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Multiple Non Semi-Trivial Solutions for Elliptic Systems

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Abstract

This is a continuation of our previous paper Chang, Wang, Zhang [6]. We investigate the multiple non semi-trivial solutions for nonlinear elliptic systems by two kinds of index theory. In particular the pseudo index theory for the $Z_2 \times Z_2$ index theory is developed

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1 Introduction

A solution (u_0, v_0) of a differential system:

$$\begin{cases} f(u, v, \nabla u, \nabla v, \cdots) = 0, \\ g(u, v, \nabla u, \nabla v, \cdots) = 0, \end{cases} \tag{1.1}$$

is called trivial if $(u_0, v_0) = (0, 0)$; it is called semi-trivial(STS, in short) if $(u_0, v_0) \neq (0, 0)$, but one of u_0 and v_0 is 0.

In this paper we continue our studies on the multiplicities of non semi-trivial solutions (NSTS, in short) in Chang, Wang and Zhang [6] for the elliptic system:

$$\begin{cases} -\Delta u = G_u(u, v), \\ -\Delta v = G_v(u, v), \end{cases} \tag{1.2}$$

where $G \in C^1(\mathbb{R}^1 \times \mathbb{R}^1)$.

Again we shall use the indices γ_1 and γ_2 for the symmetric groups $\mathcal{G}_1$ and $\mathcal{G}_2$ resp. introduced there as our main tools (see section 2 below).

The motivation for the study of the non semi-trivial solutions for nonlinear Schrödinger type systems can be seen in (e.g., [11, 2, 3, 1, 4, 12, 14, 15]) and more references therein. As a matter of fact, the notion of non semi-trivial solutions can be traced back to the so-called singular value problem for matrices. Given an $m \times n$ rectangle real matrix A, $\lambda \geq 0$ is called a singular value of A, if there exists $x \in \mathbb{R}^n \setminus \{0\}$ and $y \in \mathbb{R}^m \setminus \{0\}$ such that

$$Ax = \lambda y \quad \text{and} \quad A^\top y = \lambda x.$$

The notion of singular values has been extended to higher order tensors, see Lathauwer, de Moor, Vandewalle [9], Lim [10], Chang, Qi, Zhou [5], etc. The multiplicity of singular values for tensors was studied by critical point theory in Chang and Zhang [7].

The paper is organized as follows: In section 2, we briefly review the $\mathcal{G}_i$-genus theory for $i = 1, 2$, which was defined in [6]. In particular, the critical point Theorem and the Intersection Theorem are presented. In section 3 we have discussions on distinguishing non semi-trivial solutions from semi-trivial ones. In sections 4, examples on asymptotically linear elliptic systems are studied. Section 5 and 6 are devoted to applications of the pseudo index theory to $\mathcal{G}_1$ and $\mathcal{G}_2$- invariant functionals resp. It is a parallel development of the pseudo index theory for the Z_2-genus given by Ambrosetti and Rabinowitz. We apply the pseudo index theory to study the multiplicity of non semi-trivial solutions for superlinear elliptic systems.

2 $\mathcal{G}$-indices

Let E and F be Banach spaces, with norms $\|\circ\|_E$ and $\|\circ\|_F$ respectively. Let $\mathcal{G} = \mathcal{G}_i$ for $i = 1$ or 2 be the group actions defined as follows:

$\mathcal{G}_1 = \{id, g\}$, where g acts on $E \times F$ as

$$g(x, y) = (-x, -y), \quad \forall\, (x, y) \in E \times F,$$

and $\mathcal{G}_2 = \{id, g_1, g_2, g_1 g_2\}$, where $g_i, i = 1, 2$ act on $E \times F$ as

$$g_1(x, y) = (-x, y), \;\; g_2(x, y) = (x, -y) \;\; \forall\, (x, y) \in E \times F.$$

Let $\sum_i$ be the family of all closed $\mathcal{G}_i$-invariant sets, and $\mathcal{H}_i$ be the sets of all $\mathcal{G}_i$-equivariant continuous maps: $h : E \times F \to E \times F$, $i = 1, 2$.

Let P_1 be the projection: $E \times F \to E$, and $P_2 : E \times F \to F$, and let $h_i = P_i \circ h, i = 1, 2$. Then we have $\forall\, (x, y) \in E \times F \;\forall\, h \in \mathcal{H}_1$,

$$h(x,y) = -h(-x,-y),$$

and $\forall\,(x,y) \in E \times F\ \forall\, h \in \mathcal{H}_2$

$$h_1(x,y) = -h_1(-x,y) = h_1(x,-y),\ \ h_2(x,y) = -h_2(x,-y) = h_2(-x,y).$$

A $\mathcal{G}_1$-orbit consists of a pair of antipodal points, and a $\mathcal{G}_2$-orbit consists of four points $\{(x,y),(-x,y),(x,-y),(-x,-y)\}$ if $x \neq \theta$, $y \neq \theta$.

The $\mathcal{G}_i$ genus for $i = 1,2$ are defined as follows [6]. We point out that a $\mathcal{G}_2$ index theory was also defined in [16] whose definition is somewhat different from ours in [6] but has similar properties.

Definition 2.1 *The function $\gamma_{\mathcal{G}} : \sum \to \mathbb{Z}_+ \bigcup \{+\infty\}$ defined by*

$$\gamma_{\mathcal{G}_1}(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ min\{k \in \mathbb{Z} \,|\, \exists\, l \in \mathbb{Z}_+ \cup \{+\infty\},\ \exists h \in \mathcal{H}_1 \text{ with either } h : A \to (\mathbb{R}^k \backslash \{\theta\}) \\ \times(\mathbb{R}^l \backslash \{\theta\}) \text{ or } h : A \to (\mathbb{R}^l \backslash \{\theta\}) \times (\mathbb{R}^k \backslash \{\theta\})\}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$\gamma_{\mathcal{G}_2}(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ min\{k \in \mathbb{Z} \,|\, \exists\, l_1, l_2 \in \mathbb{Z}_+,\ \exists h \in H_2 \text{ with } h : A \to (\mathbb{R}^{l_1} \backslash \{\theta\}) \times (\mathbb{R}^{l_2} \backslash \{\theta\}), \\ k = l_1 + l_2 - 1\}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.2}$$

is called a $\mathcal{G}_i$ genus of $A \in \sum_i$, for $i = 1,2$.

The $\mathcal{G}$-genus possesses all basic properties in the index theory see [13], [6] and [16]:

1. $\gamma_{\mathcal{G}}(A) = 0$ if and only if $A = \emptyset$.
2. (monotonicity) $\forall A_1, A_2 \in \sum, A_1 \subset A_2$ implies $\gamma_{\mathcal{G}}(A_1) \leq \gamma_{\mathcal{G}}(A_2)$.
3. (continuity) $\forall\, A \in \sum$, if it is compact, then $\exists$ a $\mathcal{G}$-invariant open neighborhood N of A such that $\gamma_{\mathcal{G}}(A) = \gamma_{\mathcal{G}}(N)$.
4. (subadditivity) $\forall A_1, A_2 \in \sum, \gamma_{\mathcal{G}}(A_1 \bigcup A_2) \leq \gamma_{\mathcal{G}}(A_1) + \gamma_{\mathcal{G}}(A_2)$.
5. (hyper-invariance) $\gamma_{\mathcal{G}}(A) \leq \gamma_{\mathcal{G}}(\overline{h(A)})$, $\forall\,(A,h) \in \sum \times \mathcal{H}$.
6. (normality) $\gamma_{\mathcal{G}}(\{O\}) = 1$, where $O = \{\pm(x,y)\}$ for $\mathcal{G}_1$ and $O = \{(\pm x, \pm y)\}$ for $\mathcal{G}_2$, where $x \neq \theta, y \neq \theta$.

