

Steady States, Global Existence and Blow-up for Fourth-Order Semilinear Parabolic Equations of Cahn–Hilliard Type

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Abstract

Fourth-order semilinear parabolic equations of the Cahn–Hilliard-type

$$u_t + \Delta^2 u = \gamma u \pm \Delta(|u|^{p-1}u) \quad \text{in } \Omega \times \mathbb{R}_+, \quad (0.1)$$

are considered in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with Navier-type boundary conditions on $\partial\Omega$, or $\Omega = \mathbb{R}^N$, where $p > 1$ and γ are given real parameters. The sign “+” in the “diffusion term” on the right-hand side means the stable case, while “−” reflects the unstable (blow-up) one, with the simplest, so called *limit*, canonical model for $\gamma = 0$,

$$u_t + \Delta^2 u = \pm \Delta(|u|^{p-1}u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+. \quad (0.2)$$

The following three main problems are studied:

- (i) for the unstable model (0.1), with the $-\Delta(|u|^{p-1}u)$, existence and multiplicity of classic steady states in $\Omega \subset \mathbb{R}^N$ and their global behaviour for large $\gamma > 0$;
- (ii) for the stable model (0.2), global existence of smooth solutions $u(x, t)$ in $\mathbb{R}^N \times \mathbb{R}_+$ for bounded initial data $u_0(x)$ in the subcritical case $p \leq p_* = 1 + \frac{4}{(N-2)_+}$; and
- (iii) for the unstable model (0.2), a relation between finite time blow-up and structure of regular and *singular* steady states in the supercritical range. In particular, three distinct families of Type I and II blow-up patterns are introduced in the unstable case.

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1 Introduction and motivation for main problems: steady states, global existence, and blow-up

1.1 Models and preliminaries

In this paper, we study some properties of solutions of the following *fourth-order parabolic equation of the Cahn–Hilliard (C–H) type*:

$$u_t + \Delta^2 u = \gamma u \pm \Delta(|u|^{p-1}u) \quad \text{in } \Omega \times \mathbb{R}_+, \quad \text{where } p > 1, \gamma \in \mathbb{R}, \quad (1.1)$$

with homogeneous Navier-type boundary conditions and bounded smooth initial data,

$$u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad u(x, 0) = u_0(x) \quad \text{in } \Omega. \quad (1.2)$$

We assume that Ω is a bounded domain of \mathbb{R}^N , $N \geq 1$, with smooth boundary $\partial\Omega$ of class $C^{2+\nu}$ for some $\nu \in (0, 1)$. Here, (1.1) is a *semilinear* parabolic equation with the only nonlinearity entering as a second-order diffusion-like operator. The sign “+” in the “diffusion term” on the right-hand side of (1.1) corresponds to the stable case, while “−” reflects the unstable (blow-up) one.

Firstly, we obtain existence and multiplicity results for the steady-states of the unstable CH equation (1.1) based on a combination of analytical methods. Namely, we use variational methods, such as the fibering approach, and, based on potential operators, Lusternik–Schnirel’man category–genus theory, and others, such as homotopy approaches or perturbation theory. We specifically obtain that, depending on the parameter γ , there exists a different number of stationary solutions.

Secondly, using scaling blow-up methods, global existence and uniqueness of global classical bounded solutions for the stable *Cahn–Hilliard equation* (1.1), with the $+\Delta(|u|^{p-1}u)$, in $\mathbb{R}^N \times \mathbb{R}_+$, are shown to exist up to a critical exponent $p_* = 1 + \frac{4}{N-2}$ ($\equiv \frac{N+2}{N-2} = p_S$), $N \geq 3$, or $p_* = +\infty$ for $N = 1, 2$.

Thirdly, in the last part of the paper, different types of blow-up solutions are analysed for the unstable *Cahn–Hilliard equation* (1.1), with $-\Delta(|u|^{p-1}u)$, by using the similarity profiles associated with this unstable equation. This methodology will provide us with a direct connection with the previous analysis carried out for the multiplicity of a variational problem. However, in this particular case the problem is not variational so a homotopy/perturbation analysis must be performed.

Throughout this paper, we also state and leave several open difficult mathematical problems for these nonlinear problems and other similar ones.

There is a huge amount of publications related to equations such as (1.1). Among other models, the most popular and detailed studied ones are the Cahn–Hilliard and Sivashinsky-type equations. We refer to papers [1, 50, 53] and to surveys in [31, 17], where necessary aspects of global existence and blow-up of solutions for (1.1) are discussed in sufficient detail.

Particularly, the Sivashinsky equation is analyzed in studying phase turbulence in fluids, thermal instabilities of flame fronts, the directional solidification in alloys or the interface instability during the application of industrial beam cutting techniques. As an example, let us mention an interesting result in the context of directional solidification of a dilute binary alloy that appears in Novick-Cohen–Grinfeld [52], where the steady-states of the *Sivashinsky equation*

$$u_t = \Delta(u^2 - u - \varepsilon^2 \Delta u) - \alpha u, \quad \text{with } \alpha > 0, \tag{1.3}$$

were analyzed, focusing, specifically, on the problem of multiplicity of solutions, which is also one of the main topics in this paper.

Moreover, the classic *Cahn–Hilliard equation* describes the dynamics of a pattern formation in phase transition in alloys, glasses, and polymer solutions. This equation has been extensively studied in the past years but many questions still remain unanswered. See below discussions and details about applications and characteristics of these Cahn–Hilliard equations–type.

1.2 Main results of the paper and layout

Sections 2 is devoted to preliminaries about Cahn–Hilliard equations where discussions about applications and specific analytical characteristics of these equations are carried out.

In Section 3, we study smooth stationary solutions of (1.1) via the so called *fibering method*, obtaining existence and multiplicity results for such steady states of the problem (1.1). In other words, the solutions of the parameter dependent semilinear elliptic equation

$$-\Delta^2 u + \gamma u \pm \Delta(|u|^{p-1}u) = 0 \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

which are now regarded as steady-states of the evolution equation (1.1). In particular, we obtain that, depending on the value of the parameter γ , the unstable Cahn-Hilliard equation (1.4)₋ (with the minus sign in Δ) possesses one or several solutions or no solutions at all. The results can be summarized as follows:

- If the parameter $\gamma \leq K\lambda_1$, with $K > 0$ a positive constant and $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator, i.e., $\Delta^2 \varphi_1 = \lambda_1 \varphi_1$, then there exists at least one solution for the Cahn-Hilliard equation (1.4)₋; and;

- When the parameter is greater than the first eigenvalue of the bi-harmonic equation λ_1 , multiplied by the positive constant K , then there will not be any solution at all, if one assumes only positive solutions. However, for oscillatory solutions of changing sign the number of possible solutions increases with the value of the parameter γ . In fact, when the parameter γ goes to infinity, one has an arbitrarily large number of distinct solutions.

In Section 4, we return to the original parabolic problem of the type (1.1), and now, we concentrate on the problem of global existence and uniqueness of global classical bounded solutions in the simplest canonical *limit stable Cahn–Hilliard equation*,

$$u_t = -\Delta^2 u + \Delta(|u|^{p-1}u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Since our goal is to establish sufficient conditions of *non-blow-up* of solutions at any point in the $\{x, t\}$ -space, we consider the Cauchy problem in $\mathbb{R}^N \times \mathbb{R}_+$ with smooth bounded initial data, i.e., we take $\Omega = \mathbb{R}^N$. We prove, using a standard scaling method in Nonlinear PDE theory, that global unique solutions (non-blow-up solutions in finite time) of (1.5) exist in the subcritical (in fact, Sobolev) range

$$1 < p \leq p_* = 1 + \frac{4}{N-2} \left(\equiv \frac{N+2}{N-2} = p_S \right), \quad N \geq 3 \quad (p_* = +\infty \text{ for } N = 1, 2), \tag{1.6}$$

showing the existence of uniform a priori bounds

$$|u(x, t)| \leq C \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+.$$

In addition, by the same scaling technique, we prove that the non-autonomous C–H equation with $a(x) = |x|^\alpha > 0$ for all $x \neq 0, \alpha > 0$,

$$u_t = -\Delta^2 u + \Delta(|x|^\alpha |u|^{p-1}u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N \tag{1.7}$$

does not admit a localized blow-up at the origin $x = 0$ in a larger parameter range

$$1 < p \leq p_*(\alpha) = 1 + \frac{2(\alpha+2)}{N-2}, \quad N \geq 3. \tag{1.8}$$

However, this does not prohibit a possible blow-up at some $x_0 \neq 0$, at which the range (1.6) puts in charge again. Moreover, in the supercritical range $p > p_*$, we observe that those a priori bounds cannot be obtained through the techniques used above for the parameter range (1.6). Therefore, one cannot avoid the possibility of existence of blow-up solutions in this particular supercritical range.

Concerning the related *limit unstable C–H equation*

$$u_t = -\Delta^2 u - \Delta(|u|^{p-1}u), \tag{1.9}$$

it is well known for a long period (see a blow-up survey [31]) that solutions can blow-up for any $p > 1$. Moreover, in general, there exists a countable family of various self-similar blow-up solutions, which turned out to be positive at the critical (Fujita) exponent [17]

$$p = p_0 = 1 + \frac{2}{N}. \tag{1.10}$$

Such mass-conserving solutions, which blow-up as $t \rightarrow T^- < \infty$, can admit similarity extensions beyond, i.e., for $t > T$ (see [22] for the case (1.10)), though there remain some difficult open

mathematical problems. This can be compared to Leray’s argument of 1934 for the extension for $t > T$ of self-similar blow-up solutions (as $t \rightarrow T^-$) for the Navier–Stokes equation in \mathbb{R}^3 ; see precise Leray’s statements, references, and related comments in [23, § 2.2]¹.

Thus, in other words, for the *unstable* C–H models, the critical exponent is $p_* = 1$, so that the problem of existence and nonexistence of blow-up becomes irrelevant.

In Sections 6 and 7, as a unified issue concerning studied above stationary and global solutions of (1.9), we will try to connect possible blow-up of solutions with some features of the structure of regular and *singular* steady states, which are not bounded in L^∞ . The first blow-up type solutions of (1.9) under scrutiny in these sections are the ones obtained at the critical Sobolev exponent defined as in (1.6). The main idea behind this blow-up patterns is that the blow-up can occur via some kind of “slow” motion about its stationary solutions, to be explained in detailed later on.

Furthermore, for the final type of blow-up patterns of the unstable C–H equation, we shall adapt the techniques used to obtain blow-up patterns for the nowadays classic *semilinear heat equation*

$$u_t = -\Delta u + |u|^{p-1}u,$$

in order to ascertain these final and new blow-up structures. To do so, we are required to construct special spectral theory of linear rescaled operators involved. In particular, such a spectral analysis requires *generalized Hermite polynomial eigenfunctions* of the “adjoint” linear fourth-order operator

$$\mathbf{B}^* \equiv -\Delta^2 - \frac{1}{4}y \cdot \nabla \quad \text{in } \mathbb{R}^N.$$

Indeed, our blow-up patterns, in this particular case and in the rescaled form, will be a solution of the rescaled equation

$$Y_t = \hat{\mathbf{B}}^*Y + D(Y) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,$$

where $D(Y)$ is a quadratic perturbation of the operator $\hat{\mathbf{B}}^*$ as $Y \rightarrow 0$ and

$$\hat{\mathbf{B}}^* = -\Delta^2 - c\Delta(\frac{1}{\sqrt{2}}I) - \frac{1}{4}y \cdot \nabla - \frac{1}{2(p-1)}I,$$

with c being a certain constant. As a result, we discuss *three types* of blow-up for the unstable C–H equation (1.9).

2 Preliminary discussions of Cahn–Hilliard equations

In its origins, the Cahn–Hilliard equation was proposed as a continuum model for the description of the dynamics of pattern formation in phase transition. When a binary solution is cooled sufficiently, phase separation may occur and then proceed in two ways: either nucleation, in which nuclei of the second phase appear randomly and grow, or, in the so-called spinodal decomposition, the whole solution appears to nucleate at once and then periodic or semi-periodic structures appear. Pattern formation resulting from phase transition has been observed in alloys, glasses, and polymer solutions. From the mathematical point of view, this equation involves a fourth order elliptic operator

¹Since 1996, self-similar (Type-I) blow-up for Navier–Stokes equations was ruled out (see main references in [23, § 1.1]), so an unknown and a more complicated Type-II one seems to be necessary. Fortunately, for the C–H equation (1.9), self-similar behaviour exists in both limits $t \rightarrow T^\pm$.

and it contains a negative viscosity term. The unknown function is a scalar $u = u(x, t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}_+$ and the equation reads

$$u_t - \Delta K(u) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } K(u) := -\nu \Delta u + f(u), \quad \nu > 0, \quad (2.1)$$

and the function $f(u)$ is a polynomial of the order $2p - 1$,

$$f(u) := \sum_{j=1}^{2p-1} a_j u^j, \quad p \in \mathbb{N}, \quad p \geq 2.$$

In particular, the so-called Cahn–Hilliard equation corresponds to the case $p = 2$ and

$$f(u) := -\eta u + \mu u^3, \quad \eta, \mu > 0.$$

In the case we consider the problem in an open bounded domain Ω of \mathbb{R}^N , with a smooth boundary $\Gamma := \partial\Omega$, we can suppose the following boundary conditions. Either Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = -\frac{\partial}{\partial \mathbf{n}} K(u) = 0 \quad \text{on } \Gamma,$$

where \mathbf{n} is the unit normal forward to Γ (or Dirichlet boundary conditions). Or, assuming that $\Omega = \prod_{i=1}^N (0, L_i)$, $L_i > 0$, the periodic boundary condition

$$\varphi|_{x_i=0} = \varphi|_{x_i=L_i}, \quad i = 1, \dots, N,$$

for u and the derivatives of u at least of order ≤ 3 . The problem (2.1) can be completed with the initial-value conditions

$$u(x, 0) = u_0(x), \quad x \in \Omega.$$

The weak formulation of the problem is obtained by multiplying (2.1) by a test function $\varphi \in C_0^\infty(\Omega)$, integrating in Ω and applying the formula of integration by parts,

$$\int_{\Omega} u_t \varphi + \int_{\Omega} \nu \Delta u \Delta \varphi - \int_{\Omega} f(u) \Delta \varphi = 0. \quad (2.2)$$

Integrating again by parts yields

$$\int_{\Omega} u_t \varphi + \int_{\Omega} \nu \Delta u \Delta \varphi + \int_{\Omega} f'(u) \nabla u \cdot \nabla \varphi = 0, \quad \text{where } f' := \frac{df}{du}. \quad (2.3)$$

The dynamical system (2.1) is *gradient* and admits a Lyapunov function of the form

$$E[u](t) := \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} g(u), \quad (2.4)$$

where $g(u)$ is the primitive of $f(u)$ and assuming that u is a sufficiently regular solution of the problem. Multiplying (2.1) by $K(u)$, integrating in Ω , and applying the formula of integration by parts we find that

$$\begin{aligned} \int_{\Omega} K(u) u_t &= -\nu \int_{\Omega} \Delta u \cdot u_t + \int_{\Omega} f(u) u_t = \frac{d}{dt} E[u](t), \quad \text{and} \\ - \int_{\Omega} \Delta K(u) K(u) &= - \int_{\Gamma} \frac{\partial K(u)}{\partial \nu} K(u) \, d\Gamma + \int_{\Omega} |\nabla K(u)|^2 = \int_{\Omega} |\nabla K(u)|^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt} E[u](t) + \int_{\Omega} |\nabla K(u)|^2 = 0, \tag{2.5}$$

so that the Lyapunov function (2.4) is monotone decreasing in time. It should be pointed out that these properties can be accomplished when $\Omega = \mathbb{R}^N$.

