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Multiple Sign Changing Solutions of Nonlinear Elliptic Problems in Exterior Domains*

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Abstract

We consider the problem

$$-\Delta u + (V_\infty + V(x))u = |u|^{p-2}\,u, \quad u \in H_0^1(\Omega),$$

where Ω is an exterior domain in $\mathbb{R}^N$, $V_\infty > 0$, $V \in C^0(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > -V_\infty$ and $V(x) \to 0$ as $|x| \to \infty$. Under symmetry conditions on Ω and V, and some assumptions on the decay of V at infinity, we show that there is an effect of the topology of the orbit space of certain subsets of the domain on the number of low energy sign changing solutions to this problem.

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Key words. Nonlinear elliptic problem, unbounded domain, multiplicity of sign changing solutions.

1 Introduction

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\,u = |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is an unbounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty, and $2 < p < 2^* := \frac{2N}{N-2}$. The potential $V_\infty + V$ is assumed to satisfy

(V_0) $\quad V \in C^0(\mathbb{R}^N),\ V_\infty \in (0,\infty),\ \inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0,\ \lim_{|x|\to\infty} V(x) = 0.$

Equations of this kind arise naturally in various branches of physics and in some problems in biology as well, see for example [8, 21]. The existence of solutions to (1.1) has been extensively

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studied during the last 25 years. A detailed account is given in Cerami's survey article [11]. In what follows we make reference to the results more closely related to our study.

The main difficulty in dealing with problem (1.1) by means of variational methods is the lack of compactness. This difficulty does not appear when Ω and V are radially symmetric and we look for radial solutions [27, 8, 20]. However if, either Ω or V do not have symmetries, or if they have symmetries with finite orbits, the lack of compactness prevails.

Remarkable progress was made when P.-L. Lions introduced in [23] his concentration compactness method, which allowed to show the existence of a solution of problem (1.1) in $\mathbb{R}^N$ by a minimization argument for $V \leq 0$. This also applies in an exterior domain Ω, like the one we are considering, when $V < 0$ satisfies a suitable decay assumption at infinity. Nevertheless, when $V \geq 0$ and $\Omega \neq \mathbb{R}^N$ or when $V > 0$ and $\Omega = \mathbb{R}^N$ the question of the existence cannot be treated by minimization. To handle this situation a deeper understanding of the lack of compactness of the variational problem is needed. Benci and Cerami gave in [7] a complete description of the lack of compactness in terms of the solutions to the limit problem

$$\begin{cases} -\Delta u + V_\infty u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

associated to (1.1). This allowed them to solve the existence problem for $V \equiv 0$ when the diameter of $\mathbb{R}^N \smallsetminus \Omega$ is small enough. Bahri and Lions in [3] eliminated this restriction and, considering some decay assumptions at infinity on V, they showed the existence of a solution for $V \geq 0$. In all of these cases the solution obtained is positive.

A result concerning the existence of multiple solutions with small energy was obtained in [17] when $\Omega = \mathbb{R}^N$ and V approaches to 0 from below at infinity in a suitable way. However, the techniques employed there, provide no information on whether these solutions change sign or not. Cerami, Devillanova and Solimini established the existence of infinitely many solutions in [13] assuming that $\Omega = \mathbb{R}^N$ and V tends to zero from below at infinity at some suitable rate. Recently, Wei and Yan [29] proved the existence of infinitely many positive solutions to this problem when $\Omega = \mathbb{R}^N$ and V is a radial function tending to 0 at infinity, in a polynomial way. Without any symmetry assumptions on the potential, Cerami, Passaseo and Solimini proved in [14] an analogous result for potentials that decay very slowly.

We are interested in obtaining multiplicity of sign changing solutions. For $\Omega = \mathbb{R}^N$ and $V \equiv 0$ existence of infinitely many sign changing solutions with large symmetries was shown in [6, 24, 25]. When Ω and V have only finite symmetries, existence of a sign changing solution to problem (1.1) was shown in [12] and [10], under suitable assumptions. We shall refer to these results later in more detail.

Several multiplicity results have been obtained for the singularly perturbed problem $-\varepsilon\Delta u + (V_\infty + V(x))\, u = |u|^{p-2}u$, $u \in H^1(\mathbb{R}^N)$, for small enough $\varepsilon > 0$. It is well-known that, when $\varepsilon \to 0$, there are solutions to this problem which concentrate at critical points of the potential V, see [1, 18]. Hence, it is not surprising that the topology of certain subsets of critical points of V has an effect on the number of solutions to this problem, as has been shown for example in [16]. Even though a similar concentration phenomenon is not present in the problem we are treating here, we will prove in this article that, when looking for sign changing solutions, there is a combined effect of the topology and the symmetries of certain subsets of the domain on the number of solutions to problem (1.1).

We study the case where both Ω and V have some symmetries. To be precise, we consider a closed subgroup Γ of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and a continuous homomorphism $\phi : \Gamma \to \mathbb{Z}/2 := \{1, -1\}$. We denote by $G := \ker \phi$, by

$$\ell := \min\{\#Gx : x \in \mathbb{S}^{N-1}\},$$

and by

$$\Sigma := \{x \in \mathbb{S}^{N-1} : \#Gx = \ell\},$$

where $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, $Gx := \{gx : g \in G\}$ is the G-orbit of x and $\#Gx$ is its cardinality.

Recall that a subset X of $\mathbb{R}^N$ is Γ-invariant if $\Gamma x \subset X$ for every $x \in X$, and a function $u : X \to \mathbb{R}$ is Γ-invariant if it is constant on each Γ-orbit Γx with $x \in X$. If X is Γ-invariant and ϕ is an epimorphism (i.e. it is surjective), the group $\mathbb{Z}/2$ acts on the G-orbit space $X/G := \{Gx : x \in X\}$ of X as follows: we choose $\gamma \in \Gamma$ such that $\phi(\gamma) = -1$ and we define

$$(-1) \cdot Gx := G(\gamma x) \qquad \text{for all } x \in X.$$

This action is well defined and it does not depend on the choice of γ. We denote by

$$\Sigma_0 := \{x \in \Sigma : Gx = G(\gamma x)\}.$$

If Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$, the action of $\mathbb{Z}/2$ on its G-orbit space Z/G is free. If $Z \neq \emptyset$ the Krasnoselski genus of Z/G, denoted genus(Z/G), is defined to be the smallest $k \in \mathbb{N}$ such that there exists a continuous map $f : Z/G \to \mathbb{S}^{k-1}$ which is $\mathbb{Z}/2$-equivariant, i.e. $f((-1) \cdot Gz) = -f(Gz)$ for every $z \in Z$. We define genus$(\emptyset) := 0$.

For each subgroup K of Γ we set

$$\mu(Kz) := \begin{cases} \inf\{|\alpha_1 z - \alpha_2 z| : \alpha_1, \alpha_2 \in K,\ \alpha_1 z \neq \alpha_2 z\} & \text{if } \#Kz \geq 2, \\ 2\,|z| & \text{if } \#Kz = 1, \end{cases}$$

$$\mu_K(Z) := \inf_{z \in Z} \mu(Kz) \qquad \text{and} \qquad \mu^K(Z) := \sup_{z \in Z} \mu(Kz).$$

From now on, we will assume that Ω is Γ-invariant, that V is a Γ-invariant function and that (V_0) holds. We will also assume that $\ell < \infty$, and we will denote by c_∞ the energy of the positive solution to the limit problem (1.2). We shall prove the following result.

Theorem 1.1 *If $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$, and V satisfies the following:*

(V_1) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \qquad \textit{for all } x \in \mathbb{R}^N \textit{ with } |x| \geq r_0,$$

then problem (1.1) *has at least* genus(Z/G) *pairs of sign changing solutions $\pm u$ such that*

$$u(\alpha x) = \phi(\alpha)u(x) \qquad \textit{for all } \alpha \in \Gamma,\ \ x \in \Omega, \tag{1.3}$$

and

$$\int_\Omega |u|^p < \frac{4p}{p-2}\,\ell\, c_\infty. \tag{1.4}$$

Let us look at an example. Under analogous assumptions to those of the previous theorem, Carvalho, Maia and Miyagaki proved in [10] the existence of a solution to (1.1) satisfying (1.3) and (1.4) in the case where Γ is the group spanned by the reflection $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ on a linear subspace W of $\mathbb{R}^N$ of dimension $0 \leq \dim W < N$. In this case, taking Z as the unit sphere in the orthogonal complement of W, Theorem 1.1 asserts the existence of

$$\text{genus}(Z/G) = N - \dim W$$

pairs of solutions to this problem. Note that $\mu_\Gamma(Z) = 2$, so our assumption (V_1) is less restrictive than the one in [10] where $\lambda \in (0, \sqrt{V_\infty})$ is required.