The following theorems are known [6] [16].

Theorem 2.1 $\forall\, n,m \geq 0$,

$$\gamma_{\mathcal{G}_1}(S^n \times S^m) = min\{n,m\} + 1$$

and

$$\gamma_{\mathcal{G}_2}(S^n \times S^m) = n + m + 1.$$

Theorem 2.2 *Assume $f \in C^1(E \times F, \mathbb{R}^1)$ is bounded from below and satisfies the Palais Smale (PS) condition. For any $k \geq 1$. we define*

$$c_k = inf_{\gamma_{\mathcal{G}}(A) \geq k} max_{(x,y) \in A} f(x, y)$$

where $A \in X \bigcap \sum$. Then

1. *All c_k are critical values of f.*
2. $c_k \leq c_{k+1}, \forall k = 1, 2, \cdots,$
3. *If $c = c_{k+1} = \cdots c_{k+l}$, then $\gamma_{\mathcal{G}}(K_c) \geq l$, where K_c is the critical set with critical value c.*

This is not the exact form of the Liusternik-Schnirelmann Multiplicity Theorem, because from $\gamma_{\mathcal{G}}(K_c) \geq k$, we cannot conclude the existence of at least k distinct critical $\mathcal{G}$-orbits with critical value c. In particular, we have

Theorem 2.3 *Suppose $A \subset \sum_2$ and $\gamma_2(A) < +\infty$. Then $((E \times \{\theta\}) \bigcup (\{\theta\} \times F)) \bigcap A = \emptyset$.*

Proof. Suppose $\exists (x, \theta) \in A$ for some $x \in E \backslash \{\theta\}$. $\forall h \in \mathcal{H}_2$, if $\exists l_1, l_2 \in Z_+$ such that $h : A \to (\mathbb{R}^{l_1} \backslash \{\theta\}) \times (\mathbb{R}^{l_2} \backslash \{\theta\})$. Then from $h_2(x, \theta) = -h_2(x, \theta)$ implies $h_2(x, \theta) = \theta$, which contradicts with $\gamma_2(A) < +\infty$. Similarly, we conclude: $(\{\theta\} \times F)) \bigcap A = \emptyset$.

This means that a critical orbit $K = (\pm u_0, \theta)$ or $K = (\theta, \pm v_0)$, which correspond to semi-trivial solutions for the associate Euler Lagrange equation, has infinite $\mathcal{G}_2$ genus. In fact, the multiplicity theorem holds if $((E \times \{\theta\}) \bigcup (\{\theta\} \times F)) \bigcap K_c = \emptyset \ \forall c = c_k$, i.e., under the additional condition: if there is no semi trivial solutions on the level c, then there exist at least $\gamma_{\mathcal{G}}(K_c)$ distinct $\mathcal{G}$-critical orbits in K_c.

The proof is simple. Suppose not, say $\gamma_{\mathcal{G}}(K_c) = m$ with $((E \times \{\theta\}) \bigcup (\{\theta\} \times F)) \bigcap K_c = \emptyset$, K_c contains only $l < m$ distinct $\mathcal{G}$-critical orbits. Then by the normality and the subadditivity, this is a contradiction. Namely, we have

Theorem 2.4 *Under the assumptions of Theorem 2.2, if further, for some k, we assume*

$$((E \times \{\theta\}) \bigcup (\{\theta\} \times F)) \bigcap K_{c_k} = \emptyset$$

then K_{c_k} contains at least $\gamma_{\mathcal{G}}(K_{c_k})$ distinct $\mathcal{G}$-critical orbits.

The following intersection property has been proved for $\gamma_{\mathcal{G}_1}$. In fact, it holds for both $\gamma_{\mathcal{G}_1}$ and $\gamma_{\mathcal{G}_2}$. Although the proofs are the same, we rewrite the proof for completeness.

Theorem 2.5 *Let $X = E \times F$, and let $V = V_1 \times V_2$ be a linear subspace of X with $dim(V_i) = r_i, i = 1, 2$. Assume $A \subset \sum_1$ with $\gamma_{\mathcal{G}_1}(A) > min\{r_1, r_2\}$, or $A \subset \sum_2$ with $\gamma_{\mathcal{G}_2}(A) > r_1 + r_2 - 1$ resp. Let $V^\perp = (V_1^\perp \times F) \bigcup (E \times V_2^\perp)$ be the complement subspace of V, where $V_i^\perp$ be the complement subspace of V_i in E for $i = 1$, and in F for $i = 2$. Then*

$$A \bigcap V^\perp \neq \emptyset.$$

Proof. We define projection $P : X \to V$. Then $V_i = P_i V$ and $V_i^{\perp} = P_i(1-P)X$ for $i = 1, 2$, where P_1, P_2 are defined in the beginning of this section.

Suppose $A \bigcap V^{\perp} = \emptyset$. This means

$$\begin{cases} (V_1^{\perp} \times F) \bigcap A = \emptyset, \\ (E \times V_2^{\perp}) \bigcap A = \emptyset. \end{cases} \tag{2.3}$$

Let $Q_i = P_i P$. Then $0 \notin Q_i(A), i = 1, 2$.

Define

$$h : (x, y) \mapsto \Big(\frac{Q_1 x}{||Q_1 x||}, \frac{Q_2 y}{||Q_2 y||}\Big).$$

Then h is $\mathcal{G}_1$ as well as $\mathcal{G}_2$ equivariant, with $h : A \to V \bigcap (S^E \times S^F)$, where S^E and S^F are the unit spheres of E and F respectively .

From the hypervariance, monotonicity of the $\mathcal{G}_1$-genus, and Theorem 2.1, it follows

$$\gamma_{\mathcal{G}_1}(A) \le \gamma_{\mathcal{G}_1}(\overline{h(A)}) \le \gamma_{\mathcal{G}_1}(V \bigcap (S^{V_1} \times S^{V_2})) \le min\{r, s\},$$

and from those of $\mathcal{G}_2$-genus, we have

$$\gamma_{\mathcal{G}_2}(A) \le \gamma_{\mathcal{G}_1}(V \bigcap (S^{V_1} \times S^{V_2})) \le r + s - 1.$$

These are contradictions.

3 Non semi-trivial solutions

We present here some criteria in distinguishing solutions of (1.2) from semi-trivial solutions.

3.1 A necessary condition

Define a functional on $H_0^1(\Omega) \times H_0^1(\Omega)$,

$$J(u, v) = \int_{\Omega} \{1/2[|\nabla u|^2 + |\nabla v|^2] - G(u, v)\} dx.$$

Let $w_0 = (u_0, \theta)$ be a solution of (1.2), i.e.,

$$\begin{cases} -\Delta u_0 = G_u(u_0, \theta), \\ 0 = G_v(u_0, \theta). \end{cases} \tag{3.1}$$

By the observation, we introduce the following condition on the function G:

$$\begin{cases} G_\xi(0, \eta) \ne 0, \quad \forall\, \eta \in (-\epsilon, \epsilon) \backslash \{0\}, \\ G_\eta(\xi, 0) \ne 0, \quad \forall\, \xi \in (-\epsilon, \epsilon) \backslash \{0\} \end{cases} \tag{3.2}$$

for some $\epsilon > 0$.