Furthermore, we would finally also like to note that when (1.1) is reduced to a standard Cahn–Hilliard equation, the *limit unstable Cahn–Hilliard equation* (1.9) was studied in [17] connecting this model with various applications. In particular, if $N = 1, 2$ and $p = 3$, it arises as the limit case of the phenomenological unstable *Cahn–Hilliard equation*

$$u_t = -(\gamma u_{xx} - u^3 + \gamma_1 u)_{xx} - \gamma_2 u.$$

It is also a reduced model from solidification theory with $N = 1$ or 2 and $p = 2$ (see [4, 49]). Equations of this form arise in the theory of thermo-capillary flows in thin layers of viscous fluids with free boundaries and an anomalous dependence of the surface tension coefficient on temperature (see [1, 20]). Also, equation (1.9) occurs passing to the limit as $\gamma \rightarrow 0^+$ in the Cahn–Hilliard equation

$$u_t = \nabla \cdot (\nabla(F(u) - \Delta u)), \tag{2.6}$$

with a standard double-well potential function of the form

$$F(u) = |u|^{p-1} u - \gamma |u|^p u, \quad \text{where } \gamma > 0.$$

Another important class of fourth-order models related to (1.9) comes from the theory of thin films and general long-wave unstable equations (see [4] and [58]), where a typical quasi-linear equation takes the form

$$u_t = -(u^n u_{xxx} + u^m u_x)_x.$$

Observe that here a linear perturbation of the case $n = 0$ is treated.

3 Fiberings method for a stationary unstable C–H equation

In this section, we study the existence and multiplicity of solutions of the following stationary unstable C–H-type equation:

$$-\Delta^2 u + \gamma u - \Delta(|u|^{p-1} u) = 0 \quad \text{in } \Omega \quad (p > 1), \tag{3.1}$$

with the Navier boundary conditions as in (1.2). We will focus on achieving such results depending on the value of the parameter γ and considering the equation in a bounded domain $\Omega \subset \mathbb{R}^N$ with Navier-type boundary conditions (1.2).

Remark: on setting in \mathbb{R}^N . It turned out that, for $\gamma < 0$, the problem can be posed in the whole of \mathbb{R}^N in a class of functions properly decaying at infinity,

$$\lim_{|x| \rightarrow \infty} u(x) = 0;$$

see a brief discussion in Section 3.5 below. However, there are specific difficulties concerning a suitable functional setting of the (3.1) in \mathbb{R}^N , so this will be done in a separate paper [2], where a very wide class of solutions (critical points of a functional) is detected.

Thus, to carry out the study of (3.1) in a bounded smooth $\Omega \subset \mathbb{R}^N$, we will use the *fibering method*, introduced by S.I. Pohozaev in the 1970s [54, 55], as a convenient generalization of previous versions by Clark and Rabinowitz [11, 56] of variational approaches, and further developed by Drábek and Pohozaev [13] and others in the 1980's. In particular, recently, it was used by Brown and collaborators [8, 9] to ascertain the existence and multiplicity of solutions for equations with a variational form (in particular p-Laplacian) associated to such equation, i.e., potential operator equations, alternatively to other methods such as bifurcation theory, critical point theory and so on.

3.1 Preliminary results for the variational analysis

Firstly, observe that (3.1) is not variational in $L^2(\Omega)$, though it is variational in $H^{-1}(\Omega)$. Thus, multiplying (3.1) by $(-\Delta)^{-1}$, we obtain a nonlocal elliptic equation with the standard zero Dirichlet boundary condition²

$$-\Delta u - \gamma(-\Delta)^{-1}u - |u|^{p-1}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.2}$$

which admits a variational setting in $L^2(\Omega)$ and, hence, the fibering method can be applied. To this end, consider the following Euler functional associated to (3.2):

$$\mathcal{F}_\gamma(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\gamma}{2} \int_\Omega |(-\Delta)^{-1/2}u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}, \tag{3.3}$$

such that the solutions of (3.2) can be obtained as critical points of the C^1 functional (3.3). Note that $(-\Delta)^{-1}$ is a positive linear integral compact operator from $L^2(\Omega)$ to itself. Then, the operator $(-\Delta)^{-1/2}$ is defined as the square root of the operator $(-\Delta)^{-1}$ and it will also be referred to as a non-local compact linear operator.

Subsequently, for the functional (3.3), the following result is well-known. Hereafter, we are assuming that $H_0^1(\Omega) = W_0^{2,1}(\Omega)$.

Lemma 3.1 *The functional (3.3) is Fréchet differentiable and its Fréchet derivative is*

$$D_u \mathcal{F}_\gamma(u)\varphi := \int_\Omega \nabla u \cdot \nabla \varphi - \gamma \int_\Omega (-\Delta)^{-1/2}u \cdot (-\Delta)^{-1/2}\varphi - \int_\Omega |u|^p \varphi, \quad \varphi \in H_0^1(\Omega).$$

Proof. Let $\mathcal{F}_\gamma(u + \varphi)$ be

$$\mathcal{F}_\gamma(u + \varphi) := \frac{1}{2} \left[\int_\Omega |\nabla(u + \varphi)|^2 - \gamma \int_\Omega |(-\Delta)^{-1/2}(u + \varphi)|^2 \right] - \frac{1}{p+1} \int_\Omega |u + \varphi|^{p+1}.$$

We split the proof between two parts. The first one obtaining the Fréchet derivative for the first two terms of the functional, denoted by the functional $\mathcal{F}_{1,\gamma}$, and the second for the non-linear part, denoted by $\mathcal{F}_{2,\gamma}$. Subsequently, operating the expressions for the first two terms of the functional

²Here, as customary, $(-\Delta)^{-1}u = v$, if

$$-\Delta v = u \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Therefore, (3.2) implies that $\Delta u = 0$ on $\partial\Omega$, so that both the Navier conditions in (1.2) hold for u .

and rearranging terms yields

$$\begin{aligned} \mathcal{F}_{1,\gamma}(u + \varphi) &= \frac{1}{2} \left[\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2 + \int_{\Omega} |\nabla \varphi|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} \varphi|^2 \right] \\ &\quad + \int_{\Omega} \nabla u \cdot \nabla \varphi - \gamma \int_{\Omega} (-\Delta)^{-1/2} u \cdot (-\Delta)^{-1/2} \varphi. \end{aligned}$$

Since, $\int_{\Omega} |\nabla \varphi|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} \varphi|^2$ vanishes quite radically

$$\left| \int_{\Omega} |\nabla \varphi|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} \varphi|^2 \right| \leq K \|\varphi\|_{H_0^1(\Omega)} = o(\|\varphi\|_{H_0^1(\Omega)}),$$

as $\|\varphi\|_{H_0^1(\Omega)}$ goes to zero and for a positive constant K , we find that

$$\left| \mathcal{F}_{1,\gamma}(u + \varphi) - \mathcal{F}_{1,\gamma}(u) - \int_{\Omega} \nabla u \cdot \nabla \varphi + \gamma \int_{\Omega} (-\Delta)^{-1/2} u \cdot (-\Delta)^{-1/2} \varphi \right| = o(\|\varphi\|_{H_0^1(\Omega)}),$$

as $\varphi \rightarrow 0$ in $H_0^1(\Omega)$.

Furthermore, for the term related to the nonlinear part (the third term in the functional (3.3)), we use the Taylor’s expansion in $\varphi = 0$, such that

$$\frac{1}{p+1} |u + \varphi|^{p+1} = \frac{1}{p+1} |u|^{p+1} + |u|^p \varphi + o(|\varphi|),$$

as $\varphi \rightarrow 0$. Therefore, since $u \in H_0^1(\Omega)$, we can conclude that

$$\left| \frac{1}{p+1} \int_{\Omega} |u + \varphi|^{p+1} - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \int_{\Omega} |u|^p \varphi \right| = o(\|\varphi\|_{H_0^1(\Omega)}),$$

when φ goes to zero in $H_0^1(\Omega)$, which completes the proof.

Consequently, we have the directional derivative (Gateaux’s derivative) of the functional (3.3) as follows:

$$\frac{d}{dt} \mathcal{F}_{\gamma}(u + t\varphi)|_{t=0} = \langle \varphi, D_u \mathcal{F}_{\gamma}(u) \rangle = D_u \mathcal{F}_{\gamma}(u) \varphi. \tag{3.4}$$

Furthermore, due to (3.4), the critical points of (3.3) are weak solutions in $H_0^1(\Omega)$ for the equation (3.2). In other words, the Fréchet derivative obtained in Lemma 3.1 of the functional (3.3) is going to be zero when u is a weak solution of (3.2), i.e.,

$$D_u \mathcal{F}_{\gamma}(u) \varphi = 0. \tag{3.5}$$

We denote critical points of the functional $\mathcal{F}_{\gamma}(u)$ (3.3) as follows:

$$C_{\gamma} := \{u \in W_0^{2,1} : D_u \mathcal{F}_{\gamma}(u) \varphi = 0\}.$$

Then, as usual, the critical points of the functional $\mathcal{F}_{\gamma}(u)$ (3.3) correspond to weak solutions of the equation (3.2) and, hence, to the stationary Cahn–Hilliard equation (3.1), i.e.,

$$\int_{\Omega} \nabla u \cdot \nabla \varphi - \gamma \int_{\Omega} (-\Delta)^{-1/2} u \cdot (-\Delta)^{-1/2} \varphi - \int_{\Omega} |u|^p \varphi = 0, \tag{3.6}$$

for any $\varphi \in W_0^{2,1}(\Omega)$ (or $C_0^\infty(\Omega)$). Thus, $u \in C_\gamma$ if and only if

$$\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2 - \int_{\Omega} |u|^{p+1} = 0. \tag{3.7}$$

By classic elliptic regularity for higher-order equations (Schauder’s theory; see [3] for further details), we will then always obtain classical solutions for such equations.

For the sake of completion, we study some of the properties of the functional $\mathcal{F}_\gamma(u)$ in (3.3). To do so, the following definitions are convenient to introduce:

Definition 3.1 Given a map $\mathcal{F} : V \rightarrow \mathbb{R}^N$, where V is a Banach space, it is *weakly (sequentially) lower semicontinuous (wls)*, if, for any weakly convergent sequence $\{u_n\}$ in V , $u_n \rightharpoonup u$, as $n \rightarrow \infty$, there holds

$$\mathcal{F}(u) \leq \liminf_{n \rightarrow \infty} \mathcal{F}(u_n).$$

Definition 3.2 Given a map $\mathcal{F} : V \rightarrow \mathbb{R}^N$, where V is a Banach space, it is *weakly semicontinuous (ws)*, if, for any weakly convergent sequence $\{u_n\}$ in V , $u_n \rightharpoonup u$, as $n \rightarrow \infty$, there holds

$$\mathcal{F}(u) = \lim_{n \rightarrow \infty} \mathcal{F}(u_n).$$

Next, the following lemma is easily proved:

Lemma 3.2 *If X is a Hilbert space, then its norm is wls.*

Proof. Since the square root function is a continuous function, we find that

$$\|u\|_X^2 \leq \liminf \|u_n\|_X^2 \quad \implies \quad \|u\|_X \leq \liminf \|u_n\|_X,$$

for any sequence $\{u_n\}$ in the space X convergent to $u \in X$. Thus, firstly, we assume that $u_n \rightharpoonup u$ in X and by definition we also have that

$$0 \leq \|u_n - u\|_X^2 = \|u_n\|_X^2 - 2 \langle u_n, u \rangle_X + \|u\|_X^2,$$

where, $\langle \cdot, \cdot \rangle_X$ represents the inner product of the Hilbert space X . Hence,

$$2 \langle u_n, u \rangle_X - \|u\|_X^2 \leq \|u_n\|_X^2. \tag{3.8}$$

Moreover, owing to the convergence of the taken sequence, we can choose a subsequence of $\|u_n\|_X^2$, convergent to $\liminf \|u_n\|_X^2$. Therefore, passing to the limit (3.8), we find that

$$\|u\|_X^2 \leq \liminf \|u_n\|_X^2,$$

which concludes the proof.

Then, assuming that $u \in W_0^{2,1}(\Omega)$ is equipped with the norm

$$\|u\|_{W_0^{2,1}(\Omega)} := \left(\int_{\Omega} |\nabla u|^2 \right)^{1/2}$$

(this is possible thanks to Poincaré’s inequality) and applying Lemma 3.2 it is clear that the functional

$$u \rightarrow \int_{\Omega} |\nabla u|^2,$$

is weakly lower semicontinuous.

Furthermore, it is also easy to prove that the second and third term of the functional (3.3) are weakly semicontinuous.

Lemma 3.3 *Suppose $u \in H_0^1(\Omega)$. Then, $\int_{\Omega} |(-\Delta)^{-1/2}u|^2$ is ws.*

Proof. As performed in the proof of Lemma 3.2, we take a convergent sequence $\{u_n\}$ in $H_0^1(\Omega)$ so that $u_n \rightharpoonup u$ for some $u \in H_0^1(\Omega)$. Then, $(-\Delta)^{-1/2}u_n := f_n$, with $\{f_n\} \subset H_0^{1/2}(\Omega)$ being equicontinuous in $H_0^{1/2}(\Omega)$. Then, by the compact imbedding of $H_0^{1/2}(\Omega)$ into $L^2(\Omega)$ and by the Ascoli–Arzelá theorem we can extract a convergent subsequence $\{f_{m_i}\}$ in $L^2(\Omega)$ so that $f_{m_i} \rightarrow f$ as $m_i \rightarrow \infty$. Moreover, since the linear operator $(-\Delta)^{-1/2}$ is compact, we find that

$$\begin{aligned} f_{m_i} \rightarrow f &\implies (-\Delta)^{-1/2}u_{m_i} \rightarrow (-\Delta)^{-1/2}u \\ &\implies \int_{\Omega} |(-\Delta)^{-1/2}u_{m_i}|^2 \rightarrow \int_{\Omega} |(-\Delta)^{-1/2}u|^2. \end{aligned}$$

This completes the proof.

Lemma 3.4 *Suppose $u \in H_0^1(\Omega) = W_0^{2,1}(\Omega)$. Then, $\int_{\Omega} |u|^{p+1}$ is ws, if $p < \frac{N+2}{N-2}$.*

Proof. As demonstrated in the proof of Lemma 3.2, we take a convergent sequence $\{u_n\}$ in $W_0^{2,1}(\Omega)$ so that $u_n \rightharpoonup u$ for some $u \in W_0^{2,1}(\Omega)$. Then, standard functional analysis tells us that the functions u_n are bounded in $W_0^{2,1}(\Omega)$ and in $L^\infty(\Omega)$ by Sobolev’s inequality when $N = 1, 2$. Therefore, $\{u_n\}$ satisfies the Ascoli–Arzelá theorem, so we can extract a convergent subsequence so that $u_{n_i} \rightarrow u$ as $n_i \rightarrow \infty$ in $L^\infty(\Omega)$. Thus,

$$u_{n_i} \rightarrow u \implies \int_{\Omega} |u_{n_i}|^{p+1} \rightarrow \int_{\Omega} |u|^{p+1}.$$

Furthermore, when $N > 2$, by Sobolev’s inequality, we have the imbedding of $W_0^{2,1}(\Omega)$ into $L^q(\Omega)$, with $q = \frac{2N}{N-2}$. Then, if $p < \frac{N+2}{N-2} (= p_S)$ the imbedding of $W_0^{2,1}(\Omega)$ into $L^{p+1}(\Omega)$ is compact and by the continuity of the Nemytskii operator $f(x, u) := |u|^{p+1}$ we can extract again a convergent subsequence that proves the weakly semicontinuity for this particular case.

