Advanced Nonlinear Studies **12** (2012), 445–463

One-Signed Harmonic Solutions and Sign-Changing Subharmonic Solutions to Scalar Second Order Differential Equations

Alberto Boscaggin *

SISSA - ISAS, International School for Advanced Studies,
via Bonomea 265, 34136 Trieste - Italy
e-mail: boscaggi@sissa.it

Received 03 December 2011
Communicated by Fabio Zanolin

Abstract

Using a recent modified version of the Poincaré-Birkhoff fixed point theorem [19], we study the existence of one-signed T-periodic solutions and sign-changing subharmonic solutions to the second order scalar ODE

$$u'' + f(t, u) = 0,$$

being $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function T-periodic in the first variable and such that $f(t, 0) \equiv 0$. Partial extensions of the results to a general planar Hamiltonian systems are given, as well.

Mathematics Subject Classification. 34C25, 34B18, 37E45, 37J10.
Key words. Periodic solutions, Positive solutions, Poincaré-Birkhoff theorem.

1 Introduction

In this paper, we are mainly concerned with the existence of *one-signed* (i.e., positive or negative) *T-periodic solutions* to the second order scalar ordinary differential equation

$$u'' + f(t, u) = 0, \tag{1.1}$$

being $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function which is T-periodic in the first variable (for $T > 0$ fixed) and such that $f(t, 0) \equiv 0$. As a preliminary elementary observation - just integrate both sides of equation (1.1) - notice that no one-signed periodic solutions to (1.1) can exist when $f(t, x)x > 0$

*The author acknowledges the support of the PRIN–2009 project "Ordinary Differential Equations and Applications".

for $x \neq 0$. Accordingly, one has to prescribe some changes for the sign of $f(t, x)$ both on the positive and on the negative real half-line.

To motivate our main results, let us consider first the autonomous case $f(t, x) = f(x)$ and observe that, in this situation, there is a natural way to ensure the existence of one-signed T-periodic solutions. Indeed, if we assume that $f(x)$ is of class C^1 in a neighborhood of $x = 0$ and satisfies

$$f'(0) < 0, \qquad \text{and} \qquad \liminf_{|x|\to+\infty} f(x)\mathrm{sgn}\,(x) > 0, \tag{1.2}$$

then $f(x)$ has both a positive and a negative zero on the real line and such zeros are, of course, constant one-signed T-periodic solutions. Moreover, we stress from the beginning that, as readily seen by a direct phase-plane analysis, all "large" solutions to $u'' + f(u) = 0$ are periodic and wind the origin, so that it is natural to investigate the existence of sign-changing subharmonic solutions, as well.

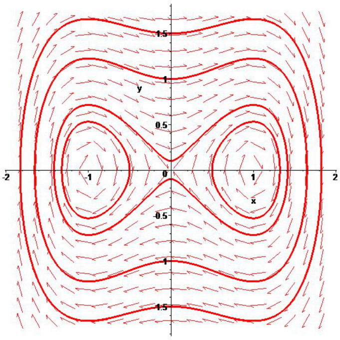

Figure 1: The phase portrait of the autonomous equation $u'' + f(u) = 0$, with $f(x) = -x + x^3$.

When trying to analyze the non-autonomous equation (1.1), more sophisticated tools have to be employed. Probably, the most classical approach in the search for one-signed solutions to boundary value problems is based on the use of some forms of the Krasnoselskii fixed point theorem on compression and expansion of cones. As explained in [24, Introduction], such a technique, which requires the conversion of the differential equation (1.1) into an integral equation via the Green's function, is particularly well-suited for separated boundary conditions, but presents some difficulties when dealing with the periodic problem. Indeed, the differential operator $u \mapsto -u''$ is not invertible with T-periodic boundary conditions, so that one is usually led to study equations of the form

$$u'' + a(t)u + g(t, u) = 0,$$

with $u \mapsto -u'' - a(t)u$ invertible, by imposing suitable conditions both on the sign of the Green's function and on the nonlinear term $g(t, x)$ or its potential $G(t, x) = \int_0^x g(t, \xi)\, d\xi$, near zero and infinity (see, for instance, [2, 17, 24] and the references therein). To the best of our knowledge, however, the known results do not cover the simple case of a natural "non-autonomous" generalization of (1.2).

In this note, we propose to consider the problem of one-signed T-periodic solutions to (1.1) using a dynamical system approach. Indeed, by looking again at the phase-plane portrait of the autonomous equation $u'' + f(u) = 0$, with $f(x)$ satisfying (1.2), we see that the origin is a (local) saddle equilibrium point, while large solutions wind the origin in the clockwise sense. Accordingly, some kind of "twist dynamics" between zero and infinity seems to appear and one can wonder if some forms of the Poincaré-Birkhoff fixed point theorem can be applied. As a matter of fact, one immediately realizes that the classical version of the theorem can not succeed. Indeed, in order to have satisfied the boundary twist condition leading to one-signed T-periodic solutions, one should have to

show that small solutions to (1.1) move in the counterclockwise sense in the phase-plane. But, also in the simple situation in which (1.1) has, near the origin, the linearized equation $u'' + q_0 u = 0$ (with $q_0 < 0$), one is led to consider - switching to polar coordinates - the equation $-\theta' = \sin^2\theta + q_0\cos^2\theta$, which always has solutions with $\theta' > 0$ as well as solutions with $\theta' < 0$.
Fortunately, a recent modified version of the Poincaré-Birkhoff fixed point theorem, proved by Margheri, Rebelo and Zanolin [19] (see also [5]), turns out to be well-suited to overcome such a difficulty and its use is the main novelty of our approach. According to the previous discussion, the key point is an essential weakening of the twist condition, at the inner boundary.

Our main results, Theorem 3.1 and Theorem 3.2, are based on two mutually independent generalizations of (1.2) to the non-autonomous case. They both exploit an assumption near $x = 0$ (see hypothesis (f_0) of Section 3), which generalizes in a natural way the condition $f'(0) < 0$. Such a condition provides information on the behavior of the solutions to the linearization (at the origin) of equation (1.1) and, accordingly, implies the inner boundary twist condition of the (modified) Poincaré-Birkhoff theorem. On the other hand, the behavior of large solutions to (1.1) (that is, the outer boundary twist condition) is ruled by a classical Landesman Lazer condition (with respect to the principal eigenvalue $\lambda_0 = 0$ of the periodic problem, see condition (f^1_∞)) in Theorem 3.1 and by a comparison with a linear sign-indefinite problem in Theorem 3.2 (see condition (f^2_∞)).

We stress that, besides one-signed T-periodic solutions, we also get the existence of infinitely many *sign-changing subharmonic solutions* (with a precise nodal characterization), as the qualitative analysis of the autonomous case suggests. It has to be noticed that the proof of the existence of subharmonic solutions is more standard and it simply relies on more classical versions of the Poincaré-Birkhoff fixed point theorem [10, 16, 21]. We finally observe that, in the situation of Theorem 3.2, a sharp information concerning the order of the subharmonics produced is available, so that, in some cases, it is possible to ensure the coexistence of one-signed and sign-changing T-periodic solutions to (1.1), as well (see Corollary 3.1).

The plan of the article is as follows. In Section 2, we present the dynamical systems preliminaries needed in the proof of our main results. As already explained, they rely on suitable versions of the Poincaré-Birkhoff theorem. It is worth noticing that we will have to deal with the linearized equation, at the origin, of equation (1.1). From a geometrical point of view, this corresponds to work in a planar annulus whose inner boundary degenerates into a single point. Such a situation has been probably the source of some inaccuracies in previous versions of the Poincaré-Birkhoff theorem (we refer to the recent paper [12] for a wide bibliography and a complete overview about the "twist theorem", with particular emphasis on some of its controversial versions), so that we have chosen to present the topic in a quite detailed way. In Section 3, we state and prove our main results, Theorem 3.1 and Theorem 3.2. In Section 4, we propose (partial) extensions of the results to a general planar Hamiltonian system

$$Jz' = \nabla_z H(t, z), \tag{1.3}$$

being $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard symplectic matrix and $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a regular function, T-periodic in the first variable and such that $\nabla_z H(t, 0) \equiv 0$. In this situation, we look for T-periodic solutions that do not wind the origin. Theorem 4.1 and Theorem 4.2 are the main results of the section. They both deal with Hamiltonian systems like (1.3) having the origin as an equilibrium point of saddle-type, in the sense of condition (H_0). Concerning the assumption at infinity, Theorem 4.2 is on the lines of Theorem 3.1, using a recent generalization of the Landesman-Lazer condition to planar systems given in [14] (see condition (H^1_∞)), while Theorem 4.1 is the counterpart of Theorem 3.2 and it is based on the comparison of the nonlinear system (1.3) with positively homogeneous Hamiltonian systems (see condition (H^1_∞)), similarly as in [3, 4]. The proofs just rely on the same arguments of the second order case.