Lemma 3.1 *Under the assumption (3.2), the system (1.2) does not have STS except the trivial solution.*

3.2 Critical values

Assume

$$\begin{cases} G_\xi(\xi,0)\xi \geq 2G(\xi,0), & \forall\, \xi, \\ G_\eta(0,\eta)\eta \geq 2G(0,\eta), & \forall\, \eta. \end{cases} \tag{3.3}$$

If $w_0 = (u_0, \theta)$ is a solution of (1.2), then we have

$$-\Delta u_0 = G_\xi(u_0, 0).$$

Multiplying u_0 on both sides of the equation, it follows

$$\int_\Omega G(u_0,0)dx \leq \frac{1}{2}\int_\Omega G_\xi(u_0,0)u_0 dx = \frac{1}{2}\int_\Omega -\Delta u_0 \cdot u_0 dx = \frac{1}{2}\int_\Omega |\nabla u_0|^2.$$

and then

$$J(w_0) = J(u_0,\theta) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 - \int_\Omega G(u_0,0) \geq 0.$$

If $w_0 = (\theta, v_0)$ is a solution of (1.2), then by the same approach we verify: $J(w_0) = J(\theta, v_0) \geq 0$. We arrive at

Lemma 3.2 *Under the assumption (3.3), if w_0 is a STS of (1.2), then $J(w_0) \geq 0$.*

Similarly, we introduce the following duel condition on G:

$$\begin{cases} G_\xi(\xi,0)\xi \leq 2G(\xi,0), & \forall\, \xi, \\ G_\eta(0,\eta)\eta \leq 2G(0,\eta), & \forall\, \eta, \end{cases} \tag{3.4}$$

and have

Lemma 3.3 *Under the assumption (3.4), if w_0 is a STS of (1.2), then $J(w_0) \leq 0$.*

4 Examples

We have given examples in Chang, Wang, Zhang [6] on the multiple non semi-trivial solutions for the elliptic system (1.2) via the above indices. In this section we shall provide more.

Example 4.1

We present an example in which the functional is neither bounded from above nor from below. Assume the function $G \in C^1(\mathbb{R}^1 \times \mathbb{R}^1)$ satisfies the following conditions:

$$\begin{array}{ll} (G_1) & G(u,v) = G(-u,-v) \text{ for all } (u,v), \\ (G_2) & G(u,v) = \lambda uv + o(u^2+v^2) \text{ as } (u,v) \to 0, \\ (G_3) & H(u,v) := G(u,v) - auv \text{ has a bounded } C^1 \text{ norm}, \end{array} \tag{4.1}$$

where $a \in (\lambda_m, \lambda_{m+1})$, $\lambda \in (\lambda_k, \lambda_{k+1})$, $m < k$, and λ_k is the k-th eigenvalue of the Laplacian on Ω with zero boundary condition.

We study the system (1.2):

$$\begin{cases} -\Delta u = G_u(u,v), \\ -\Delta v = G_v(u,v), \end{cases} \tag{4.2}$$

and introduce the functional

$$J(u,v) = \int_\Omega (\frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - G(u,v))dx.$$

First, we verify the (PS) condition for J. Let (u_n, v_n) be a (PS) sequence. Then $J'(u_n, v_n) \to 0$. We need only show that (u_n, v_n) is bounded.

Suppose $\|u_n\| \to \infty$. After subsequence, we first assume $\|v_n\|/\|u_n\| \to b < \infty$. Let $\tilde{u}_n := \frac{u_n}{\|u_n\|}$ and $\tilde{v}_n := \frac{v_n}{\|v_n\|}$. Then we may assume $\tilde{u}_n \rightharpoonup \tilde{u}$ in E, $\tilde{u}_n \to \tilde{u}$ in $L^2(\Omega)$, and $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, $\tilde{v}_n \to \tilde{v}$ in $L^2(\Omega)$. From (G_3), we may deduce that there is a $C > 0$ such that $|G_u(u,v)| \le C(|u| + |v| + 1)$ and $|G_v(u,v)| \le C(|u| + |v| + 1)$ for all (u,v). Multiplying the first equation by $u_n/\|u_n\|^2$ and integrating it over Ω, it yields

$$\|\tilde{u}_n\|^2 \le C\|\tilde{u}_n\|^2_{L^2} + C(\|v_n\|/\|u_n\|)\|\tilde{u}_n\|_{L^2}\|\tilde{v}_n\|_{L^2} + o(1).$$

From this, we see $\tilde{u} \neq 0$.

If $b = 0$, then for any $\phi \in C_0^\infty(\Omega)$, multiplying the equation by $\phi/\|u_n\|$ and integrating it over Ω, we get, by (G_3), that

$$\int_\Omega \nabla\tilde{u}_n \cdot \nabla\phi - a\tilde{v}_n\phi\|v_n\|/\|u_n\| = o(1).$$

This gives $\int_\Omega \nabla\tilde{u} \cdot \nabla\phi = 0$. A contradiction with $\tilde{u} \neq 0$.

If $b > 0$, then again we get a contradiction as follows. Using the system we have

$$\int_\Omega \nabla\tilde{u}_n \cdot \nabla\phi - a\tilde{v}_n\phi\|v_n\|/\|u_n\| = o(1),$$

and

$$\int_\Omega \nabla\tilde{v}_n \cdot \nabla\psi - a\tilde{u}_n\psi\|u_n\|/\|v_n\| = o(1).$$

Sending $n \to \infty$ we obtain

$$\int_\Omega \nabla\tilde{u} \cdot \nabla\phi - ab\tilde{v}\phi = 0, \ \forall\phi$$

and

$$\int_\Omega \nabla\tilde{v} \cdot \nabla\psi - \frac{a}{b}\tilde{u}\psi = 0, \forall\psi.$$

Since either $w = \tilde{u} + b\tilde{v}$ or $w = \tilde{u} - b\tilde{v}$ not equal to 0 and solves $-\Delta w = \pm aw$, a contradiction with the assumption $a \in (\lambda_m, \lambda_{m+1})$. Otherwise, after a subsequence, $\|v_n\|/\|u_n\| \to \infty$, it is equivalent to $\|u_n\|/\|v_n\| \to 0$. A contradiction is obtained by interchanging u_n and v_n in the previous process. Similarly, we can show $\|v_n\|$ is bounded.

Second, we define $E_k = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$. For $(u,w) \in E_k$, we have the decomposition:

$$u = \Sigma_1^k c_i\varphi_i \quad \text{and} \quad w = \Sigma_1^k d_j\varphi_j.$$

Consider the set:

$$A^+ = \{(u, u + \epsilon w) \mid u \in S_\epsilon^{k-1}, w \in S_\epsilon^{k-1}\},$$

where $S_\epsilon^{k-1} = E_k \cap S_\epsilon$, and S_ϵ is the sphere with radius $\epsilon > 0$ centered at the origin. Obviously, A^+ is $\mathcal{G}_1$ invariant. Thus, for ϵ small enough,

$$\begin{aligned} J(u,v) &= \tfrac{1}{2}[\Sigma_1^k \lambda_i c_i^2 + \Sigma_1^k \lambda_i (c_i + \epsilon d_i)^2] - \lambda \Sigma_1^k c_i (c_i + \epsilon d_i) + o(|\Sigma_i^k (c_i^2 + \epsilon d_i^2)|) \\ &= \Sigma_1^k (\lambda_i - \lambda) c_i^2 + \epsilon \Sigma_1^k (\lambda_i - \lambda) c_i d_i + \tfrac{\epsilon^2}{2} \Sigma_1^k \lambda_i \lambda_i d_i^2 + o(|\Sigma_i^k (c_i^2 + \epsilon d_i^2)|) \\ &< 0, \quad \forall\, (u,v) \in A^+. \end{aligned} \tag{4.3}$$

For sufficiently small $\epsilon > 0$, A^+ is $\mathcal{G}_1$ homeomorphic to $S^{k-1} \times S^{k-1}$, so $\gamma_{\mathcal{G}_1}(A^+) = k$.