Indeed, we observe that by Fatou’s Lemma and the continuity of the Nemytskii operator $f(x, u) := |u|^{p+1}$ it is possible to find a convergent subsequence $\{u_{n_i}\}$ such that, for any $p > 1$

$$\begin{aligned} f(x, u) = |u|^{p+1} &\leq \liminf_{n_i \rightarrow \infty} |u_{n_i}|^{p+1}, \quad \text{and} \\ \int_{\Omega} |u|^{p+1} &\leq \liminf_{n_i \rightarrow \infty} \int_{\Omega} |u_{n_i}|^{p+1}. \end{aligned}$$

3.2 Direct application of the fibering method

Subsequently, in order to apply the fibering method, we split the function $u \in W_0^{2,2}(\Omega)$ as follows (without loss of generality, we can suppose that $u \in W_0^{2,1}(\Omega)$):

$$u(x) = rv(x), \tag{3.9}$$

where $r \in \mathbb{R}$, such that $r \geq 0$, and $v \in W_0^{2,1}(\Omega)$, to obtain the so-called *fibering maps*

$$\begin{aligned} \phi_v &: \mathbb{R} \rightarrow \mathbb{R}, \\ r &\rightarrow \mathcal{F}_\gamma(rv). \end{aligned}$$

Substituting u from (3.9) into the functional (3.3), we have that

$$\phi_v(r) = \mathcal{F}_\gamma(rv) := \frac{r^2}{2} \int_\Omega |\nabla v|^2 - \frac{r^2\gamma}{2} \int_\Omega |(-\Delta)^{-1/2} v|^2 - \frac{r^{p+1}}{p+1} \int_\Omega |v|^{p+1}. \tag{3.10}$$

Thus, (3.10) defines the current fibering maps.

Note that, if $u \in W_0^{2,1}(\Omega)$ is a critical point of $\mathcal{F}_\gamma(u)$, then

$$D_u \mathcal{F}_\gamma(rv)v = \frac{\partial \mathcal{F}_\gamma(rv)}{\partial r} = 0.$$

In other words, $D_u \mathcal{F}_\gamma(rv)v = \langle D_u \mathcal{F}_\gamma(rv), v \rangle_{W_0^{2,1}(\Omega)}$. Here, we denote by $\langle \cdot, \cdot \rangle_{W_0^{2,1}(\Omega)}$ the inner product in the space $W_0^{2,1}(\Omega)$. Thus, the calculation of that derivative yields

$$\phi'_v(r) = r \int_\Omega |\nabla v|^2 - r\gamma \int_\Omega |(-\Delta)^{-1/2} v|^2 - r^p \int_\Omega |v|^{p+1}.$$

Moreover, since we are looking for non-trivial solutions (critical points), i.e., $u \neq 0$, we have to assume that $r \neq 0$. Hence,

$$\int_\Omega |\nabla v|^2 - \gamma \int_\Omega |(-\Delta)^{-1/2} v|^2 - r^{p-1} \int_\Omega |v|^{p+1} = 0, \tag{3.11}$$

and assuming that $\int_\Omega |v|^{p+1} \neq 0$, we finally arrive at

$$r^{p-1} = \frac{\int_\Omega |\nabla v|^2 - \gamma \int_\Omega |(-\Delta)^{-1/2} v|^2}{\int_\Omega |v|^{p+1}} > 0. \tag{3.12}$$

Now, calculating r from (3.12) (values of the scalar functional $r = r(v)$, where those critical points are reached) and substituting it into (3.10) gives the following functional:

$$\mathcal{G}_\gamma(v) = \mathcal{F}_\gamma(r(v)v) := \left(\frac{1}{2} - \frac{1}{p+1}\right) \frac{\left(\int_\Omega |\nabla v|^2 - \gamma \int_\Omega |(-\Delta)^{-1/2} v|^2\right)^{\frac{p+1}{p-1}}}{\left(\int_\Omega |v|^{p+1}\right)^{\frac{2}{p-1}}}. \tag{3.13}$$

According to Drábek–Pohozaev [13], $r = r(v)$ is well-defined and consequently the fibering map (3.10) possesses a unique point of monotonicity change in the case

$$\int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 > 0 \quad \text{and} \quad \int_{\Omega} |v|^{p+1} > 0. \tag{3.14}$$

This is explained in detail later on, when the analysis of the fibering maps is carried out.

Furthermore, thanks to [13, Lemma 3.2], we can assume that the Gateaux derivative of the functional \mathcal{G}_γ at the point $v \in W_0^{2,1}(\Omega)$ in the direction of v is zero, i.e.,

$$\langle D_v \mathcal{G}_\gamma(v), v \rangle_{W_0^{2,1}(\Omega)} = 0.$$

Therefore, assuming that v_c is a critical point of \mathcal{G}_γ , by the transformation carried out above, we have that a critical point $u_c \in W_0^{2,1}(\Omega)$, $u_c \neq 0$, of \mathcal{F}_γ is generated by v_c through the expression

$$u_c = r_c v_c,$$

with r_c defined by (3.12).

3.3 Multiplicity results

In the following, we shall provide a description of the fibering maps associated with (3.1). It will become clear that the essential nature of those fibering maps is determined by the sign of the terms $\gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2$ and $\int_{\Omega} |v|^{p+1}$. Since $\int_{\Omega} |v|^{p+1}$ is always non-negative, the different possibilities will depend on the value of the parameter γ . Moreover, the different zeros of those fibering maps will provide us with the critical points of the functional \mathcal{G}_γ in (3.13), and, hence, by construction, of the functional \mathcal{F}_γ given by (3.3). This is also supported by the category analysis of the functional \mathcal{F}_γ , (3.3).

Now, following [8, 9], we define the function $\omega_v : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\omega_v(r) = \int_{\Omega} |\nabla v|^2 - r^{p-1} \int_{\Omega} |v|^{p+1}. \tag{3.15}$$

By the definition of the fibering maps and their relations with the critical points of the functional (3.3), for $r > 0$,

$$\begin{aligned} rv \in C_\gamma \quad &\text{if and only if} \quad r \text{ is a solution of (3.15),} \\ \text{and} \quad \omega_v(r) &= \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2. \end{aligned} \tag{3.16}$$

Moreover,

$$\omega'_v(r) = -(p-1)r^{p-2} \int_{\Omega} |v|^{p+1},$$

and, hence, $\omega_v(r)$ is strictly decreasing, for any $r \geq 0$, since $\int_{\Omega} |v|^{p+1} > 0$. Also, we have that

$$\phi''_v(r) = \int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 - pr^{p-1} \int_{\Omega} |v|^{p+1}, \tag{3.17}$$

which can provide us with the convexity of the fibering maps depending on the increasing or decreasing function (3.16). Indeed, if $rv \in C_\gamma$, i.e., u is a critical point of the functional (3.3) that satisfies (3.7), by (3.11), we have that

$$\begin{aligned} \phi_v''(r) &= \int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 - r^{p-1} \int_{\Omega} |v|^{p+1} - (p-1)r^{p-1} \int_{\Omega} |v|^{p+1} \\ &= -(p-1)r^{p-1} \int_{\Omega} |v|^{p+1}, \end{aligned}$$

such that

$$r^{-1}\omega'_v(r) = \phi_v''(r),$$

and then, we can say that the fibering map ϕ_v is always concave. Furthermore, we define the function

$$\mathcal{H}_v(r) := \frac{r^2}{2} \int_{\Omega} |\nabla v|^2 - \frac{r^{p+1}}{p+1} \int_{\Omega} |v|^{p+1}.$$

Observe that, if the parameter γ is less than a certain value (to be specified in detail below), we find that the fibering map is positive, $\phi_v(r) > 0$, when $\int_{\Omega} |v|^{p+1} \geq 0$ (note that the opposite inequality $\int_{\Omega} |v|^{p+1} < 0$ is not possible) up to a critical value of r , i.e., for sufficiently small r 's. Then, the functional $\mathcal{H}_v(r)$ has a unique critical point at the value $r = r_{\max}$ such that

$$r_{\max} = \left(\frac{\int_{\Omega} |\nabla v|^2}{\int_{\Omega} |v|^{p+1}} \right)^{\frac{1}{p-1}},$$

and $\mathcal{H}_v(r)$ takes that maximum value at

$$\mathcal{H}_v(r_{\max}) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{\left(\int_{\Omega} |\nabla v|^2 \right)^{p+1}}{\left(\int_{\Omega} |v|^{p+1} \right)^2} \right)^{\frac{1}{p-1}}.$$

Note that $\mathcal{H}_v(r)$ is clearly increasing in the interval $(0, r_{\max})$, for sufficiently small r 's. Subsequently, by the Sobolev compact imbedding of $W_0^{2,1}(\Omega)$ into $L^{p+1}(\Omega)$, with $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and any $p > 1$ if $N = 1, 2$, we have that

$$\mathcal{H}_v(r_{\max}) \geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{1}{K_1} \right)^{\frac{1}{p-1}},$$

where $K_1 > 0$ is the constant of such imbedding. Besides, we obtain the following inequality:

$$\frac{r_{\max}^2}{2} \int_{\Omega} |(-\Delta)^{-1/2} v|^2 \leq \frac{K_2}{2} \left(\frac{\int_{\Omega} |\nabla v|^2}{\int_{\Omega} |v|^{p+1}} \right)^{\frac{2}{p-1}} \int_{\Omega} |\nabla v|^2 = \frac{K_2}{2} \left(\frac{\left(\int_{\Omega} |\nabla v|^2 \right)^{p+1}}{\left(\int_{\Omega} |v|^{p+1} \right)^2} \right)^{\frac{1}{p-1}}, \tag{3.18}$$

where K_2 is the corresponding constant for the imbedding of $H_0^1(\Omega)$ into $H_0^{1/2}(\Omega)$. Hence,

$$\frac{r_{\max}^2}{2} \int_{\Omega} v^2 \leq K_2 \frac{p+1}{p+3} \mathcal{H}_v(r_{\max}) = M \mathcal{H}_v(r_{\max}),$$

for some constant $M = K_2 \frac{p+1}{p+3} > 0$ independent of v . Thus,

$$\phi_v(r_{\max}) \geq \mathcal{H}_v(r_{\max}) - \gamma M \mathcal{H}_v(r_{\max}) = \mathcal{H}_v(r_{\max})(1 - \gamma M),$$

and, hence, $\phi_v(r_{\max}) > 0$ for all non-zero u if $\gamma < \frac{1}{M}$, providing a critical value of the parameter in obtaining the different possibilities for the existence and multiplicity of solutions, i.e., critical points, for the functional (3.3). Note that the constant M might be equivalently obtained using the expression of the first eigenvalue of the problem

$$\Delta^2 u = \lambda u, \tag{3.19}$$

under homogeneous Dirichlet boundary conditions, i.e.,

$$u = 0, \quad \nabla u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.20}$$

or Navier-type boundary conditions (1.2) imposed for the problem (3.1). Thus, a real number λ is called an eigenvalue and $u \in W_0^{2,2}(\Omega)$, $u \neq 0$, its corresponding eigenfunction if

$$\int_{\Omega} \Delta u \Delta \varphi = \lambda \int_{\Omega} u \varphi \quad \text{for any} \quad \varphi \in W_0^{2,2}(\Omega).$$

Indeed, let us observe that the first eigenvalue λ_1 is positive by definition after integration by parts

$$\lambda_1 := \min_{u \in W_0^{2,2}(\Omega)} \frac{\int_{\Omega} |\Delta u|^2}{\int_{\Omega} u^2} > 0.$$

and, in addition, for harmonic operators it is well known that the first eigenfunction is always positive too. In particular, for the Laplacian $(-\Delta) > 0$, this is Jentzsch’s classic theorem (1912) on the positivity of the first eigenfunction for linear integral operators with positive kernels (a predecessor of the Krein–Rutman theorem). However, for poly-harmonic operators $(-\Delta)^m$, with $m > 1$, the first eigenfunction ϕ_1 , associated with the eigenvalue λ_1 , is not always positive, or even unique for general domains under Dirichlet boundary conditions of the form (3.20). Both uniqueness and positivity are lost in annuli with very small inner radius (see [34] for further details and discussions). Therefore, as far as we know, apart from the particular case when the domain Ω is a ball the positivity of the eigenfunction for poly-harmonic operators is still an open problem, even when Ω is a smooth domain. Hence, in general the poly-harmonic operator $(-\Delta)^m$ in the unit ball, i.e., $\Omega = B_1$, is the only one with sign preserving solutions for the Dirichlet problem. In other words, the Green function of the poly-harmonic operator $(-\Delta)^m$ in the unit ball, with Dirichlet boundary conditions is known to be positive; see first results by Boggio (1901-05) [5, 6] (see also Elias [16] for more recent related general results and Grunau–Sweers [34]). Moreover, even for nice domains such as an ellipse the solutions might change sign. Only certain perturbations of the operator will preserve the sign of the solutions. Performing similar perturbations over the domain does not keep the positivity of the solutions either. Apart from partial results in the 2-dimensional case, under some restrictions (see [34]), the problem in higher dimensions still remains open as well as the situation for general higher order operators.

On the other hand, we note that the Dirichlet boundary conditions (3.20) do not allow us to write the eigenvalue problem (3.19) as a system of second order elliptic equations. For Navier boundary conditions as in (1.2), the first eigenfunction for the problem (3.19) is always positive by the Maximum Principle since in this particular case we can write the problem as a second order elliptic system.

For convenience, though the known results are not fully classified, we summarize the above discussion as follows:

Lemma 3.5 *Let λ_1 be the lowest eigenvalue of the problem (3.19), under homogeneous Dirichlet boundary conditions (3.20), or Navier-type boundary conditions (1.2) characterized as the minimum of the Rayleigh quotient,*

$$\lambda_1 := \min_{u \in W_0^{2,2}(\Omega)} \frac{\int_{\Omega} |\Delta u|^2}{\int_{\Omega} u^2} > 0. \tag{3.21}$$

Moreover, λ_1 is algebraically simple and it possesses an associated eigenfunction denoted by ψ_1 . Furthermore, for Navier-type boundary conditions (1.2), the eigenfunction ψ_1 is always strictly positive and unique (up to a multiplicative constant).

In addition, λ_1 is the unique and isolated eigenvalue of (3.19), and any other eigenvalue λ_k , with $k \geq 2$ of (3.19) satisfies $\lambda_k > \lambda_1$ (there is no eigenvalue less than λ_1 and in some right hand side reduced neighbourhood of λ_1 sufficiently small). Indeed, since the resolvent of the bi-harmonic operator Δ^2 is a compact linear operator in $W_0^{2,2}(\Omega)$ then, the spectrum is discrete, i.e., it might contain either infinitely many isolated eigenvalues or a finite number of isolated eigenvalues.

Remark 3.1 Note that when the operator is non-self-adjoint there are infinitely many eigenvalues. Moreover, when we have a non-self-adjoint operator it should be pointed out that the eigenvalues might be complex apart from the first one, which might be also positive. Then, the dominance of the first eigenvalue would be represented by

$$\text{Re } \tau > \lambda_1, \quad \text{for any other eigenvalue } \tau.$$

However, since the bi-harmonic operator Δ^2 is self-adjoint all the eigenvalues are real and the geometric multiplicity equals the algebraic multiplicity. Moreover, the resolvent of the bi-harmonic operator Δ^2 is a compact so, owing to [7, Theorem V I.8], the spectrum is discrete.