Notation. For $z_1, z_2 \in \mathbb{R}^2$, we write $\langle z_1 | z_2 \rangle$ to denote the Euclidean scalar product of z_1, z_2, and $|z_1|$ to denote the Euclidean norm of z_1. By $\mathcal{L}_s(\mathbb{R}^2)$, we mean the set of real symmetric matrices 2×2.

2 Classical and modified Poincaré-Birkhoff Theorem

In this section we describe the dynamical systems tools that we are going to use in order to prove our existence and multiplicity results of Section 3 and Section 4.

First of all, we recall the abstract versions of the Poincaré-Birkhoff fixed point theorem on which our entire discussion is based. To this aim, we work in the closed half-plane $\mathcal{H}^+ = \{(\theta,\rho) \in \mathbb{R}^2 \mid \rho \geq 0\}$, endowed with the standard topology. For X, Y metric spaces and $f : X \to Y$, we say that f is an *embedding* if it is an homeomorphism as a map from X onto its image $f(X) \subset Y$ in the subspace topology.

Theorem 2.1 *Let* $\widetilde{\Psi} : \mathcal{A} = \mathbb{R} \times [0, R] \subset \mathcal{H}^+ \to \mathcal{H}^+$ *be an area-preserving embedding such that:*

(i) $\widetilde{\Psi}$ *has the form*
$$\widetilde{\Psi}(\theta,\rho) = (\theta + \Theta(\theta, r), R(\theta, r)), \tag{2.1}$$
being $\Theta(\theta, r), R(\theta, r)$ *continuous functions,* 2π*-periodic in the first variable;*

(ii) $R(\theta, 0) = 0$ *for every* $\theta \in \mathbb{R}$.

Then, the following conclusions hold true:

(A) if
$$\Theta(\theta, 0) < 0, \quad \textit{and} \quad \Theta(\theta, R) > 0, \quad \textit{for every } \theta \in \mathbb{R},$$
then $\widetilde{\Psi}$ *has at least two fixed points* $(\theta^{(1)},\rho^{(1)}), (\theta^{(2)},\rho^{(2)}) \in \mathbb{R}\times\,]0, R[$ *such that*
$$(\theta^{(2)},\rho^{(2)}) - (\theta^{(1)},\rho^{(1)}) \neq (2k\pi, 0), \quad \textit{for every } k \in \mathbb{Z}; \tag{2.2}$$

(B) if, for some $\theta^* \in \mathbb{R}$,
$$\Theta(\theta^*, 0) < 0, \quad \textit{and} \quad \Theta(\theta, R) > 0, \quad \textit{for every } \theta \in \mathbb{R},$$
then $\widetilde{\Psi}$ *has at least one fixed point in* $\mathbb{R}\times\,]0, R[\,$.

Assumption (ii) expresses the invariance of the line $\mathbb{R} \times \{0\}$ under $\widetilde{\Psi}$. Condition (i), which can be clearly equivalently stated as
$$\widetilde{\Psi}(\theta + 2\pi, \rho) = \widetilde{\Psi}(\theta, \rho) + (2\pi, 0), \tag{2.3}$$
implies that $\widetilde{\Psi}$ preserves orientation, too, and it is the lift of a (unique) embedding $\Psi : \mathbb{S}^1 \times [0, R] \to \mathbb{S}^1 \times [0, +\infty[$ with respect to the covering projection
$$P : \mathcal{H}^+ \ni (\theta, \rho) \mapsto (\theta \bmod 2\pi, \rho) \in \mathbb{S}^1 \times [0, +\infty[\,,$$
that is to say, $P \circ \widetilde{\Psi} = \Psi \circ P$. By $\mathbb{S}^1$, here we mean the abstract quotient space $\mathbb{R}/(2\pi\mathbb{Z})$ and we naturally identify $\mathbb{S}^1 \times [0, R]$ with a cylinder in the half-plane $\mathcal{H}^+$. Observe that, since $\widetilde{\Psi}$ is area-preserving, then Ψ preserves the standard (Haar) measure on the cylinder. As a consequence, conclusion (A) of Theorem 2.1 directly follows from [16, Theorem 3] (see also [10, 21] for related results). On the other hand, conclusion (B) of Theorem 2.1 is exactly [19, Corollary 2] (see moreover [5, Theorem 1]).

We are now going to derive a corollary of Theorem 2.1 dealing with periodic solutions to the planar Hamiltonian system
$$Jz' = \nabla_z H(t, z), \qquad z = (u, v) \in \mathbb{R}^2, \tag{2.4}$$
where, as usual, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denotes the standard symplectic matrix. Throughout the section, we assume the following conditions to be fulfilled:

(C1) $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function, T-periodic in the first variable, such that $\nabla_z H(t,z)$ exists and is continuous on $\mathbb{R}\times\mathbb{R}^2$. Without loss of generality, moreover, we require $H(t,0)\equiv 0$.

(C2) $\nabla_z H(t,0)\equiv 0$ and there exists a continuous function $B : [0,T] \to \mathcal{L}_s(\mathbb{R}^2)$ such that

$$\lim_{z\to 0}\frac{\nabla_z H(t,z) - B(t)z}{|z|} = 0, \qquad \text{uniformly in } t\in[0,T]. \tag{2.5}$$

(C3) The uniqueness and the global continuability for the solutions to the initial value problems associated with (2.4) are guaranteed.

As a consequence of the previous assumptions, the Poincaré operator

$$\Psi : \mathbb{R}^2 \ni z \mapsto \zeta(T;z) \in \mathbb{R}^2,$$

being $\zeta(\cdot;z)$ the unique solution to (2.4) such that $\zeta(0;z) = z$, is well defined as an area-preserving (by Liouville's theorem) homeomorphism of the plane, with $\Psi(0) = 0$. Moreover, by the T-periodicity of $H(\cdot,z)$, initial values, at time $t = 0$, of T-periodic solutions to (2.4) correspond to fixed points of Ψ.

To detect such fixed points using the Poincaré-Birkhoff theorem, one usually lifts the *restriction* $\Psi|_{\mathbb{R}^2\setminus\{0\}} : \mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$ to an embedding defined on the *open* half-space $\{(\theta,\rho)\in\mathbb{R}^2 \mid \rho > 0\}$ (via polar coordinates, or related, covering projections). We refer to [21] for a detailed description, even in greater generality, of this technique. Observe, however, that in order to apply Theorem 2.1 we need a map defined on the closed half-space $\mathcal{H}^+$, which can not be obtained directly in this manner.

Here, we propose to show, with some details, how to enter in the setting of Theorem 2.1 when (2.5) is satisfied. To this aim, we define the Hamiltonian function

$$\widetilde{H}(t,\rho,\theta) = \operatorname{sgn}(\rho) H(t, \sqrt{2|\rho|}e^{i\theta}),$$

for $(t,\theta,\rho)\in\mathbb{R}\times\mathbb{R}^2$. Here, and in the following, we identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, so that we write $e^{i\theta}\in\mathbb{C}$ to denote the vector $(\cos\theta,\sin\theta)\in\mathbb{R}^2$.

Lemma 2.1 *The function $\widetilde{H}(t,\rho,\theta)$ is continuous, T-periodic in the first variable, and the partial derivatives $\widetilde{H}_\rho(t,\rho,\theta)$, $\widetilde{H}_\theta(t,\rho,\theta)$ exist and are continuous on $\mathbb{R}\times\mathbb{R}^2$.*

Proof. The continuity, and T-periodicity in the first variable, of $\widetilde{H}$, as well as the existence and continuity of $\widetilde{H}_\theta$, just follow from the assumptions on H (recall that we have assumed $H(t,0)\equiv 0$). For the same reason, $\widetilde{H}_\rho(t,\rho,\theta)$ exists and it is continuous for $\rho\neq 0$. From (2.5), however, we deduce that

$$\widetilde{H}_\rho(t,\rho,\theta) = \frac{1}{\sqrt{2|\rho|}}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid e^{i\theta}\rangle \to \langle B(t)e^{i\theta} \mid e^{i\theta}\rangle$$

for $\rho\to 0$, uniformly in $(t,\theta)\in\mathbb{R}^2$. At this point, l'Hopital's theorem implies that $\widetilde{H}_\rho(t,0,\theta)$ exists (and $\widetilde{H}_\rho(t,\cdot,\theta)$ is continuous by construction). The continuity of $\widetilde{H}_\rho$ with respect to the full variable (t,ρ,θ) is easily seen to be satisfied, too. □

We can thus consider the associated Hamiltonian system

$$\begin{cases} \rho' = \widetilde{H}_\theta(t,\rho,\theta) = \operatorname{sgn}(\rho)\sqrt{2|\rho|}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid Je^{i\theta}\rangle \\ \theta' = -\widetilde{H}_\rho(t,\rho,\theta) = \begin{cases} -\frac{1}{\sqrt{2|\rho|}}\langle \nabla_z H(t,\sqrt{2|\rho|}e^{i\theta}) \mid e^{i\theta}\rangle, & \text{for } \rho\neq 0 \\ -\langle B(t)e^{i\theta} \mid e^{i\theta}\rangle, & \text{for } \rho = 0, \end{cases} \end{cases} \tag{2.6}$$

for $(t,\rho,\theta)\in\mathbb{R}\times\mathbb{R}^2$.

