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# Asymptotic Approach to Inverse Bifurcation for Nonlinear Sturm-Liouville Problems

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**Abstract**

We consider the nonlinear inverse bifurcation problem

$$-u''(t) + f(u(t)) = \lambda u(t), \;\; u(t) > 0, \;\; t \in I := (0, 1), \;\; u(0) = u(1) = 0,$$

where $\lambda > 0$ is a positive parameter, and the nonlinear term $f(u)$ is unknown. We show that, if the asymptotic behavior of the $L^q$-bifurcation curve $\lambda = \lambda_q(\alpha)$ ($1 \le q < \infty$) of the positive solutions is understood well, then we are able to obtain $f(u)$ and to characterize the asymptotic behavior of $f(u)$ as $u \to \infty$.

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# 1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is a positive parameter. We assume that $f(u)$ satisfies the following conditions (A.1)–(A.2).

(A.1) $f(u)$ is $C^1$ for $u \ge 0$ satisfying $f(u) > 0$ for $u > 0$, $f(0) = f'(0) = 0$.

(A.2) $f(u)/u$ is strictly increasing for $u \ge 0$ and $f(u)/u \to \infty$ as $u \to \infty$.

The typical examples of $f(u)$ which satisfy (A.1) and (A.2) are as follows.

$$f(u) = u^p \quad (u \geq 0), \tag{1.4}$$

$$f(u) = u^p + u^m \quad (u \geq 0), \tag{1.5}$$

$$f(u) = u^p\left(1 - \frac{1}{1+u^2}\right) \quad (u \geq 0), \tag{1.6}$$

$$f(u) = u^5 e^u \quad (u \geq 0), \tag{1.7}$$

where $p > m > 1$ are constants.

The purpose of this paper is to establish an asymptotic approach to inverse bifurcation problem for (1.1)–(1.3). To be more precise, we assume that the nonlinear term $f(u) = u^p h(u)$ has an unknown constant $p > 1$. Then we show that, if the asymptotic formula for the $L^q$-bifurcation curve $\lambda_q(\alpha)$ as $\alpha \to \infty$ is known precisely, then we are able to find $p > 1$ and characterize the asymptotic property of $f(u)$ for $u \gg 1$. Here, $1 \leq q < \infty$ is a constant and we fix it throughout this paper. We call this idea *asymptotic approach for inverse bifurcation problems*.

We explain the background of our problem briefly. To investigate the properties of bifurcation curve for given nonlinear term $f(u)$ is called the direct problems, and has been studied intensively. We refer to [1], [2] and [7–10] for the works which treated the problems by local bifurcation theory of $L^\infty$-framework. Furthermore, since (1.1)–(1.3) is regarded as an eigenvalue problem, it seems significant to consider (1.1)–(1.3) in $L^2$-framework. Moreover, since (1.1)–(1.3) is the diffusive logistic equation of population dynamics if $f(u) = u^2$, it is also important to study (1.1)–(1.3) in $L^1$-framework. We refer to [3–6] and [11–15] for the works in this direction. However, there are few results for inverse bifurcation problems.

It should also be mentioned that our problem corresponds to linear inverse spectral problem in the following sense. The typical linear inverse spectral problem is to find the unknown potential $V$ by using the information about the corresponding eigenvalues. If we replace "unknown potential $V$" and "corresponding eigenvalues" with "unknown nonlinear term $f(u)$" and "bifurcation curve", respectively, then our nonlinear problem appears. Therefore, our problem seems to be worth considering.

Before stating our result, let us briefly recall some known facts (cf. [1]).

(i) For each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda_q(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ of (1.1)–(1.3) with $\|u_\alpha\|_q = \alpha$. Here, $\|u_\alpha\|_q$ is the $L^q$-norm of $u_\alpha$, and $\lambda_q(\alpha)$ is called $L^q$-bifurcation curve.

(ii) The set $\{(\lambda_q(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and is an unbounded curve of class $C^1$ in $\mathbf{R}_+ \times L^q(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda_q(\alpha)$ is strictly increasing for $\alpha > 0$ and $\lambda_q(\alpha) \to \infty$ as $\alpha \to \infty$.

As for the asymptotic behavior of $\lambda_q(\alpha)$ and $u_\alpha$ as $\alpha \to \infty$, it is known from [1] that

$$\frac{u_\alpha(t)}{\|u_\alpha\|_\infty} \to 1 \tag{1.8}$$

locally uniformly on $I$ as $\alpha \to \infty$. We set $g(u) := f(u)/u$. Then as $\alpha \to \infty$,

$$\lambda_q(\alpha) = g(\|u_\alpha\|_\infty) + \xi_\alpha, \tag{1.9}$$

where $\xi_\alpha = O(1)$ is the remainder term. By (1.8), we see that $\|u_\alpha\|_\infty = \alpha(1 + o(1))$ for $\alpha \gg 1$. By this and (1.9), for $\alpha \gg 1$,

$$\lambda_q(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.10}$$

Motivated by (1.10), more precise asymptotic formula for $\lambda_q(\alpha)$ as $\alpha \to \infty$ has been given in [15].

**Theorem 1.1 [15]** *Let $f(u) = u^p$, where $p > 1$ is a given constant. Then as $\alpha \to \infty$,*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + C_1 + o(1), \tag{1.11}$$

*where*

$$C_0 = \frac{2(p-1)}{q}C_2, \quad C_1 = \frac{2(p-1)}{q}C_2^2,$$
$$C_2 = \int_0^1 \frac{1-s^q}{\sqrt{1-s^2-2(1-s^{p+1})/(p+1)}}ds.$$

The formula (1.11) has been obtained first for $q = 2$ in [12] by using the relationship between $\lambda_2(\alpha)$ and the critical value associated with $\lambda_2(\alpha)$. By using the technique in [12], the following result for direct problem has been obtained in [14].