Subsequently, by the expression for the first eigenvalue λ_1 of the problem (3.19), we find an equivalent inequality to (3.18)

$$\frac{r_{\max}^2}{2} \int_{\Omega} |(-\Delta)^{-1/2} v|^2 \leq \frac{1}{2\lambda_1} \left(\frac{\int_{\Omega} |\nabla v|^2}{\int_{\Omega} |v|^{p+1}} \right)^{\frac{2}{p-1}} \int_{\Omega} |\nabla v|^2 = \frac{1}{2\lambda_1} \left(\frac{\left(\int_{\Omega} |\nabla v|^2 \right)^{p+1}}{\left(\int_{\Omega} |v|^{p+1} \right)^2} \right)^{\frac{1}{p-1}}, \tag{3.22}$$

such that

$$\frac{r_{\max}^2}{2} \int_{\Omega} |(-\Delta)^{-1/2} v|^2 \leq \frac{1}{\lambda_1} \frac{p+1}{p} \mathcal{H}_v(r_{\max}) = M_1 \mathcal{H}_v(r_{\max}),$$

with the constant $M_1 = \frac{1}{\lambda_1} \frac{p+1}{p} > 0$ independent of v but depending on the first eigenvalue λ_1 of the problem (3.19). Thus,

$$\phi_v(r_{\max}) \geq \mathcal{H}_v(r_{\max}) - \gamma M_1 \mathcal{H}_v(r_{\max}) = \mathcal{H}_v(r_{\max})(1 - \gamma M_1),$$

and, hence, $\phi_v(r_{\max}) > 0$, i.e.,

$$\mathcal{H}_v(r_{\max}) - \gamma \frac{r_{\max}^2}{2} \int_{\Omega} |(-\Delta)^{-1/2} v|^2 > 0,$$

for all non-zero u provided that $\gamma < \frac{1}{M_1} = \lambda_1 \frac{p}{p+1}$, and

$$\mathcal{H}_v(r_{\max}) > \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{1}{K_1} \right)^{\frac{1}{p-1}}.$$

These estimations for the parameter γ provide us with a critical value from which we are able to obtain the existence and multiplicity of solutions for the equation (3.2). We now discuss the different possibilities depending on the possible choices of γ , from the previous estimations. Firstly, note again that, $\int_{\Omega} |v|^{p+1} > 0$ is always positive. Hence, our analysis will focus on the value of the parameter γ .

Thus, if we assume that $\gamma < \frac{1}{M_1}$, then, since in this case

$$\mathcal{H}_v(r_{\max}) > \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{K_1}\right)^{\frac{1}{p-1}},$$

it is clear that, by the fibering method, there exists exactly one solution of (3.16). Indeed, by the definition and the analysis performed above, the fibering map $\phi_v(r)$ is a strictly increasing function for $r < r_{\max}$, and decreasing for $r > r_{\max}$. Thus, there exists a unique value of

$$r_1(v) > 0 \quad \text{such that} \quad r_1(v)v = u$$

is a critical point of the functional $\mathcal{F}_\gamma(u)$ in (3.3). Also, because $\omega'_\gamma(r_1(v)) < 0$, the unique critical point $r = r_1(v)$, that fibering map ϕ_v has, will be a local maximum, since $\phi''_v(r_1(v)) < 0$. In addition, we have that $\lim_{r \rightarrow +\infty} \phi_v = -\infty$. This kind of behaviour is shown in Figure 1.

We must point out, as we shall see below, that, after using Lusternik–Schnirel’man theory, we cannot assure that there exists a unique solution since this topological method provides us with a countable family of solutions and from the fact that the domain could be very large, the possibility of having more than one solution cannot be ruled out. Hence, to be precise, we shall say that there exists at least one solution.

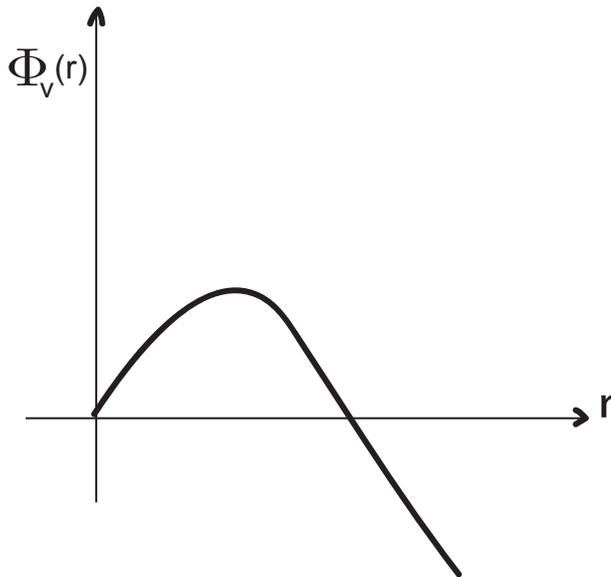


Figure 1: Profile of the fibering map for $\gamma < \frac{1}{M_1}$: a unique solution.

Moreover, if γ sufficiently large, i.e., $\gamma > \frac{1}{M_1}$, and assuming only positive solutions, there is no such critical points, since the fibering map ϕ_v is then a strictly decreasing function. However, for oscillatory solutions of changing sign, we shall show that the number of possible critical points of the functional (3.3) increases with the value of the parameter γ . Indeed, fix a value of the parameter γ bigger than $\frac{1}{M_1}$ but smaller than $\lambda_\beta - \varepsilon$, where λ_β is the β -eigenvalue of the linear bi-harmonic operator (3.19) such that

$$\psi_\beta := \sum_{|k|=\beta} c_k \hat{\psi}_k, \tag{3.23}$$

where $|\beta| > 1$, under the natural “normalizing” constraint

$$\sum_{|k|=\beta} c_k = 1.$$

Here, (3.23) represent the associated eigenfunctions to the eigenvalue λ_β and

$$\{\hat{\psi}_1, \dots, \hat{\psi}_{M_\beta}\},$$

is a basis of the eigenspace of dimension M_β . Thus, we obtain that, for a solution of the form $u = r\psi_\beta$, we will have M_β corresponding solutions similar to the one obtained in the previous case, i.e., when the parameter $\gamma < \frac{1}{M_1}$ represented by 1. Indeed, substituting $u = r\psi_\beta$ into the functional (3.3) for $\gamma = \lambda_\beta - \varepsilon$, we have

$$\mathcal{F}_\gamma(r\psi_\beta) := \frac{r^2\varepsilon}{2} \int_{\Omega} |(-\Delta)^{-1/2}\psi|^2 - \frac{r^{p+1}}{p+1} \int_{\Omega} |\psi|^{p+1},$$

and performing a similar analysis as the one done previously, we will have β -critical points (corresponding to the dimension of the eigenspace) of the above form represented by Figure 1 (see previous page). Also, we will provide details below proving that the number of solutions can even go to infinity as the parameter γ goes to infinity.

Nevertheless, to complete the problem, we add a topological analysis to the algebraic argument mentioned above. In order to estimate the number of critical points of a functional, we shall need to apply Lusternik–Schnirel’man’s (L–S) classic theory of calculus of variations. Thus, the number of critical points of the functional (3.3) will also depend on the category of the functional subset on which the fibering method is taking place.

This topological theory for potential compact operators is a natural extension of the standard minimax principles which characterize the eigenvalues of linear compact self-adjoint operators. Namely, denoting by $\lambda_1, \lambda_2, \dots$ the real eigenvalues of a self-adjoint compact operator L , ordered by their values with multiplicities, there holds:

$$\lambda_\beta = \sup_{[S^{N-1}]} \min_{v \in S^{N-1}} \langle Lv, v \rangle,$$

where S^{N-1} denotes the unit sphere in an arbitrary N -dimensional linear subspace Σ of the corresponding functional space H , and $[S^{N-1}]$ denotes the class of such spheres as Σ varies in H . Thus, applying the calculus of variations theory to an operator L , the eigenvalues of the operator L are precisely the critical values of the functional $\langle Lv, v \rangle$ on the unit ball $\partial\Sigma = \{v : \|v\| = 1\}$ of H .

The question was how/if that idea (involving eigenvalues) could be extended to more general *nonlinear potential* operators and, hence, general smooth functionals. To do so, Lusternik–Schnirel’man introduced the concept of *category* providing an estimate of the number of different

critical points of a functional on the projective spaces. However, the first problem to be faced is that one needs to find the corresponding and suitable functional subsets. Introducing the topological concept of the *genus* of a set (that we will use later on) Krasnosel’skii in the 1951 [40] (and later on studied by Borisovich in 1955; see further details in [41, p. 358]) avoided the transition to the projective spaces obtained by identifying points of the sphere which are symmetric with respect to the centre, needed to estimate the category of Lusternik–Schnirel’man. In those terms, the genus of a set provides us with a lower bound of the category. Moreover, it is clear that an estimate of the number of critical points of a functional is at the same time an estimate of the number of eigenvectors of the gradient functional (in Krasnosel’skii’s terms) and, hence, of the number of solutions of the associated nonlinear equation.

In our particular case, this functional subset is the following:

$$\mathcal{R}_{0,\gamma} = \left\{ v \in W_0^{2,1}(\Omega) : \int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 = 1 \right\}. \tag{3.24}$$

Here, “1” on the right-hand side plays no role and any positive constant would do.

According to the L–S approach (see [3, 42, 55], etc.), in order to obtain the critical points of a functional on the corresponding functional subset, $\mathcal{R}_{0,\gamma}$, one needs to estimate the category ρ of that functional subset. Thus, the category will provide us with the number of critical points that belong to the subset $\mathcal{R}_{0,\gamma}$. This depends on the value of the parameter γ . Namely, similar to [55, 29], the $\rho(\mathcal{R}_{0,\gamma})$ is given by the number of eigenvalues (with multiplicities) of the corresponding linear eigenvalue problem satisfying:

$$\rho(\mathcal{R}_{0,\gamma}) = \#\{v_{\beta} < 1\}, \quad \text{where} \tag{3.25}$$

$$-\Delta \psi_{\beta} - \gamma(-\Delta)^{-1} \psi_{\beta} = v_{\beta} \psi_{\beta} \quad \text{in } \Omega, \quad \psi_{\beta} = 0 \quad \text{on } \partial\Omega \tag{3.26}$$

(recall that the Navier condition $\Delta \psi = 0$ is then valid automatically). Thus, by studying the eigenvalue problem (3.26), a sharp estimate of the category (3.25) gets not that straightforward and easy, so we will need some extra analysis via embeddings of the corresponding functional spaces involved. However, some preliminary important conclusions from (3.26) are indeed, possible. For instance, for $\gamma > 0$ sufficiently large (in particular, $\gamma > \frac{1}{M_1}$), having on the right-hand side of (3.26) special operators of different signs,

$$-\Delta > 0 \quad \text{and} \quad -\gamma(-\Delta)^{-1} < 0 \quad (\gamma > 0)$$

(and the second one is “weaker” in the sense of compact embeddings), we have:

$$\boxed{\rho(\mathcal{R}_{0,\gamma}) \rightarrow +\infty \quad \text{as} \quad \gamma \rightarrow +\infty.} \tag{3.27}$$

Since $\rho(\mathcal{R}_{0,\gamma})$ measures, at least, a lower bound of the total number of (L–S) solutions, (3.27) clearly proves that an arbitrarily large number of various solutions can be achieved by enlarging the parameter $\gamma \gg 1$. Therefore, by [55, 29, 30] and as mentioned above, if we look for critical points of the functional $\mathcal{G}_{\gamma}(v)$ (3.13) on the set $\mathcal{R}_{0,\gamma}$ it will be necessary to estimate the category ρ of that set $\mathcal{R}_{0,\gamma}$. The critical values c_{β} and the corresponding critical points $\{v_{\beta}\}$ are:

$$c_{\beta} := \inf_{\mathcal{A} \in \mathcal{A}_{\beta}} \sup_{v \in \mathcal{A}} \mathcal{G}_{\gamma}(v) \quad (\beta = 1, 2, 3, \dots), \tag{3.28}$$

where $\mathcal{G}_\gamma(v)$ is the functional defined by (3.13) and

$$\mathcal{A}_\beta := \{\mathcal{A} : \mathcal{A} \subset \mathcal{R}_{0,\gamma}, \text{ compact subsets, } \mathcal{A} = -\mathcal{A} \text{ and } \rho(\mathcal{A}) \geq \beta\},$$

is the class of closed sets in $\mathcal{R}_{0,\gamma}$ such that, each member of \mathcal{A}_β is of genus (or category) at least β in $\mathcal{R}_{0,\gamma}$. The fact that $\mathcal{A} = -\mathcal{A}$ comes from the definition of genus (Krasnosel'skii [41, p. 358]) such that, if we denote by \mathcal{A}^* the set disposed symmetrically to the set \mathcal{A} ,

$$\mathcal{A}^* = \{v : v^* = -v \in \mathcal{A}\},$$

then, $\rho(\mathcal{A}) = 1$ when each simply connected component of the set $\mathcal{A} \cup \mathcal{A}^*$ contains neither of the pair of symmetric points v and $-v$. Furthermore, $\rho(\mathcal{A}) = \beta$ if each subset of \mathcal{A} can be covered by, a minimum, β sets of genus one, and without the possibility of being covered by $\beta - 1$ sets of genus one.

In particular, we know that the genus of an N -dimensional sphere is equal to $N + 1$. Moreover, it is known that applying an odd continuous transformation \mathbf{B} , that we define as admissible, we find that

$$\mathbf{B}(-v) = -\mathbf{B}v, \quad \rho(\mathbf{B}\mathcal{A}) \geq \rho(\mathcal{A}).$$

Hence, assuming the class of compact sets \mathcal{A}_β as subsets of the form $\mathbf{B}S^{\beta-1} \subset \mathcal{R}_{\gamma,0}$, with $S^{\beta-1}$ representing a suitable sufficiently smooth $(\beta - 1)$ -dimensional manifold (for example, the sphere) then, we can assure that in the class \mathcal{A}_β can occur sets of genus not less than β ,

$$\rho(\mathcal{A}_\beta) \geq \beta = \rho(S^{\beta-1}),$$

because $\mathcal{A}_\beta \subset \mathbf{B}S^{\beta-1}$. One cannot forget that there can also be other sets, on a different class, of genus β . As a consequence, and by definition, we find that

$$c_1 \leq c_2 \leq \dots \leq c_{l_{0,\gamma}},$$

with $l_{0,\gamma} = l_{0,\gamma}(\mathcal{R}_{0,\gamma})$ standing for the category of $\mathcal{R}_{\gamma,0}$. Indeed, taking $\varepsilon > 0$, by definition of the critical values $c_{\beta+1}$, we have that a set $A_1 \in \mathcal{A}_{\beta+1}$ exists, such that

$$\sup_{v \in A_1} \mathcal{G}_\gamma(v) < c_{\beta+1} + \varepsilon.$$

Hence, if A_1 contains a subset $A_0 \in \mathcal{A}_\beta$ such that

$$\sup_{v \in A_0} \mathcal{G}_\gamma(v) \leq \sup_{v \in A_1} \mathcal{G}_\gamma(v) < c_{\beta+1} + \varepsilon, \quad \text{and}$$

$$c_\beta = \inf_{\mathcal{A} \in \mathcal{A}_\beta} \sup_{v \in \mathcal{A}} \mathcal{G}_\gamma(v) \leq \sup_{v \in A_1} \mathcal{G}_\gamma(v) < c_{\beta+1} + \varepsilon,$$

then,

$$c_\beta < c_{\beta+1}.$$

Roughly speaking, since the dimension of the sets \mathcal{A} belonging to the classes of sets \mathcal{A}_β increases with β , this guarantees that the critical points delivering critical values (3.28) are all different.

By the analysis carried out above via the fibering method to obtain an algebraic estimate of the number of critical points for the functional (3.3), it follows that the category $l_{0,\gamma} = l_{0,\gamma}(\mathcal{R}_{0,\gamma})$ of the set $\mathcal{R}_{\gamma,0}$ is equal to the number of eigenvalues v_β of the linear operator corresponding to the

linear eigenvalue problem (3.26) depending on the relation of the eigenvalues v_β with respect to the parameter γ .