Lemma 2.2 *The uniqueness and the global continuability for the solutions to the initial value problems associated with* (2.6) *are guaranteed.*

Proof. Fix $t_0 \in \mathbb{R}$ and consider the Cauchy problem $(\rho(t_0), \theta(t_0)) = (\rho, \theta)$.
If $\rho = 0$, a standard application of Gronwall's lemma, using (2.5), implies that $\rho(t) \equiv 0$ (indeed, from (2.5) we deduce that for every $m > 0$ there exists $C_m > 0$ such that $|\nabla_z H(t, z)| \leq C_m |z|$ for $|z| \leq m$). Since $\theta'(t) = -\langle B(t)e^{i\theta(t)} \mid e^{i\theta(t)}\rangle$, the global Lipschitz continuity of the right-hand side implies that $\theta(t)$ is uniquely globally defined, too.
If $\rho \neq 0$, simple calculations show that the function $z(t) = \sqrt{2|\rho(t)|}e^{i\theta(t)}$ is a local solution to (2.4), with $z(t_0) = \sqrt{2|\rho(t_0)|}e^{i\theta(t_0)} \in \mathbb{R}^2 \setminus \{0\}$. Indeed, from (2.6) we obtain

$$Jz'(t) = \langle \nabla_z H(t, z(t)) | Je^{i\theta(t)}\rangle Je^{i\theta(t)} + \langle \nabla_z H(t, z(t)) | e^{i\theta(t)}\rangle e^{i\theta(t)},$$

and using the fact that, for every t, $\{e^{i\theta(t)}, Je^{i\theta(t)}\}$ is an orthonormal bases of $\mathbb{R}^2$, we conclude. Since the initial value problems associated with (2.6) have a unique solution and the map $(\rho, \theta) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \mapsto \sqrt{2|\rho|}e^{i\theta} \in \mathbb{R}^2 \setminus \{0\}$ is a local homeomorphism, we deduce that $\theta(t), \rho(t)$ are locally uniquely defined. Moreover, since $z(t)$ can be globally extended, never reaching the origin, we deduce that $\theta(t), \rho(t)$ can be globally extended too. □

Let us now denote by $(r(\cdot; \theta, \rho), \varphi(\cdot; \theta, \rho))$ the unique solution to (2.6) such that

$$(r(0; \theta, \rho), \varphi(0; \theta, \rho)) = (\rho, \theta).$$

Observe that, along the proof of Lemma 2.2, we have showed that

$$\zeta(t; \sqrt{2|\rho|}e^{i\theta}) = \sqrt{2|r(t; \theta, \rho)|}e^{i\varphi(t;\theta,\rho)}, \quad \rho \neq 0. \tag{2.7}$$

Moreover, with similar arguments it is possible to see that, denoting by $\zeta_B(\cdot; z)$ the unique solution to the linear Hamiltonian system $Jz' = B(t)z$ with $\zeta_B(0; z) = z$, it holds that

$$\zeta_B(t; e^{i\theta}) = \exp\left(\int_0^t \langle B(s)e^{i\varphi(s;\theta,0)} | e^{i\varphi(s;\theta,0)}\rangle\, ds\right) e^{i\varphi(t;\theta,0)} \tag{2.8}$$

We are now almost in a position to conclude. Indeed, let us define the Poincaré operator associated with (2.6), that is,

$$\widetilde{\Psi} : \mathbb{R}^2 \ni (\theta, \rho) \mapsto (\varphi(T; \theta, \rho), r(T; \theta, \rho)) \in \mathbb{R}^2.$$

The following properties hold true.

- $\widetilde{\Psi}$ is an area-preserving (by Liouville's theorem again) homeomorphism of the plane; moreover, in view of the 2π-periodicity of $\widetilde{H}(t, \rho, \cdot)$, relation (2.3) holds true:
- Since, in view of Lemma 2.2, $r(t; \theta, \rho) > 0$ for every t whenever $\rho > 0$ and $r(t; \theta, 0) \equiv 0$, we have that $\widetilde{\Psi}(\mathcal{H}^+) \subset \mathcal{H}^+$ and that $\widetilde{\Psi}(\theta, 0) = (\varphi(T; \theta, 0), 0)$ (that is, condition (*ii*) of Theorem 2.1).

Observe finally that, for k, j integer numbers, the same structural conditions are satisfied by the maps

$$\widetilde{\Psi}_{k,j} : \mathbb{R}^2 \ni (\theta, \rho) \mapsto (\varphi(kT; \theta, \rho) + 2\pi j, r(kT; \theta, \rho)) \in \mathbb{R}^2.$$

To write our final result in a more direct way, we recall the following definition.

Definition 2.1 *Let* $z = (u, v) : [t_1, t_2] \to \mathbb{R}^2$ *be a* C^1*-path such that* $z(t) \neq 0$ *for every* $t \in [t_1, t_2]$*. The* rotation number *of* $z(t)$ *around the origin, in the time interval* $[t_1, t_2]$ *is defined by*

$$\mathrm{Rot}(z(t); [t_1, t_2]) := \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{\langle Jz'(t) | z(t)\rangle}{|z(t)|^2}\, dt = \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{v(t)u'(t) - u(t)v'(t)}{u(t)^2 + v(t)^2}\, dt.$$

Whenever $z(t) = k(t)e^{i\theta(t)}$, for $k(t), \theta(t)$ functions of class C^1 and $k(t) > 0$, it is easy to see that

$$\mathrm{Rot}(z(t); [t_1, t_2]) = -\frac{1}{2\pi}\int_{t_1}^{t_2} \theta'(t)\, dt = \frac{\theta(t_1) - \theta(t_2)}{2\pi}.$$

In view of (2.7) and (2.8), we then have, for $\rho > 0$,

$$\widetilde{\Psi}_{k,j}(\theta,\rho) = (\theta + 2\pi(j - \mathrm{Rot}(\zeta(t;\ \sqrt{2\rho}e^{i\theta}); [0, kT])), r(kT; \theta, \rho)),$$

and

$$\widetilde{\Psi}_{k,j}(\theta, 0) = (\theta + 2\pi(j - \mathrm{Rot}(\zeta_B(t; e^{i\theta}); [0, kT])), 0).$$

It is clear that, if $(\theta,\rho) \in \mathbb{R}\times\,]0, R[$ is a fixed point of $\widetilde{\Psi}_{k,j}$, then $\sqrt{2\rho}e^{i\theta}$ is a fixed point of Ψ^k (the k-th iterate of Ψ), that is, $\zeta(t;\ \sqrt{2\rho}e^{i\theta})$ is a kT-periodic solution to (2.4); moreover, we further know that

$$\mathrm{Rot}(\zeta(t;\ \sqrt{2\rho}e^{i\theta}); [0, kT]) = j.$$

Observe finally that, if $(\theta^{(1)}, \rho^{(1)}), (\theta^{(2)}, \rho^{(2)}) \in \mathbb{R}\times\,]0, R[$ are fixed points of $\widetilde{\Psi}_{k,j}$ geometrically distinct in the sense (2.2), then they give rise to different kT-periodic solutions. In view of Theorem 2.1, we can thus state the following result.