Another interesting example is the following: If $N = 2n$ we identify $\mathbb{R}^N$ with $\mathbb{C}^n$ and take Γ to be the cyclic group of order $2m$ spanned by $\rho(z_1, \ldots, z_n) := (e^{\pi i/m} z_1, \ldots, e^{\pi i/m} z_n)$ and $\phi : \Gamma \to \mathbb{Z}/2$ to be the epimorphism given by $\phi(\rho) := -1$. Then $G := \ker\phi$ is the cyclic subgroup of order m spanned by

ρ^2. Since the action is free, we have that $\Sigma = \mathbb{S}^{N-1}$ and $\Sigma_0 = \emptyset$, so we may take $Z := \mathbb{S}^{N-1}$. The genus of $\mathbb{S}^{N-1}/G$ can be estimated in many cases. For example, if $m = 2^k$, Lemma 6.1 below together with Theorem 1.2 of [4] give

$$\text{genus}(\mathbb{S}^{N-1}/G) \geq \frac{N-1}{2^k} + 1.$$

Since $\mu_\Gamma(\mathbb{S}^{N-1}) = \left|e^{\pi i/m} - 1\right|$, condition (V_1) becomes more restrictive as m increases. So, if (V_1) holds for $m = 2^k$, it will also hold for $m = 2^j$ with $0 \leq j < k$. Now, if u_j is a solution provided by Theorem 1.1 for $m = 2^j$, then u_j satisfies (1.3), i.e.

$$u_j(e^{\pi il/(2^j)}z) = (-1)^l u_j(z) \qquad \forall\, l = 0, \ldots, 2^{j+1} - 1, \;\; z \in \Omega \subset \mathbb{C}^n.$$

This implies that $u_k \neq u_j$ if $k > j$. Indeed, if $k > j$ and $u_k(z) = u_j(z) \neq 0$ at some $z \in \Omega$ then, since $u_j(e^{\pi i/(2^j)}z) = -u_j(z)$ and

$$u_k(e^{\pi i/(2^j)}z) = u_k(e^{\pi i(2^{k-j})/(2^k)}z) = (-1)^{2^{k-j}} u_k(z) = u_j(z),$$

we have that $u_k(e^{\pi i/(2^j)}z) \neq u_j(e^{\pi i/(2^j)}z)$. Therefore, Theorem 1.1 provides at least

$$\sum_{j=0}^{k} \frac{N-1}{2^j} + k + 1 = (N-1)\frac{2^{k+1}-1}{2^k} + k + 1$$

pairs of solutions in this case.

On the other hand, similar actions in odd dimensions give no solutions. For example, if we take polygonal symmetry in $\mathbb{R}^3$ given by $\rho(z,t) := (e^{\pi i/m}z, t)$, $(z,t) \in \mathbb{C}\times\mathbb{R} = \mathbb{R}^3$ -as considered in [28] for a related problem- and $\phi(\rho) := -1$, then $\Sigma = \{\pm(0,0,1)\} = \Sigma_0$. So Theorem 1.1 gives no information in this case. However, if we consider the group Γ generated by ρ and the reflection $\tau(z,t) := (z,-t)$, and take $\phi(\rho) := 1$ and $\phi(\tau) := -1$, then $\Sigma = \{\pm(0,0,1)\}$ and $\Sigma_0 = \emptyset$ and Theorem 1.1 yields one pair of sign changing solutions.

For potentials with an analogous behavior at infinity, but without requiring any symmetry property on neither the domain nor the potential, in [17] it was shown that problem (1.1) has at least $\frac{N}{2} + 1$ pairs of solutions. However, the argument used there gives no precise information whether the solutions obtained change sign or not. If ϕ is an epimorphism, property (1.3) asserts that u changes sign and, as we have seen, in some cases Theorem 1.1 yields more than $\frac{N}{2} + 1$ pairs of solutions.

We shall prove also the following multiplicity result of sign changing solutions, with a different condition on the potential.

Theorem 1.2 *Let $\ell \geq 2$ and Z be a compact Γ-invariant subset of Σ. Assume that the following hold:*
(Z_0) $\text{dist}(\gamma z, Gz) > \mu(Gz)$ *for all $z \in Z$ and $\gamma \in \Gamma$ with $\phi(\gamma) = -1$.*
(V_2) *There exist $c_1 > 0$ and $\kappa > \mu^G(Z)\sqrt{V_\infty}$ such that*

$$V(x) \leq c_1 e^{-\kappa|x|} \qquad \textit{for all } x \in \mathbb{R}^N.$$

Then (1.1) *has at least* genus(Z/G) *pairs of sign changing solutions $\pm u$, which satisfy* (1.3) *and* (1.4).

Theorem 1.2 is an extension of the result obtained in [12] which states the existence of a sign changing solution to the autonomous problem $V \equiv 0$ if (Z_0) holds for some $z \in \Sigma$. Note that (Z_0) implies that $Z \subset \Sigma \setminus \Sigma_0$. Note also that condition (Z_0) cannot be realized if $N = 3$. However, we next give an example which illustrates the situation in Theorem 1.2 for higher dimensions.

We identify $\mathbb{R}^{4n}$ with $\mathbb{C}^n \times \mathbb{C}^n$ and consider the subgroup Γ of $O(4n)$ spanned by ρ and γ, where $\rho(y,z) := (e^{\pi i/m}y, e^{\pi i/m}z)$ and $\gamma(y,z) := (-\bar{z}, \bar{y})$ for $(y,z) \in \mathbb{C}^n \times \mathbb{C}^n$ and some $m \geq 3$. We define $\phi : \Gamma \to \mathbb{Z}/2$ by $\phi(\rho) = 1$, $\phi(\gamma) = -1$. Then $G := \ker\phi$ is the cyclic subgroup of order $2m$ spanned by

ρ. Since $m \geq 3$, property (Z_0) is satisfied by $Z := \mathbb{S}^{4n-1}$. Note that $\mu^G(\mathbb{S}^{4n-1}) = \left|e^{\pi i/m} - 1\right|$, hence (V_2) becomes less restrictive as m increases. We will prove in the Section 6 that genus$(\mathbb{S}^{4n-1}/G) \geq 2n+1$. Consequently, if Ω is Γ-invariant and V is Γ-invariant and satisfies (V_0) and (V_2), problem (1.1) has at least $2n + 1$ pairs of sign changing solutions $\pm u$, which satisfy (1.3) and (1.4).

This paper is organized as follows. In section 2 the variational framework for problem (1.1) is set, while in Section 2 a careful analysis of Palais-Smale sequences satisfying (1.3) is carried out. In Section 4 and 5 we prove Theorems 1.1 and 1.2. Finally, in the Appendix, estimates for the genus of an orbit space are provided.

2 The variational problem

From now on we assume without loss of generality that $V_\infty = 1$. We define

$$\|u\|_V := \left(\int_\Omega \left(|\nabla u|^2 + (1 + V(x))\, u^2\right)\right)^{1/2}. \tag{2.1}$$

Assumption (V_0) guarantees that this is a norm in $H_0^1(\Omega)$ which is equivalent to the usual one. The solutions of problem (1.1) are the critical points of the functional $J_V : H_0^1(\Omega) \to \mathbb{R}$ given by

$$J_V(u) := \frac{1}{2}\|u\|_V^2 - \frac{1}{p}|u|_p^p,$$

where $|u|_p := \left(\int_\Omega |u|^p\right)^{1/p}$ is the norm in $L^p(\Omega)$.

The homomorphism ϕ induces an orthogonal action of Γ on $H_0^1(\Omega)$ as follows: for $\gamma \in \Gamma$ and $u \in H_0^1(\Omega)$ we define $\gamma u \in H_0^1(\Omega)$ by

$$(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x).$$

The functional J_V is then Γ-invariant. So, by the principle of symmetric criticality [26], the critical points of the restriction of J_V to the fixed point space of this action, which we denote by

$$\begin{aligned} H_0^1(\Omega)^\phi : &= \{u \in H_0^1(\Omega) : \gamma u = u \ \ \forall \gamma \in \Gamma\} \\ &= \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \ \forall \gamma \in \Gamma,\ x \in \Omega\}, \end{aligned}$$

are the solutions of problem (1.1) that satisfy (1.3). Note that, if u satisfies (1.3), then u is G-invariant. Moreover, $u(\alpha x) = -u(x)$ for every $x \in \Omega$ and $\alpha \in \phi^{-1}(-1)$. Therefore, if ϕ is an epimorphism, every nontrivial solution to (1.1) which satisfies (1.3) changes sign.