Moreover, if the quadratic form

$$\int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2, \quad \text{with } v \in W_0^{2,1}$$

has Morse index $q > 0$ and, since, we know that the functional (3.3) is lower semicontinuous and coercive (proved below, in the existence section), then the equation (3.2) will have at least q distinct solutions [3, Theorem 6.7.9]. Note that, the Morse index will be precisely the dimension of the space where the corresponding form is negatively definite. This includes all the multiplicities of the eigenfunctions involved in the corresponding subspace providing a different approach for the multiplicity of solutions.

Furthermore, recall that the L–S variational aspects of construction of critical points can be closely related to structure of the “essential zeros and extrema” of the basic patterns $\{u_l\}$ of the equation (3.1). Indeed, in the standard elliptic second-order case (i.e., with no nonlocal term), in the simplest *radial* N -dimensional case, it is well known that, by Sturm’s Theorem, each solution $u_l(r)$, with $r = |x| \geq 0$ corresponding to the genus $l \geq 1$ has precisely $l - 1$ zeros (sign changes) or l isolated local extrema points. Here, we will try to derive some related results for fourth-order elliptic equations, though it cannot be done in such an impressive and rigorous manner as for the second-order equations enjoying strong Maximum Principle features.

Now, before computing the number of solutions in terms of the genus of the set $\mathcal{R}_{0,\gamma}$, let us calculate the corresponding critical value c_β of the functional (3.13) on $\mathcal{R}_{0,\gamma}$. Thus, we have that, for a given solution of (3.2) (critical point of (3.3))

$$v = Cu \in \mathcal{R}_{0,\gamma},$$

the following holds:

$$C = \frac{1}{\sqrt{\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2}}.$$

Hence, the corresponding critical value is as follows:

$$c_u = \mathcal{G}_\gamma(v) = \left(\frac{1}{2} - \frac{1}{p-1}\right) \frac{\left(\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2\right)^{p-1}}{\left(\int_{\Omega} |v|^{p+1}\right)^{\frac{2}{p-1}}}. \tag{3.29}$$

Therefore, we arrive at the following possibilities:

Genus one. For the parameter $\gamma \leq \frac{1}{M_1}$, we have previously proved that there exists at least one solution. Hence, the *genus* will be one. Indeed, taking a critical point denoted by u_1 of the functional (3.3), under the variational assumptions established for the fibering method, we have that

$$u_1 = r(v_1)v_1, \tag{3.30}$$

such that stands for the critical point of the functional (3.13). In other words, v_1 is the function in which the subsequent infimum is achieved,

$$\inf \mathcal{G}_\gamma(v) \equiv \inf \left(\frac{1}{2} - \frac{1}{p-1}\right) \frac{1}{\left(\int_{\Omega} |v|^{p+1}\right)^{\frac{2}{p-1}}}, \quad \text{with } v_1 \in \mathcal{R}_{0,\gamma}. \tag{3.31}$$

Indeed, assuming that the domain Ω is large enough, let us take a two hump structure (as done in [29])

$$\hat{v}(x) = C[v_1(x) + v_1(x + a)], \quad C \in \mathbb{R},$$

with sufficiently large $|a|$. If necessary, we also perform a slight modification of $\hat{v}(x)$ near the boundary to satisfy the boundary conditions.

Thus, since \hat{v} belongs to $\mathcal{R}_{0,\gamma}$ and, hence, $C = \frac{1}{\sqrt{2}}$ (more precisely, $C \approx \frac{1}{\sqrt{2}}$), we have

$$c_{\hat{v}} = \mathcal{G}_\gamma(\hat{v}) = 2\mathcal{G}_\gamma(v_1) > \mathcal{G}_\gamma(v_1) = c_{v_1} \equiv c_1.$$

Observe that, since $\gamma \leq \frac{1}{M_1}$, we find that

$$\left(\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2 \right) > 0.$$

Thus, for any $\hat{v} = Mv_1$ with $v_1 \in \mathcal{R}_{0,\gamma}$, such that we have that

$$\left(\int_{\Omega} |\nabla \hat{v}|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} \hat{v}|^2 \right) = M^2 > 1,$$

and, hence,

$$\mathcal{G}_\gamma(\hat{v}) > \mathcal{G}_\gamma(v_1),$$

meaning that, in the present case, a two-hump structure cannot be a L–S one.

Genus greater than two. In this particular case, we have proved that the functional (3.3) has at least two solutions once it is assumed that $\gamma > \frac{1}{M_1}$. Indeed, by that condition over the parameter γ one can say that the functional (3.13) is between at least two values, a maximum and a minimum one,

$$c_2 \leq \mathcal{G}_\gamma(v) \leq c_2^*.$$

Therefore, we will obtain at least two positive critical points for such a functional since the L–S characterization provides us with a lower bound for solutions but not exactly how many are obtained, which confirms our previous algebraic results. It should be pointed out that the situation in which there are infinitely many critical points is not ruled out.

To summarize, we state the following result:

Lemma 3.6 *The following possibilities for the number of critical points for the functional $\mathcal{F}_\gamma(u)$ (3.3) hold:*

- (i) *The elliptic problem (3.1) and, hence, (3.2), admits an arbitrarily large number of different solutions $u \in W_0^{2,2}(\Omega)$, provided that $\gamma > \frac{1}{M_1}$ such that $M_1 = \frac{1}{\lambda_1} \frac{p+1}{p} > 0$, for sufficiently large $\gamma \gg 1$.*
- (ii) *Moreover, if $\gamma > \frac{1}{M_1}$ and we consider only positive solutions, i.e., positive critical points of the functional (3.3), then, there will be no solution.*
- (iii) *And, finally, if $\gamma \leq \frac{1}{M_1}$, there exists only one critical point $r_1(v_1)v_1 = u_1$ for the functional $\mathcal{F}_\gamma(u)$ that will be a local maximum.*

Each solution is obtained as a critical point of the functional (3.3) in $W_0^{2,2}(\Omega)$.

Remark on γ -bifurcation branches: reviving the total number of solutions. It is worth mentioning that those values of the parameter γ can be used to represent a family of nontrivial solutions bifurcating from the branch of trivial solutions $(0, \gamma)$. This fact can provide us with interesting information about bifurcation near an eigenvalue of higher multiplicity. In fact, we can expect that, for multiplicity M , we still have at least M distinct one-parameter families emanating from $(0, \gamma)$. Indeed, one can see from (3.1) that those bifurcation values γ_β , numerated by a multiindex β in \mathbb{R}^N , coincide with the eigenvalues of the bi-harmonic operator in (3.19) (this time, with the original Navier boundary conditions). Then classic bifurcation-variational theory [3, 42] suggests that each such γ_β is indeed a bifurcation point from zero, and each such γ -branch (or a finite number of branches in the multiple cases) can be extended to $\gamma > \gamma_\beta$. This again gives us a precise estimate of a number of various solutions for large values of γ . Of course, this revives the same conclusions obtained earlier by the L–S analysis and the fibering method.

Indeed, using Morse index theory, available since the functional is coercive and lower semicontinuous, we know that, every time the Morse index changes, there is a bifurcation point. Since the Morse index is precisely the dimension of the space where the corresponding functional is negatively definite, including all the multiplicities of the eigenfunctions, we will obtain a bifurcation point for every eigenvalue of the bi-harmonic operator (3.19). Also, if the Morse index is infinite there will be an infinite number of bifurcation points, in clear concordance with (3.27). This analysis can provide us with a different approach in obtaining the multiplicity of the solutions for the functional (3.3).

Later on, we will discuss some bifurcation ideas in a more complicated problem, associated with non-potential operators.

3.4 Existence of solutions

Firstly, we prove that the functional (3.3) is coercive, that is crucial in obtaining the existence of solutions for the functional (3.3). Thanks to the weakly lower semicontinuity of the first two terms of the functional (3.3), we have that

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\gamma}{2} \int_{\Omega} |(-\Delta)^{-1/2} u|^2 \geq K \|u\|_{W_0^{2,1}(\Omega)}$$

for any $u \in W_0^{2,1}(\Omega)$. Note that, if $u \in C_\gamma$, then

$$\mathcal{F}_\gamma(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\int_{\Omega} |\Delta u|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u|^2 \right). \tag{3.32}$$

Therefore, by the weak lower semicontinuity via Lemma 3.2, we conclude that the functional (3.3) is coercive and bounded below, with $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and any $p > 1$ if $N = 1, 2$.

Consequently, due to the multiplicity results described in Lemma 3.6, the following theorem summarizes the existence of non-zero solutions for the functional (3.3):

Theorem 3.1 *For any $p > 1$, the existence of solutions for the boundary value problem (3.1) is as follows:*

- If $\gamma \leq \frac{1}{M_1}$, with $M_1 = \frac{1}{\lambda_1} \frac{p+1}{p} > 0$, then there exists at least one solution;
- If $\gamma > \frac{1}{M_1}$, then the elliptic problem (3.1) and, hence, (3.2), admits an arbitrarily large number of different solutions $u \in W_0^{2,2}(\Omega)$ provided that $\gamma \gg 1$. In particular, there exists no positive solutions of (3.1).

Remark 3.2 All the solutions obtained for the boundary value problem (3.1) are classical solutions by elliptic regularity for higher-order equations (see Schauder’s theory in [3] for further details).

Proof. Owing to the coercivity of the functional (3.3) and because it is also bounded below, and to the weak lower semicontinuity, there exists a maximizing sequence $\{u_n\}$ in $W_0^{2,1}(\Omega)$ for the functional (3.3) such that

$$\lim_{n \rightarrow \infty} \mathcal{F}_\gamma(u_n) = \sup_{u \in C_\gamma} \mathcal{F}_\gamma(u) > 0,$$

and, hence, $\{u_n\}$ is bounded in $W_0^{2,1}(\Omega)$. Then, by standard arguments, we can extract a convergent subsequence, denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u_1$ as $n \rightarrow \infty$ for $u_1 \in W_0^{2,1}(\Omega)$. In fact, such a convergence is strong in $W_0^{2,1}(\Omega)$.

To this end, we argue by contradiction using the fibering maps and the discussion made previously about the number of critical points for such functions. Hence, we arrive at the following situations, either there will be no positive solution if $\gamma > \frac{1}{M_1}$ (the fibering map is strictly decreasing so there will not be any critical point for the functional (3.3)), or there only exists a classical solution if $\gamma \leq \frac{1}{M_1}$, with $M_1 = \frac{1}{\lambda_1} \frac{p+1}{p} > 0$, or there are an arbitrarily large number of different solutions $u \in W_0^{2,2}(\Omega)$ if $\gamma > \frac{1}{M_1}$.

Namely, suppose that the strong convergence does not take place. Therefore, since $u_n \in C_\gamma$ and by the structure of the functional (3.3), we have that

$$\mathcal{F}_\gamma(u_n) := \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |u_n|^{p+1}. \tag{3.33}$$

By Lemma 3.6, we know that there exists a maximum if $\gamma < \frac{1}{M_1}$, where

$$\int_\Omega |u|^{p+1} > 0 \quad \text{for } u \in W_0^{2,1}(\Omega).$$

Hence, passing in (3.33) to the limit as $n \rightarrow \infty$ and using the weak semicontinuity of the third term of the functional (3.3), we find that, actually, $\int_\Omega |u_1|^{p+1} > 0$ and, consequently,

$$\mathcal{F}_\gamma(u_1) = \lim_{n \rightarrow \infty} \mathcal{F}_\gamma(u_n) = \sup_{u \in C_\gamma} \mathcal{F}_\gamma(u),$$

contradicting the nonexistence of a strong convergence in $W_0^{2,1}(\Omega)$.

On the other hand, now, again by the coercivity of the functional (3.3) and the weak lower semicontinuity, there exists a minimizing sequence $\{u_n\}$ in $W_0^{2,1}(\Omega)$ for the functional (3.3) so that

$$\lim_{n \rightarrow \infty} \mathcal{F}_\gamma(u_n) = \inf_{u \in C_\gamma} \mathcal{F}_\gamma(u).$$

Hence, by standard arguments, we can extract a convergent subsequence, denoted again by $\{u_n\}$, so that $u_n \rightharpoonup u_1$ in $W_0^{2,1}(\Omega)$, to a certain $u_1 \in W_0^{2,1}(\Omega)$.

Therefore, when $\gamma < \frac{1}{M_1}$, we have that there exists r_1 such that $\mathcal{F}_\gamma(r_1(v)) < 0$. Hence,

$$\inf_{u \in C_\gamma} \mathcal{F}_\gamma(u) < 0, \tag{3.34}$$

since the fibering map is decreasing in an interval around the value r_1 , which the minimum is achieved at. Indeed, we can write the functional $\mathcal{F}_\gamma(u)$ in (3.3) for any $u_n \in C_\gamma$

$$\mathcal{F}_\gamma(u_n) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\int_{\Omega} |\nabla u_n|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_n|^2 \right),$$

and then

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_n|^2 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\Omega} |\nabla u_n|^2 - \mathcal{F}_\gamma(u_n).$$

So, passing to the limit as $n \rightarrow \infty$, we arrive at $\gamma \int_{\Omega} |(-\Delta)^{-1/2} u_1|^2 > 0$. Moreover, since we are assuming that the convergence of $\{u_n\}$ is not strong in $W_0^{2,1}(\Omega)$, we have that

$$\int_{\Omega} |\nabla u_1|^2 < \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2. \tag{3.35}$$

Note that the sign “ \leq ” is already obtained by the lower semicontinuity. Thus, using the fibering maps (3.10) we have

$$\phi'_{v_n}(r) = r \int_{\Omega} |\nabla v_n|^2 - r\gamma \int_{\Omega} |(-\Delta)^{-1/2} v_n|^2 - r^p \int_{\Omega} |v_n|^{p+1},$$

so that $u_n = r(v_n)v_n$ and

$$\phi'_{v_1}(r) = r \int_{\Omega} |\nabla v_1|^2 - r\gamma \int_{\Omega} |(-\Delta)^{-1/2} v_1|^2 - r^p \int_{\Omega} |v_1|^{p+1},$$

with $u_1 = r_1 v_1$. From these expressions, it is easy to see that $\phi'_{v_n}(1) = 0$ for any n (u_n 's are critical points of the functional (3.3)) and thanks to the expression of $\phi''_{v_n}(r)$ in (3.17), we also find that $\phi'_{v_n}(r) < 0$ for $0 < r < 1$. Consequently, applying (3.35) and the consequences explained previously yields

$$\mathcal{F}_\gamma(r_1 u_1) < \mathcal{F}_\gamma(u_1) < \lim_{n \rightarrow \infty} \mathcal{F}_\gamma(u_n) = \inf_{u \in C_\gamma} \mathcal{F}_\gamma(u),$$

which contradicts the nonexistence of a strong convergence in $W_0^{2,1}(\Omega)$.