**Theorem 1.2 [14]** *Let $f(u) = u^p h(u)$, where $p > 1$ is a given constant. Assume that $h(u)$ is a $C^2$-function for $u \geq 0$ satisfying the following conditions.*

$$\frac{uh'(u)}{h(u)} \to 0, \quad uh'(u) \to 0 \qquad (u \to \infty). \tag{1.12}$$

*Then under the additional technical conditions on $h(u)$, the following asymptotic formula holds as $\alpha \to \infty$:*

$$\lambda_2(\alpha) = \alpha^{p-1}h(u) + C_0\sqrt{\alpha^{p-1}h(u)}(1 + o(1)). \tag{1.13}$$

The typical examples of $h(u)$ in Theorem 1.2 are:

$$h(u) = 1, \quad h(u) = 1 - \frac{1}{1+u^2}.$$

Theorem 1.2 takes us to the simple inverse problem. Assume that $h(u)$ is unknown in Theorem 1.2, but we know (1.13) holds as $\alpha \to \infty$. Then can we conclude that $h(u) = 1$ or $h(u) = 1 - 1/(1 + u^2)$ ?

Motivated by this, we consider the following inverse problem.

**Problem 1** Let $f(u) = u^p h(u)$ ($p > 1$) satisfy (A.1) and (A.2). Suppose that $h(u)$ is unknown, but it is known that as $\alpha \to \infty$,

$$\lambda_q(\alpha) \quad = \quad g(\alpha) + Ag(\alpha)^{1/2} + O(1), \tag{1.14}$$

where $A > 0$ is a given constant. Then can we find $h(u)$ ?

To state our results, we assume additional conditions (A.3) and (A.4).

(A.3) $h(u)$ is a $C^1$ function for $u > 0$, and there exists a constant $\delta_0 > 0$ such that $h(u) \geq \delta_0$ for $u > 0$. Furthermore, for an arbitrary fixed constant $0 < \epsilon \ll 1$, as $u \to \infty$,

$$\max_{\epsilon \leq s \leq 1}\left|\frac{uh'(us)}{h(u)}\right| = O((u^{p-1}h(u))^{-1/2}), \tag{1.15}$$

$$\max_{0 \leq s \leq \epsilon} s^p\left|\frac{uh'(us)}{h(u)}\right| = O((u^{p-1}h(u))^{-1/2}). \tag{1.16}$$

(A.4) There exists a constant $0 < \delta_1 \ll 1$ such that for $(1 + \delta_1)v > u > v \gg 1$,

$$f(u) = f(v) + f'(v)(u - v) + O(f(v)/v^2)(u - v)^2. \tag{1.17}$$

The typical examples of $h(u)$ (i.e. $f(u)$) satisfying (A.3) and (A.4) are:

$$h(u) = 1 \quad (f(u) = u^p),$$
$$h(u) = 1 + u^{m-p} \quad \left(f(u) = u^p + u^m,\ \ 1 < m \le \frac{p+1}{2}\right).$$

The answer to Problem 1 is as follows.

**Theorem 1.3** *Assume that all conditions in Problem 1, (A.3) and (A.4) are satisfied. Then* $f(u) = u^p h(u)$ *with* $p = 1 + (qA)/(2C_2)$ *and* $h(u) = D + d(u)$*, where* $C_2$ *is a constant in Theorem 1.1,* $A$ *is a constant in (1.14),* $D > 0$ *is an arbitrary positive constant and* $d(u) = O(u^{(1-p)/2})$ *for* $u \gg 1$.

We emphasize that in the situation of (1.14), *the exponent* $p$ *is determined by the coefficient* $A$.

We are able to consider another inverse problem, which seems simpler than Problem 1.

**Problem 2** Let $p > 1$ be a given constant. Assume that the following asymptotic formula holds as $\alpha \to \infty$:

$$\lambda_q(\alpha) = \alpha^{p-1} + A\alpha^{(p-1)/2} + O(1). \tag{1.18}$$

Then find $f(u)$.

The answer to Problem 2 is as follows.

**Theorem 1.4** *Consider Problem 2. Assume that* $A \ge C_0$. *Then for* $u \gg 1$,

$$f(u) = u^p + (A - C_0)u^{(p+1)/2} + o(u^{(p+1)/2}).$$

Theorem 1.4 is a direct consequence of Theorem 1.5 (ii) below.

It should be mentioned that the inverse problem related to Problem 2 has been treated in [16]. Precisely, the case $\lambda_q(\alpha) = \alpha^{p-1} + r(\alpha)(1 + o(1))$ with $r(\alpha) \sim \alpha^{q-1}$ $(1 < q < p, q \ne (p+1)/2)$ has been treated, and the case where $r(\alpha) \sim \alpha^{(p-1)/2}$ has not been considered.

By applying the argument used to prove Theorem 1.3 to the special case $f(u) = u^p + u^m$, we obtain the following asymptotic formula for $\lambda_q(\alpha)$ as $\alpha \to \infty$, which improves the result in [11].