The nontrivial solutions of problem (1.1) satisfying (1.3) lie on the Nehari manifold

$$\mathcal{N}^\phi := \left\{u \in H_0^1(\Omega)^\phi : u \neq 0,\ \|u\|_V^2 = |u|_p^p\right\},$$

which is of class C^2 and is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\phi$. The radial projection $\pi : H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathcal{N}^\phi$ is given by

$$\pi(u) := \left(\frac{\|u\|_V^2}{|u|_p^p}\right)^{\frac{1}{p-2}} u. \tag{2.2}$$

Observe that, for every $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$,

$$J_V(\pi(u)) = \frac{p-2}{2p}\left(\frac{\|u\|_V^2}{|u|_p^2}\right)^{\frac{p}{p-2}} = \max_{t\geq 0} J_V(tu).$$

3 The Palais-Smale condition

Benci and Cerami described the lack of compactness of the functional J_V in [7]. They showed that the Palais-Smale sequences which do not converge to a solution of problem (1.1) approach a sum of a possibly trivial solution of (1.1) plus nontrivial solutions of the limit problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{3.1}$$

translated by sequences of points in the domain which go to infinity.

We analyze next the Palais-Smale sequences belonging to $H_0^1(\Omega)^\phi$. We shall give a precise description of the relation between the symmetries of the translation points and those of the corresponding solution to the limit problem. This plays an important role in the proof of Corollary 3.1, which will be crucial for our results.

Recall that the Γ-isotropy subgroup of a point $x \in \mathbb{R}^N$ is defined as

$$\Gamma_x := \{\alpha \in \Gamma : \alpha x = x\}.$$

The Γ-orbit Γx of x is Γ-homeomorphic to the homogeneous space Γ/Γ_x. It holds true that $\Gamma_{\alpha x} = \alpha\Gamma_x\alpha^{-1}$. Thus, the conjugate groups to an isotropy group are isotropy groups, see for instance [9, 19].

Lemma 3.1 *Given a sequence (y_n) in $\mathbb{R}^N$ there exist a sequence (ζ_n) in $\mathbb{R}^N$ and a closed subgroup K of Γ such that for some subsequence of (y_n), which we still denote in the same way, the following hold:*
(a) $(\mathrm{dist}(\Gamma y_n, \zeta_n))$ *is bounded.*
(b) $\Gamma_{\zeta_n} = K$ *for all* $n \in \mathbb{N}$.
(c) If $|\Gamma/K| < \infty$ *then* $|\alpha\zeta_n - \alpha'\zeta_n| \to \infty$ *for any* $[\alpha], [\alpha'] \in \Gamma/K$ *with* $[\alpha] \neq [\alpha']$.
(d) If $|\Gamma/K| = \infty$ *then there exists a closed subgroup* K' *of* Γ *such that* $K \subset K'$, $|\Gamma/K'| = \infty$ *and* $|\alpha\zeta_n - \alpha'\zeta_n| \to \infty$ *for any* $[\alpha], [\alpha'] \in \Gamma/K'$ *with* $[\alpha] \neq [\alpha']$.

Proof. See [15, Lemma 3.2]. ∎

The energy functional $J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (3.1) is given by

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p,$$

where $\|u\|^2 := \int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)$ and $|u|_p^p := \int_{\mathbb{R}^N}|u|^p$. As usual, we identify $u \in H_0^1(\Omega)$ with its extension to $\mathbb{R}^N$ obtained by setting $u \equiv 0$ in $\mathbb{R}^N \smallsetminus \Omega$. We denote by J_0 the functional associated to problem (1.1) with $V \equiv 0$, i.e.

$$J_0(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p,$$

and by $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$. If $v \in H^1(\mathbb{R}^N)$ and $\alpha \in \Gamma$ we simply write $v\alpha$ for the composition $v \circ \alpha$.

Proposition 3.1 *Let (u_n) be a sequence in $H_0^1(\Omega)^\phi$ such that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$, $J_0(u_n) \to c > 0$ and $J_0'(u_n) \to 0$ in $H^{-1}(\Omega)$. Then there exist a sequence (ζ_n) in Ω, a closed subgroup K of finite index in Γ, a nontrivial solution v to problem* (3.1) *and a sequence (w_n) in $H_0^1(\Omega)^\phi$ such that*
(a) $\Gamma_{\zeta_n} = K$ *for all* $n \in \mathbb{N}$,
(b) $|\zeta_n| \to \infty$ *and* $|\alpha\zeta_n - \hat{\alpha}\zeta_n| \to \infty$ *if* $\hat{\alpha}\alpha^{-1} \notin K$, $\hat{\alpha}, \alpha \in \Gamma$,
(c) $v(\alpha x) = \phi(\alpha)v(x)$ *for all* $x \in \mathbb{R}^N$, $\alpha \in K$,
(d) $\left\| u_n - w_n - \sum_{[\alpha]\in\Gamma/K} \phi(\alpha)v\alpha^{-1}(\cdot - \alpha\zeta_n) \right\| \to 0,$
(e) $w_n \rightharpoonup 0$ *in* $H_0^1(\Omega)$, $J_0(w_n) \to c - |\Gamma/K|\,J_\infty(v)$ *and* $J_0'(w_n) \to 0$ *in* $H^{-1}(\Omega)$.

Proof. The sequence (u_n) is bounded in $H_0^1(\Omega)$, so

$$\frac{p-2}{2p}|u_n|_p^p = J_0(u_n) - \frac{1}{2}J_0'(u_n)\,u_n \to c > 0.$$

Lions' lemma [30, Lemma 1.21] then yields that

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 > 0.$$

We choose $y_n \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n|^2 > \delta/2$$

and, for the sequence (y_n), we choose K and (ζ_n) as in Lemma 3.1. We define $v_n(x) := u_n(x+\zeta_n)$. Passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $v_n \to v$ strongly in $L^p_{loc}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Fixing $C > 0$ such that $\mathrm{dist}(\Gamma y_n, \zeta_n) \leq C$ for every n, we have that $B_1(\alpha_n y_n) \subset B_{C+1}(\zeta_n)$ for some $\alpha_n \in \Gamma$. Since $|u_n|$ is Γ-invariant we obtain

$$\int_{B_{C+1}(0)} |v_n|^2 = \int_{B_{C+1}(\zeta_n)} |u_n|^2 \geq \int_{B_1(y_n)} |u_n|^2 > \frac{\delta}{2}.$$

This implies that $v \neq 0$. But $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, so (ζ_n) is unbounded and we may assume that $|\zeta_n| \to \infty$. A standard argument shows that v is a solution to problem (3.1) [30, Lemma 8.3].
Assertion *(b)* of Lemma 3.1 insures that, for every $\alpha \in K$, $u_n(\alpha\, x+\zeta_n) = u_n\,(\alpha(x+\zeta_n)) = \phi(\alpha)u_n(x+\zeta_n)$. Hence $v(\alpha x) = \phi(\alpha)v(x)$.
Let $\alpha_1, \alpha_2, \ldots \alpha_t \in \Gamma$ be such that $\left|\alpha_j\zeta_n - \alpha_i\zeta_n\right| \to \infty$ when $i \neq j$. Then

$$\phi(\alpha_j)v_n\alpha_j^{-1} - \sum_{i=j+1}^{t} \phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n + \alpha_j\zeta_n) \rightharpoonup \phi(\alpha_j)v\alpha_j^{-1}$$

weakly in $H^1(\mathbb{R}^N)$. By the Brezis-Lieb Lemma [30, Lemma 1.32] we have

$$\left|\phi(\alpha_j)v_n\alpha_j^{-1} - \sum_{i=j+1}^{t} \phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n + \alpha_j\zeta_n)\right|_p^p$$
$$= \left|\phi(\alpha_j)v_n\alpha_j^{-1} - \sum_{i=j}^{t}\phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n + \alpha_j\zeta_n)\right|_p^p + \left|\phi(\alpha_j)v\alpha_j^{-1}\right|_p^p + o(1).$$