Furthermore, using a different argument (but related to), owing to Lemma 3.4, we know that the third term of the functional $\mathcal{F}_\gamma(u)$ (3.3) is actually weakly semicontinuous if $p < \frac{N+2}{N-2}$, i.e., taking the same convergent subsequence in $W_0^{2,1}(\Omega)$,

$$\int_{\Omega} |u_n|^{p+1} \rightarrow \int_{\Omega} |u_1|^{p+1} = N \quad \text{as } n \rightarrow \infty.$$

Thus, we only need to prove that the first two terms are actually convergent. Namely,

$$\int_{\Omega} |\nabla u_1|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_1|^2 = \liminf_{n \rightarrow \infty} \left(\int_{\Omega} |\nabla u_n|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_n|^2 \right).$$

To do so, we argue again by contradiction, supposing that

$$\left(\int_{\Omega} |\nabla u_1|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_1|^2 \right) < \liminf_{n \rightarrow \infty} \left(\int_{\Omega} |\nabla u_n|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_n|^2 \right),$$

since, by the weak lower semicontinuity, we already know that the sign “ \leq ” is achieved. Moreover, if u_1 is actually a critical point of the functional (3.3), by (3.34), we have

$$\left(\int_{\Omega} |\nabla u_1|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_1|^2 \right) \leq \int_{\Omega} |u_1|^{p+1} = \hat{N}.$$

Thus, assuming that the inequality is not true yields

$$\left(\int_{\Omega} |\nabla u_1|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_1|^2 \right) < \hat{N}.$$

Then, it is possible to find $s_1 > 1$ such that $u_s = s_1 u_1$ and

$$\left(\int_{\Omega} |\nabla u_s|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} u_s|^2 \right) = \hat{N}.$$

However,

$$\int_{\Omega} |u_s|^{p+1} = s_1^{p+1} \int_{\Omega} |u_1|^{p+1} = s_1^{p+1} \hat{N} > \hat{N},$$

which contradicts the assumptions. Hence, $u_n \rightarrow u_1$ in $W_0^{2,1}(\Omega)$ and

$$\mathcal{F}_{\gamma}(u_1) = \lim_{n \rightarrow \infty} \mathcal{F}_{\gamma}(u_n) = \sup_{u \in C_{\gamma}} \mathcal{F}_{\gamma}(u),$$

which again contradicts the nonexistence of a strong convergence in $W_0^{2,1}(\Omega)$.

To conclude the proof, one can combine these existence results with those in Lemma 3.6 to arrive at the desired assumptions of the theorem.

Remark. Note that the first situation of existence of at least one solution for the functional is consistent with the conditions explained by (3.12) for the existence of a one turning point of the fibering map ϕ_v , denoted by (3.10). In other words,

$$\begin{aligned} &\text{either } \int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 > 0 \quad \text{and} \quad \int_{\Omega} |v|^{p+1} < 0 \text{ (unavailable);} \\ &\text{or } \int_{\Omega} |\nabla v|^2 - \gamma \int_{\Omega} |(-\Delta)^{-1/2} v|^2 < 0 \quad \text{and} \quad \int_{\Omega} |v|^{p+1} > 0. \end{aligned}$$

3.5 The variational problem in \mathbb{R}^N in two cases: $\gamma > 0$ and $\gamma < 0$

In general, the results presented above can be accomplished assuming posing the equation (3.1) in the whole \mathbb{R}^N instead of in a bounded domain $\Omega \subset \mathbb{R}^N$. However, to do so, we need to consider the integrals over \mathbb{R}^N and the functional setting over a certain weighted Sobolev space instead of $W_0^{2,2}(\Omega)$ previously assumed. Such a functional setting of the problem in \mathbb{R}^N is absolutely key in

what follows. Indeed, a proper functional setting assumes certain admissible asymptotic decay of solutions at infinity, which, for (3.1), is governed by the corresponding linearized operator.

THE CASE $\gamma > 0$. Assuming that $\gamma > 0$ and, for simplicity, the radial geometry, with $u = u(r)$, with $r = |x| \geq 0$, we then obtain, as $r \rightarrow \infty$,

$$\begin{aligned} \Delta^2 u &\equiv u^{(4)} + \frac{2(N-1)}{r} u''' + \dots = \gamma u + \dots \implies \\ u(r) &= C_1 e^{-\gamma^{1/4} r} + r^{-\frac{N-1}{2}} [C_2 \cos(\gamma^{1/4} r) + C_3 \sin(\gamma^{1/4} r)] + \dots, \end{aligned} \tag{3.36}$$

where $C_{1,2,3}$ are arbitrary constants. Overall, we observe a 3D bundle of solutions decaying at infinity, which looks rather positive. However, one can see that the second term does not look that good, and, in particular, for $N = 1$, this represents non-decaying oscillations as $r \rightarrow +\infty$, which do not belong to any suitable functional space. For $N \geq 2$, these are decaying but never belong to, say, $L^2(\mathbb{R}^N)$ (!).

Moreover, the *exponential* bundle obtained from (3.36) for $C_2 = C_3 = 0$ is *one-dimensional* only:

$$u(r) = C_1 e^{-\gamma^{1/4} r} + \dots \quad \text{as } r \rightarrow \infty, \quad C_1 \in \mathbb{R}. \tag{3.37}$$

Obviously, this 1D bundle is not enough to, say, “shoot from infinity” *two* symmetry boundary conditions at the origin:

$$u'(0) = u'''(0) = 0, \tag{3.38}$$

since, algebraically, at least two parameters are needed to satisfy (3.38). Of course, the exponentially decaying solutions (3.37) are the best possible and belong to any reasonable functional space naturally involved.

If the whole 3D bundle in (3.36) is involved (note that this can contradict any reasonable variational setting, but we are not precise in that here), then the shooting problem becomes *overdetermined*: using *three* parameters $C_{1,2,3}$ to satisfy *two* boundary conditions (3.38). The set of solutions is then expected not to be discrete and should be represented via continuous curves. This case is more artificial and is less interesting.

THE CASE $\gamma < 0$. This case is more promising. Indeed, calculating the admissible asymptotics from (3.36) yields a *two-dimensional* exponential bundle:

$$u(r) = e^{-r|\gamma|^{1/4}/\sqrt{2}} [C_1 \cos(\frac{|\gamma|^{1/4}}{\sqrt{2}} r) + C_2 \sin(\frac{|\gamma|^{1/4}}{\sqrt{2}} r)] + \dots, \quad C_{1,2} \in \mathbb{R}. \tag{3.39}$$

Matching with two symmetry boundary conditions (3.38) yields a well-posed and well-balanced algebraic “2D–2 shooting problem”. A similar (but easier and without non-local terms) fourth-order problem was studied in [30, § 6]. It was shown that (for the analogy of the present case $\gamma < 0$) such a problem can admit a countable set of countable families of solutions, where only the first infinite family is the L–S one. We expect that several properties and results of that study can be translated to the nonlocal problem under consideration, but this will require some additional work to be done in a separate paper [2].

4 Global existence for the stable Cahn–Hilliard equation

Next, returning to the fourth-order non-stationary parabolic models presented at the beginning of this paper, without loss of generality, we consider the Cauchy problem for the stable equation (1.5),

with bounded and, if necessary, exponentially decaying at infinity initial function $u_0(x)$. We are going to apply a scaling method, which gets rid in the limit of any lower-order or other perturbations in the PDE's, leaving only the main principal operators and nonlinearities that might be responsible for a finite time blow-up singularity. Therefore, to reveal key aspects of the method, we can consider this maximally simplified model (1.5). Our main goal is to prove the following:

Theorem 4.1 *The Cauchy problem (1.5) in the parameter range (1.6) admits a unique global classical solution, and moreover, it is uniformly bounded:*

$$|u(x, t)| \leq C \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+. \tag{4.1}$$

Proof. It consists of four steps.

STEP I: A PRIORI BOUNDS ON SMOOTH SOLUTIONS. This is a pretty standard step in nonlinear PDE theory. Writing (1.5) in the pseudo-parabolic form,

$$(-\Delta)^{-1}u_t = \Delta u - |u|^{p-1}u, \tag{4.2}$$

and multiplying by u_t in the metric of $L^2(\mathbb{R}^N)$ yields:

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{p+1} \int |u|^{p+1} \right) = -\|u_t\|_{H^{-1}}^2 \leq 0. \tag{4.3}$$

In particular, this shows that, on smooth solutions, (1.5) is a gradient dynamical system admitting a positive definite Lyapunov function, so that a number of strong results from this area are available (see e.g., [35]), though the key L^∞ -estimate still remains uncertain.

Integrating (4.3) over an arbitrary interval $(0, T)$ yields the following *a priori* bounds on smooth solutions:

$$\|\nabla u(t)\|_2^2 \leq C \quad \text{and} \quad \int |u(t)|^{p+1} \leq C \quad \text{for all } t > 0. \tag{4.4}$$

STEP II: PROVING NON-BLOW-UP BY SCALING. SUBCRITICAL RANGE. Here, we follow [33]; see also [28]. Namely, arguing by contradiction, we assume that there exist sequences $\{t_k\} \rightarrow T^-$, $\{x_k\} \subset \mathbb{R}^N$, and $\{C_k\}$ such that

$$\sup_{\mathbb{R}^N \times [0, t_k]} |u(x, t_k)| = |u(x_k, t_k)| = C_k \rightarrow +\infty. \tag{4.5}$$

In other words, the solution blows-up in finite time. We next perform the change

$$u_k(x, t) \equiv v(x_k + x, t_k + t) = C_k v_k(y, s), \quad \text{where } x = a_k y, \quad t = a_k^4 s, \tag{4.6}$$

and the sequence $\{a_k\} \rightarrow 0$ is chosen in such a manner that the *a priori* estimates (4.4) hold for the sequence $\{v_k(y, s)\}$. In particular,

$$\|\nabla u(t)\|_2^2 = C_k^2 a_k^{N-2} \int \|\nabla v(s)\|^2 \quad \text{and} \quad \int |u(t)|^{p+1} = C_k^{p+1} a_k^N \int |v(s)|^{p+1}.$$

This gives respectively (both choices eventually lead to the same result):

$$a_k = C_k^{-\frac{2}{N-2}} \quad (N \geq 3) \quad \text{and} \quad a_k = C_k^{-\frac{p+1}{N}}. \tag{4.7}$$

Note that, after such a scaling, the rescaled functions $v_k(y, s)$ are defined on the intervals

$$s \in \left[-\frac{t_k}{a_k^4}, \frac{T-t_k}{a_k^4} \right). \tag{4.8}$$

As usual, such a rescaling near blow-up time, in the limit $k \rightarrow \infty$, leads to the so-called *ancient solutions* (i.e., defined for all $s < 0$) in Hamilton’s notation [36]. Various scalings have been typical techniques of reaction-diffusion theory for many years; see different forms of its application in [57, 32].

Substituting (4.6) into equation (1.5) yields that $v_k(y, s)$ satisfies a perturbed equation

$$(v_k)_s = -\Delta^2 v_k + \delta_k \Delta(|v_k|^{p-1} v_k) \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \quad \text{where} \tag{4.9}$$

$$\delta_k = C_k^{\gamma_1}, \quad \gamma_1 = p - 1 - \frac{4}{N-2} \quad \text{and} \quad \delta_k = C_k^{\gamma_2}, \quad \gamma_2 = p - 1 - \frac{2(p+1)}{N}, \tag{4.10}$$

respectively. One can see that

$$\delta_k \rightarrow 0, \quad \text{if } \gamma_{1,2} < 0 \implies p < p_*. \tag{4.11}$$

We next perform a backward shifting in time technique by fixing $s_0 > 0$ large enough (this is possible in the time-interval in (4.8) since $a_k \rightarrow 0$), and setting $\bar{v}_k(s) = v_k(s - s_0)$. Then, by construction, we have that

$$|\bar{v}_k(s)| \leq 1 \quad \text{and} \quad (4.4) \text{ for } v_k(y, s) \text{ hold on } (0, s_0), \tag{4.12}$$

so that $\{\bar{v}_k(s)\}$ is a family of uniformly bounded classical solutions of the uniformly parabolic equation (4.9) with bounded smooth coefficients. By classic parabolic regularity theory [15, 19], we have that the sequence $\{\bar{v}_k\}$ is uniformly bounded and equicontinuous on any compact subset of $\mathbb{R}^N \times (0, s_0)$. Indeed, the necessary uniform gradient bound can be obtained from the integral equation of (4.9), or by other usual regularity methods for uniformly parabolic equations.

Therefore, by the Ascoli–Arzelá theorem, along a certain subsequence, $\bar{v}_k(s) \rightarrow \bar{v}(s)$ uniformly on compact subsets of $\mathbb{R}^N \times (0, s_0)$. Passing to the limit in equation (4.9) and using that the scaling parameter satisfy $\delta_k \rightarrow 0$, yields that $\bar{v}(s)$ is a bounded weak solution and, hence, a classical solution of the Cauchy problem for the linear *bi-harmonic equation*

$$\bar{v}_s = -\Delta^2 \bar{v}, \quad \text{with data} \quad |\bar{v}_0| \leq 1, \quad \|\nabla \bar{v}_0\|_2 \leq C, \quad \int |\bar{v}_0|^{p+1} \leq C. \tag{4.13}$$

We next represent the solutions as follows:

$$\bar{v}(s_0) = b(s_0) * \bar{v}_0 \equiv s_0^{-\frac{N}{4}} \int_{\mathbb{R}^N} F\left(\frac{y-z}{s^{1/4}}\right) \bar{v}_0(z) \, dz \equiv s_0^{-\frac{N}{4}} \int_{\mathbb{R}^N} (\nabla)^{-1} F\left(\frac{y-z}{s^{1/4}}\right) \nabla \bar{v}_0 \, dz. \tag{4.14}$$

Finally, using the Hölder inequality in the convolution yields:

$$|\bar{v}(s_0)| \leq s_0^{-\frac{N-1}{4}} C \ll 1 \quad (N \geq 2) \tag{4.15}$$

for all $s_0 \gg 1$. Hence, the same holds for $\sup_y |\bar{v}_k(y, s_0)|$ for $k \gg 1$, from whence comes the contradiction with the assumption $\sup_y |v_k(y, s_0)| = 1$. Thus, $v(x, t)$ does not blow-up and remains bounded for all $t > 0$ (but not uniformly still, as required by (4.1)).

STEP III: PROVING NON-BLOW-UP BY SCALING. CRITICAL CASE. For $p = p_*$, we have that $\delta_k \equiv 1$, so that (4.9) for $\{v_k\}$ takes the unperturbed form

$$v_s = -\Delta^2 v + \Delta(|v|^{p-1} v) \quad \text{in } \mathbb{R}^N \times \mathbb{R}. \tag{4.16}$$

Since, as we have seen, (4.16) is a smooth gradient system with a monotone operator in H^{-1} , so that zero is the only equilibrium, we have that, for any regular enough global solution $v(y, s)$,

$$v(y, s) \rightarrow 0 \quad \text{as } s \rightarrow +\infty \tag{4.17}$$

uniformly in \mathbb{R}^N . Then, as above, by passing to the limit $k \rightarrow \infty$, we then obtain existence of an *ancient* solution $\bar{v}(y, s)$ satisfying

$$\bar{v}(s) \rightarrow 0 \quad \text{as } s \rightarrow -\infty \quad \text{and} \quad \|\bar{v}(0)\|_\infty = 1. \tag{4.18}$$

However, such a solution is obviously nonexistent, since then (4.16) for $s \ll -1$, where $|\bar{v}(s)| \ll 1$, becomes an asymptotically small perturbation of the linear equation (4.13), so that the same argument applies.

STEP IV: UNIFORM BOUNDEDNESS. Assuming now that $C_k \rightarrow +\infty$ and $t_k \rightarrow +\infty$ and performing the same scaling and passing to the limit yield the result.

This completes the proof of the theorem.

Remark. For the non-autonomous C–H equation (1.7), one can derive similar global *a priori* estimates (4.3). However, since the translations in x are not allowed now, we can perform a proof of non-blow-up at the fixed point $x = 0$. Then, all the arguments apply if we set $x_k = 0$ (or $x_k \approx 0$ close enough), and the only difference is that α will enter some exponents, so that we will have

$$a_k = C_k^{-\frac{p+1}{\alpha+N}}, \quad \gamma_1 = p - 1 - \frac{2(\alpha+2)}{N-2}, \quad \gamma_2 = p - 1 - \frac{(p+1)(\alpha+2)}{\alpha+N}, \quad \text{etc.},$$

which will eventually lead to the critical exponent in (1.8). Indeed, this non-blow-up is a conventional one, since (1.8) does not prevent blow-up at any neighbouring points $x \neq 0$, for which the range (1.6) remains correct.