Since $\forall\, A \in \sum_1$, by Theorem 2.3, if $j > m$ and $\gamma_{\mathcal{G}_1}(A) \geq j$, then $A \cap (E_m \oplus E_m)^\perp$ is a nonempty set. If $(u,v) \in A \cap (E_m \oplus E_m)^\perp$, writing $(u,v) = (u_1 + u_2, v_1 + v_2)$ with $u_1 \in E_m$, $v_1 \in E_m$, and $u_2, v_2 \in E_m^\perp$ we have either $u_1 = 0$ or $v_1 = 0$. Note that (G_3) implies that there is $C > 0$ such that $G(u,v) \leq C + auv$ for all (u,v). If $u_1 = 0$ we have $J(u,v) \geq J(u_2, v_2) - C_1$ for some C_1 independent of (u,v), then

$$J(u,v) \geq inf_{E_m^\perp \oplus E_m^\perp} J(u,v) - C_1 > -\infty.$$

The case $v_1 = 0$ is similar.

Now, $\forall\, k \geq 1$, we define

$$c_k = inf_{\gamma_{\mathcal{G}_1}(A) \geq k} max_{(u,v)\in A} J(u,v).$$

Since

$$max_{(u,v)\in A^+} J(u,v) < 0,$$

we have $-\infty < c_{m+1} \leq c_{m+2} \leq \cdots \leq c_k < 0$. Consequently, by Theorem 2.2, the functional J has at least $k - m$ pairs of critical points with negative critical values, which are solutions of the above system.

According to the assumption (3.3), after Lemma 3.1, all critical values for semi trivial solutions are nonnegative, therefore those solutions we obtained by minimax principle are truly non semi-trivial. Similarly, we can handle the case when $\lambda \in (-\lambda_{k+1}, -\lambda_k)$. In summary, we have:

Theorem 4.1 *Under the assumptions $(G_1), (G_2), (G_3)$ and (3.3) (or 3.2), if $\exists\, k \in \mathbb{N}$ with $k > m$ such that $|\lambda| \in (\lambda_k, \lambda_{k+1})$, then the system (4.2) possesses at least $k - m$ distinct pairs of non semi-trivial solutions.*

Remark 4.1 The above example is a correction of Example 4 in Chang, Wang and Zhang [6].

Remark 4.2 In Theorem 4.1, if the assumption (G_2) is replaced by

$$(G_2') \;\; G(u,v) = \frac{1}{2}(\alpha u^2 + \beta v^2) + o(u^2 + v^2) \text{ as } (u,v) \to 0,$$

where $\alpha \in (\lambda_k, \lambda_{k+1})$, $\beta \in (\lambda_l, \lambda_{l+1})$. We have

Theorem 4.2 *Under the assumptions (G_1), (G_2'), (G_3) and (3.3) (or 3.2), if $min\{k,l\} > m$, then the system (1.2) possesses at least $min\{k,l\} - m$ distinct pairs of NST solutions.*

Proof. The proof is similar to that of Theorem 4.1. The only modification is to reconstruct a $\mathcal{G}_1$ invariant subset $A^+ \subset J_0$ with $\gamma_{\mathcal{G}_1}(A^+) = min\{k, l\}$. To this end, we use the subspace $E_k = span\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ and $F_l = span\{\varphi_1, \varphi_2, \cdots, \varphi_l\}$. For $(u, v) \in E_k \times F_l$, we have the decomposition:

$$u = \Sigma_1^k c_i \varphi_i, \quad v = \Sigma_1^l d_j \varphi_j.$$

Define the set:

$$A^+ = \{(u, v) \in E^k \times F^l \mid \|u\| = \|v\| = \varepsilon.\}$$

where $\| \circ \|$ is the $H_0^1(\Omega)$ norm, $\varepsilon > 0$ is small and is to be determined. Thus

$$J(u, v) = \Sigma_1^k (1 - \frac{\alpha}{\lambda_i}) c_i^2 + \Sigma_1^l (1 - \frac{\beta}{\lambda_j}) d_j^2 + o(\Sigma_1^\infty (c_q^2 + d_q^2)) < 0, \quad \forall\, (u, v) \in A^+.$$

Since A^+ is homeomorphic to $S^{k-1} \times S^{l-1}$, the desired result $\gamma_{\mathcal{G}_1}(A^+) = min\{k, l\}$ follows from Theorem 2.1.

5 Positive critical values for $\mathcal{G}_1$ invariant functionals

We have pointed out previously that those unconstraint multiple solution problems studied by $\mathcal{G}_1$ index theory can also be studied by the $\mathbb{Z}_2$ genus. In this section, we shall use the pseudo index theory for the $\mathbb{Z}_2$ genus to study $\mathcal{G}_1$ invariant functionals with positive critical values by the following example.

We study the multiple NST solution problem on $H_0^1(\Omega) \times H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$.

$$\begin{cases} -\Delta u = \mu_1 v^3 + 3\mu_2 u^2 v, \\ -\Delta v = \mu_2 u^3 + 3\mu_1 v^2 u. \end{cases} \tag{5.1}$$

The associated functional reads as:

$$J(u, v) = \int_\Omega (\frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - [\mu_1 u v^3 + \mu_2 u^3 v]) dx.$$

We assume that for $j = 1, 2, \mu_j > 0$.

1. The functional satisfies the (PS) condition.

Verification: Let $\{(u_j, v_j)\}$ be a $(PS)_c$ sequence, i.e.,

$$J(u_j, v_j) \to c,$$

and

$$\begin{cases} \int_\Omega \nabla u_j \nabla \phi - \mu_1 v_j^3 \phi - 3\mu_2 u_j^2 v_j \phi = o(\|\phi\|), \\ \int_\Omega \nabla v_j \nabla \psi - \mu_2 u_j^3 \psi - 3\mu_1 v_j^2 u_j \psi = o(\|\psi\|). \end{cases} \tag{5.2}$$

It follows

$$\int_\Omega (|\nabla u_j|^2 + |\nabla v_j|^2) - 4[\mu_1 u_j v_j^3 + \mu_2 u_j^3 v_j]) dx = o(\|u_j\| + \|v_j\|).$$

Thus

$$\int_\Omega 1/4(|\nabla u_j|^2 + |\nabla v_j|^2) = o(\|u_j\| + \|v_j\|) + c + o(1),$$

and then $||u_j||$ and $||v_j||$ are bounded, there exist subsequences

$$\begin{cases} u_j \rightharpoonup \tilde{u},\ v_j \rightharpoonup \tilde{v}, \\ u_j \to \tilde{u}\ v_j \to \tilde{v},\ (L^p(\Omega)),\ p \in (2,6), \\ u_j v_j^3 \to \tilde{u}\tilde{v}^3,\ u_j^3 v_j \to \tilde{u}^3\tilde{v},\ (L^p(\Omega)),\ p \in (1, 3/2). \end{cases} \tag{5.3}$$

By using (5.2), we obtain: $u_j \to \tilde{u}\ v_j \to \tilde{v},\ H_0^1(\Omega)$.