The change of variable $y = x - \alpha_j\zeta_n$ yields

$$\left|u_n - \sum_{i=j+1}^{t} \phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n)\right|_p^p = \left|u_n - \sum_{i=j}^{t} \phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n)\right|_p^p + |v|_p^p + o(1),$$

and iterating this equality we obtain

$$|u_n|_p^p - \left|u_n - \sum_{i=1}^{t} \phi(\alpha_i)v\alpha_i^{-1}(\cdot - \alpha_i\zeta_n)\right|_p^p = t\,|v|_p^p + o(1).$$

From the last expression we deduce that

$$\frac{2p}{p-2}c \geq t\,|v|_p^p\,.$$

Hence, assertion *(d)* of Lemma 3.1 implies that $|\Gamma/K| < \infty$, i.e. K has finite index in Γ. Thus assertion *(c)* of Lemma 3.1 allows us to take $t := |\Gamma/K|$. Finally, we choose $R > 0$ such that $\left(\mathbb{R}^N \smallsetminus \Omega\right) \subset B_R(0)$, and a radially symmetric cut off function $\chi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \chi(x) \leq 1, \chi(x) = 0$ if $|x| \leq R$ and $\chi(x) = 1$ if $|x| \geq 2R$. We define

$$w_n(x) := u_n(x) - \sum_{i=1}^{|\Gamma/K|} \phi(\alpha_i)\chi(x)v\left(\alpha_i^{-1}(x - \alpha_i\zeta_n)\right).$$

Then $w_n \in H_0^1(\Omega)^\phi$ and standard arguments show that (w_n) satisfies *(d)* and *(e)*. ■

We shall say that J_V *satisfies the ϕ-equivariant Palais-Smale condition $(PS)_c^\phi$ at the level c* if every sequence (v_n) such that

$$v_n \in H_0^1(\Omega)^\phi, \qquad J_V(v_n) \to c, \qquad J_V'(v_n) \to 0 \text{ in } H^{-1}(\Omega), \tag{3.2}$$

contains a convergent subsequence in $H_0^1(\Omega)$. The proposition above gives us a level below which the functional J_V satisfies the Palais-Smale condition.

Corollary 3.1 *J_V satisfies condition $(PS)_c^\phi$ for all $c < |\Gamma/G|\,\ell\, c_\infty$.*

Proof. Let (v_n) be a sequence which satisfies (3.2). Since (v_n) is bounded in $H_0^1(\Omega)^\phi$, a subsequence satisfies that $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)^\phi$, $v_n \to v_0$ strongly in $L_{loc}^p(\Omega)$ and $v_n(x) \to v_0(x)$ a.e. in Ω. Defining $u_n := v_n - v_0$ we have that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)^\phi$. Furthermore, $J_0(u_n) \to d := c - J_V(v_0)$, $J_0'(u_n) \to 0$ in $H^{-1}(\Omega)$ and v_0 is a solution of (1.1) [30, Lemma 8.2].
If $d \leq 0$ then $u_n \to 0$ strongly in $H_0^1(\Omega)$. If $d > 0$ there exist $\zeta_n \in \Omega$, a closed subgroup K of finite index in Γ, a nontrivial solution v of (3.1) and a sequence (w_n) in $H_0^1(\Omega)^\phi$ with properties *(a)*-*(e)* of Proposition 3.1. In particular,

$$J_0(u_n) = J_0(w_n) + |\Gamma/K|\, J_\infty(v) + o(1).$$

Consequently,

$$c \geq d \geq |\Gamma/K| J_\infty(v).$$

We now distinguish two cases: We note that if $K \subset G$ then $|\Gamma/K| = |\Gamma/G|\,|G/K| = |\Gamma/G|\,(\#G\zeta_n) \geq |\Gamma/G|\,\ell$. Therefore, $c \geq |\Gamma/G|\,\ell\, c_\infty$. If $K \not\subseteq G$ then ϕ is an epimorphism. Property *(c)* of Proposition 3.1 asserts that v changes sign and, consequently $J_\infty(v) \geq 2\, c_\infty$. In addition, $|\Gamma/K| = |G/(K \cap G)| = \#G\zeta_n \geq \ell$. Thus, $c \geq 2\,\ell\, c_\infty = |\Gamma/G|\,\ell\, c_\infty$. In both cases, we obtain a contradiction to our hypothesis. Therefore, $u_n \to 0$ strongly in $H_0^1(\Omega)$. ■

We denote by ∇J_V the gradient of J_V with respect to the scalar product $\langle\cdot,\cdot\rangle_V$ which induces the norm (2.1), and by $\nabla_{\mathcal{N}^\phi} J_V(u)$ the orthogonal projection of $\nabla J_V(u)$ onto the tangent space $T_u\mathcal{N}^\phi$ to the Nehari manifold $\mathcal{N}^\phi$ in the point $u \in \mathcal{N}^\phi$. We say that J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}^\phi$ if every sequence (u_n) such that

$$u_n \in \mathcal{N}^\phi, \qquad J_V(u_n) \to c, \qquad \nabla_{\mathcal{N}^\phi} J_V(u_n) \to 0, \tag{3.3}$$

has a convergent subsequence in $H_0^1(\Omega)$.

Corollary 3.2 *J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}^\phi$ for all $c < |\Gamma/G|\,\ell\, c_\infty$.*

Proof. Let (u_n) be a sequence which satisfies (3.3). In view of Corollary 3.1, we just need to prove that $\nabla J_V(u_n) \to 0$.

If $u \in \mathcal{N}^\phi$, the tangent space $T_u\mathcal{N}^\phi$ is the subspace of $H_0^1(\Omega)^\phi$ which is orthogonal to $\nabla F(u)$, where $F(u) := \|u\|_V^2 - |u|_p^p$. Note that, since F is Γ-invariant, $\nabla F(u) \in H_0^1(\Omega)^\phi$. The same is true for J_V. We express $\nabla J_V(u_n)$ as

$$\nabla J_V(u_n) = \nabla_{\mathcal{N}^\phi} J_V(u_n) + t_n \nabla F(u_n), \qquad t_n \in \mathbb{R}. \tag{3.4}$$

By taking the scalar product of the above equality with u_n one gets

$$\langle \nabla_{\mathcal{N}^\phi} J_V(u_n), u_n\rangle_V = \langle \nabla J_V(u_n), u_n\rangle_V - t_n \langle \nabla F(u_n), u_n\rangle_V = t_n(p-2)\|u_n\|_V^2 \geq c_1 t_n,$$

with $c_1 > 0$. By the hypotheses, $\langle \nabla_{\mathcal{N}^\phi} J_V(u_n), u_n\rangle_V \to 0$, and so $t_n \to 0$. On the other hand, there exists a constant $c_2 > 0$ such that

$$\left|\langle \nabla F(u_n), v\rangle_V\right| \leq 2\|u_n\|_V\|v\|_V + p\,|u_n|_p^{p-1}\,|v|_p \leq c_2\|v\|_V$$

for all $v \in H_0^1(\Omega)$. This shows that $(\nabla F(u_n))$ is bounded. Thus, from identity (3.4), $\nabla J_V(u_n) \to 0$ follows. ■

4 Proof of Theorem 1.1

Let $\omega \in H^1(\mathbb{R}^N)$ be the unique positive solution to problem (3.1) which is radially symmetric about the origin. It is well-known (see [8, 22]) that there exist positive constants b_0, b_1 such that

$$\lim_{|x|\to\infty} \left|D^i\omega(x)\right| |x|^{\frac{N-1}{2}} \exp|x| = b_i \qquad \text{for } i = 0, 1. \tag{4.1}$$

Let Z be a Γ-invariant subset of $\Sigma \setminus \Sigma_0$ and set $\lambda \in (0, \mu_\Gamma(Z))$. We choose $\varepsilon \in (0, \frac{\mu_\Gamma(Z)-\lambda}{\mu_\Gamma(Z)+\lambda})$ and a radially symmetric cut off function $\chi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ if $|x| \leq 1-\varepsilon$ and $\chi(x) = 0$ if $|x| \geq 1$. For $S > 0$ we define $\omega^S \in H^1(\mathbb{R}^N)$ by

$$\omega^S(x) := \chi\left(\frac{x}{S}\right)\omega(x).$$

Then, $\omega^S \to \omega$ in $H^1(\mathbb{R}^N)$ as $S \to \infty$. Using (4.1) we obtain the following asymptotic estimates

$$\left|\|\omega\|^2 - \left\|\omega^S\right\|^2\right| = O(e^{-2(1-\varepsilon)S}), \qquad \left||\omega|_p^p - |\omega^S|_p^p\right| = O(e^{-p(1-\varepsilon)S}) \tag{4.2}$$

as $S \to \infty$, see [17, Lemma 2]. We define $\rho := \frac{\mu_\Gamma(Z)+\lambda}{4}$, and we consider the function $(\omega^{\rho R})_{Ry} \in H^1(\mathbb{R}^N)$ given by

$$(\omega^{\rho R})_{Ry}(x) := \omega^{\rho R}(x - Ry).$$

Note that $\operatorname{supp}(\omega^{\rho R})_{Ry} \subset B_{\rho R}(Ry)$.