**Theorem 1.5** *Let* $f(u) = u^p + u^m$ $(p > m > 1)$.
*(i) Assume that* $m > (p+1)/2$. *Then as* $\alpha \to \infty$.

$$\begin{aligned}\lambda_q(\alpha) &= \alpha^{p-1} + \alpha^{m-1} + C_0\alpha^{(p-1)/2} \\ &\quad + \left\{\frac{2C_2}{q}\left(m - \frac{p+1}{2}\right) + \frac{2(p-1)\beta}{q}\right\}\alpha^{m-(p+1)/2}(1 + o(1)),\end{aligned} \tag{1.19}$$

*where* $C_0$, $C_2$ *are constants defined in Theorem 1.1, and* $\beta$ *is a positive constant given by*

$$\beta := \int_0^1 \frac{2(1 - s^q)\{(1 - s^{m+1})/(m+1) - (1 - s^{p+1})/(p+1)\}}{(1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}))^{3/2}} ds. \tag{1.20}$$

*(ii) Assume that* $m = (p+1)/2$. *Then as* $\alpha \to \infty$,

$$\begin{aligned}\lambda_q(\alpha) &= \alpha^{p-1} + \alpha^{(p-1)/2} + C_0\alpha^{(p-1)/2} + \frac{2(p-1)\beta}{q} + \frac{2(p-1)(p+q-1)}{q^2}C_2^2 \\ &\quad + o(1).\end{aligned} \tag{1.21}$$

**Remark 1.6** (i) The condition (A.3) comes from (1.12) and seems stronger than (1.12), but this is only due to a technicality. Indeed, if we consider $f(u) = u^5 e^u$ and $q = 2$, then $g(u) = u^4 e^u$ does not satisfy (A.3), and we know from [13] that as $\alpha \to \infty$

$$\lambda_2(\alpha) = \alpha^4 e^\alpha + \frac{\pi}{4}\alpha^3 e^{\alpha/2} + \frac{\pi}{4}u^2 e^{\alpha/2}(1 + o(1)), \tag{1.22}$$

which is different from (1.14). Therefore, (1.14) does not hold without (A.3).

(ii) It is also clear that (A.3) does not hold for $f(u) = u^p + u^m$ $(p > m > (p+1)/2)$, and (1.19) does not coincide with (1.14). Indeed, $g(u) = u^{p-1} + u^{m-1}$ and it follows from (1.19) that $A$ in (1.14) should be $C_0$. Then by Taylor expansion, (1.14) is

$$\begin{aligned} \lambda_q(\alpha) &= \alpha^{p-1} + \alpha^{m-1} + C_0(\alpha^{p-1} + \alpha^{m-1})^{1/2} + O(1) \\ &= \alpha^{p-1} + \alpha^{m-1} + C_0\alpha^{(p-1)/2} + \frac{1}{2}C_0\alpha^{m-(p+1)/2}(1 + o(1)). \end{aligned} \tag{1.23}$$

Then by direct calculation, we easily find that the coefficients of $\alpha^{m-(p+1)/2}$ in (1.19) and (1.23) satisfy

$$\frac{2C_2}{q}\left(m - \frac{p+1}{2}\right) + \frac{2(p-1)\beta}{q} < \frac{1}{2}C_0 = \frac{C_2(p-1)}{q}. \tag{1.24}$$

Therefore, (A.3) is necessary for our conclusion.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.3 under the condition that the key lemma (Lemma 2.2) is valid. In Section 3, we prove Lemma 2.2. Theorem 1.5 will be proved in Section 4

# 2 Proof of Theorem 1.3

In what follows, $C$ denotes various positive constants independent of $\alpha \gg 1$. We write $(\lambda, u_\alpha)$ for a unique solution pair of (1.1)–(1.3) with $\|u_\alpha\|_q = \alpha$. We begin with the fundamental tools which play important roles in what follows. It is well known that

$$u_\alpha(t) = u_\alpha(1-t), \quad t \in I, \quad \|u_\alpha\|_\infty = u_\alpha\left(\frac{1}{2}\right), \tag{2.1}$$

$$u_\alpha'(t) > 0, \quad 0 \le t < \frac{1}{2}. \tag{2.2}$$

Multiply (1.1) by $u_\alpha'(t)$. Then

$$\left(u_\alpha''(t) + \lambda u_\alpha(t) - f(u_\alpha(t))\right) u_\alpha'(t) = 0.$$

This along with (2.1) implies that

$$\begin{aligned} \frac{1}{2}u_\alpha'(t)^2 + \frac{1}{2}\lambda u_\alpha(t)^2 - F(u_\alpha(t)) &= \text{constant} \\ &= \frac{1}{2}\lambda\|u_\alpha\|_\infty^2 - F(\|u_\alpha\|_\infty), \quad (\text{put } t = 1/2) \end{aligned} \tag{2.3}$$

where $F(u) := \int_0^u f(s)ds$. We set

$$L_\alpha(\theta) = \lambda(\|u_\alpha\|_\infty^2 - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)). \tag{2.4}$$

This along with (2.2) and (2.3) implies that for $0 \le t \le 1/2$

$$u'_\alpha(t) = \sqrt{L_\alpha(u_\alpha(t))}. \tag{2.5}$$

By this and (2.1), we obtain

$$\begin{aligned}
\|u_\alpha\|_\infty^q - \alpha^q &= 2\int_0^{1/2} \frac{(\|u_\alpha\|_\infty^q - u_\alpha^q(t))u'_\alpha(t)}{\sqrt{L_\alpha(u_\alpha(t))}}dt \\
&= 2\int_0^{\|u_\alpha\|_\infty} \frac{(\|u_\alpha\|_\infty^q - \theta^q)}{\sqrt{L_\alpha(\theta)}}d\theta \\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \int_0^1 \frac{1-s^q}{\sqrt{B_\alpha(s)}}ds \qquad (2.6)\\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}} \left\{ \int_0^1 \frac{1-s^q}{\sqrt{J(s)}}ds + \int_0^1 \left( \frac{1-s^q}{\sqrt{B_\alpha(s)}} - \frac{1-s^q}{\sqrt{J(s)}} \right) ds \right\} \\
&= \frac{2\|u_\alpha\|_\infty^q}{\sqrt{\lambda}}(C_2 + M_\alpha),
\end{aligned}$$

where

$$J(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}), \tag{2.7}$$

$$B_\alpha(s) := 1 - s^2 - \frac{2}{\lambda\|u_\alpha\|_\infty^2}(F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)), \tag{2.8}$$

$$M_\alpha := \int_0^1 \left( \frac{1-s^q}{\sqrt{B_\alpha(s)}} - \frac{1-s^q}{\sqrt{J(s)}} \right) ds. \tag{2.9}$$