2. Recall the pseudo index theory for the $\mathbb{Z}_2$ genus(see [13]). Let E be a Banach space and let $f \in C^1(E, \mathbb{R}^1)$ satisfy the (PS) condition. Let

$$\Sigma^* = \{K |\ K \text{is compact and symmetric}\},$$

$$\Lambda = \{h| \text{ is an odd homeomorphism satisfying } h(B_1) \subset J^{-1}[0, \infty)\},$$

where B_1 is the unit ball centered at θ in E. Let γ be the $\mathbb{Z}_2$ genus. Define

$$\gamma^*(K) = inf_{h\in\Lambda}\gamma(K \cap h(\partial B_1)), \quad \forall\, K \in \Sigma^*,$$

$$\Gamma_n = \{K \in \Sigma^* \,|\, \gamma^*(K) \geq n\},$$

and

$$c_n^* = inf_{K\in\Gamma_n} sup_{x\in K} f(x). \tag{5.4}$$

One has the Multiplicity Theorem

Theorem 5.1 *Assume that $f \in C^1(E, \mathbb{R}^1)$ is an odd function satisfying the Palais Smale Condition. Assume $\Gamma_n \neq \emptyset$, and if c_n^* defined in (5.4) is finite, then c_n^* is a critical value of J. Moreover, if $c = c_{n+1}^* = \cdots = c_{n+k}^*$, then the critical set K_c with critical value c has $\gamma(K_c) \geq m$.*

3. The multiple solutions for J with positive critical values.

It is easily seen that $J(u, v) = J(-u, -v)$, i.e., J is an even functional. Let $E_j = span\{\varphi_1, \varphi_2. \cdots, \varphi_j\}$ where φ_k is the kth eigenfunction of $-\Delta$ on $H_0^1(\Omega)$. We define

$$\Pi_j = \{(u, u) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, u \in E_j\}.$$

Note that

$$J|_{\Pi_j} = \int_\Omega [|\nabla u|^2 - (\mu_1 + \mu_2)u^4]dx, \quad \forall\, w = (u, u) \in \Pi_j,$$

there exists $R_j > 0$ such that $J^{-1}[0, \infty) \cap \Pi_j \subset \Pi_j \cap B_{R_j}$. where B_r is the $r-$ ball in $H_0^1(\Omega) \times H_0^1(\Omega.)$.

Let

$$K_j = \Pi_j \cap B_{R_j},$$

we claim:

$$\gamma^*(K_j) \geq j.$$

Indeed, $\forall\, h \in \Lambda,\ h(B_1) \subset J^{-1}[0, \infty)$,

$$K_j \cap h(\partial B_1) \subset \Pi_j \cap h(\partial B_1) \subset K_j \cap h(\partial B_1),$$

i.e.,

$$K_j \cap h(\partial B_1) = \Pi_j \cap h(\partial B_1).$$

Therefore

$$\gamma^*(K_j) = inf_{h\in\Lambda}\gamma(\Pi_j \cap h(\partial B_1)) = j,$$

provided by Borsuk Ulam Theorem.

Since K_j are compact, c_j^* is finite. Moreover, we claim that there exists $\alpha > 0$ such that $c_j^* \geq \alpha\ \forall\, j$. In fact, from the special feature of J, there exists $\rho > 0, \alpha > 0$ such that

$$J|_{(u,v)\in\partial B_\rho} \geq \alpha.$$

In particular, we take $h_0 = \rho id$, then $h_0 \in \Lambda$. Since $\forall\, K \in \Gamma_j,\ K \cap \partial B_\rho = K \cap h_0(\partial B_1) \neq \emptyset$, therefore

$$c_j^* = inf_{K\in\Gamma_n} sup_{(u,v)\in K} J(u,v) \geq inf_{(u,v)\in\partial B_\rho} J(u,v) \geq \alpha.$$

4. NSTS

It remains to see if the solutions obtained are NST. To the special structure of the system, among all STS, there is only the trivial solution. Indeed, (3.2) is satisfied. Therefore all solutions we obtained are NSTS. There are infinitely many NSTS for the system.

Remark 5.1 Extensions:

It is easy to see that the method works for many other elliptic systems with $\mathcal{G}_1$ symmetry. For example

$$G(u,v) = F(u,v) + h(uv),$$

where

1. $F(u,v) = \mu_1|u|^{p+1} + \mu_2|v|^{q+1}, 1 < p, q < 2^* - 1, \mu_1 > 0, \mu_2 > 0$, or
2. $F(u,v) = |uv|^p uv,\ \ 0 < 2p < 2^* - 2,$

and $h \in C^1(\mathbb{R}^1)$ satisfies

1. $h(0) = 0,\ h'(0) \in (0, \lambda_1)$
2. $\exists\, M > 0$ such that $|h(t)| + |h'(t)| \leq M,\ \ \forall\, t.$

These conditions are used to ensure the (PS) condition, (3.2), superliner at (θ,θ) and at infinity.

6 γ_2- pseudo index and applications

Now, we extend the pseudo index theory for the Z_2-genus to γ_2. For a given functional $J \in C^1(E \times F, \mathbb{R}^1)$, we introduce

$$\Sigma_2^* = \{K \in \Sigma_2 \,|\, K \text{is compact}\},$$

$$\Lambda = \{h \in \mathcal{H}_2 \,|\, \text{is a homeomorphism satisfying } h(B_1^E \times B_1^F) \subset J^{-1}[0,\infty)\}$$

where B_r^E is the ball with radius r centered at the origin in the Banach space E, similarly, we use the notation B_r^F. Let

$$H^* = \{\eta \in \mathcal{H}_2 \,|\, \text{is a homeomorphism satisfying}\, \eta|_{J_0} = id|_{J_0}\}.$$

Thus, $\forall\, \eta \in H^*, \eta : J^{-1}[0,\infty) \to J^{-1}[0,\infty)$.

For example, if η is an equivariant deformation derived from a non increasing flow of the functional J, and is identity on J_0, then $\eta \in H^*$. It is easily seen: $\eta^{-1} : \Lambda \to \Lambda$, if $\eta \in H^*$.

Now, we define

$$\gamma_2^*(K) = inf_{h\in\Lambda}\, \gamma_2(K \cap h(S^E \times S^F)) \ \ \forall\, K \in \Sigma_2^*,$$

where S^E is the unit sphere centered at the origin in the Banach space E, similar notation applies to S^F.

The following basic properties for pseudo indices hold:

1. $\forall\, A \in \Sigma_2^*,\ B \in \Sigma_2,\ \eta \in H^*, \overline{A\backslash B} \in \Sigma_2^*,\ \overline{\eta(A)} \in \Sigma_2^*$.
2. (monotonicity) $\forall A_1, A_2 \in \Sigma_2^*, A_1 \subset A_2$ implies $\gamma_2^*(A_1) \le \gamma_2^*(A_2)$.
3. (subadditivity) $\forall A_1 \in \Sigma_2^*,\ A_2 \in \Sigma_2, \gamma_2^*(\overline{A_1\backslash A_2}) \ge \gamma_2^*(A_1) - \gamma_2(A_2)$.
4. (hyper-invariance) $\gamma_2^*(A) \le \gamma_2^*(\overline{\eta(A)}),\ \forall\, A \in \Sigma_2^*,\ \eta \in H^*$.