Lemma 4.1 *If (V_1) holds, there exist $C_1 > 0$ and $r_1 > 0$ such that $(\omega^{\rho R})_{Ry} \in H_0^1(\Omega)$ and*

$$J_V\left(t(\omega^{\rho R})_{Ry}\right) \leq c_\infty - C_1 e^{-\lambda R} \qquad \textit{for all } y \in Z,\ R \geq r_1 \textit{ and } t \geq 0.$$

Proof. We may assume without loss of generality that the $r_0 > 0$ of condition (V_1) also satisfies $\left(\mathbb{R}^N \setminus B_{r_0}(0)\right) \subset \Omega$. Note that, since $\mu_\Gamma(Z) \leq 2$, $\rho \in (0, 1)$. Therefore, there exists $r_1 > 0$ such that $R - \rho R \geq r_0$ provided $R \geq r_1$, which implies that $(\omega^{\rho R})_{Ry} \in H_0^1(\Omega)$ for all $y \in Z$ and $R \geq r_1$.

Since $\omega^{\rho R} \to \omega$ in $H^1(\mathbb{R}^N)$ as $R \to \infty$, we may choose $t_1 > t_0 > 0$ and $r_1 > 0$ such that the following also holds:

$$\max_{t\geq 0} J_V\left(t(\omega^{\rho R})_{Ry}\right) = \max_{t\in[t_0,t_1]} J_V\left(t(\omega^{\rho R})_{Ry}\right) \qquad \text{for all } y \in Z \text{ and } R \geq r_1.$$

By condition (V_1), for all $t \in [t_0, t_1]$, $y \in Z$ and $R \geq r_1$, we have

$$\int_\Omega V(x) \left| t\omega^{\rho R}(x - Ry) \right|^2 dx = \int_{|x| \leq \rho R} V(x + Ry) \left| t\omega^{\rho R}(x) \right|^2 dx$$
$$\leq -\left(c_0 t^2 \int_{|x| \leq \rho R} e^{-\lambda|x+Ry|} |\omega(x)|^2 \, dx \right)$$
$$\leq -\left(c_0 t_0^2 \int_{\mathbb{R}^N} e^{-\lambda|x|} |\omega(x)|^2 \, dx \right) e^{-\lambda R} = -C_2 e^{-\lambda R}.$$

Using the estimates (4.2) we conclude that, choosing $r_1 > 0$ even larger if necessary, there exists $C_1 > 0$ such that

$$J_V\left(t(\omega^{\rho R})_{Ry}\right) = \frac{1}{2} \left\| t(\omega^{\rho R})_{Ry} \right\|^2 + \frac{1}{2} \int_\Omega V(x) \left| t(\omega^{\rho R})_{Ry} \right|^2 dx - \frac{1}{p} \left| t(\omega^{\rho R})_{Ry} \right|_p^p$$
$$= \frac{1}{2} \|t\omega\|^2 - \frac{1}{p} |t\omega|_p^p + O(e^{-2(1-\varepsilon)\rho R}) - C_2 e^{-\lambda R}$$
$$= J_\infty(t\omega) - C_1 e^{-\lambda R} \leq c_\infty - C_1 e^{-\lambda R},$$

because $c_\infty = \max_{t \geq 0} J_\infty(t\omega)$ and $2(1 - \varepsilon)\rho > \frac{\mu_\Gamma(Z)+\lambda}{2}\left(1 - \frac{\mu_\Gamma(Z)-\lambda}{\mu_\Gamma(Z)+\lambda}\right) = \lambda.$ ■

Proof of Theorem 1.1. $J_V : \mathcal{N}^\phi \to \mathbb{R}$ is an even function, which is bounded from below and satisfies $(PS)_c^\phi$ on $\mathcal{N}^\phi$ for all $c < |\Gamma/G| \, \ell \, c_\infty = 2 \, \ell \, c_\infty$. Consequently, if $d < 2 \, \ell \, c_\infty$, J_V has at least

$$\text{genus}(\mathcal{N}^\phi \cap J_V^d)$$

pairs of critical points $\pm u$ with critical value $J_V(u) \leq d$, where $J_V^d := \{u \in H_0^1(\Omega) : J_V(u) \leq d\}$. We fix $R \geq r_1$ and $d := 2 \, \ell \, (c_\infty - C_1 e^{-\lambda R})$ with C_1 and r_1 as in Lemma 4.1.

Notice that $\#\Gamma y = 2 \, \ell$ for all $y \in Z$, as $Z \subset \Sigma \smallsetminus \Sigma_0$. Observe also that, if $[\alpha] \neq [\beta]$ in $\Gamma/\Gamma_y \cong \Gamma y$, then $|R\alpha y - R\beta y| \geq \mu_\Gamma(Z)R > 2\rho R$. For this reason, $(\omega^{\rho R})_{R\alpha y}$ and $(\omega^{\rho R})_{R\beta y}$ have disjoint support, for each $y \in Z$. We define

$$\theta(y) := \pi \left(\sum_{[\alpha] \in \Gamma/\Gamma_y} \phi(\alpha) (\omega^{\rho R})_{R\alpha y} \right) \tag{4.3}$$

where π is the radial projection onto $\mathcal{N}^\phi$, see (2.2). So, it holds true that

$$J_V(\theta(y)) = 2 \, \ell \, \max_{t \geq 0} J_V(t(\omega^{\rho R})_{Ry}) \leq d.$$

The map $\theta : Z \to \mathcal{N}^\phi \cap J_V^d$ defined by (4.3) is continuous. Moreover, $\theta(gy) = \theta(y)$ for all $g \in G$ and $\theta(\gamma y) = -\theta(y)$ if $\phi(\gamma) = -1$. Therefore, θ induces a continuous map $\hat{\theta} : Z/G \to \mathcal{N}^\phi \cap J_V^d$, given by $\hat{\theta}(Gy) := \theta(y)$, which satisfies $\hat{\theta}((-1) \cdot Gy) = -\hat{\theta}(Gy)$ for all $y \in Z$. This yields that

$$\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}^\phi \cap J_V^d)$$

and concludes the proof. ■

5 Proof of Theorem 1.2

For $\zeta \in \mathbb{R}^N$ we define

$$I(\zeta) := \int_{\mathbb{R}^N} \omega^{p-1}(x)\omega(x - \zeta)dx \qquad \text{and} \qquad A(\zeta) := \int_{\mathbb{R}^N} V^+(x)\omega^2(x - \zeta)dx.$$

In order to describe the asymptotic behavior of I we will use the following result of Bahri and Li.

Lemma 5.1 *Let $f \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h \in C^0(\mathbb{R}^N)$ be radially symmetric functions satisfying*

$$\lim_{|x|\to\infty} f(x)\,|x|^b\, e^{d|x|} = \tau \qquad \text{and} \qquad \int_{\mathbb{R}^N} |h(x)|\,(1+|x|^b)e^{d|x|}dx < \infty$$

for $d \geq 0$, $b \geq 0$ and $\tau \in \mathbb{R}$. Then

$$\lim_{|y|\to\infty} \left(\int_{\mathbb{R}^N} f(x+y)h(x)dx \right) |y|^b\, e^{d|y|} = \tau \int_{\mathbb{R}^N} h(x)e^{-dx_1}dx.$$

Proof. See Proposition 1.2 in [2]. ■

As ω is radially symmetric, from (4.1) and Lemma 5.1 we deduce

$$\lim_{|\xi|\to\infty} I(\xi)\,|\xi|^{\frac{N-1}{2}}\, e^{|\xi|} = \nu > 0.$$

Let $S > 0$ be such that

$$\frac{\nu}{2} \leq I(\xi)\,|\xi|^{\frac{N-1}{2}}\, e^{|\xi|} \leq \frac{3\nu}{2} \qquad \text{if} \ \ |\xi| \geq S. \tag{5.1}$$