Corollary 2.1 *Assume $(C_1), (C_2), (C_3)$ to be satisfied and let $\zeta(t; z), \zeta_B(t; z)$ be defined as in the previous discussion. Then, the following conclusions hold true:*

(A) if

$$\mathrm{Rot}(\zeta_B(t; z); [0, kT]) < j, \quad \textit{for every } z \in \mathbb{R}^2, \textit{ with } |z| = 1$$

and there exists $R > 0$ such that

$$\mathrm{Rot}(\zeta(t; z); [0, kT]) > j, \quad \textit{for every } z \in \mathbb{R}^2, \textit{ with } |z| = R,$$

then there exist two (distinct) kT-periodic solutions $z^{(1)}_{k,j}(t), z^{(2)}_{k,j}(t)$ to (2.4), such that, for $i = 1, 2$,

$$\mathrm{Rot}(z^{(i)}_{k,j}(t); [0, kT]) = j;$$

(B) if there exists $z^* \in \mathbb{R}^2$ with $|z^*| = 1$, such that

$$\mathrm{Rot}(\zeta_B(t; z^*); [0, kT]) < j,$$

and there exists $R > 0$ such that

$$\mathrm{Rot}(\zeta(t; z); [0, kT]) > j, \quad \textit{for every } z \in \mathbb{R}^2, \textit{ with } |z| = R,$$

then there exists a kT-periodic solutions $z_{k,j}(t)$ to (2.4), such that

$$\mathrm{Rot}(z_{k,j}(t); [0, kT]) = j.$$

Remark 2.1 It is possible to see (see, for instance, [8, pp. 523-524]) that, whenever a kT-periodic solution (with $k > 1$) satisfies $\mathrm{Rot}(z(t); [0, kT]) = j$ and k, j are relatively prime integers (namely, their greatest common divisor is 1), then $z(t)$ is not lT-periodic for any integer $l = 1, \ldots, k-1$. In this case, we say that $z(t)$ is a *subharmonic solution of order k* to (2.4). Notice that subharmonic solutions of order k to (2.4) correspond to fixed points of Ψ^k which are not fixed points of Ψ^l, for $l = 1, \ldots, k-1$. Referring to conclusion (A) of Corollary 2.1, we also remark that, as pointed out in the proof of [22, Theorem 5], it is possible to show that $z^{(1)}_{k,j}(\cdot) \not\equiv z^{(2)}_{k,j}(\cdot + lT)$ for every integer $l = 1, \ldots, k-1$, that is $z^{(1)}_{k,j}(t)$, $z^{(2)}_{k,j}(t)$ do not belong to the same periodicity class. This corresponds to the fact that the orbits, through Ψ, of $z_1 := z^{(1)}_{k,j}(0)$ and $z_2 := z^{(2)}_{k,j}(0)$, namely $O_1 := \{z_1, \Psi(z_1), \ldots, \Psi^{k-1}(z_2)\}$ and $O_2 := \{z_2, \Psi(z_2), \ldots, \Psi^{k-1}(z_2)\}$, are disjoint.

We conclude this section recalling that the scalar equation (1.1) can be written as an equivalent Hamiltonian systems of the form (2.4), via the position $z = (u, v)$, $H(t, u, v) = (1/2)v^2 + \int_0^u f(t, x)\,dx$. In such a case, kT-periodic solutions $z(t) = (u(t), v(t))$ to (2.4) with $\mathrm{Rot}(z(t); [0, kT]) = j$ correspond to kT-periodic solutions $u(t)$ to (1.1) with $2j$ zeros in the interval $[0, kT[$.

3 The main results

In this section, we state and prove our main results (Theorem 3.1 and Theorem 3.2) dealing with the existence of one-signed harmonic (i.e., T-periodic) solutions and sign-changing subharmonic solutions to the second order scalar ODE

$$u'' + f(t, u) = 0. \tag{3.1}$$

We first give the statement of both the results, together with some brief comments on the assumptions and on the relationship with the existing literature; the proofs are postponed at the end of this section. For the precise definition of subharmonic solution employed, we refer the reader to Remark 2.1.

Throughout the section, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which is T-periodic in the first variable (so that, when passing to the equivalent planar Hamiltonian system (2.4) as described at the end of Section 2, assumption (C_1) is satisfied). Moreover, we assume the following condition:

(f_0) *$f(t, 0) \equiv 0$ and there exists a continuous function $q_0 : [0, T] \to \mathbb{R}$, with*

$$q_0(t) \leq 0, \quad \text{for every } t \in [0, T], \qquad \text{and} \qquad \int_0^T q_0(t)\,dt < 0,$$

such that

$$\lim_{x \to 0} \frac{f(t, x)}{x} = q_0(t), \qquad \text{uniformly in } t \in [0, T].$$

Writing equation (3.1) in Hamiltonian form, we immediately see that assumption (C_2) is fulfilled with $B(t) = \begin{pmatrix} q_0(t) & 0 \\ 0 & 1 \end{pmatrix}$.

We state the first main result.

Theorem 3.1 *Suppose that the uniqueness for the solutions to the Cauchy problems associated with* (3.1) *is guaranteed. Moreover, assume condition* (f_0) *and*

(f^1_∞) *it holds that*

$$\lim_{|x| \to +\infty} \frac{f(t, x)}{x} = 0, \qquad \text{uniformly in } t \in [0, T], \tag{3.2}$$

and

$$\int_0^T \limsup_{x \to -\infty} f(t, x)\,dt < 0 < \int_0^T \liminf_{x \to +\infty} f(t, x)\,dt. \tag{3.3}$$

Then the following conclusions hold true:

(i) there exist a positive T-periodic solution $u_p(t)$ and a negative T-periodic solution $u_n(t)$ to (3.1)*,*

(ii) there exists an integer $k^* \geq 1$ such that, for every integer $k \geq k^*$, there exist at least two subharmonic solutions $u_k^{(1)}(t), u_k^{(2)}(t)$ of order k to (3.1)*, not belonging to the same periodicity class, and with exactly two zeros in the interval* $[0, kT[$.

Assumption (3.2) is a sublinearity condition, in the x-variable, for the nonlinear term $f(t,x)$ (and we recall that it implies the global continuability for the solutions to (3.1)), while (3.3) is the well-known Landesman-Lazer condition (with respect to the principal eigenvalue $\lambda_0 = 0$ of the T-periodic problem). Notice that, in order for the integrals in (3.3) to make sense, it is implicitly assumed that there exists $\eta \in L^1(0,T)$ such that

$$f(t,x)\mathrm{sgn}\,(x) \geq \eta(t), \qquad \text{for every } t \in [0,T],\ x \in \mathbb{R}.$$

Under similar assumptions to those of Theorem 3.1, the existence of two nontrivial T-periodic solutions to (3.1) is proved in [23, Theorem 1] via Morse theory techniques, but no information concerning their sign is given.

We now state the second main result.

Theorem 3.2 *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (3.1) *are guaranteed. Moreover, assume condition* (f_0) *and*

(f^2_∞) *there exist* $q_\infty \in L^\infty(0,T)$, *with*

$$\int_0^T q_\infty(t)\,dt > 0,$$

such that

$$\liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \geq q_\infty(t), \quad \text{uniformly in } t \in [0,T]. \tag{3.4}$$

Then the following conclusions hold true:

(i) there exist a positive T-periodic solution $u_p(t)$ and a negative T-periodic solution $u_n(t)$ to (3.1),

(ii) there exists an integer $k^* \geq 1$ such that, for every integer $k \geq k^*$, there exists an integer m_k such that, for every integer j relatively prime with k and such that $1 \leq j \leq m_k$, there exist at least two subharmonic solutions $u^{(1)}_{j,k}(t), u^{(2)}_{j,k}(t)$ of order k to (3.1), *not belonging to the same periodicity class, and with exactly $2j$ zeros in the interval $[0,kT[$; moreover, we have the estimate*

$$m_k \geq n_k := \mathcal{E}^-\left(\frac{k}{2\pi}\frac{\int_0^T q_\infty(t)\,dt}{\sqrt{\mathrm{ess\,sup}_{[0,T]} q_\infty(t)}}\right), \tag{3.5}$$

where, for $r > 0$, we denote by $\mathcal{E}^-(r)$ the greatest integer strictly *less than r.*

Assumption (3.4) is a comparison of the nonlinear equation (3.1) with a linear non-autonomous problem, $u'' + q_\infty(t)u = 0$. Notice that only a condition on the mean value of the weight function $q_\infty(t)$ is required. In the definite-sign case (together with an assumption near the origin related to (f_0)), the existence of one-signed T-periodic solutions to (3.1) is proved in [15, Proposition 3.3]. Moreover, in [15, Theorem 4.1], some other T-periodic solutions are provided. Theorem 3.2 gives more information in this direction. Indeed, from (3.5) it is possible to estimate in a sharp way the order of the subarmonics produced, so that multiple sign-changing T-periodic solutions can appear depending on the "size" of the weight function $q_\infty(t)$. In particular, we can state the following corollary.

Corollary 3.1 *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (3.1) *are guaranteed. Moreover, assume condition* (f_0) *and, for a suitable integer $m \geq 1$ and $q_\infty(t) \equiv q_\infty > 0$,*

$$\liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \geq q_\infty > \lambda_m, \quad \text{uniformly in } t \in [0,T],$$

being $\lambda_m := \left(\frac{2m\pi}{T}\right)^2$ *the m-th eigenvalue of the differential operator* $u \mapsto -u''$ *with T-periodic boundary conditions. Then, equation* (3.1) *has a positive T-periodic solution, a negative T-periodic solution and, for every integer j with* $1 \leq j \leq m$, *two sign-changing T-periodic solutions with exactly* $2j$ *zeros in the interval* $[0, T[$.