The proofs are standard, we omit them.

We denote

$$\Gamma_n^* = \{K \in \Sigma_2^* \,|\, \gamma_2^*(K) \ge n\}\ \forall\, n \in \mathbb{N},$$

and in case $\Gamma_n^* \ne \emptyset$, we define

$$c_n = inf_{K\in\Gamma_n^*}\, sup_{(u,v)\in K} J(u,v). \tag{6.1}$$

Firstly, we extend Theorem 2.1. For all $(j,k) \in \mathbb{N}\times\mathbb{N}$, let E_j, F_k be j,k dimensional linear subspaces of E and F respectively.

Lemma 6.1 *If $h \in \mathcal{H}_2$ is a homeomorphism, then $h(S^E \times S^F) \subset (E\backslash\{\theta\}) \times (F\backslash\{\theta\})$.*

Proof. We prove it by contradiction. Suppose that there exists $(u,v) \in S^E \times S^F$, such that $h(u,v) = (x,\theta)$, or $h(u,v) = (\theta,y)$ for some $(x,y) \in (E\backslash\{\theta\}) \times (F\backslash\{\theta\})$. By the $\mathcal{G}_2$ -equivariance of h, we have

$$h(u,-v) = (x,\theta),\ \text{ or } h(-u,v) = (\theta,y),\ \text{ resp.}$$

But h is a homeomorphism, this is a contradiction.

Lemma 6.2 *If $h \in \mathcal{H}_2$ is a homeomorphism, and*

$$A = (E_j \times F_k) \cap h(S^E \times S^F),$$

then $\gamma_2(A) = j + k - 1$.

Proof. 1° After Lemma 6.1,

$$i : (E_j \times F_k) \cap h(S^E \times S^F) \to (E_j\backslash\{\theta\}) \times (F_k\backslash\{\theta\})$$

is an injection. Therefore

$$\gamma_2(A) \le j + k - 1.$$

2° $\forall\,(x, y) \in S^E \times S^F$, let $Z_1(y) = h_1(S^E \times \{y\}), Z_2(x) = h_2(\{x\} \times S^F)$. They are symmetric subsets in E and F respectively. Define

$$B = (E_j \cap Z_1(y)) \times (F_k \cap Z_2(x)).$$

We claim: $\gamma_2(B) \geq j + k - 1$. In fact, suppose it is not true. then there exists a $\mathcal{G}_2$-equivariant map $\varphi = (\varphi_1, \varphi_2)$ such that

$$\varphi : B \to (\mathbb{R}^l \backslash \{\theta\}) \times (\mathbb{R}^m \backslash \{\theta\})$$

with $l + m < k + j$. Since

$$\varphi_1|_{E_j \backslash \{\theta\}} : E_j \cap Z_1(y) \to \mathbb{R}^l \backslash \{\theta\},$$

and

$$\varphi_2|_{F_k \backslash \{\theta\}} : F_k \cap Z_2(x) \to \mathbb{R}^m \backslash \{\theta\},$$

are odd mappings, according to Borsuk Ulam Theorem, we have

$$l \geq j, \;\; m \geq k.$$

and then $l + m \geq k + j$. This is a contradiction.

From, $B \subset A$, it follows

$$\gamma_2(A) \geq \gamma_2(B) \geq j + k - 1.$$

Lemma 6.3 $\forall\,(j, k) \in \mathbb{N}^2$*, if the set* $(E_j \times F_k) \cap J^{-1}[0, \infty)$ *is bounded, then* $\Gamma_n^* \neq \emptyset$*, where* $n = j + k - 1$.

Proof. Set $K = (E_j \times F_k) \cap (B_R^E \times B_R^F)$, where $R > 0$ is chosen such that

$$(E_j \times F_k) \cap J^{-1}[0, \infty) \subset K.$$

Thus $\forall\, h \in \Lambda$,

$$(E_j \times F_k) \cap h(B_1^E \times B_1^F) \subset (E_j \times F_k) \cap J^{-1}[0, \infty) \subset K,$$

and then

$$K \cap h(S^E \times S^F) = (E_j \times F_k) \cap h(S^E \times S^F).$$

According to Lemma 6.2,

$$\gamma_2^*(K) = inf_{h \in \Lambda} \gamma_2(K \cap h(S^E \times S^F)) = j + k - 1.$$

Thus, $K \in \Gamma_n^*$.

Applying the general pseudo index theory, see Rabinowitz [13], we obtain the following

Theorem 6.1 *Assume that* $J \in C^1(E \times F, \mathbb{R}^1)$ *is a* $\mathcal{G}_2$*-invariant function satisfying the Palais Smale Condition. Assume* $\Gamma_k^* \neq \emptyset$*, and if* c_k *defined in (6.1) is finite. Then* c_k *is a critical value of* J*. Moreover, if* $c = c_{k+1} = \cdots = c_{k+m}$*, then the critical set* K_c *with critical value* c *has* $\gamma_2(K_c) \geq m$.

Now we turn to an example in the applications of the $\mathcal{G}_2$ pseudo index theory. Let us assume $\alpha, \beta > 0,\ 1 < p, q < 2^* - 1$ and $G \in C^1(\mathbb{R}^1 \times \mathbb{R}^1)$ satisfying

$$\begin{array}{ll} (G_1) & G(x,y) = G(-x,y) = G(x,-y)\ \forall\,(x,y) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ (G_2) & \exists\, M > 0,\ |G(x,y)| + |\nabla G(x,y)| \leq M, \\ (G_3) & G(0,0) = 0, \\ (G_4) & G(u,v) \leq 0. \end{array} \tag{6.2}$$

We consider the functional on $(E\backslash\{\theta\}) \times (E\backslash\{\theta\})$, where $E = H^1_0(\Omega)$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain:

$$J(u,v) = \int_\Omega [1/2(|\nabla u|^2 + |\nabla v|^2) - \frac{|u|^{p+1}}{p+1} - \frac{|v|^{q+1}}{q+1} - G(u,v)]dx - \frac{\alpha}{||u||} - \frac{\beta}{||v||},$$

where $||u|| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ is the E norm. The Euler Lagrange equation reads as:

$$\begin{cases} -\Delta(1 + \frac{\alpha}{||u||^3})u = |u|^{p-1}u + G_u(u,v), \\ -\Delta(1 + \frac{\beta}{||v||^3})v = |v|^{p-1}v + G_v(u,v). \end{cases} \tag{6.3}$$

1. The functional J satisfies the Palais Smale Condition. In fact, let (u_j, v_j) be a $(PS)_c$ sequence, i.e.,

$$\begin{array}{rl} J(u_j,v_j) = & \int_\Omega [1/2(|\nabla u_j|^2 + |\nabla v_j|^2) - \frac{|u_j|^{p+1}}{p+1} - \frac{|v_j|^{q+1}}{q+1} - G(u_j,v_j)]dx \\ & -\frac{\alpha}{||u_j||} - \frac{\beta}{||v_j||} = c + o(1) \end{array} \tag{6.4}$$

and

$$\begin{cases} \int [\nabla u_j \nabla\phi - |u_j|^{p-1}u_j\phi - G_u(u_j,v_j)\phi + \frac{\alpha \nabla u_j \nabla \phi}{||u_j||^3}]dx = o(||\phi||)\ \ \forall\,\phi \in E \\ \int [\nabla v_j \nabla\psi - |v_j|^{p-1}v_j\psi - G_v(u_j,v_j)\psi + \frac{\beta \nabla v_j \nabla \psi}{||v_j||^3}]dx = o(||\psi||)\ \ \forall\,\psi \in E. \end{cases} \tag{6.5}$$