5 A short discussion on blow-up in the supercritical stable model

Thus, in the supercritical range $p > p_*$, the scaling argument establishing uniform L^∞ -bounds (4.1) from weaker Sobolev and L^{p+1} -estimates (4.4) does not apply, and finite time blow-up of some solutions becomes plausible.

Consider the non-autonomous model (1.7). Then blow-up at $x = 0$ is impossible in the range (1.8). As a first step towards blow-up scenarios (at $x = 0$) for (1.7) with $p > p_*(\alpha)$, one should consider self-similar solutions of the standard form:

$$u_S(x, t) = (T - t)^{-\gamma} f(y), \quad \gamma = \frac{\alpha+2}{4(p-1)}, \quad y = x/(T - t)^{\frac{1}{4}}, \quad t < T, \tag{5.1}$$

where $T > 0$ is the corresponding blow-up time. Then f in (5.1) solves the following elliptic equation:

$$-\Delta^2 f - \frac{1}{4} y \cdot \nabla f - \gamma f + \Delta(|y|^\alpha |f|^{p-1} f) = 0 \quad \text{in } \mathbb{R}^N. \tag{5.2}$$

Thus, we arrive at a “stationary” problem (5.2) for blow-up profiles $f(y)$. This returns us to previous Section 3. However, one can immediately observe that the corresponding steady problem is not variational, due to the presence of an extra linear first-order operator, increasing the difficulty of the analysis and not allowing the techniques used in Section 3.

Moreover, proving nonexistence of any nontrivial solutions of (5.2) in some subrange of $p > p_*(\alpha)$ is an important but still difficult open problem, at least in general.

However, there exist other more involved scenarios of blow-up, which we now start to develop for the corresponding unstable C–H equation. For (1.9), such scenarios are “more generic” and easier to implement, though, partially, in a non-rigorous way. We still do not know whether such scenarios can be applied to the stable equation (1.7).

6 Two types of blow-up in the unstable C–H equation

For simplicity, we now again fix $\alpha = 0$ and consider the corresponding unstable C–H equation (1.9). The first type of possible blow-up patterns remains the same and will be shown as follows.

6.1 Self-similar blow-up

This occurs according to formulae as in (5.1) with $\alpha = 0$:

$$u_S(x, t) = (T - t)^{-\gamma} f(y), \quad \gamma = \frac{1}{2(p-1)}, \quad y = x/(T - t)^{\frac{1}{4}}, \quad t < T, \tag{6.1}$$

where the similarity profile $f(y)$ solves the elliptic equation

$$-\Delta^2 f - \frac{1}{4} y \cdot \nabla f - \frac{1}{2(p-1)} f - \Delta(|f|^{p-1} f) = 0 \quad \text{in } \mathbb{R}^N. \tag{6.2}$$

Let us discuss properties of blow-up similarity profiles in greater detail, which is necessary for future extensions. Thus:

(i) In the critical “mass Fujita” case (1.10), equation (6.2) admits a countable family of positive blow-up patterns $\{f_k(y) > 0\}_{k \geq 0}$, with exponential decay as $y \rightarrow \infty$. In the radial setting, these are obtained from a third-order ODE derived on integration [17, § 2-4]. This is a simpler case. Moreover, such Type I blow-up patterns generate Dirac’s delta as final-time profiles,

$$u(x, T^-) = c_k \delta(x) \quad \text{in } \mathbb{R}^N \quad (c_k > 0). \tag{6.3}$$

(ii) There exists another critical dipole exponent, e.g., $p_1 = 2$ for $N = 1$ (or $p_1 = 1 + \frac{2}{N+1}$), where the ODE for $N = 1$ again reduces to the third order and admits extended study [17, § 5].

(iii) For general $p > 1$, $p \neq p_0$, p_1 , (6.2) is truly a fourth-order elliptic equation or an ODE in the radial setting. Then construction of proper blow-up profiles $f(y)$ requires taking into account non-exponentially decaying asymptotic bundles. E.g., for $N = 1$, this means considering a 2D bundle of the form

$$f(y) = Ay^{-\frac{2}{p-1}} + \dots + Cy^{-\frac{1}{3}} e^{-a_0 y^{4/3}} + \dots \quad \text{as } y \rightarrow +\infty, \quad a_0 = 3 \cdot 2^{-\frac{8}{3}}, \tag{6.4}$$

where A and C are arbitrary parameters. In Appendix A, we discuss a 2D shooting strategy, which requires constructing a solution of (6.2) for $p \neq p_0$ by using both parameters A and C in (6.4).

In (6.4), the last term is responsible for exponentially decaying functions, while the first one gives an algebraic decay. It follows from (6.4) that, for any $A \neq 0$,

$$f \in L^1(\mathbb{R}) \quad \text{for } p \in (1, p_0) \quad \text{and} \quad f \notin L^1(\mathbb{R}) \quad \text{for } p > p_0. \tag{6.5}$$

Since the mass evolution of the similarity solutions (6.1) is given by

$$\int_{\mathbb{R}^N} u_S(x, t) \, dx = (T - t)^{\frac{N(p-p_0)}{4(p-1)}} \int_{\mathbb{R}^N} f(y) \, dy, \tag{6.6}$$

the mass conservation implies that

$$\int f(y) = 0 \quad \text{for all } p \in (1, p_0). \tag{6.7}$$

Thus, the following holds:

$$A = 0 \text{ in (6.4), iff } p = p_0 = 1 + \frac{2}{N},$$

i.e., in other cases, $f(y)$ cannot in general have exponential decay. This changes the blow-up asymptotics for $p \neq p_0$: (6.1), (6.4) imply the limit $t \rightarrow T^-$

$$u(x, T^-) = A|x|^{-\frac{2}{p-1}}, \quad A \neq 0, \quad \text{so that} \tag{6.8}$$

$$u(x, T^-) \in L^1_{\text{loc}}(\mathbb{R}^N), \quad \text{if } p > p_0; \quad u(x, T^-) \notin L^1_{\text{loc}}(\mathbb{R}^N), \quad \text{if } p < p_0. \tag{6.9}$$

We refer to [17], where most of the results are obtained for $N = 1$. In Figure 2, we present numerical results showing how the blow-up similarity profiles $f(y)$ changes sign for $p < p_0$, so that we expect that

$$A(p) < 0 \quad \text{for } p \in (1, p_0), \quad A(p_0) = 0, \quad \text{and } A(p) > 0 \quad \text{for } p > p_0. \tag{6.10}$$

Main ideas and shooting techniques of construction admit natural extensions to higher dimensions, where mathematical justifications become much more difficult, especially in the critical and supercritical Sobolev range

$$p \geq p_S = \frac{N+2}{(N-2)_+}. \tag{6.11}$$

In particular, it can be expected that these p -branches of similarity profiles may blow-up as $p \rightarrow p_S^-$, without proper extension beyond. Therefore, for $p \geq p_S$, another scenario of blow-up is necessary, to be introduced later on.

6.2 A branching approach to blow-up similarity profiles

Indeed, the problem (6.2) is not variational. However, it can be viewed as a perturbation of a variational one, which we have dealt with above. Let us introduce the following family of operators:

$$\mathbf{A}_\mu f \equiv -\Delta^2 f - \mu y \cdot \nabla f - \frac{1}{2(p-1)} f - \Delta(|f|^{p-1} f) = 0 \quad \text{in } \mathbb{R}^N, \tag{6.12}$$

where $\mu \in [0, \frac{1}{4}]$ is a parameter. Indeed, for $\mu = 0$, we arrive at the variational problem (3.1) in \mathbb{R}^N , with (this is key for existence!)

$$\gamma = -\frac{1}{2(p-1)} < 0 \quad (\text{cf. Section 3.5}). \tag{6.13}$$

Therefore, we expect that there exists a *branching* of solutions of (6.12) from critical L–S points at $\mu = 0$. This can be established in a reasonably standard way (see related examples in [29, 30]). This can also then guarantee existence of an arbitrarily large number of solutions (if the L–S family for $\mu = 0$ is countable) at least for sufficiently small $\mu > 0$. The principle difficulty is then a extension of these solutions up to the necessary value $\mu = \frac{1}{4}$ dictated by the similarity blow-up equation (6.2). Such a global extension problem remains open and represents a fundamental problem in nonlinear non-potential operator theory. Numerical methods even for the case $N = 1$ (or the radial one in \mathbb{R}^N) are also rather delicate in the present case. We will treat such a problem in [2].

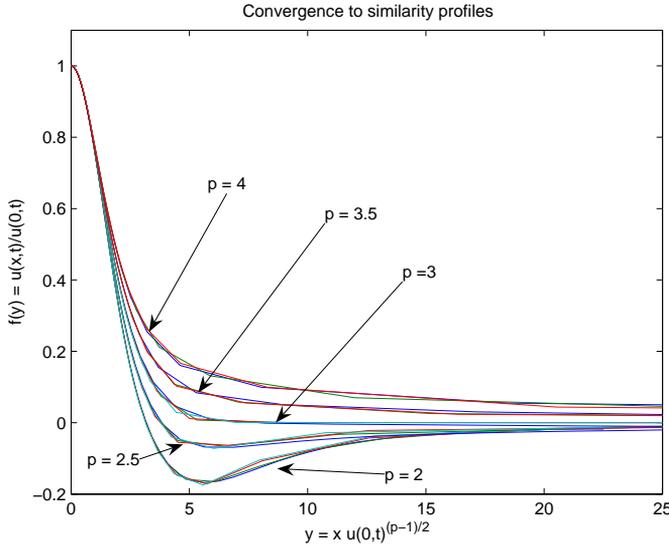


Figure 2: Similarity profiles satisfying (6.2) for $N = 1$ obtained via numerical solving the PDE (1.9) and scaling, [17, § 5].

6.3 Remark on similarity extension beyond blow-up: towards Leray’s scenario (1934)

Let us point out another important feature of self-similar blow-up under the presence of the mass conservation. Namely, self-similar blow-up such as (6.1) admits self-similar extensions beyond via *global* similarity solutions

$$u_S^+(x, t) = (t - T)^{-\gamma} F(y), \quad \gamma = \frac{1}{2(p-1)}, \quad y = x/(t - T)^{\frac{1}{4}}, \quad t > T, \tag{6.14}$$

where the similarity profile F now solves a slightly different elliptic equation

$$-\Delta^2 F + \frac{1}{4} y \cdot \nabla F + \gamma F - \Delta(|F|^{p-1} F) = 0 \quad \text{in } \mathbb{R}^N. \tag{6.15}$$

Then, for “initial data” (6.3), we look for solutions of (6.15) with exponential decay, while, for that in (6.8), one needs $F(y)$ with the *same* constant $A \neq 0$, as inherited from blow-up evolution via $f(y)$ with the asymptotics (6.4).

It is key that the ODE (6.15) in the radial setting has another 3D bundle at infinity, e.g., for $N = 1$,

$$F(y) = Ay^{-\frac{2}{p-1}} + \dots + y^{-\frac{1}{3}} e^{-\frac{a_0}{2} y^{4/3}} [B \cos(\frac{a_0 \sqrt{3}}{2} y^{4/3}) + C \sin(\frac{a_0 \sqrt{3}}{2} y^{4/3})] + \dots, \tag{6.16}$$

where $A \neq 0$ is fixed by the pre-history as $t \rightarrow T^-$, but two parameters B and C are left to shoot necessary two symmetry conditions at the origin

$$F'(0) = F'''(0) = 0.$$

Note that, in general, such a construction beyond blow-up can assume existence of a certain “mass defect”, when the parameters for $t < T$ and $t > T$ are not entirely consistent; see [17, § 4] and [22] for the case $p = p_0$. Indeed, solvability of the above shooting/matching problem on construction of suitable *extension pairs* $\{f(y), F(y)\}$ for blow-up solutions of the C–H equation must be accompanied by advanced numerical methods. For the fourth-order RD equation in 1D (a non-conservative model)

$$u_t = -u_{xxxx} + |u|^{p-1}u \quad \text{in } \mathbb{R} \times \mathbb{R}, \tag{6.17}$$

for such a blow-up/extension study, see [10, 24, 26].

This construction can be connected with Leray’s scenario of self-similar blow-up/extension proposed in 1934 for the Navier–Stokes equations in \mathbb{R}^3 , [43, p. 245].

Thus, we next turn our attention to another new type of blow-up for the unstable C–H equation (1.9), which is directly associated with its *stationary* solutions.

6.4 “Quasi-stationary” Type II(LN) blow-up in the critical case

We consider the unstable equation (1.9) in the *critical Sobolev case*

$$p = p_S = \frac{N+2}{N-2} \quad \text{for } N \geq 3. \tag{6.18}$$

Then, the stationary equation

$$\Delta_\xi W + W^p = 0, \quad \xi \in \mathbb{R}^N, \quad W(0) = d > 0 \quad (p = p_S = \frac{N+2}{N-2}) \tag{6.19}$$

is known to admit a 1D family of classic Loewner–Nirenberg (L–N) conformally invariant exact solutions [44] (1974). The corresponding symmetries of (6.19) were earlier detected by Ibragimov in 1968 [38]. These solutions are given explicitly by

$$W_0(\xi) = d \left[\frac{N(N-2)}{N(N-2) + d^{\frac{2}{p-1}} |\xi|^2} \right]^{\frac{N-2}{2}} > 0 \quad \text{in } \mathbb{R}^N, \quad d > 0, \tag{6.20}$$

and exhibit a number of uniqueness and other exceptional properties concerning equation (6.19); see [24, § 6] for extra details and references.

The idea of such **Type II(LN)** blow-up patterns (according to a classification in [24]) consists of noting that blow-up can occur via some “slow” motion about the stationary solutions (6.20). This formally means the following non-stationary parameter time-dependence:

$$d = d(t) \rightarrow +\infty \quad \text{as } t \rightarrow T^-. \tag{6.21}$$

For the standard second-order reaction-diffusion equation

$$u_t = \Delta u + |u|^{p-1}u, \quad p = p_S, \tag{6.22}$$

such a blow-up scenario was proposed in [18].

Construction of such Type II blow-up patterns of non-self-similar kind consists of few steps.

STEP I: STANDARD RESCALING. We first use standard full similarity scaling in (1.9):

$$u(x, t) = (T - t)^{-\frac{1}{2(p-1)}} v(y, \tau), \quad y = \frac{x}{(T-t)^{1/4}}, \quad \tau = -\ln(T - t) \rightarrow +\infty, \quad t \rightarrow T^-. \tag{6.23}$$

Then $v(y, \tau)$ solves the following rescaled parabolic equation:

$$v_\tau = \mathbf{A}(v) \equiv -\Delta^2 v - \frac{1}{4} y \cdot \nabla v - \frac{1}{2(p-1)} v - \Delta(|v|^{p-1}v) \quad \text{in } \mathbb{R}^N \times (\tau_0, \infty), \tag{6.24}$$

where $\tau_0 = -\ln T$, and \mathbf{A} is the stationary elliptic operator in (6.2), so that similarity profiles are just stationary solutions of (6.24).

