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An Example of Chaos for a Cubic Nonlinear Schrödinger Equation with Periodic Inhomogeneous Nonlinearity

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Abstract

Using a topological approach we prove the existence of infinitely many periodic solutions, as well as the presence of symbolic dynamics for second order nonlinear differential equations of the form

$$-\ddot{u} - \mu u + g(t)h(u) = 0$$

where $\mu > 0$ is a given constant and $g : \mathbb{R} \to \mathbb{R}$ is a periodic positive weight function. Our main application concerns the study of the case in which $h(u)$ is a cubic nonlinearity. Such a choice is motivated by previous investigations dealing with the nonlinear Schrödinger equation $i\psi_t = -\frac{1}{2}\psi_{xx} + g(x)|\psi|^2\psi$.

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1 Introduction and statement of the main results

The present paper is devoted to the study of periodic solutions (harmonics and subharmonics) and solutions with a complex behavior for a class of nonautonomous second order nonlinear equations of the form

$$-\ddot{u} - \mu u + g(t)h(u) = 0, \tag{1.1}$$

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where $\mu > 0$ is a given constant and $g : \mathbb{R} \to \mathbb{R}_0^+ :=]0, +\infty)$ is a bounded periodic positive weight function with $\inf_{\mathbb{R}} g > 0$. For the nonlinear term, we assume that

$$h(s) := f(s)s,$$

with $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying

$$f(0) = 0, \quad f'(s)s > 0, \ \forall\, s \neq 0$$

and such that $f(\pm\infty) = +\infty$. An example of a nonlinearity satisfying the above assumptions is given by

$$h(s) = |s|^{p-1} s, \quad \text{for } p > 1, \tag{1.2}$$

which represents a typical model widely investigated in the literature. The interest in studying such class of equations comes from various physical models in different fields, as well as in the investigation of stationary solutions of reaction diffusion equations in a one-dimensional non-homogeneous environment (medium). In particular, the present paper is motivated by recent results [1, 2, 18] concerning some nonlinear Schrödinger equations arising in physical contexts as nonlinear optics and mathematical models for Bose-Einstein condensates. To make the presentation of our theorems more transparent and thus avoid some unnecessary technicalities, we'll focus our attention to the special case in which $h(s)$ is a *cubic nonlinearity*. All the results can be easily extended to a function $h(s)$ as in (1.2) and even to more general nonlinearities (see Remark 4.3). The choice of $h(s) = s^3$ is also justified by the search of some special solutions to the cubic nonlinear Schrödinger equation with inhomogeneous nonlinearity

$$i\psi_t = -\frac{1}{2}\psi_{xx} + g(x)|\psi|^2\psi, \tag{1.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a T-periodic and bounded function (not necessarily continuous). With this respect, our research is partially inspired by a paper of Belmonte-Beitia and Torres [2] where the authors look for solitary wave solutions of (1.3) of the form

$$\psi(x, t) = e^{i\lambda t}\phi(x),$$

where $\phi(x)$ is a solution of the second order nonlinear ordinary differential equation in $\mathbb{R}$

$$-\frac{1}{2}\phi_{xx} + \lambda\phi + g(x)\phi^3 = 0. \tag{1.4}$$

In [2] the coefficient $g(x)$ of the nonlinearity is an even function. Moreover, the following condition is assumed:

$$0 < g_{\min} \leq g(x) \leq g_{\max}\,. \tag{1.5}$$

The sign of the parameter λ plays a crucial role for the qualitative properties of the solutions. As observed in [2, Theorem 1], the dynamics is simpler when $\lambda \geq 0$. Indeed, the following result holds.

Proposition 1.1 *([2]). Assume* (1.5) *and let* $\lambda \geq 0$. *Then the only bounded solution of* (1.4) *is the trivial one,* $\phi = 0$.

In view of Proposition 1.1 in the sequel we shall assume $\lambda < 0$ (see (1.9) below). Without the hypothesis (1.5), namely when the coefficient $g(x)$ is allowed to change its sign, we can have nontrivial bounded solutions independently on the values of λ (see Proposition 1.2 below).

We shall prove the existence of a large number of periodic solutions, as well as bounded solutions with a complex behavior for equation (1.4), that is the presence of symbolic dynamics in a sense that will be precisely defined later. To this end we apply a dynamical system approach and hence the spatial variable $x \in \mathbb{R}$ for the function $\phi(x)$ will be formally regarded as a time-variable. Accordingly, equation (1.4) will be equivalently written as

$$\ddot{u} + 2ku - 2g(t)u^3 = 0, \tag{1.6}$$

for $k := -\lambda$, thus entering in the setting of (1.1). A solution $u(t)$ for equation (1.6) corresponds to a solution $\phi(x) := u(t)$, $x = t$ of (1.4). Besides its connection to (1.3), equation (1.6) provides a further motivation of our study. Precisely, when $g(t) > 0$ and $k > 0$, equation (1.6) can be seen as a periodic perturbation of the classical second-order equation

$$\ddot{u} + (1 + \kappa^2)u - 2\kappa^2 u^3 = 0 \tag{1.7}$$

defining the *Jacobi elliptic sine function* [10, p.8].

In [17], Terracini and Verzini obtained a general result about the existence of solutions with complex oscillatory behavior for the equation

$$-\ddot{u} = \alpha(t)u^3 + mu + h(t), \tag{1.8}$$

when the coefficient $\alpha(t)$ changes its sign and $m \in \mathbb{R}$ is an arbitrary constant. In particular, the following result can be deduced as a consequence from [17].

Proposition 1.2 *Let $\alpha, h : \mathbb{R} \to \mathbb{R}$ be T-periodic and bounded. Suppose that $\alpha(t)$ changes its sign in the sense that there are $2(n+1)$ consecutive closed intervals I_i^+ and I_i^- such that $[0, T] = \bigcup_{i=0}^{n}(I_i^+ \cup I_i^-)$ with $\alpha > 0$ on $\mathrm{int}(I_i^+)$ and $\alpha < 0$ on $\mathrm{int}(I_i^-)$. Then there exist T-periodic solutions of (1.8) having an arbitrarily large number of zeros in each I_i^+ and at least one zero in each I_i^-.*

Clearly, Proposition 1.2 applies to equation (1.6) for any $k \in \mathbb{R}$ provided that g changes its sign in the same manner like $-\alpha(t)$ in [17]. For the case $k = 0$, results related to Proposition 1.2, and thus applicable to (1.6) when $g(t)$ is a sign-changing term, have been also obtained in [3, 13, 14]. In all these papers the complex oscillatory solutions are large, in the sense that the number of zeros of a solution $u(t)$ in an interval I_i^+ where $\alpha(t)$ is positive (and, correspondingly, $g(t)$ is negative) increases with $|u(t_0)| + |\dot{u}(t_0)|$, with $t_0 \in \mathrm{int}(I_i^+)$. In particular, if one looks for solutions with a high number of oscillations, then $|u|_\infty$ as well as $|\dot{u}|_\infty$ must be large. In what follows we confine ourselves to the case in which the coefficient of the cubic nonlinearity is a periodic stepwise function. Such a choice is also motivated by the model of Bose-Einstein condensates (shortly denoted as BECs) studied in [16], where the authors consider the case

$$g(x) = \begin{cases} g_0\,, & \text{if } 0 \le \mathrm{mod}(x, L) < L_1 \\[2ex] g_1\,, & \text{if } L_1 \le \mathrm{mod}(x, L) < L. \end{cases}$$

In this model the parameter $L > 0$ is a period for the space variable and the piecewise constant function $g(x)$ is negative (positive) for attractive (repulsive) interatomic interactions. We shall focus our investigation to the case in which $g(x)$ is positive (like in [2]). With respect to the model given in [16], the positivity of g refers to the case of repulsive BECs. In such a situation we take $\lambda < 0$ (consistently with Proposition 1.1). This leads us to consider (1.6) with

$$k > 0 \tag{1.9}$$

and $g(t)$ a *positive* (nonconstant) T-periodic function of the form

$$g(t) = \begin{cases} g_-\,, & \text{if } 0 \le t < T_1 \\[2ex] g_+\,, & \text{if } T_1 \le t < T_1 + T_2 \end{cases} \qquad T_1 + T_2 = T, \tag{1.10}$$

with
$$0 < g_- < g_+ \,. \tag{1.11}$$
With such a set of assumptions, the results we are going to present are not a consequence of Proposition 1.2 or analogous results in [3, 13, 14, 17]. Indeed, contrary to the above cited articles where large oscillatory solutions are found, we look for periodic solutions having constant sign or a small number of zeros. Moreover, all the solutions we find will be confined in a prescribed interval, namely $[-\sqrt{k/g_-}, \sqrt{k/g_-}]$, which depends on the coefficients of the equation. A uniform bound for the derivative of the solutions will be provided, too.