**Lemma 2.1** *$f'(\alpha) \le C\alpha^{p-1}$ for $\alpha \gg 1$.*

*Proof.* Let $0 < \epsilon \ll 1$ be fixed. For $\alpha \gg 1$, we put $u = (1+\epsilon)\alpha$ and $v = \alpha$ in (A.4). Then by (A.2),

$$\begin{aligned}
f'(\alpha)\epsilon\alpha &= f(\alpha(1+\epsilon)) - f(\alpha) + O(f(\alpha)) \le Cf(\alpha(1+\epsilon)) \qquad (2.10)\\
&\le Cg(\alpha(1+\epsilon))\alpha \le C\alpha^p h(\alpha(1+\epsilon)).
\end{aligned}$$

Now we show that $h(u)$ is bounded for $u \gg u_0 > 0$. Put $s = 1$ in (1.15) to obtain

$$\left| \int_{u_0}^u \frac{h'(y)}{\sqrt{h(y)}}dy \right| \le \int_{u_0}^u \left| \frac{h'(y)}{\sqrt{h(y)}} \right| dy \le C\int_{u_0}^u \frac{1}{y^{(p+1)/2}}dy \le C.$$

This implies that for $u \gg u_0$

$$\sqrt{h(u)} \le \sqrt{h(u_0)} + C \le C. \tag{2.11}$$

By this and (2.10), we obtain our conclusion. ■

By (1.10), (A.3) and Lemma 2.1, for $\alpha \gg 1$,

$$C^{-1}\alpha^{p-1} \le \lambda \le C\alpha^{p-1}, \tag{2.12}$$

$$C^{-1}\alpha^p \le f(\alpha) \le C\alpha^p, \tag{2.13}$$

$$C^{-1}\alpha^{p-1} \le g(\alpha) \le C\alpha^{p-1}. \tag{2.14}$$

The following Lemma 2.2 plays essential roles to prove Theorem 1.3.

**Lemma 2.2** $M_\alpha = O(g(\alpha)^{-1/2})$ *as* $\alpha \to \infty$.

We tentatively accept this lemma and prove Theorem 1.3. Lemma 2.2 will be proved in Section 3.

*Proof of Theorem 1.3.* By (2.6), Lemma 2.2 and Taylor expansion, for $\alpha \gg 1$,

$$\begin{aligned} \|u_\alpha\|_\infty &= \alpha\left(1 - \frac{2}{\sqrt{\lambda}}(C_2 + M_\alpha)\right)^{-1/q} \\ &= \alpha\left(1 + \frac{2}{q\sqrt{\lambda}}(C_2 + M_\alpha) + \frac{2(q+1)}{q^2\lambda}(C_2 + M_\alpha)^2(1 + o(1))\right). \end{aligned} \tag{2.15}$$

By this, (1.9), (1.14), (1.17), (2.12), (2.14) and Lemmas 2.1 and 2.2,

$$\begin{aligned} \lambda &= \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + \xi_\alpha \\ &= \frac{1}{\alpha}\left(1 - \frac{2}{q\sqrt{\lambda}}(C_2 + M_\alpha) + O(\alpha^{1-p})\right) \\ &\quad \times\left(f(\alpha) + \frac{2\alpha}{q\sqrt{\lambda}}f'(\alpha)(C_2 + M_\alpha) + O(\alpha)\right) + \xi_\alpha \\ &= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q\sqrt{\lambda}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + M_\alpha\frac{2C_2}{q\sqrt{\lambda}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1) \\ &= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right)(g(\alpha) + Ag(\alpha)^{1/2} + O(1))^{-1/2} + O(1) \\ &= \frac{f(\alpha)}{\alpha} + \frac{2C_2}{q\sqrt{g(\alpha)}}\left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1). \\ &= g(\alpha) + Ag(\alpha)^{1/2} + O(1). \end{aligned} \tag{2.16}$$

This implies that for $\alpha \gg 1$

$$f'(\alpha) - r\frac{f(\alpha)}{\alpha} = O(\sqrt{g(\alpha)}), \tag{2.17}$$

where $r := 1 + (qA)/(2C_2)$. By using (2.13) and (2.14), we solve (2.17) directly, and easily obtain from (2.13) that $r = p$, and for $\alpha \gg 1$

$$f(\alpha) = D\alpha^p + O(\alpha^{(p+1)/2}), \tag{2.18}$$

where $D > 0$ is an arbitrary constant. ■

# 3 Proof of Lemma 2.2

Let an arbitrary $0 < \epsilon \ll 1$ be fixed. For $0 \le s \le 1$, by (2.7) and (2.8), we put

$$\begin{aligned} K_\alpha(s) &:= J(s) - B_\alpha(s) \\ &= \frac{2}{\lambda\|u_\alpha\|_\infty^2}\{F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)\} - \frac{2}{p+1}(1 - s^{p+1}). \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} M_\alpha &= \int_0^1 \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)}+\sqrt{B_\alpha(s)})}ds \\ &= \int_{1-\epsilon}^1 \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)}+\sqrt{B_\alpha(s)})}ds \\ &\quad + \int_\epsilon^{1-\epsilon} \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)}+\sqrt{B_\alpha(s)})}ds \\ &\quad + \int_0^\epsilon \frac{(1-s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)}+\sqrt{B_\alpha(s)})}ds \\ &:= M_{1,\alpha} + M_{2,\alpha} + M_{3,\alpha}. \end{aligned} \tag{3.2}$$