It follows that

$$\begin{cases} \int [|\nabla u_j|^2 - |u_j|^{p+1} - G_u(u_j,v_j)u_j]dx + \frac{\alpha}{||u_j||} = o(||u_j||) \\ \int [|\nabla v_j|^2 - |v_j|^{q+1} - G_v(u_j,v_j)v_j]dx + \frac{\beta}{||v_j||} = o(||v_j||). \end{cases} \tag{6.6}$$

From (6.6) and (G_2), we conclude the existence of $\delta > 0$, such that

$$||u_j|| \geq \delta, ||v_j|| \geq \delta. \tag{6.7}$$

By adding (6.4) with (6.6), we obtain

$$\begin{array}{rl} & (\frac{1}{2} - \frac{1}{p+1})||u_j||^2 \\ & +(\frac{1}{2} - \frac{1}{q+1})||v_j||^2 - \int_\Omega [G(u_j,v_j) - \frac{1}{p+1}G_u(u_j,v_j)u_j - \frac{1}{q+1}G_v(u_j,v_j)v_j]dx \\ & -(1 + \frac{1}{p+1})\frac{\alpha}{||u_j||} - (1 + \frac{1}{q+1})\frac{\beta}{||v_j||} \\ = & o(||u_j|| + ||v_j||) + c + o(1). \end{array} \tag{6.8}$$

Applying the assumption (G_2) and (6.7), we conclude the boundedness of (u_j, v_j) in $H^1_0 \times H^1_0$. Therefore

$$\begin{cases} u_j \rightharpoonup \tilde{u}, \ v_j \rightharpoonup \tilde{v}, \ (H_0^1(\Omega)), \\ (u_j, v_j) \to (\tilde{u}, \tilde{v}) \ \ (L^{p+1} \times L^{q+1}), \\ \nabla G(u_j, v_j) \to \nabla G(\tilde{u}, \tilde{v}) \ \ (L^2 \times L^2). \end{cases} \tag{6.9}$$

Also, after a subsequence, $\|u_j\| \to \xi$, $\|v_j\| \to \eta$, $\xi, \eta > 0$. Since

$$\begin{cases} (1 + \frac{\alpha}{\|u_j\|^3})u_j = (-\Delta)^{-1}(|u_j|^{p-1}u_j + G_u(u_j, v_j)), \\ (1 + \frac{\beta}{\|v_j\|^3})v_j = (-\Delta)^{-1}(|v_j|^{q-1}v_j + G_v(u_j, v_j)) \end{cases} \tag{6.10}$$

are strongly convergent in $H_0^1 \times H_0^1$, and $1 + \frac{\alpha}{\|u_j\|^3} \to 1 + \frac{\alpha}{\xi^3}$, $1 + \frac{\beta}{\|v_j\|^3} \to 1 + \frac{\beta}{\eta^3}$, we proved the strongly convergent of (u_j, v_j). The (PS) condition is verified.

2. For any finitely dimensional linear subspace $U \subset H_0^1$, $V \subset H_0^1$, it is not difficult to verify:

$$J(u, v) \to -\infty, \ \ \text{as } (u, v) \in U \times V, \ \|u\|^2 + \|v\|^2 \to \infty.$$

There exists $R > 0$ such that $(U \times V) \cap J^{-1}[0, \infty) \subset K := (U \times V) \cap (B_R \times B_R)$. Applying Lemma 6.3,

$$\gamma_2^*(K) \geq dim(U) + dim(V) - 1.$$

Thus, $\forall\, n \in \mathbb{N}, \Gamma_n^* \neq \emptyset$. Since K is compact,

$$c_n = inf_{K \in \Gamma_n^*} sup_{(u,v) \in K} J(u, v)$$

is finite.

3. Now we turn to verify: $c_n > 0\, \forall\, n \in \mathbb{N}$. Denote

$$C_r = sup_{\|u\|=1} \frac{|u|_{r+1}^{r+1}}{r+1}, 1 \leq r \leq 2^* - 1.$$

Lemma 6.4 *Assume* (G_4) *and*

$$0 < \alpha < \frac{1}{8(4C_p)^{\frac{3}{p-2}}}, \ 0 < \beta < \frac{1}{8(4C_q)^{\frac{3}{q-2}}}. \tag{6.11}$$

There exists $\rho_0 > 0$ *such that*

$$J(u, v) \geq 1/4\rho_0^2, \quad as\, \|u\| = \|v\| = \rho_0.$$

Proof. Let

$$f(t) = 1/2t^2 - C_p t^{p+1} - \alpha/t,$$

and choose

$$\rho_0 \in ((8\alpha)^{1/3}, \frac{1}{(4C_p)^{1/(p-1)}}).$$

Then

$$f(\rho_0) \geq (1/2 - C_p\rho_0^{p-1})\rho_0^2 - \alpha/\rho_0 \geq 1/4\rho_0^2 - \alpha/\rho_0 \geq 1/8\rho_0^2.$$

Applying to the functional J, the estimate follows.

Theorem 6.2 *Under the assumptions $(G_1) - (G_4)$ and (6.11), the system (6.3) possesses infinitely many NSTS with positive critical values.*

As the last example of this paper, we turn to the asymptotically linear case. In order to avoid some technical difficulties, we study ordinary differential systems. Let λ_j be the j-th eigenvalue of the differential operator $-\frac{d^2}{dt^2}$ on the interval $(0, T)$ with Dirichlet boundary condition. We make some assumptions on $G \in C^1(\mathbb{R}^1 \times \mathbb{R}^1)$:
(G_1) $G(x,y) = G(-x,y) = G(x,-y)$ $\forall\,(x,y) \in \mathbb{R}^1 \times \mathbb{R}^1$,
(G_2) $\exists R_0 > 0$, $G(x,y) = 1/2(ax^2 + by^2)$, $\quad x^2 + y^2 \geq R_0^2$,
(G_3) $\exists R_0 > r_0 > 0$, $G(x,y) = 1/2(cx^2 + dy^2)$, $\quad x^2 + y^2 \leq r_0^2$,
(G_4) $p < k, q < l$, $a \in (\lambda_k, \lambda_{k+1})$, $b \in (\lambda_l, \lambda_{l+1})$, $c \in (\lambda_p, \lambda_{p+1})$, $d \in (\lambda_q, \lambda_{q+1})$.

Denote $E = H_0^1(0,T)$. One studies the following functional on $(E\backslash\{\theta\}) \times (E\backslash\{\theta\})$:

$$J(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \int_0^T G(u,v)dt - \frac{\alpha}{\|u\|} - \frac{\beta}{\|v\|}, \tag{6.12}$$

where $\|u\|^2 = \int_0^T |\dot{u}|^2 dt$. We assume $\alpha > 0$ and $\beta > 0$.

The verification of the (PS) Condition is the same as that in Theorem 6.2.

Let $\varphi_1, \varphi_2, \varphi_3, \cdots$, be the eigenfunctions with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and let $E_j = span\{\varphi_1, \varphi_2, \cdots, \varphi_j\}$.