We study equation (1.6) in the phase plane by means of the equivalent system
$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2g(t)u^3 \,. \end{cases} \tag{1.12}$$
In the special case when $g(t)$ is a constant function, say $g(t) = \bar{g}$, $\forall\, t \in \mathbb{R}$, system (1.12) is conservative with first integral (energy)
$$E(x, y) := \frac{1}{2}y^2 + kx^2 - \frac{1}{2}\bar{g}x^4.$$
The equilibrium points are the origin which is a local center and two saddle points $(\pm\sqrt{k/\bar{g}}, 0)$ which are connected by heteroclinic orbits (see Figure 1). For a T-periodic non-constant function $g(\cdot)$ sat-

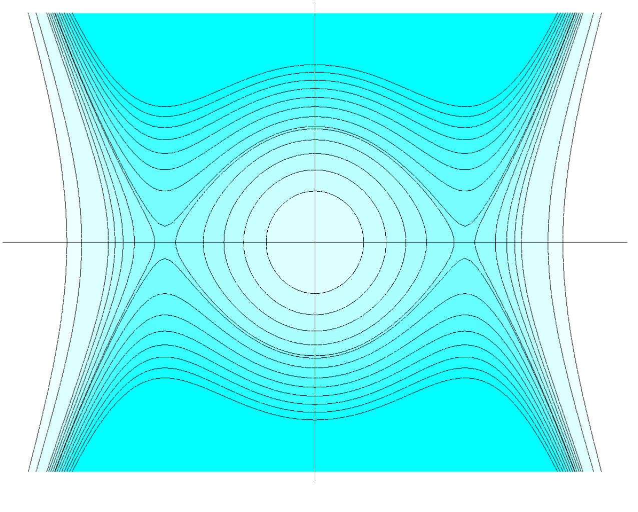

Figure 1: Phase-portrait of system (1.12). A darker color represents a higher value of the energy of the orbits.

isfying (1.5), Belmonte-Beitia and Torres [2] consider a comparison with two auxiliary autonomous equations that we write as
$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2g_{\min}u^3 \end{cases} \tag{1.13}$$
and
$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2g_{\max}u^3 \,. \end{cases} \tag{1.14}$$
Setting
$$\xi_- := \sqrt{\frac{k}{g_{\min}}}, \quad \xi_+ := \sqrt{\frac{k}{g_{\max}}}, \tag{1.15}$$

with $0 < \xi_+ < \xi_-$, we have that system (1.13) and system (1.14) have the saddle points in $(\pm\xi_-, 0)$ and $(\pm\xi_+, 0)$, respectively. Moreover, $\alpha(t) := \xi_+$ and $\beta(t) := \xi_-$ are a lower solution and an upper solution (constant) for equation (1.6). Hence, from the theory of upper and lower solutions [4] it follows that unstable [19] T-periodic solutions $\tilde{u}(\cdot)$ and $\tilde{v}(\cdot)$ of (1.6) exist such that

$$-\xi_- \le \tilde{u}(t) \le -\xi_+ \quad \text{and } \xi_+ \le \tilde{v}(t) \le \xi_-, \ \forall\, t \in \mathbb{R} \tag{1.16}$$

(see [2, Proposition 1]). In order to better understand the dynamics for system (1.12), we consider its associated Poincaré map

$$\Psi : \mathbb{R}^2 \supseteq \mathrm{dom}\Psi \to \mathbb{R}^2, \quad z_0 \mapsto \zeta(T; 0, z_0),$$

where $\zeta(\cdot) = \zeta(\cdot; t_0, z_0)$ is the unique noncontinuable solution of (1.12) with the initial condition $\zeta(t_0) = z_0$. For $t_0 = 0$, we usually take the simplified notation $\zeta(t; z_0) := \zeta(t; 0, z_0)$.

For a coefficient $g(t)$ satisfying condition (1.5) Ψ is not defined on the whole plane. In fact, solutions presenting a blow up in arbitrarily small time intervals occur for suitable choices of initial conditions (see [21] and the references therein). In any case, the domain $\mathrm{dom}\Psi$ of Ψ is an open set and Ψ is a homeomorphism of $\mathrm{dom}\Psi$ onto its image. The possibility of having blow up of the solutions and, consequently, the fact that the Poincaré map is not defined on $\mathbb{R}^2$ would suggest some caution in the use of our dynamical systems approach. In order to overcome such difficulty, we can truncate the function

$$s \mapsto 2ks - 2g(t)s^3$$

outside the interval $[-\xi_-, \xi_-]$. In this manner we have a locally Lipschitz bounded nonlinearity and the global existence of the solutions is guaranteed. Along the proof of our main result we will produce periodic and chaotic-like solutions which take values in $[-\xi_-, \xi_-]$ and, therefore, are solutions of (1.6). A similar approach, based on classical upper and lower solutions arguments, was adopted in [22] for a different equation involving a cubic nonlinearity. For the reader's convenience we have reported in the Appendix the full details which justify our approach. As a consequence, from now on, when we study equation (1.6) we don't worry about possible blow up for the solutions and consider the Poincaré map as globally defined.

For a weight function $g(t)$ of the form (1.10), observe that Ψ can be described as the superposition of the Poincaré maps associated to autonomous the differential systems

$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2u^3 g_- \end{cases} \tag{1.17}$$

and

$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2u^3 g_+ . \end{cases} \tag{1.18}$$

More precisely, for a given positive parameter $\bar{g}$, the equation

$$\begin{cases} \dot{u} = y \\ \dot{y} = -2ku + 2\bar{g}u^3 \end{cases} \tag{1.19}$$

defines a local dynamical system in the plane. For any initial point $z_0 \in \mathbb{R}^2$, let $t \mapsto \Psi^t_{\bar{g}}(z_0)$ be the (noncontinuable) solution $(u(t), y(t))$ of (1.19) with $(u(0), y(0)) = z_0$ which is defined on a open maximal interval of existence. With these positions, we have that

$$\Psi = \Psi^{T_2}_{g_+} \circ \Psi^{T_1}_{g_-} .$$

In other words, the dynamics for system (1.12) is obtained by following the orbits of (1.17) along the time interval $[0, T_1]$ and then switching to those of (1.18) for the time interval $[0, T_2]$. Such procedure is then repeated by T-periodicity, recalling also that $T = T_1 + T_2$. To denote the saddle equilibrium points for system (1.17) and (1.18) we follow the same notation like in (1.15) and set

$$\xi_- := \sqrt{\frac{k}{g_-}}, \quad \xi_+ := \sqrt{\frac{k}{g_+}}, \tag{1.20}$$

with $0 < \xi_+ < \xi_-$, as above. Thus we have completed the introduction of our basic setting and we

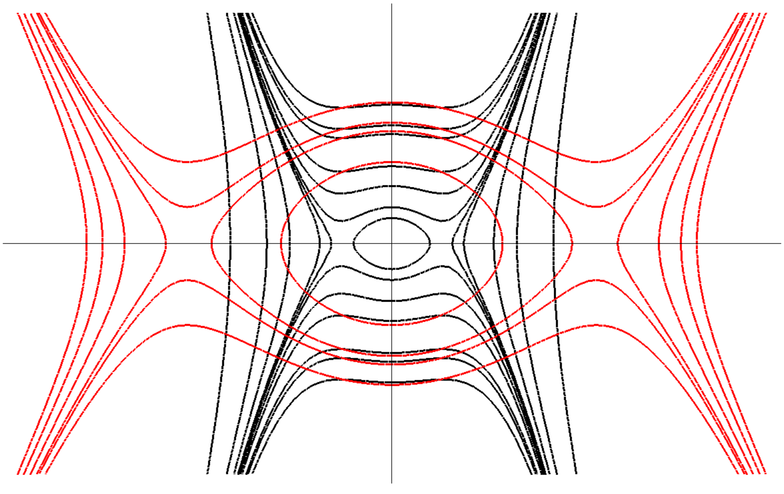

Figure 2: Overlapping the phase-portrait of systems (1.18) (darker lines) and (1.17) for $k = 1$ and $g_+ = 4$ and $g_- = 1/4$. For typographical reasons we have used a slightly different x- and y-scaling.

are ready to present our main result for equation (1.6).

Theorem 1.1 *Let g_-, g_+ satisfy* (1.11) *and let $k > 0$. Then there exist T_1^* and T_2^* such that, for each $T_1 \geq T_1^*$ and $T_2 \geq T_2^*$, equation* (1.6) *has infinitely many periodic solutions as well as solutions presenting a complex dynamics with $u(t) \in]-\xi_-, \xi_-[$. More precisely, given $g : \mathbb{R} \to \mathbb{R}$ a T-periodic function satisfying* (1.10) *with $T = T_1 + T_2$, there exists a compact set $\Lambda \subseteq]0, \xi_-[\times]0, +\infty)$ which is invariant for the Poincaré map Ψ and such that for each two-sided sequence $(s_n)_{n\in\mathbb{Z}} \in \Sigma_2 := \{0, 1\}^{\mathbb{Z}}$ there exists at least one initial point $z \in \Lambda$ such that the solution $\zeta(t; z) = (u(t), \dot{u}(t))$ has the following behavior:*

- *if $s_n = 0$, then on the interval $]nT, nT + T_1[$, $0 < u(t) < \xi_-$ and $\dot{u}(t)$ vanishes exactly once;*
- *if $s_n = 1$, then on the interval $]nT, nT + T_1[$, $\dot{u}(t)$ has precisely three zeros and $u(t)$ vanishes exactly twice;*
- *on each interval $]nT + T_1, (n+1)T[$, $u(t)$ belongs to $]0, \xi_-[$, is strictly convex and $\dot{u}(t)$ vanishes exactly once (this happens independently on the value of s_n);*
- *if the sequence $(s_n)_n$ is k-periodic, then the corresponding solution is kT-periodic.*