**Lemma 3.1** *For* $\alpha \gg 1$

$$|M_{1,\alpha}| = O(g(\|u_\alpha\|_\infty)^{-1/2}). \tag{3.3}$$

*Proof.* By (3.1),

$$\frac{K'_\alpha(s)}{2} = -\frac{f(\|u_\alpha\|_\infty s)}{\lambda \|u_\alpha\|_\infty} + s^p. \tag{3.4}$$

This along with (1.9) implies that

$$\frac{K'_\alpha(1)}{2} = \frac{\xi_\alpha}{\lambda}. \tag{3.5}$$

Since $f(u) = g(u)u$, for $1-\epsilon \le s \le 1$, by (1.9), (3.4) and Taylor expansion,

$$\begin{aligned} \frac{K''_\alpha(s)}{2} &= -\frac{f'(\|u_\alpha\|_\infty s)}{\lambda} + ps^{p-1} \\ &= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty) + \xi_\alpha} + ps^{p-1} \\ &= -\frac{g'(\|u_\alpha\|_\infty s)\|u_\alpha\|_\infty s + g(\|u_\alpha\|_\infty s)}{g(\|u_\alpha\|_\infty)}\left(1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1+o(1))\right) \\ &\quad + ps^{p-1}. \end{aligned} \tag{3.6}$$

We put

$$H(s,u) = ps^{p-1}\frac{h(us)}{h(u)} + us^p\frac{h'(us)}{h(u)}. \tag{3.7}$$

For $u \gg 1$,

$$g'(u) = (p-1)u^{p-2}h(u) + u^{p-1}h'(u). \tag{3.8}$$

By this, (3.6) and (3.7), we obtain

$$\begin{aligned} \frac{K''_\alpha(s)}{2} &= -H(s,\|u_\alpha\|_\infty)\left(1 - \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}(1+o(1))\right) + ps^{p-1}. \\ &= ps^{p-1}\left(1 - \frac{h(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)}\right) - \|u_\alpha\|_\infty s^p \frac{h'(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)} \\ &\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}H(s,\|u_\alpha\|_\infty)(1+o(1)). \end{aligned} \tag{3.9}$$

By this, (1.9), (1.15), (2.14) and mean value theorem, for $1-\epsilon < s < s_1 < s_2 < 1$, we obtain

$$\begin{aligned} \frac{K''_\alpha(s_1)}{2} &= ps_1^{p-1}\left(1-\frac{h(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty))}\right) - \|u_\alpha\|_\infty s_1^p \frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)} \\ &\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} H(s_1, \|u_\alpha\|_\infty)(1+o(1)) \\ &= ps_1^{p-1}\left(\frac{h'(\|u_\alpha\|_\infty s_2)}{h(\|u_\alpha\|_\infty)}\right)\|u_\alpha\|_\infty(1-s_1) - \|u_\alpha\|_\infty s_1^p \frac{h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)} \\ &\quad + \frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)} H(s_1, \|u_\alpha\|_\infty)(1+o(1)) \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2})) + O\left(\frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}\right) \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2})). \end{aligned} \tag{3.10}$$

Since $K_\alpha(1) = 0$, by (3.5), (3.10) and Taylor expansion, for $1-\epsilon \le s \le 1$,

$$\begin{aligned} \frac{K_\alpha(s)}{2} &= \frac{1}{2}\left(K_\alpha(1) + K'_\alpha(1)(s-1) + \frac{1}{2}K''_\alpha(s_1)(s-1)^2\right) \\ &= \frac{\xi_\alpha}{2\lambda}(s-1) + O(g(\|u_\alpha\|_\infty)^{-1/2}))(s-1)^2. \end{aligned} \tag{3.11}$$

By this, (2.7), (2.8), (3.1), (3.11) and Taylor expansion, for $1-\epsilon \le s \le 1$,

$$J(s) \ge (p-1-\delta_1)(1-s)^2, \tag{3.12}$$

$$B_\alpha(s) = J(s) - K_\alpha(s) \ge \frac{\xi_\alpha}{\lambda}(1-s) + \frac{\delta_1}{2}(1-s)^2. \tag{3.13}$$

Then by (2.14), (3.2), (3.11)-(3.13), we obtain

$$\begin{aligned} |M_{1,\alpha}| &\le \int_{1-\epsilon}^1 \frac{(1-s^q)|K_\alpha(s)|}{J(s)\sqrt{B_\alpha(s)}} ds \\ &\le C\int_{1-\epsilon}^1 \frac{(\xi_\alpha/(\lambda) + O(g(\|u_\alpha\|_\infty^{-1/2}))(1-s)}{\sqrt{(\xi_\alpha/\lambda)(1-s) + (\delta_1/2)(1-s)^2}} ds \\ &= C\int_{1-\epsilon}^1 \sqrt{\frac{\xi_\alpha}{\lambda}}\frac{1}{\sqrt{1-s}} ds + O(g(\|u_\alpha\|_\infty)^{-1/2})\int_{1-\epsilon}^1 \frac{1-s}{\sqrt{(\delta_1/2)(1-s)^2}} ds \\ &\le C\left(\sqrt{\frac{\xi_\alpha}{\lambda}} + O(g(\|u_\alpha\|_\infty)^{-1/2})\right) = O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned} \tag{3.14}$$

