $\mathcal{M}_n, \mathcal{M}_{n,x}, \mathcal{M}_{n,xx}$ and $\mathcal{M}_{n,t}$ converge uniformly on compact subsets of $\mathbb{R}^2$ to $\mathcal{M}, \mathcal{M}_x, \mathcal{M}_{xx}$ and $\mathcal{M}_t$ respectively. Note that the coefficients $f_j^{(n)}(t,x)$ are just linear combinations of at most r among the coefficients $u_j^{(n)}(t,x)$ of the polynomial $U_n(t,x,\lambda)$ (plus, in the case of $f_k^{(n)}(t,x)$, a uniformly convergent bounded function). This implies that for every $t \in \mathbb{R}$, the coefficients $f_j^{(n)}(t,x)$ converge to functions $f_j(t,x)$ uniformly on compact subsets of $\mathbb{R}$. Also, $f_{j,x}^{(n)}(t,x)$ converge to $f_{j,x}(t,x)$ uniformly on compact subsets of $\mathbb{R}$, for every $t \in \mathbb{R}$. Moreover, since the polynomials $U_n(t,x,\lambda)$ and $U_{n,x}(t,x,\lambda)$ are bounded by quantities which depend only on λ, the same holds for the coefficients $u_j^{(n)}(t,x)$ and $u_{j,x}^{(n)}(t,x)$, and hence for the c oefficients $f_j^{(n)}(t,x)$ and $f_{j,x}^{(n)}(t,x)$. All these observations lead to the conclusion that $U_r^{(n)}(t,x)$ and $U_{r,x}^{(n)}(t,x)$ converge to functions $U_r(t,x)$ and $U_{r,x}(t,x)$ for every $\lambda \in \mathbb{C}$. Moreover, for every fixed $\lambda \in \mathbb{C}$, the functions $U_r^{(n)}(t,x)$ and $U_{r,x}^{(n)}(t,x)$ are uniformly bounded on compact subsets of $\mathbb{R}^2$. Now, it is clear that $A_n(t,x,\lambda)$ is uniformly bounded on compact subsets of $\mathbb{R}^2$ (and for every fixed $\lambda \in \mathbb{C}$).

We rewrite (5.1) as

$$\begin{aligned} B_n(t,x,\lambda) - B_n(t,0,\lambda) &= \int_0^x [A_n(t,s,\lambda), B_n(t,s,\lambda)]ds \\ A_n(t,x,\lambda) - A_n(0,x,\lambda) &= \int_0^t (B_{r,x}^{(n)}(s,x,\lambda) - [A_n(s,x,\lambda), B_r^{(n)}(s,x,\lambda)])ds, \end{aligned}$$

and by applying the bounded convergence theorem we obtain

$$\begin{aligned} B(t,x,\lambda) - B(t,0,\lambda) &= \int_0^x [A(t,s,\lambda), B(t,s,\lambda)]ds \\ A(t,x,\lambda) - A(0,x,\lambda) &= \int_0^t (B_{r,x}(s,x,\lambda) - [A(s,x,\lambda), B_r(s,x,\lambda)])ds. \end{aligned}$$

The Sturm-Liouville hierarchy is then solved when the initial data lies in $\mathcal{R}_{\Lambda_0}$. The solution of the r-th order Sturm-Liouville equation defines a curve $t \mapsto a(t,x) = (p(t,x), q(t,x), y(t,x)) \in \mathcal{R}_{\Lambda_0}$. Theorem 5.1 is proved.

We make one concrete example of a new family of evolution equations we will be able to solve. Set $r = 1, k = 0$, $p(t,x) \equiv 1$ and fix $\mathcal{M}(t,x)$. Then we have $\tilde{U}_1(t,x,\lambda) = U_1(t,x,\lambda) = f_1(t,x)\lambda + f_0(t,x)$. We have

$$f_1(t,x) = \frac{c_1}{\sqrt{y(t,x)}},$$

while

$$f_0(t,x) = \frac{1}{\mathcal{M}(t,x)}\left(c_0 + \int_0^x \mathcal{M}_t(t,s)ds\right).$$

The Sturm-Liouville evolution equations of the first order are the given by

$$\begin{cases} 2q_t(t,x) = 2q_x(t,x)f_0(t,x) + 4q(t,x)f_{0,x}(t,x) + f_{0,xxx}(t,x) \\ \\ 2y_t(t,x) = -2c_1q_x(t,x)f_1(t,x) + 2y_x(t,x)f_0(t,x) - 4q(t,x)f_{1,x}(t,x) + \\ \qquad\qquad\qquad +4q_x(t,x)f_{0,x}(t,x) + f_{1,xxx}(t,x) \end{cases}$$