The constants T_1^* and T_2^* can be estimated from the coefficients of the equation (see Remark 4.1). The proof of Theorem 1.1 is based on some topological arguments from the theory of *topological horseshoes*. Under such name one usually describes suitable variants of the classical Smale's horseshoe theory to a more general setting in which some hyperbolicity assumptions are neglected. A consequence of the Smale's horseshoe theory [12, 20] ensures the existence of a compact invariant set Λ for a planar diffeomorphism Φ such that $\Phi|_\Lambda$ is *conjugate* to the Bernoulli shift on two symbols. In recent years several authors have proposed different approaches to tackle situations in which the conditions involved in Smale method are replaced by topological assumptions of crossing type (see, for instance, [7] and the references therein). In the present paper we follow a technique introduced in [15] and further developed in some subsequent articles (see [8, 9] for recent accounts on such topic). In order to make our work self-contained and for the reader's convenience, we have recalled in Section 2 the main abstract tools which are employed in the proof of Theorem 1.1. A special feature of our approach is due to the fact that for certain geometrical configurations, like those occurring in the proof of Theorem 1.1, the result turns out to be stable with respect to small perturbations of the maps which are involved. In our case, a small perturbation of the Poincaré map can be produced by allowing a T-periodic coefficient which is close in the L^1-norm on $[0, T]$ to the periodic stepwise function $g(t)$ defined in (1.10). More precisely, the following result holds.

Theorem 1.2 *Let g_-, g_+ satisfy* (1.11) *and let $k > 0$. Then there exist T_1^* and T_2^* such that for each $T_1 > T_1^*$ and $T_2 > T_2^*$ there is $\varepsilon = \varepsilon(T_1, T_2) > 0$ such that equation*

$$\ddot{u} + 2ku - 2a(t)u^3 = 0, \tag{1.21}$$

has infinitely many periodic solutions as well as solutions presenting a complex dynamics, with $u(t) \in]-\xi_-, \xi_-[$, provided that $a : \mathbb{R} \to \mathbb{R}$ is a T-periodic function satisfying $\int_0^T |a(t) - g(t)|\, dt < \varepsilon$, with $g : \mathbb{R} \to \mathbb{R}$ a T-periodic function satisfying (1.10).

In Theorem 1.2 we can permit discontinuities in $a(t)$ provided that solutions are considered in the Carathéodory sense. With this respect, a periodic weight function $a(t)$ which is locally Lebesgue integrable can be considered as well. On the other hand, we are allowed to take an arbitrarily smooth function $a(t)$ which approximates the step function $g(t)$. The stability of Theorem 1.1 with respect to small perturbations of the Poincaré map is not confined to the coefficient of the nonlinearity. For instance, we can obtain the same result for a perturbed equation of the form

$$\ddot{u} + c\dot{u} + 2ku - 2a(t)u^3 = e(t, u, \dot{u}),$$

provided that c, $\int_0^T |a(t) - g(t)|\, dt$ and $|e(\dots)|$ are sufficiently small.

2 Topological tools

In this section we recall some basic facts about the *stretching along the paths* approach which we use in the proof of Theorem 1.1. Such approach, which can be considered as a variant of the theory of *topological horseshoes*, as developed by Kennedy and Yorke [7], provides the existence of periodic points and chaotic-like dynamics for some maps defined on suitable planar domains that we call *topological rectangles*. Prior to the presentation of our technique, we explain in which sense we mean that a given map presents chaotic dynamics.

Definition 2.1 *Let X be a metric space, let $\Phi : D_\Phi(\subseteq X) \to X$ be a map and $\mathcal{D} \subseteq D_\Phi$ a nonempty set. Assume also that $p \geq 2$ is an integer. We say that Φ induces chaotic dynamics on p symbols in the set $\mathcal{D}$ if there exist p nonempty pairwise disjoint compact sets*

$$\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_{p-1} \subseteq \mathcal{D}$$

such that, for each two–sided sequence of p symbols

$$(s_i)_{i\in\mathbb{Z}} \in \Sigma_p := \{0, \dots, p-1\}^{\mathbb{Z}},$$

there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ with

$$w_i \in \mathcal{K}_{s_i} \quad \text{and} \quad w_{i+1} = \Phi(w_i), \quad \forall\, i \in \mathbb{Z} \tag{2.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k−periodic sequence (that is, $s_{i+k} = s_i$, $\forall i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k−periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (2.1).

Note that, as a particular consequence of our definition, we have that for each $i \in \{0, \dots, p-1\}$ there is at least one fixed point of Φ in $\mathcal{K}_i$. It can be also proved [9] that if the map Φ fulfills Definition 2.1 and is also *continuous and injective on*

$$\mathcal{K} := \bigcup_{i=0}^{p-1} \mathcal{K}_i \subseteq \mathcal{D}$$

(As in the case of Section 3 where Φ is the Poincaré map associated to a planar system), then there exists a nonempty compact set

$$\Lambda \subseteq \mathcal{K}$$

which is invariant for Φ (i.e., $\Phi(\Lambda) = \Lambda$) and such that $\Phi|_\Lambda$ is semiconjugate to the two-sided Bernoulli shift σ on p symbols

$$\sigma : \Sigma_p \to \Sigma_p, \quad \sigma(\,(s_i)_{i\in\mathbb{Z}}\,) = (s_{i+1})_{i\in\mathbb{Z}},$$

according to the commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\Phi"] \arrow[d, "g"'] & \Lambda \arrow[d, "g"] \\ \Sigma_p \arrow[r, "\sigma"'] & \Sigma_p \end{tikzcd}$$

where g is a continuous and surjective function. Moreover, as a consequence of Definition 2.1 we can take Λ such that it contains as a dense subset the periodic points of Φ and such that the counterimage (by the semiconjugacy g) of any periodic sequence in Σ_p contains a periodic point of Φ (see [9] for the details). As usual, we denote by Σ_p the set of two-sided sequence of two symbols with its standard metric [20].

We recall that the semiconjugation to the Bernoulli shift (even without reference to the periodic points) is one of the typical requirements for chaotic dynamics as it implies a positive topological entropy for the map $\Phi|_\Lambda$.

As a next step, we introduce some definitions which are peculiar of our technique. By *path* γ in a metric space X we mean a continuous mapping $\gamma : [t_0, t_1] \to X$ and we set $\bar{\gamma} := \gamma([t_0, t_1])$. Without loss of generality we'll usually take $[t_0, t_1] = [0, 1]$. By a *sub–path* σ of γ we mean just the restriction of γ to a compact subinterval of its domain. An *arc* is the homeomorphic image of the compact interval $[0, 1]$. We define an *oriented rectangle* in $\mathbb{R}^2$ as a pair

$$\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-),$$

where $\mathcal{R} \subseteq \mathbb{R}^2$ is homeomorphic to the unit square $[0, 1]^2$ (we usually refer to $\mathcal{R}$ as a *topological rectangle*) and

$$\mathcal{R}^- := \mathcal{R}^-_l \cup \mathcal{R}^-_r$$

is the disjoint union of two disjoint compact arcs $\mathcal{R}^-_l, \mathcal{R}^-_r \subseteq \partial\mathcal{R}$ (which are called the components or sides of $\mathcal{R}^-$). We also denote by $\mathcal{R}^+$ the closure of $\partial\mathcal{R} \setminus (\mathcal{R}^-_l \cup \mathcal{R}^-_r)$ which is the union of two compact arcs $\mathcal{R}^+_d$ and $\mathcal{R}^+_u$. Given an oriented rectangle $(\mathcal{R}, \mathcal{R}^-)$ it is always possible to prove the existence of a homeomorphism $h : [0, 1]^2 \to \mathcal{R} = h([0, 1]^2) \subseteq X$ such that $h(\{0\}\times[0, 1]) = \mathcal{R}^-_l$, $h(\{1\}\times[0, 1]) = \mathcal{R}^-_r$, $h([0, 1] \times \{0\}) = \mathcal{R}^+_d$ and $h([0, 1] \times \{1\}) = \mathcal{R}^+_u$. This can be proved using some classical results from plane topology like the Schönflies theorem [11]. In Section 3 we construct topological rectangles as planar regions obtained by the intersections sets of the form $\{a \le H(x, y) \le b\}$ where H is a Hamiltonian (typically, the energy associated to some second-order equation). The subscripts l, r, u, d stands, conventionally, for *left, right, up, down*.

We end our list of definitions recalling the concept of *stretching along the paths*. Suppose that $\Phi : D_\Phi(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$ is a planar map. Let $\widehat{\mathcal{M}} := (\mathcal{M}, \mathcal{M}^-)$ and $\widehat{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-)$ be oriented rectangles.