■

**Lemma 3.2** *$M_{2,\alpha} = O(g(\|u_\alpha\|_\infty)^{-1/2})$ as $\alpha \to \infty$.*

*Proof.* Since $f(u) = u^p h(u)$, by (3.1), for $0 \le s \le 1-\epsilon$,

$$\begin{aligned} K_\alpha(s) &= \frac{1}{\lambda\|u_\alpha\|_\infty^2}\int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^p h(t)dt - \frac{1}{p+1}(1-s^{p+1}) \\ &= \frac{1}{(p+1)\lambda\|u_\alpha\|_\infty^2}\left\{\left[t^{p+1}h(t)\right]_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} - \int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt\right\} \\ &\quad - \frac{1}{p+1}(1-s^{p+1}). \end{aligned} \tag{3.15}$$

Since $\xi_\alpha > 0$ in (1.9), by (1.15), for $\epsilon \le s \le 1-\epsilon$,

$$\begin{aligned} \frac{1}{\lambda\|u_\alpha\|_\infty^2}\left|\int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt\right| &\le \frac{1}{h(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty^{p+1}}\int_{\|u_\alpha\|_\infty s}^{\|u_\alpha\|_\infty} |t^{p+1}h'(t)|dt \\ &\le \max_{\epsilon\le s\le 1}\left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s)}{h(\|u_\alpha\|_\infty)}\right|(1-s) \qquad (3.16)\\ &= O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned}$$

By this, (1.9), (1.15), (2.12) and mean value theorem, for $\epsilon \le s < s_1 < 1-\epsilon$,

$$\begin{aligned} \left|\frac{K_\alpha(s)}{2}\right| &\le \frac{1}{(p+1)\lambda\|u_\alpha\|_\infty^2}\left\{\|u_\alpha\|_\infty^{p+1}h(\|u_\alpha\|_\infty) - \|u_\alpha\|_\infty^{p+1}s^{p+1}h(\|u_\alpha\|_\infty s)\right\} \\ &\quad + O(g(\|u_\alpha\|_\infty)^{-1/2}) - \frac{1}{p+1}(1-s^{p+1}) \qquad (3.17)\\ &\le \frac{1}{p+1}(1-s^{p+1})\left(\frac{\|u_\alpha\|_\infty^{p-1}h(\|u_\alpha\|_\infty)}{\lambda} - 1\right) \\ &\quad + \frac{\|u_\alpha\|_\infty^{p-1}s^{p+1}}{\lambda(p+1)}(h(\|u_\alpha\|_\infty) - h(\|u_\alpha\|_\infty s)) + O(g(\|u_\alpha\|_\infty)^{-1/2}) \\ &\le \frac{\xi_\alpha}{(p+1)\lambda}(1-s^{p+1}) + \left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)}\right| + O(g(\|u_\alpha\|_\infty)^{-1/2}) \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned}$$

Note that for $0 \le s \le 1-\epsilon$,

$$J(s) \ge \delta_2 > 0. \qquad (3.18)$$

By this and (3.17), for $\epsilon \le s \le 1-\epsilon$ and $\alpha \gg 1$,

$$B_\alpha(s) \ge J(s) - K_\alpha(s) \ge \frac{\delta_2}{2} > 0. \qquad (3.19)$$

Then by this, (3.2), (3.17) and (3.18),

$$|M_{2,\alpha}| \le C\int_\epsilon^{1-\epsilon} |K_\alpha(s)|(1-s^q)ds = O(g(\|u_\alpha\|_\infty)^{-1/2}).$$

■

**Lemma 3.3** *$M_{3,\alpha} = O(g(\|u_\alpha\|_\infty)^{-1/2})$ as $\alpha \to \infty$.*

*Proof.* For $0 < s \le \epsilon$, by Taylor expansion,

$$K_\alpha(s) = K_\alpha(0) + K_\alpha'(s_1)s, \qquad (3.20)$$

where $0 < s_1 < s$. We first estimate $K_\alpha(0)$. By (3.15),

$$\begin{aligned} K_\alpha(0) &= \frac{1}{\lambda\|u_\alpha\|_\infty^2(p+1)}\left[\|u_\alpha\|_\infty^{p+1}h(\|u_\alpha\|_\infty) - \int_0^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt\right] \\ &\quad - \frac{1}{p+1} \\ &= \frac{1}{p+1}\left(\frac{\|u_\alpha\|_\infty^{p-1}h(\|u_\alpha\|_\infty)}{\lambda} - 1\right) \\ &\quad - \frac{1}{(p+1)\lambda\|u_\alpha\|_\infty^2}\int_0^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt. \qquad (3.21) \end{aligned}$$

We put

$$\frac{1}{\lambda\|u_\alpha\|_\infty^2}\int_0^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt = I + II \tag{3.22}$$
$$:= \frac{1}{\lambda\|u_\alpha\|_\infty^2}\int_{\|u_\alpha\|_\infty\epsilon}^{\|u_\alpha\|_\infty} t^{p+1}h'(t)dt + \frac{1}{\lambda\|u_\alpha\|_\infty^2}\int_0^{\|u_\alpha\|_\infty\epsilon} t^{p+1}h'(t)dt.$$

By (3.16), we see that $I = O(g(\|u_\alpha\|_\infty)^{-1/2})$. Put $t = \|u_\alpha\|_\infty s$ to obtain by (1.16),

$$\begin{aligned} |II| &= \frac{1}{\lambda\|u_\alpha\|_\infty^2}\left|\int_0^\epsilon \|u_\alpha\|_\infty^{p+2}s^{p+1}h'(\|u_\alpha\|_\infty s)ds\right| \\ &\le \frac{\|u_\alpha\|_\infty^{p+2}\max_{0\le s\le\epsilon}|s^p h'(\|u_\alpha\|_\infty s)|}{h(\|u_\alpha\|_\infty)\|u_\alpha\|_\infty^{p+1}}\int_0^\epsilon sds \\ &\le C\frac{\max_{0\le s\le\epsilon}|s^p\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s)|}{h(\|u_\alpha\|_\infty)} \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned} \tag{3.23}$$