Lemma 6.5 *For all $i \leq k,\ j \leq l$*

$$J(u,v) \to -\infty \ \ as\, \|u\| + \|v\| \to \infty, (u,v) \in E_i \times E_j.$$

Proof. There is a constant C such that

$$J(u,v) \leq 1/2(\|u\|^2 - a|u|_2^2) + 1/2(\|v\|^2 - b|u|_2^2) + C.$$

Now,

$$\|u\|^2 \leq \lambda_k |u|_2^2,\ \ \|v\|^2 \leq \lambda_l |v|_2^2,$$

$\lambda_k < a$, $\lambda_l < b$, and on any finite dimensional space all norms are equivalent, the conclusion follows. In particular, there exist $R > 0$ such that $(E_i \times E_j) \cap J^{-1}[0,\infty) \subset (E_i \times E_j) \cap (B_R \times B_R)$.

Now we modify the definition of the $\mathcal{G}_2$ pseudo index by using

$$\Lambda^* = \{h \in \mathcal{H}_2, h(B_1 \times B_1) \subset J^{-1}[0,\infty) \cup (B_R \times B_R)\}$$

to replace Λ in the definition of γ_2^*. Again, we see

$$\eta^{-1} : \Lambda^* \to \Lambda^* \ \ \forall\,\eta \in H^*,$$

because

$$\eta^{-1} \circ h(B_1 \times B_1) \subset \eta^{-1}(J^{-1}[0,\infty) \cup ((B_R \times B_R) \cap J_0)) \subset J^{-1}[0,\infty) \cup (B_R \times B_R),$$

provided $\eta|_{J_0} = id|_{J_0}$.

To this modified $\mathcal{G}_2$ pseudo index, now, we define the set Γ_n^*, and then

$$c_n^* = inf_{K \in \Gamma_n^*} Sup_{(u,v) \in K} J(u,v).$$

The following lemma holds:

Lemma 6.6

$$\Gamma_n^* \neq \emptyset, \quad \forall\, 0 < n \le k + l - 1$$

Proof. Let $K = (E_i \times E_j) \cap (B_R \times B_R)$, we show: $\gamma_2^*(K) \ge i + j - 1,\ \forall\, i \le k, j \le l$. The proof is the same as in Lemma 6.3. It follows

$$\Gamma_{i+j-1}^* \neq \emptyset.$$

Next we show: $\exists\, \varepsilon_0 > 0$ such that $c_n^* \ge \varepsilon_0\ \forall\, n \ge p + q$. To this end let us introduce a function with 2 parameters

$$\Phi(\xi, \eta; t) = -\xi/t + \eta t^2.$$

The following lemma is easy to verify.

Lemma 6.7 *If $\eta < 0$, then*

$$max_{t\in(0,\infty)}\Phi = \Phi(\sigma) = -3/2\xi^{2/3}(-2\eta)^{1/3},$$

where $\sigma = \xi^{1/3}(-2\eta)^{-1/3}$.

If $\eta > 0$, then $\Phi'(t) > 0$, and for any $r > 0$, $max_{t\in[0,r]}\Phi = -\xi/r + \eta r^2$.

By Schwarz inequality, we have the embedding

$$|u|_\infty \le (T/2)^{1/2}\, \|u\|.$$

Thus, if $r_0 \ge (T/2)^{1/2}\, r$, then for $\|u\|^2 + \|v\|^2 \le r^2$,

$$\begin{aligned} J|_{E_p^\perp \times E_q^\perp} &\ge \Phi(\alpha, 1/2(1 - c/\lambda_{p+1}); \|u\|) + \Phi(\beta, 1/2(1 - d/\lambda_{q+1}); \|v\|), \\ J|_{E_p^\perp \times E_q} &\ge \Phi(\alpha, 1/2(1 - c/\lambda_{p+1}); \|u\|) + \Phi(\beta, 1/2(1 - d/\lambda_1); \|v\|), \\ J|_{E_p \times E_q^\perp} &\ge \Phi(\alpha, 1/2(1 - c/\lambda_1); \|u\|) + \Phi(\beta, 1/2(1 - d/\lambda_{q+1}); \|v\|). \end{aligned} \tag{6.13}$$

If there exist $\rho_0 \in (0, r_0)$ and $\varepsilon_0 > 0$ such that

$$r_0^2 \ge \tfrac{2}{T}(2\rho_0^2 + \alpha^{2/3}(\tfrac{c}{\lambda_1} - 1)^{-2/3} + \beta^{2/3}(\tfrac{d}{\lambda_1} - 1)^{-2/3}), \tag{6.14}$$

and

$$\begin{aligned} min\{&\tfrac{1}{2}(1 - \tfrac{c}{\lambda_{p+1}})\rho_0^2 - \tfrac{\beta}{\rho_0} - \tfrac{3}{2}\alpha^{2/3}(\tfrac{c}{\lambda_1} - 1)^{1/3}, \\ &\tfrac{1}{2}(1 - \tfrac{d}{\lambda_{q+1}})\rho_0^2 - \tfrac{\alpha}{\rho_0} - \tfrac{3}{2}\beta^{2/3}(\tfrac{d}{\lambda_1} - 1)^{1/3}\} \ge \varepsilon_0 \end{aligned} \tag{6.15}$$

then we may choose

$$\sigma_1 = \alpha^{1/3}(c/\lambda_1 - 1)^{-1/3},$$

$$\sigma_2 = \beta^{1/3}(d/\lambda_1 - 1)^{-1/3},$$

and obtain

$$J|_{(S_{\rho_0}^{E_p^\perp} \times S_{\rho_0}^{E_q^\perp}) \cup (S_{\rho_0}^{E_p^\perp} \times S_{\sigma_2}^{E_q}) \cup (S_{\sigma_1}^{E_p} \times S_{\rho_0}^{E_q^\perp})} \ge \varepsilon_0. \tag{6.16}$$

Now, to any set $A \in \Gamma_n^*, n \ge p + q$, after Theorem 2.5,

$$A \cap h(\partial B_1 \times \partial B_1) \cap ((E_p^\perp \times E) \cup (E \times E_q^\perp)) \neq \emptyset \ \ \forall\, h \in \Lambda^*.$$

In particular, we choose

$$h(u, v) = (\sigma_1 u_1 + \rho_0 u_2, \sigma_2 v_1 + \rho_0 v_2),$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ are orthogonal decompositions of u and v, $u_1 \in E_p, u_2 \in E_p^\perp$, and $v_1 \in E_q, v_2 \in E_q^\perp$. This means

$$A \cap (S_{\sigma_1}^{E_p} \times S_{\rho_0}^{E_q^\perp} \cup S_{\rho_0}^{E_p^\perp} \times S_{\sigma_2}^{E_q} \cup S_{\rho_0}^{E_p^\perp} \times S_{\rho_0}^{E_q^\perp}) \neq \emptyset.$$

By (6.16) we proved

$$c_n^* = inf_{A \in \Gamma_n^*} J(u, v) \geq \varepsilon_0.$$

According to Theorem 6.1 we arrive at

Theorem 6.3 *Under the assumptions* $(G_1) - (G_4)$ *and (6.14) and (6.15), the functional (6.12) possesses* $k + l - p - q - 1$ *non semi-trivial critical points of positive critical values.*

We remark that (6.14) and (6.15) are easily satisfied when we choose α, β small.

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