Therefore,

$$\frac{I(\zeta)}{I(\xi)} \leq 3e^{-(|\zeta|-|\xi|)} \qquad \text{if} \ \ |\zeta| \geq |\xi| \geq S. \tag{5.2}$$

Lemma 5.2 *Let $M \in (0, 2)$. If $V(x) \leq ce^{-\kappa|x|}$ for all $x \in \mathbb{R}^N$ with $c > 0$ and $\kappa > M$, then*

$$\lim_{|\zeta|\to\infty} A(\zeta)e^{M|\zeta|}\,|\zeta|^{\frac{N-1}{2}} = 0.$$

Proof. Throughout this proof c will denote possibly distinct positive constants that are independent of ζ. Let us fix $\varepsilon \in (0, 1)$ such that $\kappa(1-\varepsilon) > M$. Then

$$\begin{aligned}\int_{B_{\varepsilon|\zeta|}(\zeta)} V^+(x)\omega^2(x-\zeta)e^{M|\zeta|}\,|\zeta|^{\frac{N-1}{2}}\,dx &\leq c\,e^{-(\kappa(1-\varepsilon)-M)|\zeta|}\,|\zeta|^{\frac{N-1}{2}} \int_{\mathbb{R}^N} \omega^2(x)dx \\ &= c\,e^{-(\kappa(1-\varepsilon)-M)|\zeta|}\,|\zeta|^{\frac{N-1}{2}}\,. \end{aligned} \tag{5.3}$$

On the other hand, according to (4.1), for $x \in \mathbb{R}^N \smallsetminus B_{\varepsilon|\zeta|}(\zeta)$ and $|\zeta|$ large enough,

$$\omega^2(x-\zeta) \leq c\,e^{-2|x-\zeta|}|x-\zeta|^{-(N-1)}.$$

Therefore, making the change of variable $y = \frac{x}{|\zeta|}$ and defining $z := \frac{\zeta}{|\zeta|}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon|\zeta|}(\zeta)} V^+(x)\omega^2(x-\zeta)e^{M|\zeta|}\,|\zeta|^{\frac{N-1}{2}}\,dx \\ &\qquad \leq c\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon|\zeta|}(\zeta)} \frac{e^{-(\kappa|x|+2|x-\zeta|-M|\zeta|)}\,|\zeta|^{\frac{N-1}{2}}}{|x-\zeta|^{N-1}}dx \\ &\qquad = c\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon}(z)} \frac{e^{-|\zeta|(\kappa|y|+2|y-z|-M)}\,|\zeta|^{\frac{N+1}{2}}}{|y-z|^{N-1}}dy. \end{aligned} \tag{5.4}$$

Set $\kappa_0 := \min\{\kappa, 2\}$ and fix $\delta \in (0, 1)$ such that $\delta\kappa_0 > M$. Then

$$\kappa\,|y| + 2\,|y-z| - M \geq \kappa_0(|y| + |y-z| - \delta) + (\delta\kappa_0 - M) \geq \delta\kappa_0 - M > 0.$$

Taking into account that $\max_{t\in\mathbb{R}} e^{-dt}t^{\frac{N+1}{2}} = \left(\frac{N+1}{2e}\right)^{\frac{N+1}{2}} d^{-\frac{N+1}{2}}$ for $d > 0$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_\varepsilon(z)} \frac{e^{-|\zeta|(\kappa|y|+2|y-z|-M)}\,|\zeta|^{\frac{N+1}{2}}}{|y-z|^{N-1}}dy\\
&\quad\le e^{-(\delta\kappa_0-M)|\zeta|}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)} \frac{e^{-|\zeta|\kappa_0(|y|+|y-z|-\delta)}\,|\zeta|^{\frac{N+1}{2}}}{|y-z|^{N-1}}dy\\
&\quad\le e^{-(\delta\kappa_0-M)|\zeta|}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)} \frac{dy}{(\kappa_0(|y|+|y-z|-\delta))^{\frac{N+1}{2}}|y-z|^{N-1}}\\
&\quad= ce^{-(\delta\kappa_0-M)|\zeta|}.
\end{aligned} \tag{5.5}$$

Now the assertion of Lemma 5.2 follows from inequalities (5.3), (5.4) and (5.5). ∎

Lemma 5.3 *If $h \in C_c^0\left(\mathbb{R}^N\right)$ is radially symmetric and $q > 1$, then*

$$\lim_{|\zeta|\to\infty}\left(\int_{\mathbb{R}^N} h(x)\omega^q(x-\zeta)dx\right)|q\zeta|^{\frac{N-1}{2}}\,e^{q|\zeta|} = 0.$$

Proof. This is an immediate consequence of Lemma 5.1 and expression (4.1). ∎

Lemma 5.4 *(i) If $p \ge 2$ and $a_1, a_2, \ldots, a_l \ge 0$, then*

$$\left|\sum_{i=1}^{l} a_i\right|^p \ge \sum_{i=1}^{l} a_i^p + (p-1)\sum_{i\ne j} a_i^{p-1}a_j.$$

(ii) If $p \ge 2$ and $a, b \ge 0$, then

$$|a-b|^p \ge a^p + b^p - pab\left(a^{p-2}+b^{p-2}\right).$$

Proof. See Lemma 4 in [12]. ∎

For $y \in \mathbb{R}^N$ we define

$$\sigma_y := \sum_{[\alpha]\in\Gamma/\Gamma_y} \phi(\alpha)\omega_{\alpha y} \in H^1(\mathbb{R}^N)^\phi.$$

We choose $R_0 > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B_{R_0}(0)$, and a radially symmetric cut off function $\chi \in C^\infty(\mathbb{R}^N)$ such that $0 \le \chi(x) \le 1$, $\chi(x) = 0$ if $|x| \le R_0$ and $\chi(x) = 1$ if $|x| \ge 2R_0$. Notice that $\chi\sigma_y \in H_0^1(\Omega)^\phi$. Note also that, for all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}
\|\chi u\|_V^2 &= \int_{\mathbb{R}^N}\left(|\chi\nabla u + u\nabla\chi|^2 + (1+V(x))\,|\chi u|^2\right) \qquad (5.6)\\
&= \int_{\mathbb{R}^N}\chi^2\left(|\nabla u|^2 + (1+V(x))|u|^2\right) + \int_{\mathbb{R}^N}\left(|\nabla\chi|^2 - \frac{1}{2}\Delta(\chi^2)\right)u^2\\
&\le \|u\|_V^2 - \int_{\mathbb{R}^N}(\chi\,\Delta\chi)u^2,\\
|\chi u|_p^p &= |u|_p^p + \int_{\mathbb{R}^N}(\chi^p-1)\,|u|^p. \qquad (5.7)
\end{aligned}$$

Proposition 5.1 *Let $\ell \ge 2$, let Z be a compact Γ-invariant subset of $\Sigma\setminus\Sigma_0$ that satisfies condition (Z_0) of* Theorem 1.2. *If V satisfies (V_2), then there exist $C_1 > 0$ and $R_1 > 0$ such that*

$$\frac{\|\chi\,\sigma_{Rz}\|_V^2}{|\chi\,\sigma_{Rz}|_p^2} \le (2\,\ell\,\|\omega\|^2)^{\frac{p-2}{p}} - C_1e^{-2R}R^{-\frac{N-1}{2}} \qquad \textit{for all } z \in Z \textit{ and } R \ge R_1.$$

Proof. Since ω is a solution to problem (3.1), for any $y, y' \in \mathbb{R}^N$ one has that

$$\int_{\mathbb{R}^N} [\nabla\omega(x-y)\cdot\nabla\omega(x-y') + \omega(x-y)\omega(x-y')]dx = \int_{\mathbb{R}^N} \omega(x-y)^{p-1}\omega(x-y')\,dx = I(y'-y).$$

We fix $\gamma \in \Gamma$ with $\phi(\gamma) = -1$ and set

$$\varepsilon_y := \sum_{\substack{[\alpha],[\beta]\in G/G_y \\ [\alpha]\neq[\beta]}} I(\beta y - \alpha y), \qquad \widehat{\varepsilon}_y := \sum_{[\alpha],[\beta]\in G/G_y} I(\beta y - \alpha\gamma y).$$

Since ω is radially symmetric, one has that $\varepsilon_y = \varepsilon_{\gamma y}$.