Remark 3.1 In [15], the emphasis is on the case when double resonance at infinity occurs, namely when

$$\lambda_h \leq \liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \leq \limsup_{|x|\to+\infty} \frac{f(t,x)}{x} \leq \lambda_{h+1},$$

for some integer $h \geq 1$. When the existence and multiplicity of periodic solutions to equations like (3.1) is studied with functional analytic techniques, this situation presents peculiar difficulties due to a lack of compactness, so that some additional conditions (like the Landesman-Lazer one) have to be added. We stress that, with the Poincaré-Birkhoff fixed point theorem approach, the situation is very different since the estimates of the rotation numbers can be performed in any case, and no nonresonance assumptions are needed. However, as shown in [4], the Landesman-Lazer condition can be useful to obtain sharper multiplicity results. For instance, according to [4, Corollary 4.1], the conclusion of Corollary 3.1 still holds true whenever $\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = \lambda_m$ and the classical Landesman-Lazer condition, with respect to the m-th eigenvalue, is fulfilled.

Remark 3.2 As for the existence of one-signed T-periodic solutions, a common interpretation of both Theorem 3.1 and Theorem 3.2 is provided by the concept of nonresonance with respect to the principal eigenvalue $\lambda_0 = 0$ of the T-periodic problem. From this point of view, (f_0) is a nonuniform nonresonance assumption at the origin (see [20] and [4, Remark 3.4]), while (f^1_∞), (f^2_∞) both require $f(t,x)$ to be nonresonant with respect to λ_0 at infinity. In particular, (f^2_∞) directly imposes that $f(t,x)/x$ is "far away" from zero for $|x|$ large (even if in a quite mild sense), while, in (f^1_∞) a Landesman-Lazer condition is added in the sublinear case $f(t,x)/x \to 0$.

Results providing the existence of positive solutions in terms of nonresonance conditions with respect to the principal eigenvalue are very common in literature, dealing with the Dirichlet problem [7, 18].

We now give the proofs of the results.

Proofs of Theorem 3.1 and Theorem 3.2. We first state, in three separate claims, some consequences of assumptions (f_0), (f^1_∞) and (f^2_∞) from the point of view of the rotation numbers of the solutions. The technical proofs are postponed until the end of the section. To this aim, let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, T-periodic in the first variable. For simplicity of exposition, we will always assume that $g(t, 0) \equiv 0$ and that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with

$$u'' + g(t,u) = 0 \tag{3.6}$$

are guaranteed.

Claim 1. *For every integer* $k \geq 1$ *and for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* $u'' + q_0(t)u = 0$, *it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, kT]) < 1; \tag{3.7}$$

moreover, if $|u(0)| = 1$ *and* $u'(0) = 0$, *then*

$$\mathrm{Rot}((u(t), u'(t)); [0, T]) < 0. \tag{3.8}$$

Claim 2. *Let* $g(t,x)$ *satisfy assumption* (f^1_∞). *Then there exist an integer* $k^* \geq 1$ *and* $\widetilde{R} > 0$ *such that, for every integer* $k \geq k^*$ *and for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* (3.6), *and with* $\sqrt{u(0)^2 + u'(0)^2} = \widetilde{R}$, *it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, kT]) > 1; \tag{3.9}$$

moreover, there exists $R > 0$ *such that, for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* (3.6)*, and with* $\sqrt{u(0)^2 + u'(0)^2} = R$*, it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, T]) > 0. \tag{3.10}$$

Claim 3. *Let* $g(t, x)$ *satisfy assumption* (f^2_∞)*. Then there exist an integer* $k^* \geq 1$ *such that, for every integer* $k \geq k^*$*, there exists* $\widetilde{R}_k > 0$ *such that for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* (3.6)*, and with* $\sqrt{u(0)^2 + u'(0)^2} = \widetilde{R}_k$*, it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, kT]) > n_k \geq 1; \tag{3.11}$$

moreover, there exists $\hat{R} > 0$ *such that, for every* $u : \mathbb{R} \to \mathbb{R}$ *solving* (3.6)*, and with* $\sqrt{u(0)^2 + u'(0)^2} = \hat{R}$*, it holds that*

$$\mathrm{Rot}((u(t), u'(t)); [0, T]) > 0. \tag{3.12}$$

We are now ready to prove Theorem 3.1 and Theorem 3.2. We first deal with conclusion (*i*), both in case of Theorem 3.1 and Theorem 3.2, proving the existence of a positive T-periodic solution $u_p(t)$. The existence of a negative T-periodic solution $u_n(t)$ follows using a similar argument. Define the function

$$\widetilde{f}(t, x) = \mathrm{sgn}\,(x) f(t, |x|) = \begin{cases} f(t, x) & \text{for } x \geq 0 \\ -f(t, -x) & \text{for } x < 0 \end{cases}$$

and consider the auxiliary equation

$$u'' + \widetilde{f}(t, u) = 0. \tag{3.13}$$

Notice that $\widetilde{f}(t, x)$ satisfies assumption (f_0) - with the same $q_0(t)$ - as well. In view of (3.8) and (3.10) for Theorem 3.1, and of (3.8) and (3.12) for Theorem 3.2, the existence of a T-periodic solution, say $u_p(t)$, to (3.13) satisfying

$$\mathrm{Rot}((u_p(t), u_p'(t)); [0, T]) = 0 \tag{3.14}$$

follows from (*B*) of Corollary 2.1, for $k = 1$ and $j = 0$. Moreover, (3.14) implies that $u_p(t)$ has constant sign and, since $\widetilde{f}(t, x)$ is odd in the x-variable, we can assume that it is positive. Hence $u_p(t)$ solves (3.1), thus concluding the proof.

We now pass to conclusion (*ii*). In the case of Theorem 3.1, the thesis follows directly by applying (*A*) of Corollary 2.1, with $k \geq k^*$ and $j = 1$, to equation (3.1), in view of (3.7) and (3.9). In the case of Theorem 3.2, the thesis follows again from (*A*) of Corollary 2.1, with $k \geq k^*$ and $1 \leq j \leq n_k$, to equation (3.1), in view of (3.7) and (3.11). □

We end the section with the proofs of our technical claims.

Proof of Claim 1. The proof of (3.7) is quite classical (following from the fact that $q_0(t) \leq 0$) and it will be omitted.

Concerning (3.8) (and focusing, for instance, on the case $u(0) = 1$), we preliminarily observe that $u(t)$ has to be convex on $[0, T]$. Indeed, setting $J = \{s \in [0, T] \mid u''(t) \geq 0, t \in [0, s]\}$, it is clear that J is a nonempty (since $0 \in J$) closed interval. Suppose by contradiction that $t^* := \sup J < T$; then $u(t^*) \geq 1$, so that $u(t) > 0$ in a right neighborhood of t^*. Accordingly, $u''(t) = -q(t)u(t) \geq 0$ in a right neighborhood of t^*, a contradiction. As a consequence, $u(t) \geq 1$ and $u'(t) \geq 0$ for every $t \in [0, T]$, so that

$$u'(T) = -\int_0^T q_0(s)u(s)\,ds > 0,$$

from which the claim follows. □

Proof of Claim 2. For the proof of (3.9), we refer to [13, Lemma 4.3] (see also [9]).