Definition 2.2 *Let $\mathcal{H} \subseteq \mathcal{M} \cap D_\Phi$ be a compact set. We say that $(\mathcal{H}, \Phi)$ stretches $\widehat{\mathcal{M}}$ to $\widehat{\mathcal{N}}$ along the paths and write*

$$(\mathcal{H}, \Phi) : \widehat{\mathcal{M}} \Longleftrightarrow\!\!\!\!\to \widehat{\mathcal{N}},$$

if the following conditions hold:

- *Φ is continuous on $\mathcal{H}$;*
- *for every path $\gamma : [a, b] \to \mathcal{M}$ such that $\gamma(a) \in \mathcal{M}_l^-$ and $\gamma(b) \in \mathcal{M}_r^-$ (or $\gamma(a) \in \mathcal{M}_r^-$ and $\gamma(b) \in \mathcal{M}_l^-$), there exists a subinterval $[t', t''] \subseteq [a, b]$ such that*

$$\gamma(t) \in \mathcal{H}, \quad \Phi(\gamma(t)) \in \mathcal{N}, \quad \forall\, t \in [t', t'']$$

  and, moreover, $\Phi(\gamma(t'))$ and $\Phi(\gamma(t''))$ belong to different components of $\mathcal{N}^-$.

In the special case in which $\mathcal{H} = \mathcal{M}$, we simply write

$$\Phi : \widehat{\mathcal{M}} \Longleftrightarrow\!\!\!\!\to \widehat{\mathcal{N}}.$$

In applications of this theory to differential equations, we usually take as Φ a Poincaré map. In this framework, it may happen that Φ allows a natural splitting as the composition of two maps. This often occurs for nonautonomous equations in which a variation of some time-dependent coefficients can produce a valuable change in the qualitative behavior of the solutions. The next result, taken from [8, Theorem 2.1], provides the existence of periodic points and chaotic-like dynamics when such splitting is assumed.

Theorem 2.1 *Let $\Phi_1 : D_{\Phi_1}(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$ and $\Phi_2 : D_{\Phi_2}(\subseteq \mathbb{R}^2) \to \mathbb{R}^2$ be continuous maps and let $\widehat{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$, $\widehat{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ be oriented rectangles. Suppose that the following conditions are satisfied:*

◊ *there exist $m \geq 1$ pairwise disjoint compact sets $\mathcal{H}_0, \dots, \mathcal{H}_{m-1} \subseteq \mathscr{A} \cap D_{\Phi_1}$ such that*

$$(\mathcal{H}_i, \Phi_1) : \widehat{\mathscr{A}} \Longleftrightarrow\!\!\!\!\to \widehat{\mathscr{B}}, \quad \text{for } i = 0, \dots, m-1\,;$$

◊ *there is a compact set $\mathcal{K} \subseteq \mathscr{B} \cap D_{\Phi_2}$ such that $(\mathcal{K}, \Phi_2) : \widehat{\mathscr{B}} \Longleftrightarrow\!\!\!\!\to \widehat{\mathscr{A}}$.*

Then we have:

- *If $m = 1$ then the map $\Phi := \Phi_2 \circ \Phi_1$ has at least a fixed point $z \in \mathcal{H} := \mathcal{H}_0$ such that $\Phi(z) \in \mathcal{K}$.*

- *If $m \geq 2$, then the map $\Phi := \Phi_2 \circ \Phi_1$ induces chaotic dynamics on m symbols in the set*

$$\mathcal{H}^* := \bigcup_{j=0,\dots,m-1} \mathcal{H}'_j \quad \text{for } \mathcal{H}'_j := \mathcal{H}_j \cap \Phi_1^{-1}(\mathcal{K}).$$

  Moreover, for each sequence of m symbols

$$\mathbf{s} = (s_n)_n \in \{0, \dots, m-1\}^{\mathbb{N}},$$

  there exists a compact connected set $C_{\mathbf{s}} \subseteq \mathcal{H}'_{s_0}$ with

$$C_{\mathbf{s}} \cap \mathscr{A}_d^+ \neq \emptyset, \quad C_{\mathbf{s}} \cap \mathscr{A}_u^+ \neq \emptyset$$

  and such that, for every $w \in C_{\mathbf{s}}$, there exists a sequence $(y_n)_n$ with $y_0 = w$ and

$$y_n \in \mathcal{H}'_{s_n}, \quad \Phi(y_n) = y_{n+1}, \ \forall\, n \geq 0.$$

3 Technical estimates and proofs

We split this section in several parts. At the beginning we describe the geometry associated to systems (1.18) and (1.17) and introduce two oriented rectangles $\widehat{\mathscr{A}}$ and $\widehat{\mathscr{B}}$. Then we prove the stretching along the path property for the Poincaré maps. Finally, we conclude with the proofs of Theorem 1.1 and Theorem 1.2.

3.1 Phase plane analysis and technical estimates

Let us fix two numbers η_+ , η_- such that

$$\xi_+ < \eta_+ < \eta_- < \xi_- \,. \tag{3.1}$$

We are interested in describing the rectangular regions obtained by the intersections of the energy level lines of systems (1.18) and (1.17) passing through the points $(\xi_+, 0)$, $(\eta_+, 0)$ and $(\eta_-, 0)$, $(\xi_-, 0)$, respectively. More precisely, we introduce two parameters $v \in [\xi_+, \eta_+]$ and $w \in [\eta_-, \xi_-]$ and look for the pairs (x, y) such that

$$\begin{cases} \mathcal{E}_+(x,y) := y^2 + 2kx^2 - g_+x^4 = 2kv^2 - g_+v^4 \\ \mathcal{E}_-(x,y) := y^2 + 2kx^2 - g_-x^4 = 2kw^2 - g_-w^4 \,. \end{cases} \tag{3.2}$$

By symmetry, we can look for the solutions of (3.2) with $x > 0$ and $y > 0$. In this manner, we find, for

$$(w, v) \in \mathscr{R} := [\eta_-, \xi_-] \times [\xi_+, \eta_+],$$

$$x = \mathcal{L}_1(w, v) := \left(\frac{2kw^2 - g_-w^4 - (2kv^2 - g_+v^4)}{g_+ - g_-} \right)^{1/4}$$

and

$$y = \mathcal{L}_2(w, v) := \left(2kw^2 - g_-w^4 - (2k\,\mathcal{L}_1(w, v)^2 - g_-\,\mathcal{L}_1(w, v)^4) \right)^{1/2}.$$

We define the set

$$\mathscr{A} := \mathcal{L}(\mathscr{R}), \quad \text{for } \mathcal{L}(w, v) := (\mathcal{L}_1(w, v), \mathcal{L}_2(w, v))$$

and also the set $\mathscr{B}$ as the symmetric of $\mathscr{A}$ with respect to the x-axis.
By definition, $\mathscr{A}$ and $\mathscr{B}$ are topological rectangles (in fact, observe that $\mathcal{L} : \mathscr{R} \to \mathscr{A}$ is a homeomorphism). Now we choose an orientation for such regions as follows.

$$\mathscr{A}_l^- := \mathscr{A} \cap \{(x, y) : \mathcal{E}_-(x, y) = \mathcal{E}_-(\eta_-, 0)\},$$

$$\mathscr{A}_r^- := \mathscr{A} \cap \{(x, y) : \mathcal{E}_-(x, y) = \mathcal{E}_-(\xi_-, 0)\}$$

and

$$\mathscr{B}_l^- := \mathscr{B} \cap \{(x, y) : \mathcal{E}_+(x, y) = \mathcal{E}_+(\eta_+, 0)\},$$

$$\mathscr{B}_r^- := \mathscr{B} \cap \{(x, y) : \mathcal{E}_+(x, y) = \mathcal{E}_+(\xi_+, 0)\}.$$

Before proceeding further, we make some observations about the trajectories of systems (1.17) and (1.18) in the strip $[-\xi_-, \xi_-] \times \mathbb{R}$. With respect to system (1.17), let us consider an energy level line of the form

$$\Gamma_-^c := \{(x, y) \in [-\xi_-, \xi_-] \times \mathbb{R} : \mathcal{E}_-(x, y) = c\}, \quad \text{for } c \in]0, c_-^*[\,,$$

with

$$c_-^* := \mathcal{E}_-(\xi_-, 0).$$

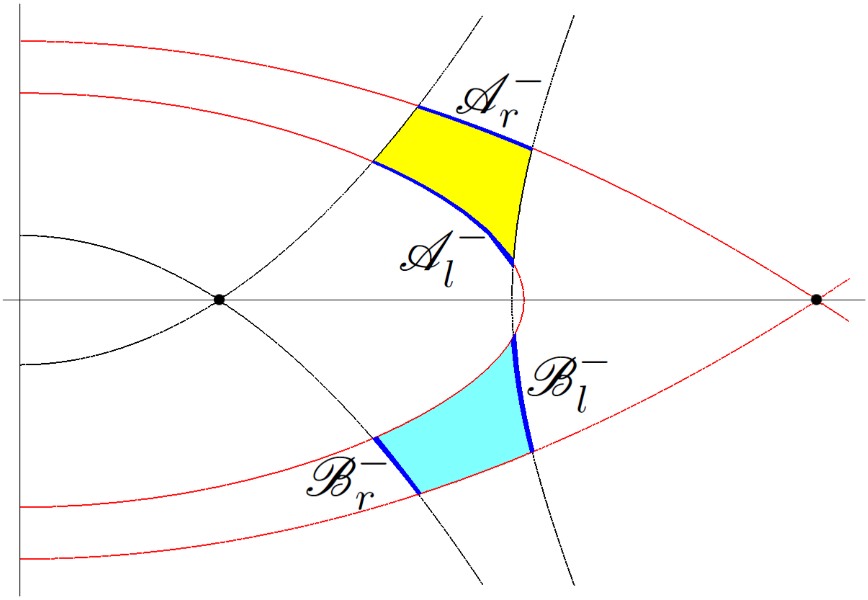

Figure 3: Example of the regions $\mathscr{A}$ and $\mathscr{B}$, where we have put in evidence the boundary components $\mathscr{A}_l^-$ and $\mathscr{A}_r^-$, as well as $\mathscr{B}_l^-$ and $\mathscr{B}_r^-$, respectively.