As another example, choose $r = k = 1$ and $p(t,x) \equiv 1$. Fix $\mathcal{M}(t,x)$. Then we have

$$f_1(t,x) = \frac{1}{\mathcal{M}(t,x)}\left(c_1 - c_0\left(\sum_{i=1}^{\infty} \frac{1}{s_i(t,x)}\right)\right),$$

$$f_0 = \frac{c_0}{\mathcal{M}(t,x)}.$$

In this case, the Sturm-Liouville evolution equations of the first order are

$$\begin{cases} y_t(t,x) = y_x(t,x)f_1(t,x) + 2y(t,x)f_{1,x}(t,x), \\ \\ 2q_t(t,x) = 2q_x(t,x)f_1(t,x) - y_x(t,x)f_0(t,x) + 4q(t,x)f_{1,x}(t,x) - \\ \qquad\qquad -4y(t,x)f_{0,x}(t,x) + f_{1,xxx}(t,x) \end{cases}$$

We finish the discussion by making some observations. First of all, we should clarify how to obtain the K-dV and the CH equations as particular cases of the construction above. What could seem singular is the fact that we do not fix either q or y, but we choose only $p(t,x)$ and $\mathcal{M}(t,x)$. So, let $k = 1$ and fix $p(t,x) = 1$. Let $s_i(t,x)$ be solutions of the equations in (5.9) which we modify by setting

$$\frac{\mathcal{M}(t,x)\prod_{i=1}^{\infty} s_i(t,x)}{k_\infty(0)} = 2.$$

This implies that $y(t,x) = 1$. Then we proceed as above. The relation (5.10) can be rewritten as follows

$$U_{r,x}(\lambda) = \sum_{i=1}^{\infty} (U_r(s_i) - U_r(\lambda)) \frac{s_{i,x}}{\lambda - s_i}.$$

The recursion formulas for the coefficients $f_j(t,x)$ now give

$$f_r = c_r,$$

$$f_{r-1} = c_{r-1} + c_r\varsigma_1(\mathcal{S}),$$

$$\vdots$$

$$f_0 = \sum_{i=0}^{r} c_i\varsigma_i(\mathcal{S}),$$

where $c_0, \ldots, c_r$ are constants. This is exactly the K-dV hierarchy of evolution equations, and indeed, if $r = 1$, we obtain

$$f_1 = c_1, \qquad f_0(t,x) = c_0 + c_1\varsigma_1(\mathcal{S}(t,x)) = \tilde{c}_0 + \frac{c_1}{2}q(t,x).$$

The first order K-dV equation in given by using (5.4):

$$\begin{cases} f_1 = c_1 \\ f_0 = \tilde{c}_0 + \dfrac{c_1}{2}q \\ 2q_t = 2q_x f_0 + 4q f_{0,x} - f_{0,xxx} \end{cases}.$$

The last equation in this system translates to

$$2q_t = 2q_x\left(\tilde{c}_0 + \frac{c_1}{2}q\right) + 4q\frac{c_1 q_x}{2} - \frac{c_1}{2}q_{xxx},$$

that is

$$q_t = \frac{3}{2}c_1 q q_x - \frac{c_1}{4}q_{xxx} + \tilde{c}_0 q_x,$$

which is a version of the well-known K-dV equation (by letting $\tilde{c}_0 = 0$ we obtain the classical K-dV equation).

Things are simpler when we wish to reproduce the CH-hierarchy, and especially the CH-equation. In this case, we set $p(t,x) = 1$ and $\mathcal{M}(t,x) = 2$. Then one uses the Riccati equation and obtains

$$m_+(t,x,0) + m_-(t,x,0) = 0, \qquad q(t,x) = 1,$$

which is the desired setting for the CH-hierarchy. The related spectral problem is based on the so-called acoustic equation $-\varphi'' + \varphi = \lambda y \varphi$. So, let us choose $k = r$. The relation defining $U_r(t,x)$ translates to

$$U_r(\lambda) = \sum_{i=0}^{\infty} \left[\frac{\lambda^r}{s_i^r} U_r(s_i) - U_r(\lambda) \right] \frac{\lambda s_{i,x}}{s_i(\lambda - s_i)}.$$

Clearly, the relations (5.12) translate to

$$f_r(t,x) = \frac{1}{2} \left(\sum_{i=0}^{r} c_i \varsigma_{r-i}(\mathcal{S}^{-1}(t,x)) \right),$$

$$f_j(t,x) = \frac{1}{2} \left(\sum_{i=0}^{j} c_i \varsigma_{j-i}(\mathcal{S}^{-1}(t,x)) \right), \quad j = 0, \ldots, r-1.$$

Now, the CH-equation of order r will be given by

$$y_t = y_x f_r + 2y f_{r,x}.$$

In the case when $r = 1$, we obtain the CH-equation of the first order we described at the beginning of this section. Note that, in this case, one obtains

$$u_1(t,x) = c_1 - c_0 \left(\sum_{i=1}^{\infty} \frac{1}{s_i(t,x)} \right).$$

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