For the proof of (3.10), we first recall that from (3.3) we can deduce, using the definition of inferior limit and the monotone convergence theorem, the existence of a constant $M > 0$ and functions $h_+, h_- \in L^1(0,T)$ with

$$\int_0^T h_-(t)\,dt < 0 < \int_0^T h_+(t)\,dt, \tag{3.15}$$

such that

$$g(t,x)x \geq h_+(t)x^+ - h_-(t)x^-, \qquad \text{for every } |x| \geq M \tag{3.16}$$

(being, as usual, $x^+(t) := \max\{x(t),0\}$ and $x^-(t) := \max\{-x(t),0\}$). We now observe that, in view of a standard compactness argument based on the global continuability (the so-called elastic property, cf. [22, Lemma 10]), it is enough to prove the existence of $R^* > 0$ such that (3.10) holds true whenever $\sqrt{u(t)^2 + u'(t)^2} \geq R^*$ for every $t \in [0,T]$. To see this, assume by contradiction that there exists a sequence of functions $u_n(t)$ solving (3.6), with $\sqrt{u_n(t)^2 + u_n'(t)^2} \geq n$ for every $t \in [0,T]$, such that, for n large enough,

$$2\pi \mathrm{Rot}((u_n(t), u_n'(t)); [0,T]) = \int_0^T \frac{g(t,u_n(t))u_n(t) + u_n'(t)^2}{u_n(t)^2 + u_n'(t)^2}\,dt \leq 0. \tag{3.17}$$

Setting $x_n(t) = \frac{u_n(t)}{\|u_n\|_{C^1}}$, we have that $x_n(t)$ solves

$$x_n''(t) + \frac{g(t,u_n(t))}{\|u_n\|_{C^1}} = 0.$$

By standard arguments, in view of (3.2), it is seen that there exists a nonzero $x \in H^2(0,T)$ such that, up to subsequences, $x_n \to x$ strongly in $C^1([0,T])$, with $x(t)$ solving the equation $x''(t) = 0$. We deduce that it has to be $x(t) = at + b$, for suitable $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$.
We now distinguish two cases. If $a \neq 0$, then we have that

$$\int_0^T \frac{u_n'(t)^2}{u_n(t)^2 + u_n'(t)^2}\,dt = \int_0^T \frac{x_n'(t)^2}{x_n(t)^2 + x_n'(t)^2}\,dt \to \int_0^T \frac{x'(t)^2}{x(t)^2 + x'(t)^2}\,dt > 0$$

and, using (3.2) again,

$$\int_0^T \frac{g(t,u_n(t))u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt = \int_0^T \frac{g(t,u_n(t))x_n(t)}{\|u_n\|_{C^1}(x_n(t)^2 + x_n'(t)^2)}\,dt \to 0.$$

Recalling (3.17), we get a contradiction.

If $a = 0$ (and assuming for instance that $b > 0$), we have that $u_n(t) \to +\infty$ uniformly so that, using (3.16), we get, for n large,

$$\begin{aligned} 0 \;&\geq\; 2\pi \mathrm{Rot}((u_n(t), u_n'(t)); [0,T]) \geq \int_0^T \frac{g(t,u_n(t))u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt \\ &\geq\; \int_0^T \frac{h_+(t)u_n(t)}{u_n(t)^2 + u_n'(t)^2}\,dt. \end{aligned}$$

Multiplying by $\|u_n\|_{C^1}$ and passing to the limit, we obtain

$$0 \geq \liminf_{n\to+\infty} \int_0^T \frac{h_+(t)x_n(t)}{x_n(t)^2 + x_n'(t)^2}\,dt = \frac{1}{b}\int_0^T h_+(t)\,dt,$$

which is a contradiction in view of (3.15). □

Proof of Claim 3. For this proof, we will use the following variant of the rotation number of Definition 2.1: for real numbers $\mu, \nu > 0$ and a C^1-path $z = (u,v) : [t_1,t_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every $t \in [t_1,t_2]$, we set

$$\mathrm{Rot}_{\mu,\nu}(z(t); [t_1,t_2]) := \frac{\sqrt{\mu\nu}}{2\pi} \int_{t_1}^{t_2} \frac{v(t)u'(t) - u(t)v'(t)}{\mu u(t)^2 + \nu v(t)^2}\,dt.$$

The choice $\mu = \nu = 1$ leads to the usual notion of rotation number (which we continue to denote simply by Rot); the only property which we will use (see [22]) is that, for every integer j,

$$\mathrm{Rot}(z(t); [t_1, t_2]) \leqslant j \iff \mathrm{Rot}_{\mu,\nu}(z(t); [t_1, t_2]) \leqslant j. \tag{3.18}$$

We first deal with (3.11). As a first step, define k^* to be the least integer such that $n_{k^*} \geq 1$ and fix $k \geq k^*$. Setting $Q = \operatorname{ess\,sup}_{[0,T]} q_\infty(t)$, it is possible to choose $\delta > 0$ such that

$$\frac{k}{2\pi\sqrt{Q}} \int_0^T (q_\infty(t) - \delta)\, dt > n_k;$$

accordingly, there exists $M > 0$ such that, for every $t \in [0, T]$ and $x \in \mathbb{R}$,

$$g(t, x)x \geq (q_\infty(t) - \delta)x^2 - M.$$

Whenever $u : \mathbb{R} \to \mathbb{R}$ solves (3.6) with $u(t)^2 + u'(t)^2 \neq 0$, we can compute

$$\begin{aligned}
\mathrm{Rot}_{1,1/Q}((u(t), u'(t)); [0, kT]) \quad \geq \quad & \frac{1}{2\sqrt{Q}\pi} \int_0^{kT} \frac{\frac{1}{Q}(q_\infty(t) - \delta)u'(t)^2}{u(t)^2 + \frac{1}{Q}u'(t)^2}\, dt \\
& + \frac{1}{2\sqrt{Q}\pi} \int_0^{kT} \frac{(q_\infty(t) - \delta)u(t)^2}{u(t)^2 + \frac{1}{Q}u'(t)^2}\, dt \\
& - \frac{M}{2\sqrt{Q}\pi} \int_0^{kT} \frac{dt}{u(t)^2 + \frac{1}{Q}u'(t)^2} \\
\geq \quad & \frac{k}{2\pi\sqrt{Q}} \int_0^T (q_\infty(t) - \delta)\, dt \\
& - \frac{M}{2\sqrt{Q}\pi} \int_0^{kT} \frac{dt}{u(t)^2 + \frac{1}{Q}u'(t)^2} > n_k,
\end{aligned}$$

whenever $\sqrt{u(t)^2 + u'(t)^2}$ is uniformly large. By standard compactness arguments based on the global continuability (the so-called elastic property, cf. [22, Lemma 10]), we can deduce that the previous relation holds true whenever $\sqrt{u(0)^2 + u'(0)^2} = \widetilde{R}_k$, for $\widetilde{R}_k > 0$ suitably chosen. In view of (3.18), we conclude.

The same arguments as above also show that, if $k = 1$, then (3.12) holds true. □

4 The planar Hamiltonian system

In this section, we give partial generalizations of the result of Section 3 to a planar Hamiltonian system

$$Jz' = \nabla_z H(t, z) \qquad z = (u, v) \in \mathbb{R}^2, \tag{4.1}$$

where the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is supposed to satisfy condition (C_1) of Section 2. We introduce our central assumption, which reads as follows:

(H_0) $\nabla_z H(t, 0) \equiv 0$ *and there exist a continuous function* $B : [0, T] \to \mathcal{L}_s(\mathbb{R}^2)$ *and* $z^* \in \mathbb{R}^2$, *with* $|z^*| = 1$, *such that*

$$\langle B(t)z^* | z^* \rangle < 0, \quad \textit{for every } t \in [0, T], \tag{4.2}$$

and

$$\lim_{z \to 0} \frac{\nabla_z H(t, z) - B(t)z}{|z|} = 0, \qquad \textit{uniformly in } t \in [0, T]. \tag{4.3}$$

Some remarks about the hypothesis are in order. First of all, condition (4.3) just means that the Hamiltonian system (4.1) can be linearized at zero (that is, condition (C_2) of Section 3). On the other hand, (4.2) specifies the nature of $z = 0$ as an equilibrium point of (4.1). In particular, notice that when $B(t) \equiv B \in \mathcal{L}_s(\mathbb{R}^2)$, then (4.2) is satisfied if and only if at least one of the (real) eigenvalues of the symmetric matrix B is negative. In this case, one can have two possibilities:

- both the eigenvalues are nonpositive, so that the solutions to the linear autonomous Hamiltonian system $Jz' = Bz$ are equilibrium points or rotate around the origin counterclockwise;
- one eigenvalue is negative and the other one is positive, so that the origin is a saddle equilibrium point for the system $Jz' = Bz$.

From our point of view, the common feature of this two different dynamical scenarios is that not all the solutions to $Jz' = Bz$ move in the clockwise sense; anyway, the second possibility seems to be the most interesting so that, with slight abuse in terminology, we will say that systems satisfying (H_0) have an equilibrium point of *saddle-type*.

Notice that, when (4.1) comes from the second order equation (3.1), then condition (H_0) is a strengthening of condition (f_0): indeed, it is satisfied for $B(t) = \begin{pmatrix} q_0(t) & 0 \\ 0 & 1 \end{pmatrix}$ and $z^* = (1, 0)$, provided that $q_0(t) < 0$ for every $t \in [0, T]$.