The set Γ_-^c is a periodic orbit of (1.17) and we denote by $\tau_-(c)$ its fundamental period. By standard computations, it follows that that

$$\lim_{c\to 0^+} \tau_-(c) = \frac{2\pi}{\sqrt{2k}} \quad \text{and} \quad \lim_{c\to c_-^*} \tau_-(c) = +\infty.$$

An explicit expression for the period map of any closed orbit can be given by a quadrature formula involving the potential

$$\mathcal{F}_-(x) := kx^2 - \frac{1}{2}g_-x^4,$$

as

$$\tau_-(c) = 2\int_{-w}^{w} \frac{d\,x}{\sqrt{c - 2\mathcal{F}_-(x)}}\,, \quad \text{with } 2\mathcal{F}_-(\pm w) = \mathcal{E}_-(w,0) = c,$$

for $0 < w < \xi_-$.

We also denote by $\tau_-^\#$ the minimal period of the line with energy

$$c_-^\# := \mathcal{E}_-(\eta_-,0),$$

passing through the point $(\eta_-,0)$. Notice that $\tau_-(c)$ and therefore also $\tau_-^\#$ can be explicitly computed in terms of Jacobi elliptic functions (see Remark 4.1), since $\mathcal{F}_-(x)$ is a quartic polynomial. With respect to system (1.18), with associate potential function

$$\mathcal{F}_+(x) := kx^2 - \frac{1}{2}g_+x^4,$$

let us consider an energy level line of the form

$$\Gamma_+^c := \{(x,y) \in [\xi_+,\xi_-]\times\mathbb{R} : \mathcal{E}_+(x,y) = c\}, \quad \text{for } c \in\,]d_-,d_+[\,,$$

with

$$d_- := \mathcal{E}_+(\xi_-,0) = 2\mathcal{F}_+(\xi_-) < d_+ := \mathcal{E}_+(\xi_+,0) = 2\mathcal{F}_+(\xi_+)$$

(observe that $\mathcal{F}_+$ is strictly decreasing on $[\xi_+, +\infty)$). All the sets Γ_+^c are orbit paths which connect two symmetric points $(\xi_-, -p)$ and (ξ_-, p) with

$$p = p(c) := \sqrt{c - d_-}$$

and passing through a point $(v, 0)$ with $v \in]\xi_+, \xi_-[$. The time needed to run from $(\xi_-, -p)$ to (ξ_-, p) along Γ_+^c can be computed as

$$\omega(c) = 2 \int_v^{\xi_-} \frac{dx}{\sqrt{c - 2\mathcal{F}_+(x)}}, \quad \text{with } 2\mathcal{F}_+(v) = \mathcal{E}_+(v, 0) = c,$$

for $v \in]\xi_+, \xi_-[$. It is easy to see that that

$$\lim_{c \to d_-} \omega(c) = 0 \quad \text{and} \quad \lim_{c \to d_+} \omega(c) = +\infty.$$

We denote by $\omega^\#$ the running time of the orbit path $\Gamma_+^{c_+^\#}$ passing through the point $(\eta_+, 0)$, with energy $c_+^\# := \mathcal{E}_+(\eta_+, 0)$.

We conclude this preliminary part with some elementary remarks about the mutual position of the energy level lines of $\mathcal{E}_-$ and $\mathcal{E}_+$. Define the functions

$$f_-(x) := 2k - 2g_- x^2, \quad f_+(x) := 2k - 2g_+ x^2$$

for $x \geq 0$. It is obvious that $f_\pm(x) > 0$ if and only if $0 < x < \xi_\pm$ and that $f_+(x) < f_-(x)$ for all $x > 0$. Let us consider the positive semi-orbit $\{\Psi_{g_+}^t(v, 0) : t \geq 0\}$ of system (1.18) passing through a point $(v, 0)$ with $\xi_+ < v < \xi_-$. Such semi-orbit is contained in the energy level line $\mathcal{E}_+(x, y) = c$ for $c = \mathcal{E}_+(v, 0)$. Its points can be represented as the graph of the strictly increasing function $y = \sqrt{2(\mathcal{F}_+(v) - \mathcal{F}_+(x))}$ which is defined for $x \geq v$ (recall also that $\mathcal{F}_+(x) = \int_0^x f_+(s) s\, ds$ is strictly decreasing on $[\xi_+, +\infty)$). Hence, if $(\bar{x}, \bar{y})$ is any point of the orbit with $\bar{x} \geq \xi_-$, then $\bar{y} \geq p(c) = (\mathcal{E}_+(v, 0) - d_-)^{1/2} = \sqrt{2(\mathcal{F}_+(v) - \mathcal{F}_+(\xi_-))} > 0$. In this case, we claim that

$$\mathcal{E}_-(\bar{x}, \bar{y}) > c_-^* = \mathcal{E}_-(\xi_-, 0). \tag{3.3}$$

To prove such assertion, we have

$$\begin{aligned}
\frac{1}{2}(\mathcal{E}_-(\bar{x}, \bar{y}) - \mathcal{E}_-(\xi_-, 0)) &= \frac{1}{2}\bar{y}^2 + \mathcal{F}_-(\bar{x}) - \mathcal{F}_-(\xi_-) \\
&= \mathcal{F}_+(v) - \mathcal{F}_+(\bar{x}) + \mathcal{F}_-(\bar{x}) - \mathcal{F}_-(\xi_-) \\
&> \mathcal{F}_+(\xi_-) - \mathcal{F}_+(\bar{x}) + \mathcal{F}_-(\bar{x}) - \mathcal{F}_-(\xi_-) \\
&= \int_{\xi_-}^{\bar{x}} (f_-(s) - f_+(s)) s\, ds \geq 0
\end{aligned}$$

and therefore (3.3) holds.

3.2 Proof of the stretching along the path property

After these preliminary estimates, we are now in a position to prove the stretching along the path property for the Poincaré maps. We consider at first system (1.17). We fix a positive integer $j \geq 1$. If

$$T_1 \geq \frac{2j+1}{2} \tau_-^\# \tag{3.4}$$

we have that for each point $z_0 \in \mathscr{A}_l^-$ the solution $(u(t; z_0), \dot{u}(t; z_0)) = \zeta(t; z_0)$ makes at least $j + \frac{1}{2}$ winds around the origin in the phase plane in the interval $[0, T_1]$. Hence, both u and $\dot{u}$ have at least $2j + 1$ simple zeros in the interval $[0, T_1]$. On the other hand, since the set $\mathscr{A}_r^-$ is contained in the heteroclinic orbit of (1.17) connecting $(-\xi_-, 0)$ to $(\xi_-, 0)$, in the upper part of the phase plane, we have that for every $z_0 \in \mathscr{A}_r^-$, it follows that $u(t; z_0) > 0$ and $\dot{u}(t; z_0) > 0$ for all $t \in [0, T_1]$. As a consequence, we can define some subsets of $\mathscr{A}$ as follows

- ⋆ H_0 is the set of points $z \in \mathscr{A}$ such that $u(t,z) \geq 0$ and $\partial_t u(t;z) = \dot{u}(t;z)$ vanishes exactly once on $[0, T_1]$;
- ⋆ H_j (for $j \geq 1$) is the set of points $z \in \mathscr{A}$ such that $\partial_t u(t,z)$ vanishes exactly $2j+1$ times on $[0, T_1]$ and $u(T_1;z) \geq 0$.

We can give an alternative definition of these sets by introducing a system of polar coordinates (θ, ρ) in order to count the number of turns around the origin for the solutions of (1.17). Hence, for each $z \in \mathscr{A}$ and $0 \leq t \leq T_1$, we set

$$\zeta(t;z) = (\rho(t;z)\cos(\theta(t;z)), \rho(t;z)\sin(\theta(t;z))).$$

One can easily check that $\partial_t \theta(t;z) < 0$. Hence we can define

$$H_i := \{z \in \mathscr{A} : -\frac{\pi}{2} - 2i\pi \leq \theta(T_1;z) \leq -2i\pi\}, \qquad \text{for } i \in \mathbb{N}. \tag{3.5}$$

With this description, it is clear that the sets H_i's are closed. Suppose that (3.4) is satisfied for some $j \geq 1$. In this case,

$$\theta(T_1;z_0) < -\frac{\pi}{2} - 2j\pi, \quad \forall z_0 \in \mathscr{A}_l^-,$$

while

$$\theta(t;z_0) > 0, \quad \forall z_0 \in \mathscr{A}_r^- \text{ and for every } t \in [0, T_1].$$

Therefore, the sets $H_0, H_1, \ldots H_j$ are all nonempty.