Observe that the function $y \mapsto \mu(Gy)$ is continuous in Σ. Hence, as Z is compact, $\mu_G(Z)$ and $\mu^G(Z)$ are attained in Z. For simplicity, we shall denote them

$$m := \mu_G(Z) = \min_{z\in Z} \mu(Gz) \qquad \text{and} \qquad M := \mu^G(Z) = \max_{z\in Z}\mu(Gz).$$

Furthermore, property (Z_0) implies that there exists $a > 0$ such that

$$\mathrm{dist}(\gamma z, Gz) - \mu(Gz) \geq a \qquad \text{for all } z\in Z \text{ and } \gamma\in\Gamma \text{ with } \phi(\gamma) = -1. \tag{5.8}$$

Let $z \in Z$. We choose $\alpha_z, \beta_z \in G$ such that $|\alpha_z z - \beta_z z| = \mu(Gz)$ and denote $\xi_z := \alpha_z z - \beta_z z$. From (5.6), (5.7) and Lemma 5.4 we deduce

$$\|\chi\,\sigma_{Rz}\|_V^2 \leq 2\,\ell\,\|\omega\|^2 + 2\varepsilon_{Rz} - 2\widehat{\varepsilon}_{Rz} + \int_{\mathbb{R}^N} V^+\sigma_{Rz}^2 - \int_{\mathbb{R}^N}(\chi\,\Delta\chi)\sigma_{Rz}^2,$$

$$|\chi\,\sigma_{Rz}|_p^p \geq 2\,\ell\,|\omega|_p^p + 2(p-1)\varepsilon_{Rz} - c\widehat{\varepsilon}_{Rz} + \int_{\mathbb{R}^N}(\chi^p - 1)\,|\sigma_{Rz}|^p,$$

with $c > 0$.

In what follows c_i denotes a positive constant independent of z and R. If $R \geq \frac{S}{m}$ inequality (5.1) yields that

$$c_2 e^{-2R}(2R)^{-\frac{N-1}{2}} \leq \varepsilon_{Rz} \leq c_3 e^{-Rm}(Rm)^{-\frac{N-1}{2}},$$

inequalities (5.2) and (5.8) imply that

$$\frac{\widehat{\varepsilon}_{Rz}}{\varepsilon_{Rz}} \leq \sum_{[\alpha],[\beta]\in G/G_z} \frac{I(R\beta z - R\alpha\gamma z)}{I(R\xi_z)} \leq 3\sum_{[\alpha],[\beta]\in G/G_z} e^{-R(|\beta z - \alpha\gamma z| - \mu(Gz))} \leq c_4 e^{-Ra},$$

whereas inequality (5.1) gives

$$\frac{\int_{\mathbb{R}^N} V^+\sigma_{Rz}^2}{\varepsilon_{Rz}} \leq c_5 \sum_{[\alpha]\in\Gamma/\Gamma_z} \frac{A(R\alpha z)}{I(R\xi_z)} \leq \frac{2c_4}{\nu}\sum_{[\alpha]\in\Gamma/\Gamma_z} A(R\alpha z)e^{MR}(MR)^{\frac{N-1}{2}}.$$

Property (Z_0) guarantees that $M \in (0,2)$. From Lemma 5.2 it follows then that

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} V^+\sigma_{Rz}^2}{\varepsilon_{Rz}} = 0$$

uniformly in $z \in Z$. Analogously, using Lemma 5.3 we conclude that

$$\lim_{R\to\infty}\frac{\int_{\mathbb{R}^N}(\chi\,\Delta\chi)\sigma_{Rz}^2}{\varepsilon_{Rz}} = 0 \qquad \text{and} \qquad \lim_{R\to\infty}\frac{\int_{\mathbb{R}^N}(\chi^p-1)\,|\sigma_{Rz}|^p}{\varepsilon_{Rz}} = 0$$

uniformly in $z \in Z$. Consequently, there exists $R_1 \geq \frac{S}{m}$ such that

$$\begin{aligned}\frac{\|\chi\,\sigma_{Rz}\|_V^2}{|\chi\,\sigma_{Rz}|_p^2} &\leq \frac{2\,\ell\,\|\omega\|^2 + 2\,\varepsilon_{Rz} + o(\varepsilon_{Rz})}{\left(2\,\ell\,\|\omega\|^2 + 2(p-1)\,\varepsilon_{Rz} + o(\varepsilon_{Rz})\right)^{2/p}} \\ &\leq (2\,\ell\,\|\omega\|^2)^{\frac{p-2}{p}} - c_6\,\varepsilon_{Rz} \\ &\leq (2\,\ell\,\|\omega\|^2)^{\frac{p-2}{p}} - c_7 e^{-2R}\,(2R)^{-\frac{N-1}{2}} \qquad \text{for all } z \in Z,\ R \geq R_1.\end{aligned}$$

This finishes the proof. ∎

Proof of Theorem 1.2. We fix $R \geq R_1$ and define

$$d := \frac{p-2}{2p}\left[(2\,\ell\,\|\omega\|^2)^{\frac{p-2}{p}} - C_1 e^{-2R} R^{-\frac{N-1}{2}}\right]^{\frac{p}{p-2}}$$

with R_1 and C_1 as in Proposition 5.1. Then, for all $z \in Z$,

$$J_V(\pi(\chi\sigma_{Rz})) = \frac{p-2}{2p}\left(\frac{\|\chi\,\sigma_{Rz}\|_V^2}{|\chi\,\sigma_{Rz}|_p^2}\right)^{\frac{p}{p-2}} \leq d < 2\,\ell\,c_\infty,$$

where π is the radial projection onto $\mathcal{N}^\phi$, see (2.2). The map $\theta : Z \to \mathcal{N}^\phi \cap J_V^d$ defined by $\theta(z) := \pi(\chi\sigma_{Rz})$ is continuous. Moreover, $\theta(gz) = \theta(z)$ for all $g \in G$ and $\theta(\gamma z) = -\theta(z)$ if $\phi(\gamma) = -1$. Consequently, θ induces a continuous map $\hat{\theta} : Z/G \to \mathcal{N}^\phi \cap J_V^d$, given by $\hat{\theta}(Gz) := \theta(z)$, which satisfies $\hat{\theta}((-1)\cdot Gz) = -\hat{\theta}(Gz)$ for all $z \in Z$. This implies that

$$\text{genus}(Z/G) \leq \text{genus}(\mathcal{N}^\phi \cap J_V^d).$$

Arguing as in the proof of Theorem 1.1 we conclude that J_V has at least genus(Z/G) pairs of critical points $\pm u$ with critical value $J_V(u) \leq d$. ∎

6 The genus of an orbit space

In Theorem 1.1 and Theorem 1.2 the number of solutions is given in terms of the genus of the orbit space Z/G. We shall give estimates for it in terms of the Γ-genus of Z.

Let us recall the notion of Γ-genus, see [5] for further details. Let Γ be a compact Lie group. The *join* of the Γ-spaces $X_1, \ldots, X_m$ is the space

$$X_1 * \cdots * X_m := \left\{[s_1, x_1, \ldots, s_m, x_m] : s_i \in [0,1],\ \sum_{i=1}^m s_i = 1,\ x_i \in X_i\right\}$$

where $[s_1, x_1, \ldots, s_m, x_m] = [t_1, y_1, \ldots, t_m, y_m]$ if, for each $i = 1, ..., m$, either $s_i = t_i$ and $x_i = y_i$ or $s_i = t_i = 0$. This is again a Γ-space with the action

$$\alpha[s_1, x_1, \ldots, s_m, x_m] := [s_1, \alpha x_1, \ldots, s_m, \alpha x_m].$$

The Γ-*genus* of a nonempty Γ-space X is the smallest $m \in \mathbb{N}$ such that there exist closed subgroups $\Gamma_1, ..., \Gamma_m$ of Γ with $\Gamma_i \neq \Gamma$ and a continuous Γ-equivariant map

$$f : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m,$$

i.e. $f(\alpha x) = \alpha f(x)$ for all $x \in X$, $\alpha \in \Gamma$. We denote it by Γ-genus(X). If no such map exists we set Γ-genus$(X) := \infty$.

If $\Gamma = \mathbb{Z}/2$ then $\mathbb{Z}/2 * \cdots * \mathbb{Z}/2 \cong \mathbb{S}^{m-1}$ with the action given by multiplication, so that the $\mathbb{Z}/2$-genus is just the Krasnoselskii genus.