Remark 4.1 It can be interesting to recall that to every linear Hamiltonian system $Jz' = B(t)z$ (even in dimension greater than two) can be assigned an integer number $i(-B)$, the celebrated *Conley-Zehnder index* [6] (also named Maslov index by some authors; we write here $i(-B)$ to be consistent with the definition of the Conley-Zehnder index as given in [1], where the Hamiltonian system is written with the symplectic matrix on the right-hand side). Such an object, whose definition relies on the intersection theory of Lagrangian subspaces in a symplectic vector space, often plays a fundamental role when trying to study the existence and multiplicity of periodic solutions to (nonlinear) Hamiltonian systems via critical points techniques. In the particular case of planar Hamiltonian system and in the T-nonresonant case (i.e., when the only T-periodic solution to $Jz' = B(t)z$ is the trivial one) a detailed analysis of the relationship between the Conley-Zehnder index and the rotation numbers of the solutions in the plane is available [19]. In particular, according to [19, Lemma 4], we can deduce that (4.2)-(4.3) imply that $i(-B) \leq 0$ (indeed, when $i(-B) \geq 1$ then every solution to $Jz' = B(t)z$ has a negative rotation number, and we will show (see Claim 1 at the end of the section) that this is not the case).

We now state our first main result. To this aim, let us denote by $\mathcal{P}$ the class of the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 and strictly positive, i.e. for every $\lambda > 0$ and for every $z \in \mathbb{R}^2 \setminus \{0\}$,

$$0 < V(\lambda z) = \lambda^2 V(z).$$

We recall (see [11]) that the origin is a global isochronous center for the autonomous Hamiltonian system

$$Jz' = \nabla V(z), \qquad \text{with } V \in \mathcal{P}, \tag{4.4}$$

that is to say, all the nontrivial solutions to (4.4) are periodic with the same minimal period (and rotate around the origin in the clockwise sense).

Theorem 4.1 combines, for the nonlinear planar Hamiltonian systems (4.1), an equilibrium point of saddle-type with an asymptotic dynamics like that of (4.4).

Theorem 4.1 *Suppose that the uniqueness and the global continuability for the solutions to the Cauchy problems associated with* (4.1) *are guaranteed. Moreover, assume condition* (H_0) *and*

(H_∞^2) *there exist $V \in \mathcal{P}$ and $\gamma \in L^1(0,T)$, with*

$$\int_0^T \gamma(t)\,dt > 0,$$

such that

$$\liminf_{|z|\to\infty} \frac{\langle \nabla_z H(t,z)|z\rangle}{2V(z)} \geq \gamma(t), \quad \textit{uniformly in } t \in [0,T].$$

Then the following conclusions hold true:

(i) there exists a T-periodic solution $z_0(t)$ to (4.1) *such that*

$$\mathrm{Rot}(z_0(t);[0,T]) = 0.$$

(ii) there exists an integer $k^* \geq 1$ such that, for every integer $k \geq k^*$, there exists an integer m_k such that, for every integer j relatively prime with k and such that $1 \leq j \leq m_k$, there exist at least two subharmonic solutions $z^{(1)}_{j,k}(t), z^{(2)}_{j,k}(t)$ of order k to (4.1)*, not belonging to the same periodicity class, with ($i = 1,2$)*

$$\mathrm{Rot}(z^{(i)}_{j,k}(t);[0,kT]) = j.$$

Moreover, we have the estimate

$$m_k \geq n_k := \mathcal{E}^-\left(\frac{k\int_0^T \gamma(t)\,dt}{\int_{\{V\leq 1\}} dx\,dy}\right), \tag{4.5}$$

where, for $r > 0$, we denote by $\mathcal{E}^-(r)$ the greatest integer strictly *less than r.*

Theorem 4.1 can be seen as the counterpart of Theorem 3.2 for a planar Hamiltonian system. Indeed, we refer to the first part of the section for the relationship between assumption (H_0) and assumption (f_0); on the other hand, it can be seen (see [3, Corollary 3.1] for a similar computation) that assumption (f_∞^2) implies assumption (H_∞^2). Notice, however, that here, at point (i), only *one* T-periodic solution is obtained.

We now state our second main result. To this aim, we denote by $\mathcal{P}^*$ the class of the C^1-functions $V:\mathbb{R}^2 \to \mathbb{R}$ which are positively homogeneous of degree 2 and nonnegative, i.e. for every $\lambda > 0$ and for every $z \in \mathbb{R}^2 \setminus \{0\}$,

$$0 \leq V(\lambda z) = \lambda^2 V(z).$$

Of course, $\mathcal{P} \subset \mathcal{P}^*$. Systems of the type

$$Jz' = \nabla V(z), \qquad \text{with } V \in \mathcal{P}^* \setminus \mathcal{P}, \tag{4.6}$$

have been recently considered in [14] as a possible generalization of the scalar second order equation $u'' = 0$ which, indeed, turns out to be equivalent to system (4.6) with $V(x,y) = \frac{1}{2}y^2 \in \mathcal{P}^* \setminus \mathcal{P}$. In [14, Theorem 3.2], a Landesman-Lazer type condition is introduced to ensure the solvability of the T-periodic problem associated with a sublinear perturbation of (4.6). Here we show - in the spirit of [4, Proposition 4.1] - that such a condition provides enough information about the rotation numbers of large solutions to nonlinear Hamiltonian systems which are asymptotic at infinity to systems of the type (4.6). Accordingly, combining again with assumption (H_0), we get the existence of a nontrivial T-periodic solution with zero rotation number.

Theorem 4.2 *Suppose that the uniqueness for the solutions to the Cauchy problems associated with* (4.1) *is guaranteed. Moreover, assume condition* (H_0) *and*

(H^1_∞) *there exists* $V \in \mathcal{P}^*$*, with* $\nabla V(z)$ *locally Lipschitz continuous, such that*

$$\lim_{|z|\to+\infty} \frac{\nabla_z H(t,z) - \nabla V(z)}{|z|} = 0, \qquad \text{uniformly in } t \in [0,T]. \tag{4.7}$$

Moreover, setting $R(t,z) = \nabla_z H(t,z) - \nabla V(z)$*, suppose that:*

- *for every* $t \in [0,T]$*, for every* $z \in \mathbb{R}^2$ *with* $|z| \leq 1$ *and for every* $\lambda > 1$*,*

$$\langle R(t,\lambda z)|z\rangle \geq \eta(t), \tag{4.8}$$

for a suitable $\eta \in L^1(0,T)$*,*

- *for every* $\xi \in \mathbb{R}^2 \setminus \{0\}$ *satisfying* $\nabla V(\xi) = 0$*,*

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle > 0. \tag{4.9}$$

Then there exists a T*-periodic solution* $z_0(t)$ *to* (4.1) *such that*

$$\mathrm{Rot}(z_0(t);[0,T]) = 0.$$

Theorem 4.2 is a partial generalization of Theorem 3.1. Indeed, (4.9) is a Landesman-Lazer type condition; in particular, according to [14, Remark 4.1], assumption (H^1_∞) implies assumption (f^1_∞). However, only one T-periodic solution is obtained and no conclusions about subharmonic solutions are provided, at all.

We now sketch the proofs of our results.

Proofs of Theorem 4.1 and Theorem 4.2. The conclusions follow from the following claims, similarly as in the proofs of Theorem 3.1 and Theorem 3.2. The main difference is that here we always work directly on equation (4.1) (and, as a consequence, only one T-periodic solution is obtained).

Claim 1. *For every integer* $k \geq 1$ *and for every* $z : \mathbb{R} \to \mathbb{R}$ *solving* $Jz' = B(t)z$*, it holds that*

$$\mathrm{Rot}(z(t)); [0,kT]) < \frac{1}{2}, \tag{4.10}$$

moreover, if $z(0) = z^*$*, then*

$$\mathrm{Rot}(z(t);[0,T]) < 0. \tag{4.11}$$

To see this, we just write the solution $z(t)$ as $z(t) = \sqrt{2\rho(t)}e^{i\theta(t)}$, with $\rho(t), \theta(t)$ of class C^1 and $\rho(t) > 0$ (compare with Definition 2.1), and set $z^* = e^{i\theta^*}$, with $\theta^* \in [0, 2\pi[$. In view of (2.6), assumption (H_0) implies that $-\theta'(t) < 0$ whenever $z(t) = \lambda z^*$ for $\lambda \neq 0$, namely $\theta(t) \equiv \theta^* \mod \pi$. As a consequence, $\theta(t_2) < \theta(t_1) + \pi$ for every $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, so that (4.10) follows. With the same arguments, we also deduce the validity of (4.11). □