By this, (1.9), (2.12) and (3.22),

$$|K_\alpha(0)| \le \frac{\xi_\alpha}{(p+1)\lambda} + O(g(\|u_\alpha\|_\infty)^{-1/2}) = O(g(\|u_\alpha\|_\infty)^{-1/2}). \tag{3.24}$$

By (1.9), (1.16), (2.12), (2.14), (3.4) and mean value theorem, for $s_1 < s_2 < 1$,

$$\begin{aligned} &\left|\frac{K_\alpha'(s_1)}{2}\right| \\ &= \left|s_1^p - \frac{f(\|u_\alpha\|_\infty s_1)}{\lambda\|u_\alpha\|_\infty}\right| \\ &= \left|\frac{\|u_\alpha\|_\infty^p s_1^p h(\|u_\alpha\|_\infty) + s_1^p\|u_\alpha\|_\infty\xi_\alpha - \|u_\alpha\|_\infty^p s_1^p h(\|u_\alpha\|_\infty s_1) - \xi_\alpha\|u_\alpha\|_\infty s_1}{\lambda\|u_\alpha\|_\infty}\right| \\ &\le s_1^p\left|\frac{h(\|u_\alpha\|_\infty) - h(\|u_\alpha\|_\infty s_1)}{h(\|u_\alpha\|_\infty)}\right| + C\frac{\xi_\alpha}{\lambda} \\ &\le s_1^p\left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s_2)}{h(\|u_\alpha\|_\infty)}\right|(1-s_1) + C\frac{\xi_\alpha}{\lambda} \\ &\le s_2^p\left|\frac{\|u_\alpha\|_\infty h'(\|u_\alpha\|_\infty s_2)}{h(\|u_\alpha\|_\infty)}\right| + C\frac{\xi_\alpha}{\lambda} \\ &= O(g(\|u_\alpha\|_\infty)^{-1/2}). \end{aligned} \tag{3.25}$$

By this, (3.20) and (3.24), for $0 \le s \le \epsilon$ and $\alpha \gg 1$

$$|K_\alpha(s)| = O(g(\|u_\alpha\|_\infty)^{-1/2}). \tag{3.26}$$

By this and (3.18), for $0 \le s \le \epsilon$ and $\alpha \gg 1$,

$$B_\alpha(s) = J(s) - K_\alpha(s) \ge \delta_2 - O(g(\|u_\alpha\|_\infty)^{-1/2}) \ge \frac{\delta_2}{2}. \tag{3.27}$$

By this, (3.2), (3.18) and (3.26),

$$|M_{3,\alpha}| \le C\int_0^\epsilon |K_\alpha(s)|ds = O(g(\|u_\alpha\|_\infty)^{-1/2}). \tag{3.28}$$

■

*Proof of lemma 2.2.* Since $\alpha = \|u_\alpha\|_\infty(1 + o(1))$, Lemma 2.2 follows from (2.14) and Lemmas 3.1–3.3. ■

# 4 Proof of Theorem 1.5

Let $f(u) = u^p + u^m$ $(p > m \ge (p+1)/2 > 1)$.

**Lemma 4.1** $M_\alpha = \beta\alpha^{m-p}(1 + o(1))$ *as* $\alpha \to \infty$.

*Proof.* We recall the notations (2.6)–(2.9) and (3.1). Then

$$\begin{aligned} B_\alpha(s) &:= 1 - s^2 \\ &\quad - \frac{2}{\lambda}\left\{\frac{1}{p+1}\|u_\alpha\|_\infty^{p-1}(1 - s^{p+1}) + \frac{1}{m+1}\|u_\alpha\|_\infty^{m-1}(1 - s^{m+1})\right\}. \end{aligned} \tag{4.1}$$

By this, (1.9) and (2.7),

$$\begin{aligned} K_\alpha(s) &= J(s) - B_\alpha(s) \\ &= -\frac{2}{p+1}\frac{\|u_\alpha\|_\infty^{m-1} + \xi_\alpha}{\lambda}(1 - s^{p+1}) + \frac{2}{m+1}\frac{\|u_\alpha\|_\infty^{m-1}}{\lambda}(1 - s^{m+1}). \end{aligned} \tag{4.2}$$

Then

$$\begin{aligned} M_\alpha &= \int_0^1 \frac{(1 - s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds \\ &= M_{4,\alpha} + M_{5,\alpha} \\ &= -\frac{2\xi_\alpha}{(p+1)\lambda}\int_0^1 \frac{(1 - s^q)(1 - s^{p+1})}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds \\ &\quad + \frac{2\|u_\alpha\|_\infty^{m-1}}{\lambda}\int_0^1 \frac{(1 - s^q)\{(1 - s^{m+1})/(m+1) - (1 - s^{p+1})/(p+1)\}}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds. \end{aligned} \tag{4.3}$$

Let $0 < \epsilon \ll 1$ be fixed. We put

$$\begin{aligned} M_{41,\alpha} + M_{42,\alpha} &:= -\frac{2\xi_\alpha}{(p+1)\lambda}\int_{1-\epsilon}^1 \frac{(1 - s^q)(1 - s^{p+1})}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds \\ &\quad - \frac{2\xi_\alpha}{(p+1)\lambda}\int_0^{1-\epsilon} \frac{(1 - s^q)(1 - s^{p+1})}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds. \end{aligned} \tag{4.4}$$

Then by the same argument as that to obtain (3.14),

$$|M_{41,\alpha}| \le C\frac{\xi_\alpha}{\lambda}\int_{1-\epsilon}^1 \frac{(1 - s^q)(1 - s^{p+1})}{\sqrt{\xi_\alpha(1 - s)/\lambda J(s)}}ds \le C\sqrt{\frac{\xi_\alpha}{\lambda}} \le C\lambda^{-1/2}. \tag{4.5}$$