Let $\gamma : [0,1] \to \mathscr{A}$ be a continuous path such that $\gamma(0) \in \mathscr{A}_l^-$ and $\gamma(1) \in \mathscr{A}_r^-$ and let us consider the image of γ through the Poincaré map $\Psi_{g_-}^{T_1}$. To this aim, we consider the auxiliary function

$$\sigma(t,s) = (\sigma_1(t,s), \sigma_2(t,s)) := \zeta(t;\gamma(s)), \quad \text{for } s \in [0,1],\ t \in [0, T_1].$$

By definition, $\sigma_1(t,s) = u(t;\gamma(s))$ and $\sigma_2(t,s) = \frac{\partial}{\partial t} u(t;\gamma(s))$. We also denote by $\theta(t,s)$ the angular function associated to $\zeta(t;\gamma(s))$. As we have already observed, $\sigma_1(t,1) > 0$ and $\sigma_2(t,1) > 0$ for all $t \in [0, T_1]$. Moreover, $\theta(T_1, 1) > 0$. On the other hand, from (3.4), $\sigma_1(t,0)$ and $\sigma_2(t,0)$ have at least $2j+1$ simple zeros in $[0, T_1]$. Moreover, $\theta(T_1, 0) < -(4j+1)\pi/2$. Thus we find that the range of the map

$$[0,1] \ni s \mapsto \theta(T_1, s)$$

covers the interval $[-(4j+1)\pi/2, 0]$. Using the fact that $\mathscr{B}$ is contained in the interior of the fourth quadrant, by an elementary continuity argument, we can find $j+1$ closed subintervals $[s_i', s_i'']$ for $i = 0, \ldots, j$, with

$$0 < s_j' < s_j'' < \cdots < s_i' < s_i'' < s_{i-1}' < s_{i-1}'' < \cdots < s_0' < s_0'' < 1,$$

such that

$$\sigma(T_1, s) = \Psi_{g_-}^{T_1}(\gamma(s)) \in \mathscr{B}, \quad \forall s \in [s_i', s_i''],$$

with

$$\sigma(T_1, s_i') \in \mathscr{B}_r^-, \quad \sigma(T_1, s_i'') \in \mathscr{B}_l^-, \qquad \text{for } i = 0, \ldots, j$$

and, moreover,

$$\gamma(s) \in H_i, \quad \forall s \in [s_i', s_i''].$$

In this manner, we have proved that

$$(H_i, \Psi_{g_-}^{T_1}) : \widehat{\mathscr{A}} \Leftrightarrow\!\!\!\!\!\!\!\longrightarrow \widehat{\mathscr{B}}, \qquad \text{for } i = 0, \ldots, j.$$

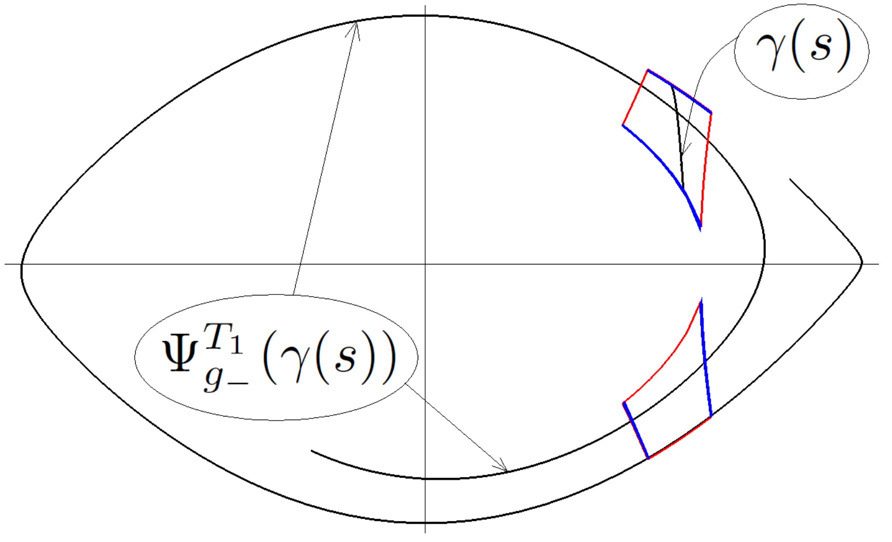

Figure 4: The stretching of the path $\gamma(s)$, connecting $\mathscr{A}_l^-$ to $\mathscr{A}_r^-$, through the Poincaré map $\Psi_{g_-}^{T_1}$. For our figure (drawn with a slightly different $x-$ and $y-$ scaling) we have taken the coefficients $k = 1, g_+ = 4, g_- = 1/4$ and $T_1 = 12.7$. In this case $\Psi_{g_-}^{T_1}(\gamma(s))$ performs at least *three crossings* of the region $\mathscr{B}$. For typographical reasons we have not represented the whole image $\{\Psi_{g_-}^{T_1}(\gamma(s)) : s \in [0,1]\}$ and only two crossings appear.

We switch now to system (1.18). If

$$T_2 \geq \omega^{\#} \tag{3.6}$$

holds, then for each point $z_0 \in \mathscr{B}_l^-$ the solution $(u(t;z_0), \dot{u}(t;z_0)) = \zeta(t;z_0)$ satisfies

$$\mathcal{E}_+(\zeta(t;z_0)) = c_+^{\#}, \quad \forall\, t \in [0,T_2], \quad \text{with } u(T_2;z_0) > \xi_-\,.$$

In this case, since the positive semi-orbit of z_0 crosses the abscissa in the phase plane at the point $(\eta_+, 0)$ with $\xi_+ < \eta_+ < \xi_-$, from (3.3) we conclude that

$$\mathcal{E}_-(\zeta(T_2;z_0)) > c_-^*\,, \quad \forall\, z_0 \in \mathscr{B}_l^-\,,$$

with $\dot{u}(T_2;z_0) > 0$.

On the other hand, if $z_0 \in \mathscr{B}_r^-$, the solution $(u(t;z_0), \dot{u}(t;z_0)) = \zeta(t;z_0)$ is on the energy level line of the equilibrium point $(\xi_+, 0)$, hence $u(t;z_0) \in]\xi_+, \xi_-[$ and $\dot{u}(t;z_0) < 0$, for every $t \in [0,T_2]$.

Let $\gamma : [0,1] \to \mathscr{B}$ be a continuous path such that $\gamma(0) \in \mathscr{B}_l^-$ and $\gamma(1) \in \mathscr{B}_r^-$ and let us consider the image of γ through the Poincaré map $\Psi_{g_+}^{T_2}$. Using the fact that the set

$$\{(x,y) : \mathcal{E}_+(\eta_+, 0) \leq \mathcal{E}_+(x,y) \leq \mathcal{E}_+(\xi_+, 0)\}$$

is invariant for the flow of (1.18) and

$$\mathcal{E}_-(\zeta(T_2;\gamma(0))) > c_-^*, \quad \partial_t u(T_2;\gamma(0)) > 0 \quad \text{and} \quad \partial_t u(T_2;\gamma(1)) < 0,$$

we can find a closed interval $[s_0, s_1] \subseteq [0,1]$ such that $\Psi_{g_+}^{T_2}(\gamma(s)) \in \mathscr{A}$ for all $s \in [s_0, s_1]$ and

$$\Psi_{g_+}^{T_2}(\gamma(s_0)) \in \mathscr{A}_r^-\,, \quad \Psi_{g_+}^{T_2}(\gamma(s_1)) \in \mathscr{A}_l^-\,.$$

Thus, also the stretching property for $\Psi_{g_+}^{T_2}$ is proved, namely

$$(\Psi_{g_+}^{T_2}) : \widehat{\mathscr{B}} \Leftrightarrow\!\!\!\to \widehat{\mathscr{A}}.$$

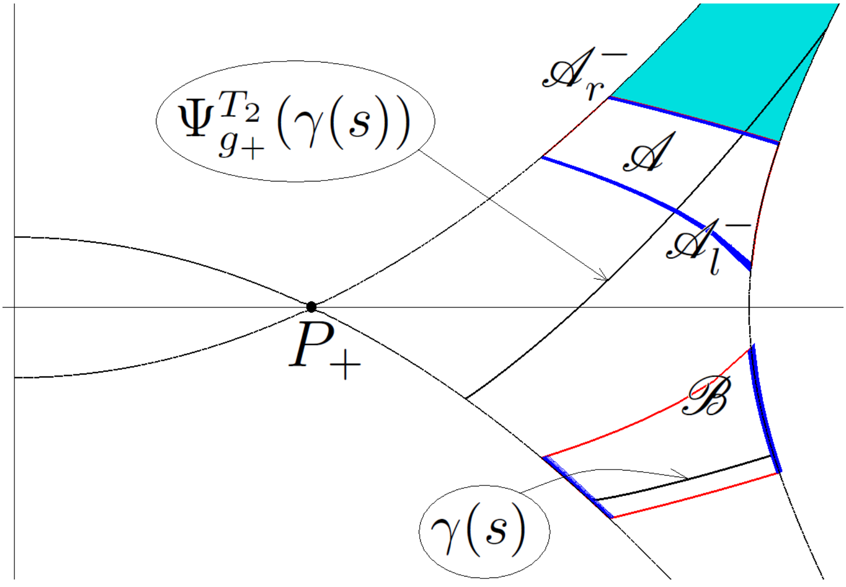

Figure 5: The stretching of the path $\gamma(s)$, connecting $\mathscr{B}_l^-$ to $\mathscr{B}_r^-$, through the Poincaré map $\Psi^{T_2}_{g_+}$. For our figure (drawn with a slightly different $x-$ and $y-$ scaling) we have taken the coefficients $k = 1, g_+ = 4, g_- = 1/4$ (as in Figure 4) and $T_2 = 0.22$. In this case $\Psi^{T_2}_{g_+}(\gamma(s))$ performs one crossing of the region $\mathscr{A}$. We have depicted with a darker color the set $\{(x,y) : \mathcal{E}_+(\eta_+,0) \leq \mathcal{E}_+(x,y) \leq \mathcal{E}_+(\xi_+,0),\ \mathcal{E}_-(x,y) \geq c^*_-,\ y > 0\}$ and denoted by P_+ the equilibrium point $(\xi_+,0)$.