Let Γ and Λ be compact Lie groups, $\phi : \Gamma \to \Lambda$ be a continuous epimorphism, $K := \ker\phi$ and X a Γ-space. Then Λ acts on the orbit space X/K as follows: for each $x \in X$, $\alpha \in \Lambda$ and some $\gamma \in \Gamma$ such that $\phi(\gamma) = \alpha$ we define

$$\alpha \cdot Kx := K(\gamma x). \tag{6.1}$$

This action is well defined because K is a normal subgroup of Γ. The quotient map $q : X \to X/K$ satisfies that $q(\gamma x) = \phi(\gamma) \cdot q(x)$ for any $\gamma \in \Gamma$, $x \in X$. The following result holds:

Lemma 6.1 Γ-*genus*$(X) = \Lambda$-*genus*(X/K).

Proof. Let $\Lambda_1, ..., \Lambda_m$ be closed subgroups of Λ, $\Lambda_i \neq \Lambda$, and let $f : X/K \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m$ be a continuous Λ-equivariant map. We define $\Gamma_i := \{\gamma \in \Gamma : \phi(\gamma) \in \Lambda_i\}$. Then ϕ induces homeomorphisms $\phi_i : \Gamma/\Gamma_i \to \Lambda/\Lambda_i$ that satisfy $\phi_i(\gamma\Gamma_i) = \phi(\gamma)\Lambda_i$, which in turn induce an homeomorphism

$$\phi_1 * \cdots * \phi_m : \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m,$$

defined in the obvious way. The map $F : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m$ given by $F := (\phi_1 * \cdots * \phi_m)^{-1} \circ f \circ q$ is continuous and Γ-equivariant. Hence, Γ-genus$(X) \leq \Lambda$-genus(X/K).

Conversely, let $\Gamma_1, ..., \Gamma_m$ be closed subgroups of Γ, $\Gamma_i \neq \Gamma$, and let $F : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m$ be a continuous Γ-equivariant map. We define $\Lambda_i := \phi(\Gamma_i)$ and set $\phi_1 * \cdots * \phi_m$ as above. Observe that $(\phi_1 * \cdots * \phi_m) \circ F$ is continuous and constant on $q^{-1}(Kx)$ for each $x \in X$. Hence, it induces a map $f : X/K \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m$ which is continuous and Λ-equivariant. Therefore, Λ-genus$(X/K) \leq \Gamma$-genus(X). ∎

Let us look at an example. Let Γ be the subgroup of $O(4n)$ spanned by ρ and γ, where

$$\rho(y, z) := (e^{\pi i/m}y, e^{\pi i/m}z), \qquad \gamma(y, z) := (-\bar{z}, \bar{y}), \qquad \forall (y, z) \in \mathbb{C}^n \times \mathbb{C}^n \equiv \mathbb{R}^{4n},$$

$\bar{z} = (\bar{z}_1, ..., \bar{z}_n)$ and $\bar{z}_i$ is the conjugate of z_i. Note that ρ is of order $2m$, γ is of order 4, $\rho^m = \gamma^2$ and $\gamma\rho = \rho^{-1}\gamma$. Let us consider the homomorphism $\phi : \Gamma \to \mathbb{Z}/2$ given by $\phi(\rho) = 1$ and $\phi(\gamma) = -1$. Then $G := \ker\phi$ is the cyclic subgroup spanned by ρ. The following holds:

Proposition 6.1 *(a) genus*$(\mathbb{S}^{4n-1}/G) \geq 2n + 1$.
(b) If $m \geq 3$ *then dist*$(\gamma x, Gx) > \mu(Gx)$ *for all* $x \in \mathbb{S}^{4n-1}$.

Proof. *(a)* Let us consider the cyclic subgroup of order 4 of Γ spanned by γ and denote it by $\mathbb{Z}/4$. The kernel of the restriction of ϕ to $\mathbb{Z}/4$ is the group $K = \{1, \gamma^2\}$. Lemma 6.1, together with Theorem 1.2 of [4], yields

$$\text{genus}(\mathbb{S}^{4n-1}/K) = \mathbb{Z}/4\text{-genus}(\mathbb{S}^{4n-1}) \geq 2n + \frac{1}{2}.$$

As $K \subset G$ the quotient map $\mathbb{S}^{4n-1}/K \to \mathbb{S}^{4n-1}/G$ is well defined and is $\mathbb{Z}/2$-equivariant for the action defined in (6.1). Therefore,

$$\text{genus}(\mathbb{S}^{4n-1}/G) \geq \text{genus}(\mathbb{S}^{4n-1}/K).$$

Combining both inequalities one obtains the assertion.
(b) $\gamma x \cdot \rho^j x = 0$ and, consequently, $\left|\gamma x - \rho^j x\right| = \sqrt{2}$ for any $x \in \mathbb{S}^{4n-1}$, $j = 1, ..., 2m$. On the other hand, $\mu(Gx) = \left|e^{\pi i/m} - 1\right| < \sqrt{2}$ if $m \geq 3$. ∎

References

[1] A. Ambrosetti, M. Badiale, and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. **140** (1997), 285–300.

[2] A. Bahri and Y.Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^N$, Rev. Mat. Iberoamericana **6** (1990), 1–15.

[3] A. Bahri and P.-L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 365–413.

[4] T. Bartsch, *On the genus of representation spheres*, Comment. Math. Helv. **65** (1990), 85–95.

[5] T. Bartsch, Topological methods for variational problems with symmetries, Lecture Notes in Mathematics **1560**, Springer-Verlag, Berlin, 1993.

[6] T. Bartsch and M. Willem, *Infinitely many nonradial solutions of a Euclidean scalar field equation*, J. Funct. Anal. **117** (1993), 447–460.

[7] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rational Mech. Anal. **99** (1987), 283–300.

[8] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations I and II*, Arch. Rational Mech. Anal. **82** (1983), 313–345 and 347–375.

[9] G.E. Bredon, Introduction to compact transformation groups, Pure and Applied Mathematics **46**, Academic Press, New York - London, 1972.

[10] J.S. Carvalho, L.A. Maia, and O.H. Miyagaki, *Antisymmetric solutions for the nonlinear Schrödinger equation*, Differential Integral Equations **24** (2011), 109–134.

[11] G. Cerami, *Some nonlinear elliptic problems in unbounded domains*, Milan J. Math. **74** (2006), 47–77.

[12] G. Cerami and M. Clapp, *Sign changing solutions of semilinear elliptic problems in exterior domains*, Calc. Var. Partial Differential Equations **30** (2007), 353–367.

[13] G. Cerami, G. Devillanova, and S. Solimini, *Infinitely many bound states for some nonlinear scalar field equations*, Calc. Var. Partial Differential Equations **23** (2005), 139–168.

[14] G. Cerami, D. Passaseo, and S. Solimini, *Infinitely many positive solutions to some escalar field equations with non-symmetric coefficients*, Comm. Pure Appl. Math., to appear.

[15] S. Cingolani, M. Clapp, and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. angew. Math. Phys. **63** (2012), 233–248.

[16] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. **10** (1997), 1–13.

[17] M. Clapp and T. Weth, *Multiple solutions of nonlinear scalar field equations*, Comm. Partial Differential Equations **29** (2004), 1533–1554.

[18] M. del Pino and P.L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations **4** (1996), 121–137.

[19] T. tom Dieck, Transformation groups, De Gruyter Studies in Mathematics **8**, Walter de Gruyter, Berlin - New York, 1987.

[20] M.J. Esteban and P.-L. Lions, *Existence and nonexistence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982/83), 1–14.

[21] P.C. Fife, Mathematical aspects of reacting and diffusing systems, Lecture Notes in Biomathematics **28**, Springer-Verlag, Berlin-New York, 1979.

[22] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud. **7a**, Academic Press (1981), 369–402.

[23] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I and II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145 and 223–283.

[24] S. Lorca and P. Ubilla, *Symmetric and nonsymmetric solutions for an elliptic equation on* $\mathbb{R}^N$, Nonlinear Anal. **58** (2004), 961–968.

[25] M. Musso, F. Pacard, and J. Wei, *Finite-energy sign-changing solutions with dihedral symmetry for the stationary non linear Schrödinger equation*, Journal of European Mathematical Society, to appear.

[26] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.

[27] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[28] J. Wei, T. Weth, *Nonradial symmetric bound states for a system of coupled Schrödinger equations*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **18** (2007), 279–293.

[29] J. Wei and S. Yan, *Infinitely many positive solutions for the nonlinear Schrödinger equations in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations **37** (2010), 423–439.

[30] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications **24**, Birkhäuser, Boston, 1996.