Furthermore, by (3.18), (3.19) and (3.27)

$$|M_{42,\alpha}| \le C\frac{\xi_\alpha}{\lambda}\int_0^{1-\epsilon}(1 - s^q)(1 - s^{p+1})ds \le C\lambda^{-1}. \tag{4.6}$$

We now calculate $M_{5,\alpha}$. For $0 \le s \le 1$, we put

$$k(s) := \frac{2}{m+1}(1 - s^{m+1}) - \frac{2}{p+1}(1 - s^{p+1}). \tag{4.7}$$

Then it is easy to see that for $1 - \epsilon \le s \le 1$

$$0 \le k(s) \le C(1 - s)^2. \tag{4.8}$$

Since (3.12) and (3.13) are still valid, by (4.8), for $1 - \epsilon \le s \le 1$,

$$\frac{(1 - s^q)|k(s)|}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} \le C\frac{(1 - s^q)(1 - s)^2}{(1 - s)^3} \le C. \tag{4.9}$$

Since it is clear that the integrand of $M_{5,\alpha}$ is bounded for $0 \le s \le 1 - \epsilon$, by (4.9) and Lebesgue's convergence theorem, as $\alpha \to \infty$,

$$\begin{aligned} \frac{M_{5,\alpha}}{\|u_\alpha\|_\infty^{m-1}/\lambda} &= 2\int_0^1 \frac{(1 - s^q)\{(1 - s^{m+1})/(m + 1) - (1 - s^{p+1})/(p + 1)\}}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})}ds \\ &\to \beta. \end{aligned}$$

By this and (4.4)–(4.6), we obtain that $M_\alpha = \beta\alpha^{m-p}(1 + o(1))$. ■

*Proof of Theorem 1.5.* By the same argument as that in [15, Appendix], we see that $\xi_\alpha = O(\alpha^{p-1}e^{-C\alpha^{(p-1)/2}})$. By (1.9), (2.15), Lemma 4.1 and Taylor expansion,

$$\begin{aligned} \lambda &= \|u_\alpha\|_\infty^{p-1} + \|u_\alpha\|_\infty^{m-1} + \xi_\alpha \\ &= \alpha^{p-1}\left(1 - \frac{2}{\sqrt{\lambda}}(C_2 + M_\alpha)\right)^{-(p-1)/q} \\ &\quad + \alpha^{m-1}\left(1 - \frac{2}{\sqrt{\lambda}}(C_2 + M_\alpha)\right)^{-(m-1)/q} + \xi_\alpha \\ &= \alpha^{p-1}\left\{1 + \frac{2(p - 1)}{q\sqrt{\lambda}}(C_2 + M_\alpha)\right. \\ &\qquad\qquad \left. + \frac{2(p - 1)(p + q - 1)}{q^2\lambda}(C_2 + M_\alpha)^2(1 + o(1))\right\} \\ &\quad + \alpha^{m-1}\left\{1 + \frac{2(m - 1)}{q\sqrt{\lambda}}(C_2 + M_\alpha)\right. \\ &\qquad\qquad \left. + \frac{2(m - 1)(m + q - 1)}{q^2\lambda}(C_2 + M_\alpha)^2(1 + o(1))\right\} + \xi_\alpha \\ &= \alpha^{p-1} + \alpha^{q-1} + \frac{2C_2}{q\sqrt{\lambda}}\{(p - 1)\alpha^{p-1} + (m - 1)\alpha^{m-1}\} \\ &\quad + \frac{2M_\alpha}{q\sqrt{\lambda}}\{(p - 1)\alpha^{p-1} + (m - 1)\alpha^{m-1}\} + \frac{2(p - 1)(p + q - 1)}{q^2}C_2^2 + o(1) + \xi_\alpha. \end{aligned} \tag{4.10}$$

By Taylor expansion and (1.10),

$$\begin{aligned} \lambda^{-1/2} &= (\alpha^{p-1} + \alpha^{m-1}(1 + o(1)))^{-1/2} \\ &= \alpha^{(1-p)/2}\left(1 - \frac{1}{2}\alpha^{m-p} + o(\alpha^{m-p})\right). \end{aligned} \tag{4.11}$$

Then by (4.11) and (4.12),

$$
\begin{aligned}
\lambda &= \alpha^{p-1} + \alpha^{m-1} \\
&\quad + \frac{2C_2}{q}\alpha^{(1-p)/2}\left(1 - \frac{1}{2}\alpha^{m-p} + o(\alpha^{m-p})\right)\{(p-1)\alpha^{p-1} + (m-1)\alpha^{m-1}\} \\
&\quad + \frac{2M_\alpha}{q}\alpha^{(1-p)/2}\left(1 - \frac{1}{2}\alpha^{m-p} + o(\alpha^{m-p})\right)\{(p-1)\alpha^{p-1} + (m-1)\alpha^{m-1}\} \\
&\quad + \frac{2(p-1)(p+q-1)}{q^2}C_2^2 + o(1) + \xi_\alpha \\
&= \alpha^{p-1} + \alpha^{m-1} + \frac{2C_2(p-1)}{q}\alpha^{(p-1)/2} \\
&\quad + \left\{\frac{2C_2}{q}\left(m - \frac{p+1}{2}\right) + \frac{2(p-1)\beta}{q}\right\}\alpha^{m-(p+1)/2} \\
&\quad + o(\alpha^{m-(p+1)/2}) + \frac{2(p-1)(p+q-1)}{q^2}C_2^2 + o(1).
\end{aligned}
\tag{4.12}
$$

By this, we obtain (1.19) and (1.21). ∎

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