Claim 2. *Let* $H(t,z)$ *satisfy assumption* (H^1_∞)*. Then there exists* $R > 0$ *such that, for every* $z : \mathbb{R} \to \mathbb{R}$ *solving* (4.1)*, and with* $|z(0)| = R$*, it holds that*

$$\mathrm{Rot}(z(t);[0,T]) > 0. \tag{4.12}$$

Observe first that, in view of the global continuability for the solutions to (4.1) (which follows from (4.7), since $\nabla V(z)$ grows at most linearly), it is enough to show that there exists $R^* > 0$ such that (4.12) holds true whenever $|z(t)| \geq R^*$ for every $t \in [0,T]$. To see this, assume by contradiction that there exists a sequence of functions $z_n(t)$ solving (4.1), with $|z_n(t)| \geq n$ for every $t \in [0,T]$, such that, for n large enough,

$$2\pi\mathrm{Rot}(z_n(t);[0,T]) = \int_0^T \frac{\langle \nabla V(z_n(t)) + R(t,z_n(t))|z_n(t)\rangle}{|z_n(t)|^2}\, dt \leq 0. \tag{4.13}$$

For further convenience, recall also that, in view of Euler's formula, we have

$$\int_0^T \frac{\langle \nabla V(z_n(t)) + R(t,z_n(t)) | z_n(t)\rangle}{|z_n(t)|^2}\,dt = \int_0^T \frac{2V(z_n(t))}{|z_n(t)|^2}\,dt + \int_0^T \frac{\langle R(t,z_n(t)) | z_n(t)\rangle}{|z_n(t)|^2}\,dt. \tag{4.14}$$

Setting $w_n(t) = \frac{z_n(t)}{\|z_n\|_\infty}$, the function $w_n(t)$ satisfies

$$Jw_n'(t) = \nabla V(w_n(t)) + \frac{R(t,z_n(t))}{\|z_n\|_\infty},$$

and, by standard arguments using (4.7), it is seen that there exists $0 \not\equiv w \in H^1(]0,T[\,;\mathbb{R}^2)$ such that $w_n \to w$ uniformly, with $w(t)$ solving $Jw'(t) = \nabla V(w(t))$. Notice that, since $\nabla V(z)$ is locally Lipschitz continuous and $w(t) \not\equiv 0$, it has to be $w(t) \neq 0$ for every $t \in [0,T]$; moreover, $\nabla V(w(t)) = 0$ if and only if $V(w(t)) = 0$.
We now distinguish two cases. If $w(t)$ is nonconstant, then, using (4.7), we get

$$\int_0^T \frac{\langle R(t,z_n(t)) | z_n(t)\rangle}{|z_n(t)|^2}\,dt = \int_0^T \frac{\langle R(t,z_n(t)) | w_n(t)\rangle}{\|z_n\|_\infty |w_n(t)|^2}\,dt \to 0.$$

On the other hand, since $V(w(t)) \equiv V(w(0)) > 0$ (otherwise $w(0)$ would be an equilibrium point for $Jz' = \nabla V(z)$ and, by uniqueness, $w(t)$ should be constant), we have

$$\int_0^T \frac{2V(z_n(t))}{|z_n(t)|^2}\,dt = \int_0^T \frac{2V(w_n(t))}{|w_n(t)|^2}\,dt \to \int_0^T \frac{2V(w(t))}{|w(t)|^2}\,dt > 0.$$

In view of (4.13) and (4.14), we have a contradiction.

If $w(t)$ is constant, it has to be $w(t) \equiv \xi$, for a suitable $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that $\nabla V(\xi) = 0$. In this case, we have, using (4.14) and the nonnegativeness of V,

$$0 \geq 2\pi \mathrm{Rot}(z_n(t);[0,T]) \geq \int_0^T \frac{\langle R(t,z_n(t)) | z_n(t)\rangle}{|z_n(t)|^2}\,dt.$$

Multiplying by $\|z_n\|_\infty$, we get

$$0 \geq \int_0^T \frac{\langle R(t,\|z_n\|_\infty w_n(t)) | w_n(t)\rangle}{|w_n(t)|^2}\,dt,$$

so that, since $w_n(t) \to \xi$ uniformly, using (4.8) and Fatou's lemma we obtain

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle R(t,\|z_n\|_\infty w_n(t)) | w_n(t)\rangle\,dt,$$

contradicting (4.9). □

Claim 3. *Let $H(t,z)$ satisfy assumption (H^2_∞). Then there exist an integer $k^* \geq 1$ such that, for every integer $k \geq k^*$, there exists $\widetilde{R}_k > 0$ such that, for every $z : \mathbb{R} \to \mathbb{R}$ solving* (4.1)*, and with $|z(0)| = \widetilde{R}_k$, it holds that*

$$\mathrm{Rot}(z(t);[0,kT]) > n_k \geq 1;$$

moreover, there exists $\hat{R} > 0$ such that, for every $z : \mathbb{R} \to \mathbb{R}$ solving (4.1)*, and with $|z(0)| = \hat{R}$, it holds that*

$$\mathrm{Rot}(z(t);[0,T]) > 0.$$

This can be proved exactly as in [3, Theorem 3.1] (by exchanging the role of zero and infinity). □

Acknowledgements

I wish to thank Maurizio Garrione for many fruitful discussions about the relationship between Landesman-Lazer conditions and rotation numbers.

References

[1] A. Abbondandolo, Morse Theory for Hamiltonian systems, Chapman & Hall/CRC Research Notes in Mathematics **425**, Boca Raton, 2001.

[2] F.M. Atici and G.Sh. Guseinov, *On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions*, J. Comput. Appl. Math. **132** (2001), 341–356.

[3] A. Boscaggin, *Subharmonic solutions of planar Hamiltonian systems: a rotation number approach*, Adv. Nonlinear Stud. **11** (2011), 77–103.

[4] A. Boscaggin and M. Garrione, *Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré-Birkhoff theorem*, Nonlinear Anal. **74** (2011), 4166–4185.

[5] J. Campos, A. Margheri, R. Martins and C. Rebelo, *A note on a modified version of the Poincaré-Birkhoff theorem*, J. Differential Equations **203** (2004), 55–63.

[6] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984), 207–253.

[7] D.G. De Figueiredo, *Positive solutions of semilinear elliptic problems*, Differential equations (São Paulo, 1981), pp. 34–87, Lecture Notes in Math. **957**, Springer, Berlin, 1982.

[8] T. Ding and F. Zanolin, *Subharmonic solutions of second order nonlinear equations: a time-map approach*, Nonlinear Anal. **20** (1993), 509–532.

[9] T. Ding and F. Zanolin, *Periodic solutions and subharmonic solutions for a class of planar systems of Lotka-Volterra type*, World Congress of Nonlinear Analysts '92, Vol. I-IV (Tampa, FL, 1992) (Berlin), de Gruyter, Berlin, (1996), 95–106.

[10] W.Y. Ding, *Fixed point of twist mappings and periodic solutions of ordinary differential equations*, Acta Math. Sinica **25** (1982) 227–235 [Chinese].

[11] A. Fonda, *Positively homogeneous Hamiltonian systems in the plane*, J. Differential Equations **200** (2004), 162–184.

[12] A. Fonda, M. Sabatini and F. Zanolin, *Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré-Birkhoff Theorem*, to appear in Topol. Methods Nonlinear Anal.

[13] A. Fonda, Z. Schneider and F. Zanolin, *Periodic oscillations for a nonlinear suspension bridge model*, J. Comput. Appl. Math **52** (1994), 113–140.

[14] M. Garrione, *Resonance at the first eigenvalue for first order systems in the plane: vanishing Hamiltonians and the Landesman-Lazer condition*, Differential Integral Equations **25** (2012), 505–526.

[15] L. Gasiński and N. Papageorgiou, *A multiplicity theorem for double resonant periodic problems*, Adv. Nonlinear Stud. **10** (2010), 819–836.

[16] P. Le Calvez and J. Wang, *Some remarks on the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **138** (2010), 703–715.

[17] R. Ma, C. Gao and R. Chen, *Existence of positive solutions of nonlinear second-order periodic boundary value problems*, Bound. Value Probl. **2010**, 18 pp.

[18] R. Manásevich, F.I. Njoku and F. Zanolin, *Positive solutions for the one-dimensional p-Laplacian*, Differential Integral Equations **8** (1995), 213–222.

[19] A. Margheri, C. Rebelo and F. Zanolin, *Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, J. Differential Equations **183** (2002), 342–367.

[20] J. Mawhin and J.R. Ward, *Nonresonance and existence for nonlinear elliptic boundary value problems*, Nonlinear Anal. **5** (1981), 677–684.

[21] C. Rebelo, *A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems*, Nonlinear Anal. **29** (1997), 291–311.

[22] C. Rebelo and F. Zanolin, *Multiplicity results for periodic solutions of second order ODEs with asymmetric nonlinearities*, Trans. Amer. Math. Soc. **348** (1996), 2349–2389.

[23] J. Su and L. Zhao, *Multiple periodic solutions of ordinary differential equations with double resonance*, Nonlinear Anal. **70** (2009), 1520–1527.

[24] P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003), 643–662.