3.3 Conclusion of the proof

Now, we can apply Theorem 2.1 for $\Phi_1 = \Psi^{T_1}_{g_-}$, $\Phi_2 = \Psi^{T_2}_{g_+}$ and $\mathcal{H}_i = H_i$ for $i = 0,\dots, j = m-1$. In particular, for $j = 1$, we obtain a chaotic dynamics on two symbols for the Poincaré map Ψ according to Definition 2.1. The fact that the solutions we find have the precise behavior described in Theorem 1.1 is an evident consequence of the phase plane analysis that we have performed along the previous steps. This concludes the proof of Theorem 1.1 which holds for any choice of (T_1^*, T_2^*) with

$$T_1^* \geq (3/2)\tau^\#_-\,, \quad T_2^* \geq \omega^\# \tag{3.7}$$

(see (3.4) and (3.6)).

Concerning the proof of Theorem 1.2 it is sufficient to observe that the above argument persists under small perturbation in the coefficients governing the equation. More precisely, once we have fixed $T_1 > T_1^*$ and $T_2 > T_2^*$, for T_1^* and T_2^* as in the proof of Theorem 1.1, an application of the theorem of continuous dependence of the solutions [6] guarantees the validity of Theorem 1.2 for a suitable $\varepsilon > 0$. □

4 Remarks and variants of the main result

Remark 4.1 In view of the formulas (3.7) and (3.4), in order to estimate the time T_1^* we need to know the time $\tau^\#_-$, where such time is of the form

$$\tau_-(c) = 2\int_{-w}^{w} \frac{d\,x}{\sqrt{c - 2\mathcal{F}_-(x)}}\,, \tag{4.1}$$

with $2\mathcal{F}_-(\pm w) = \mathcal{E}_-(w,0) = c$, for $w = \eta_- < \xi_-$. Using the fact that

$$\mathcal{F}_-(x) := kx^2 - \frac{1}{2}g_-x^4,$$

we obtain that

$$c - 2\mathcal{F}_-(x) = (w^2 - x^2)(2k - w^2 g_- - g_- x^2).$$

As a consequence, we can express $\tau_-(c)$ as

$$\tau_-(c) = \frac{4}{\sqrt{g_-}} \int_0^w \frac{dx}{\sqrt{(w^2 - x^2)(\rho^2 - x^2)}}, \text{ with } \rho^2 := \frac{2k}{g_-} - w^2.$$

The theory of elliptic integrals allows us to conclude that

$$\tau_-(c) = \frac{4}{\sqrt{2k - w^2 g_-}} \operatorname{sn}^{-1}(1, \varrho) = \frac{4}{\sqrt{2k - w^2 g_-}} \int_0^{\pi/2} \frac{du}{\sqrt{1 - \varrho^2 \sin^2 u}}$$

for

$$\varrho := \left(\frac{2k}{g_- w^2} - 1\right)^{-1/2},$$

where $\operatorname{sn}(t, v)$ is the Jacobi elliptic sine function of modulus $v \in]0, 1[$. Observe that $\varrho \in]0, 1[$ by the choice $0 < w = \eta_- < \xi_-$. Similar considerations apply to the evaluation of the time $\omega(c)$ and, consequently, to estimate the time T_2^* (see, e.g., [5]). In conclusion, for a fixed value of the coefficients k and $g_\pm$, one can numerically estimate the constants T_1^* and T_2^* which are not necessarily "large numbers".

Remark 4.2 The proof of Theorem 1.1 (as well as Theorem 1.2) has been performed by suitably choosing the rectangular regions $\mathscr{A}$ and $\mathscr{B}$ where the abstract topological result can be applied. With obvious modifications one could repeat the same argument by selecting two rectangular regions $\mathscr{A}'$ and $\mathscr{B}'$ which are symmetric to $\mathscr{A}$ and $\mathscr{B}$ (with respect to the origin), as in Figure 6. The corresponding constants T_1^* and T_2^* will be exactly the same we have obtained along the proof of Theorem 1.1. The only difference will be in the conclusion, concerning the behavior of the solutions. A less obvious fact, although a straightforward consequence of the argument of the proof, is that

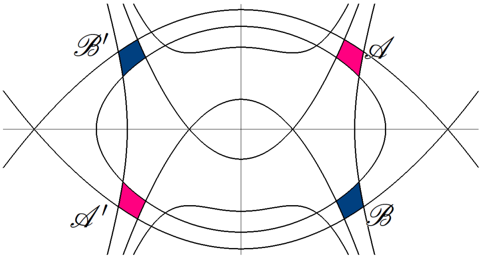

Figure 6: Examples of the regions where Theorem 2.1 applies. The topological rectangles $\mathscr{A}$ and $\mathscr{B}$ are those chosen for the proof of Theorem 1.1. The choice of the regions $\mathscr{A}'$ and $\mathscr{B}'$ is possible too, leading to a symmetric result.

also other ways to apply Theorem 2.1 are admissible. For instance, a possible choice is described in Figure 7 for the pair $(\mathscr{A}, \mathscr{B})$ and for the symmetric one $(\mathscr{A}', \mathscr{B}')$. In general, for such new regions, the constants T_1^* and T_2^* will be not the same found for the sets of Figure 6. Also the behavior of the solutions will be completely different.

Remark 4.3 Even if we have developed all the computations for an equation with a cubic nonlinear term like (1.4), we can easily adapt our technique to more general second order equations of the form

$$-\ddot{u} + \lambda u + g(t) f(u) u = 0, \quad \lambda < 0, \tag{4.2}$$

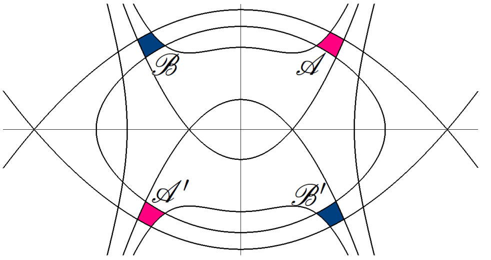

Figure 7: Examples of alternative regions where Theorem 2.1 applies.

by assuming $f : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function, such that $f'(0) = 0$, $sf'(s) > 0$ for all $s \neq 0$, and satisfying

$$f(\pm\infty) > \frac{|\lambda|}{g_-}.$$

Such kind of nonlinearities has been recently considered in [1] for $f(s)$ an even function. Indeed, what is really necessary in our approach is that equation (4.2) has an associated phase-portrait similar to that depicted in Figure 2 and the symmetry with respect to the y-axis is not needed. Clearly, for a general nonlinearity $f(s)$ the period $\tau_-(c)$ and the time $\omega(c)$ will be expressed by formulas involving the potentials

$$\mathcal{F}_\pm(x) = \frac{|\lambda|}{2}x^2 - g_\pm \int_0^x f(s)s\,ds$$

and without an explicit knowledge of the analytic form of $f(s)$ we cannot provide more effective estimates as in Remark 4.1.

Appendix

We consider the second order scalar ODE

$$\ddot{x} + h(t,x) = 0, \tag{A.1}$$

where the function $h = h(t,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions [6], that is $h(\cdot, s)$ is measurable for all $s \in \mathbb{R}$, $h(t,\cdot)$ is continuous for almost every $t \in \mathbb{R}$ and, for every compact interval I and each constant $r > 0$ there is a measurable function $\rho = \rho_{I,r}$ with $\rho \in L^1(I, \mathbb{R}^+)$ such that $|h(t,s)| \leq \rho(t)$ for almost every $t \in I$ and every $s \in [-r, r]$. Solutions of (A.1) are considered in the generalized sense, as functions in $W^{2,1}_{\text{loc}}$.

Lemma 4.1 *Suppose there are two constants α, β, with $\alpha < \beta$ such that, for almost every $t \in \mathbb{R}$ it holds that*

$$h(t,s) \geq 0, \ \ \textit{for} \ \ s \leq \alpha \qquad \textit{and} \ \ \ h(t,s) \leq 0, \ \ \textit{for} \ \ s \geq \beta. \tag{A.2}$$

Suppose that $x(\cdot)$ is a solution of (A.1) *defined on $\mathbb{R}$ for which there exists a two-sided sequence $(t_n)_{n\in\mathbb{Z}}$ with $t_n \to \pm\infty$ for $n \to \pm\infty$ such that*

$$\alpha \leq x(t_n) \leq \beta, \quad \forall\, t_n\,. \tag{A.3}$$

Then, $\alpha \leq x(t) \leq \beta$, for all $t \in \mathbb{R}$. In particular, if $x(t)$ is a periodic solution of (A.1) *such that $x(t_0) \in [\alpha, \beta]$ for some $t_0 \in \mathbb{R}$, then $x(t) \in [\alpha, \beta]$, for all $t \in \mathbb{R}$.*

Proof. Without loss of generality, we can assume that the sequence t_n is strictly increasing with $n \in \mathbb{Z}$. Given an arbitrary $\bar{t} \in \mathbb{R}$ we choose $\sigma < \bar{t} < \tau$, with $\sigma, \tau \in \{t_n\}$. We claim that $\alpha \leq x(\bar{t}) \leq \beta$. Suppose, by contradiction, that $x(\bar{t}) > \beta$ and let $]\sigma_0, \tau_0[\, \subseteq\,]\sigma, \tau[$ be a maximal open interval containing $\bar{t}$ and such that $x(t) > \beta$ for all $t \in]\sigma_0, \tau_0[$. Moreover, by the maximality of the interval and the fact that $x(\sigma), x(\tau) \leq \beta$, we have that $x(\sigma_0) = x(\tau_0) = \beta$ with $\dot{x}(\sigma_0) \geq 0 \geq \dot{x}(\tau_0)$. Integrating the equation on $[\sigma_0, t] \subseteq [\sigma_0, \tau_0]$ and using the fact that $-h(\xi, x(\xi)) \geq 0$ on that interval, yields $\dot{x}(t) \geq \dot{x}(\sigma_0) \geq 0$. Similarly, integrating on $[t, \tau_0] \subseteq [\sigma_0, \tau_0]$, we obtain $\dot{x}(t) \leq \dot{x}(\tau_0) \leq 0$. Hence, $x(t) = \beta$ for all $t \in [\sigma_0, \tau_0]$, a contradiction. Similarly, we check that $x(\bar{t}) \geq \alpha$.

If $x(t)$ is periodic (of some period $L > 0$) and $x(t_0) \in [\alpha, \beta]$ for some t_0 , we can apply the first part of the result with $t_n = t_0 + nL$. □

Our result is a slight modification of standard facts from the theory of upper and lower solutions (cf. [4]). Indeed, α and β are constant lower and upper solutions for (A.1). Actually, a more general variant of lemma 4.1 holds if we assume the existence of two $W^{2,1}_{\mathrm{loc}}$-functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ with $\alpha \leq \beta$ such that

$$\ddot{\alpha}(t) + h(t, \alpha(t)) \geq 0 \geq \ddot{\beta}(t) + h(t, \beta(t)), \qquad \text{for a.e. } t \in \mathbb{R}.$$

In this case, if, instead of (A.2), we assume

$$h(t, s) \geq h(t, \alpha(t)), \quad \text{for } s \leq \alpha(t) \quad \text{and} \quad h(t, s) \leq h(t, \beta(t)), \quad \text{for } s \geq \beta(t),$$

the same argument guarantees that $\alpha(t) \leq x(t) \leq \beta(t)$, for all $t \in \mathbb{R}$ provided that $\alpha(t_n) \leq x(t_n) \leq \beta(t_n)$, $\forall\, t_n$ (for t_n as above). Lemma 4.1 can be applied as follows. Let $\alpha < \beta$ be two constants such that

$$h(t, \alpha) \geq 0 \geq h(t, \beta), \quad \text{for a.e. } t \in \mathbb{R} \tag{A.4}$$

and define the truncated function

$$\tilde{h}(t, s) = \begin{cases} h(t, \alpha) & \text{for } s \leq \alpha \\ h(t, s) & \text{for } \alpha \leq s \leq \beta \\ h(t, \beta) & \text{for } s \geq \beta. \end{cases}$$

We consider the modified differential equation

$$\ddot{x} + \tilde{h}(t, x) = 0. \tag{A.5}$$

Observe that all the solutions of (A.5) are globally defined. Then we have:

Corollary 4.1 *Assume* (A.4) *and let $x(t)$ be a solution of* (A.5) *such that* (A.3) *holds for a two-sided sequence $(t_n)_n$ with $t_n \to \pm\infty$ for $n \to \pm\infty$. Then $x(\cdot)$ is a solution of* (A.1) *with $\alpha \leq x(t) \leq \beta$ for all $t \in \mathbb{R}$.*

The conclusion of Corollary 4.1 holds also for the equation $\ddot{x} + c\dot{x} + h(t, x) = 0$ and its modification with $\tilde{h}(t, x)$.

References

[1] J. Belmonte-Beitia and J. Cuevas, *Existence of dark solitons in a class of stationary nonlinear Schrödinger equations with periodically modulated nonlinearity and periodic asymptotics*, J. Math. Phys. **52** (2011), 032702, 9 pp.

[2] J. Belmonte-Beitia and P. J. Torres, *Existence of dark soliton solutions of the cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity*, J. Nonlinear Math. Phys. **15** (2008), suppl. 3, 65-72.

[3] A. Capietto, W. Dambrosio and D. Papini, *Superlinear indefinite equations on the real line and chaotic dynamics*, J. Differential Equations **181** (2002), 419-438.

[4] C. De Coster and P. Habets, Upper and Lower Solutions in the Theory of ODE Boundary Value Problems: Classical and Recent Results. *In:* Non-linear Analysis and Boundary Value Problems for Ordinary Differential Equations (Udine), pp. 178, CISM Courses and Lectures, **371**, Springer, Vienna, 1996.

[5] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, Fifth edition, Translation edited and with a preface by Alan Jeffrey, Academic Press, Inc., Boston, MA, 1994.

[6] J.K. Hale, Ordinary Differential Equations. Second Edition, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[7] J. Kennedy and J.A. Yorke, *Topological horseshoes*, Trans. Amer. Math. Soc. **353** (2001), 2513–2530.

[8] A. Margheri, C. Rebelo and F. Zanolin, *Chaos in periodically perturbed planar Hamiltonian systems using linked twist maps*, J. Differential Equations **249** (2010), 3233-3257.

[9] A. Medio, M. Pireddu and F. Zanolin, *Chaotic dynamics for maps in one and two dimensions: a geometrical method and applications to economics*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19** (2009), 3283-3309.

[10] K.R. Meyer and G.R. Hall, Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Applied Mathematical Sciences, **90**, Springer-Verlag, New York, 1992.

[11] E.E. Moise, Geometric Topology in Dimensions 2 and 3, Graduate Texts in Mathematics, Vol. **47**, Springer - Verlag, New York 1977.

[12] J. Moser, *Stable and Random Motions in Dynamical Systems. With special emphasis on celestial mechanics*, Hermann Weyl Lectures, The Institute for Advanced Study. Ann. of Math. Stud., vol. **77**, Princeton University Press, Princeton, NJ, 1973.

[13] D. Papini, *Prescribing the nodal behaviour of periodic solutions of a superlinear equation with indefinite weight*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 43-63.

[14] D. Papini and F. Zanolin, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15** (2000), 203-233.

[15] D. Papini and F. Zanolin, *Periodic points and chaotic-like dynamics of planar maps associated to nonlinear Hill's equations with indefinite weight*, Georgian Math. J. **9** (2002), 339-366.

[16] A. S. Rodrigues, P. G. Kevrekidis, M. A. Porter, D. J. Frantzeskakis, P. Schmelcher, and A. R. Bishop, *Matter-wave solitons with a periodic, piecewise-constant scattering length*, Phys. Rev. A **78** 013611 (2008), 1–10.

[17] S. Terracini and G. Verzini, *Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities*, Nonlinearity **13** (2000), 1501-1514.

[18] P.J. Torres and V.V. Konotop, *On the existence of dark solitons in a cubic-quintic nonlinear Schrdinger equation with a periodic potential*, Comm. Math. Phys. **282** (2008), 1-9.

[19] A. Ureña, *Dynamics of periodic second-order equations between an ordered pair of lower and upper solutions*, Advanced Nonlinear Studies **11** (2011), 675-694.

[20] S. Wiggins, *Global Bifurcations and Chaos. Analytical methods*, Applied Mathematical Sciences, **73**, Springer-Verlag, New York, 1988.

[21] Z. Xie, *Uniqueness and blow-up rate of large solutions for elliptic equation* $-\Delta u = \lambda u - b(x)h(u)$, J. Differential Equations **247** (2009), 344-363.

[22] C. Zanini and F. Zanolin, *Complex dynamics in a nerve fiber model with periodic coefficients*, Nonlinear Anal. Real World Appl. **10** (2009), 1381-1400.

[23] C. Zanini and F. Zanolin, *Periodic solutions for a class of second order ODEs with a Nagumo cubic type nonlinearity*, Discrete Contin. Dynam. Systems **32** (2012), 4045–4067.