STEP II: SPECTRAL THEORY AND GENERALIZED HERMITE POLYNOMIALS. We need a detailed knowledge of spectral properties of the non-self-adjoint linear operator, which appeared in (6.24), i.e., for

$$\mathbf{B}^* = -\Delta^2 - \frac{1}{4} y \cdot \nabla \quad \text{in } L^2_{\rho^*}(\mathbb{R}^N), \quad \rho^*(y) = e^{-a|y|^{4/3}}, \quad a \in (0, 3 \cdot 2^{-\frac{8}{3}}). \tag{6.25}$$

This ‘‘adjoint’’ Hermite operator has a number of good spectral properties [14]:

Lemma 6.1 $\mathbf{B}^* : H^4_{\rho^*}(\mathbb{R}^N) \rightarrow L^2_{\rho^*}(\mathbb{R}^N)$ is a bounded linear operator with the spectrum

$$\sigma(\mathbf{B}^*) = \{\lambda_\beta = -\frac{|\beta|}{4}, \quad |\beta| = 0, 1, 2, \dots\} \quad (= \sigma(\mathbf{B}), \quad \mathbf{B} = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I). \tag{6.26}$$

Eigenfunctions $\psi_\beta^*(y)$ are $|\beta|$ th-order generalized Hermite polynomials:

$$\psi_\beta^*(y) = \frac{1}{\sqrt{\beta!}} [y^\beta + \sum_{j=1}^{|\beta|/4} \frac{1}{j!} (\Delta)^{2j} y^\beta], \quad |\beta| = 0, 1, 2, \dots, \tag{6.27}$$

and the subset $\{\psi_\beta^*\}$ is complete in $L^2_{\rho^*}(\mathbb{R}^N)$.

As usual, if $\{\psi_\beta\}$ is the adjoint basis of eigenfunctions of the adjoint (in the dual metric of L^2) operator

$$\mathbf{B} = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I \quad \text{in } L^2_\rho(\mathbb{R}^N), \quad \rho = \frac{1}{\rho^*}, \tag{6.28}$$

with the same spectrum (6.26), the bi-orthonormality condition holds in $L^2(\mathbb{R}^N)$:

$$\langle \psi_\mu, \psi_\nu^* \rangle = \delta_{\mu\nu} \quad \text{for any } \mu, \nu. \tag{6.29}$$

We recall that the generating formula for eigenfunctions ψ_β is as follows [14]:

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y), \tag{6.30}$$

where $F(y)$ is the rescaled kernel of the fundamental solution of the bi-harmonic equation

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{so that} \tag{6.31}$$

$$b(x, t) = t^{-\frac{N}{4}} F(y), \quad y = \frac{x}{t^{1/4}}, \quad \text{and } \mathbf{B}F = 0, \quad \int F = 1. \tag{6.32}$$

STEP III: FORMAL CONSTRUCTION OF TYPE II(LN) BLOW-UP PATTERNS FOR $p = p_S$. Let $v(y, \tau)$ be the rescaled solution of (6.24) in, say, radial geometry at the moment (which is not essential, since the patterns can be essentially non-radial). This is the precise specification of the class of blow-up patterns we are dealing with: we assume that $v(y, \tau)$ behaves for $\tau \gg 1$ closely to the stationary manifold composed of the explicit equilibria (6.20), i.e., for some unknown function $\varphi(\tau)$, the ‘‘singular part’’ of the solution takes the form

$$v(y, \tau) = \varphi(\tau) W_0(\varphi^{\frac{p-1}{2}}(\tau)y) + \dots, \quad \text{where } \varphi(\tau) \rightarrow +\infty \quad \text{as } \tau \rightarrow +\infty. \tag{6.33}$$

Such a representation (6.33) of $v(y, \tau)$ is then assumed to be uniformly valid as $\tau \rightarrow +\infty$ on any compact subsets in the new variable $\zeta = \varphi^{\frac{p-1}{2}}(\tau)y$.

Next, it follows that, on the solutions (6.33) in terms of the original rescaled variable y

$$|v(y, \tau)|^{p-1}v(y, \tau) \rightarrow \frac{e_N}{\varphi(\tau)} \delta(y) \quad \text{as } \tau \rightarrow +\infty \tag{6.34}$$

in the sense of distributions, where $e_N > 0$ is the constant

$$e_N = \int_{\mathbb{R}^N} W_0^p(\zeta) d\zeta. \tag{6.35}$$

Therefore, on this manifold of solutions, the rescaled equation (6.24) takes asymptotically the form

$$v_\tau = \mathbf{A}(v) \equiv -\Delta^2 v - \frac{1}{4} y \cdot \nabla v - \frac{N-2}{8} v - \frac{e_N}{\varphi(\tau)} \Delta \delta(y) + \dots \quad \text{for } \tau \gg 1. \tag{6.36}$$

Using Lemma 6.1, we are looking for Type II patterns of the simplest eigenfunction expansion over generalized Hermite polynomials, i.e., assuming that

$$v_\beta(y, \tau) = c_\beta(\tau)\psi_\beta^*(y) + \dots \quad \text{as } \tau \rightarrow +\infty. \tag{6.37}$$

Actually, this means looking for a kind of stable (or centre) subspace behaviour for the equation (6.24). Then, as usual, substituting (6.37) into (6.36) and multiplying by the adjoint eigenfunction ψ_β (recalling the bi-orthonormality (6.29)) yield the following asymptotic equation for the expansion coefficient:

$$\dot{c}_\beta = -\alpha_\beta c_\beta + \frac{h_\beta}{\varphi(\tau)} + \dots, \tag{6.38}$$

where $\alpha_\beta = \frac{2|\beta|+N-2}{8} > 0$. The crucial coefficient h_β is calculated as follows:

$$h_\beta = -e_N \langle \Delta \delta, \psi_\beta \rangle = -e_N \langle \delta, \Delta \psi_\beta \rangle = -e_N (\Delta \psi_\beta)(0) \neq 0. \tag{6.39}$$

According to (6.30), the non-vanishing condition (6.39) imposes essential restrictions on the multi-indexes β and admissible eigenfunctions $\psi_\beta(y)$ in (6.30), for which such blow-up patterns can actually exist. Note that, according to (6.30), the Laplacian in (6.39)

$$\Delta \psi_\beta \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} (D_{y_1 y_1}^2 D^\beta F + D_{y_2 y_2}^2 D^\beta F + \dots + D_{y_N y_N}^2 D^\beta F)$$

is indeed a linear combinations of N other eigenfunctions of \mathbf{B} . The non-vanishing condition in (6.39) requires that all the components of the corresponding multiindex β should be even.

Let us return to the crucial ‘‘dynamical system’’ (6.38), where we show the leading equation accompanying infinitely many others corresponding to further stable subspaces. The asymptotic equation contains two unknowns, the actual coefficient $c_\beta(\tau)$ and the corresponding scaling function $\varphi = \varphi_\beta(\tau)$ from (6.37). Obviously, the key issue now is to establish an asymptotic relation between them for $\tau \gg 1$, which will allow a proper ‘‘balance’’ that is necessary for existence of such a blow-up pattern. On one hand, this looks like a standard procedure by assuming that the singular component in (6.37) actually determines the evolution of the expansion coefficient $c_\beta(\tau)$. Under this hypothesis, we then have, in a standard manner, by using (6.30), for $\tau \gg 1$,

$$\begin{aligned} c_\beta(\tau) &\sim \langle \varphi(\tau)W_0(\varphi^{\frac{p-1}{2}}(\tau)y), \psi_\beta \rangle = \varphi(\tau) \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \langle W_0(\varphi^{\frac{p-1}{2}}(\tau)y), D^\beta F \rangle \\ &= \varphi(\tau) \frac{1}{\sqrt{\beta!}} \langle D_y^\beta W_0(\varphi^{\frac{p-1}{2}}(\tau)y), F(y) \rangle. \end{aligned} \tag{6.40}$$

Finally, changing the variable in the last integral by setting $z = \varphi^{\frac{p-1}{2}}(\tau)y$, we arrive at the following integral relation:

$$c_\beta(\tau) \sim [\varphi(\tau)]^{1+\frac{(p-1)(|\beta|-N)}{2}} \int_{\mathbb{R}^N} D_z^\beta W_0(z) F(z\varphi^{-\frac{p-1}{2}}(\tau)) dz. \tag{6.41}$$

However, resolving uncertainties in (6.41) as $\tau \rightarrow +\infty$ is not that easy. Moreover, it is not clear that the projection integral operator onto the eigenspace in (6.41) gives a correct link between these two functions, since, in some cases, extra integrals over subsets in the rescaled variables of *Outer Regions* (which we do not study here) should be taken into account. Recall that even in the second-order case of the RD equation (6.22), where standard Hermite polynomials and classic spectral theory occur, a sufficiently sharp obtaining of all the time factors is not always possible, [18], especially in higher dimensions $N \geq 7$, with no results obtained at all.

Therefore, instead of dealing with a singular integral such as in (6.41), we apply another, simpler, but more qualitative and rough (but sufficient for our goals) method of “balancing” the expansions, which, in some cases of blow-up reaction-diffusion theory, led to rigorous results; cf. various examples in [32, Ch. 4–11]. Since, for higher-order parabolic equations, we do not have any chance of getting more justified formal expansions, we are allowed to concentrate on a principal issue of balancing the asymptotic expansion, without trying to perform further matching of (6.33) with outer regions. This can be very difficult even for the second-order case [18], where some parameter ranges require further analysis and even new ideas.

Thus, as usual in blow-up approaches, existence of such a blow-up pattern requires a certain balance of the two leading terms on the right-hand side of (6.38), i.e., one needs

$$\alpha_\beta c_\beta(\tau) \sim \frac{h_\beta}{\varphi(\tau)} \implies c_\beta(\tau) \sim \frac{h_\beta}{\alpha_\beta \varphi(\tau)} \quad \text{for } \tau \gg 1, \tag{6.42}$$

where the sign “ \sim ” assumes omitting other multipliers of slower behaviour. Overall, this means that we can use the following *ansatze* for the expansion coefficient:

$$c_\beta(\tau) = \frac{\kappa(\tau)}{\varphi(\tau)}, \tag{6.43}$$

where $\kappa(\tau)$ is a slow varying function as $\tau \rightarrow +\infty$ in comparison with $\varphi(\tau)$. Substituting (6.43) into (6.38) yields

$$\frac{\dot{\kappa}}{\varphi} - \kappa \frac{\dot{\varphi}}{\varphi^2} = -\alpha_\beta \frac{\kappa}{\varphi} + \frac{h_\beta}{\varphi} + \dots \tag{6.44}$$

This gives the only κ -independent balance:

$$-\kappa \frac{\dot{\varphi}}{\varphi^2} \sim -\alpha_\beta \frac{\kappa}{\varphi} \implies \frac{\dot{\varphi}}{\varphi} \sim \alpha_\beta \implies \varphi = \varphi_\beta(\tau) \sim e^{\alpha_\beta \tau}. \tag{6.45}$$

The slow varying function $\kappa(\tau)$ cannot be determined from such a simple matching and requires further difficult asymptotic analysis of projections like (6.41) or other approaches.

Thus, up to slower scaling factors, we get a countable family of such patterns with

$$\varphi_\beta(\tau) \sim e^{\alpha_\beta \tau} + \dots \quad \text{and} \quad c_\beta(\tau) \sim e^{-\alpha_\beta \tau} + \dots \quad \text{for } \tau \gg 1, \quad |\beta| \geq 0. \tag{6.46}$$

The expansion (6.37) actually assumes dealing with a 1D eigenspace, which is the case for $|\beta| = 0$ only, where $\lambda_0 = 0$ is simple. For any $k = |\beta| \geq 1$, a more general, than (6.37) eigenfunction expansion should be taken into account:

$$v_\beta(y, \tau) = \sum_{|\beta|=k} c_\beta(\tau) \psi_\beta^*(y) + \dots, \tag{6.47}$$

which leads to more difficult dynamical systems for the coefficients $\{c_\beta(\tau)\}_{|\beta|=k}$ (to say nothing of the multiple projection integrals, which replace (6.41)), but eventually can induce more exiting Type II blow-up patterns.

Overall, bearing in mind the scaling in (6.33), this yields a possibility of constructing a countable family of distinct complicated Type II blow-up structures, where most of them are not radially symmetric. To reveal the actual space-time and changing sign structures of such Type II patterns, special matching procedures apply. In [18], this analysis has been performed in the radial geometry for (6.22), though still no rigorous justification of the existence of such blow-up scenarios in \mathbb{R}^N was achieved. In [48], existence of related radial nonnegative blow-up patterns was encouraged by putting zero Dirichlet data on the boundary of a shrinking ball. This boundary constraint indeed essentially simplifies the problem in comparison with those in \mathbb{R}^N .

Thus, the first Fourier coefficient in (6.37) or general expansions on multi-dimensional eigenspaces imply a complicated structure of the pattern about the formed Dirac’s $\delta(y)$ according to (6.34). However, since these expansions are given by generalized Hermite polynomials $\{\psi_\beta^*\}$, this matching is expected not to impose more difficulties as those in [24, § 4].

As a related extension issue beyond blow-up, let us note that the singular part (6.33), with the factors (6.46), creates at $t = T$ a very weak singularity such that, in the sense of distributions, its singular part is as follows:

$$u(x, T) \sim (\delta(x))^\gamma, \quad \text{where } \gamma = \frac{2}{N(p-1)} = \frac{N-2}{2N} < \frac{1}{p-1} < 1. \tag{6.48}$$

Therefore, for such Type II blow-up patterns, by classic parabolic theory, for equation (1.9) with such a “weak singular” data (6.48) at $t = T$,

$$\text{there exists a unique continuation of such blow-up solutions for } t > T \tag{6.49}$$

locally in time (*incomplete blow-up*), and such solutions are bounded and classical therein.

Finally, we again comment on the fact that regular *stationary solutions* are key for existence of such Type II blow-up patterns.

7 Extra Type II blow-up patterns for the unstable C–H equation: linearization about singular steady state and matching

Such new Type II blow-up patterns for the semilinear heat equation (6.22) were constructed earlier in [37]; see [46] for extra details. We apply this method to the higher-order equation (1.9), which will require completely different spectral theory and related mathematical tools of matching.

7.1 Singular stationary solution (SSS)

Consider the stationary equation in (6.19) in the range

$$p > p_N = \frac{N}{N-2} \quad \text{for } N \geq 3. \tag{7.1}$$

Then, as is well known, there exists the explicit radial *singular steady state* (SSS) of the standard scaling invariant form

$$U(y) = C_* |y|^{-\mu}, \quad \text{where } \mu = \frac{2}{p-1}, \quad C_* = D^{\frac{1}{p-1}}, \quad D = \frac{2}{p-1} (N - 2 - \frac{2}{p-1}) > 0. \tag{7.2}$$

7.2 Linearization in Inner Region I: discrete spectrum via Hardy’s inequality

We next perform linearization in (6.24) about the SSS by setting:

$$v = U + Y \implies Y_\tau = \hat{\mathbf{B}}^* Y + \mathbf{D}(Y), \tag{7.3}$$

where $\mathbf{D}(Y)$ is a quadratic perturbation as $Y \rightarrow 0$ and

$$\hat{\mathbf{B}}^* = \mathbf{H}^* - \frac{1}{4} y \cdot \nabla - \frac{1}{2(p-1)} I, \quad \mathbf{H}^* = -\Delta^2 - c\Delta(\frac{1}{|y|^2} I), \quad c = pD. \tag{7.4}$$

Here, \mathbf{H}^* contains the main singular terms at the origin $y = 0$. Similar to Lemma 6.1, the operator $\hat{\mathbf{B}}^*$ at infinity admits a proper functional setting in the same metric of $L^2_{\rho^*}$. However, it is also singular at the origin $y = 0$, where its setting depends on the principal part \mathbf{H}^* .

Proposition 7.1 *The symmetric operator \mathbf{H}^* admits a Friedrich’s self-adjoint extension with the domain $H^4_0(B_1)$, discrete spectrum, and compact resolvent in $L^2(B_1)$, where $B_1 \subset \mathbb{R}^N$ is the unit ball, iff*

$$c = pD \leq c_H = \frac{(N-2)^2}{4}. \tag{7.5}$$

Proof. Indeed, (7.5) is just a corollary of the classic *Hardy inequality*

$$\frac{(N-2)^2}{4} \int_{B_1} \frac{u^2}{|y|^2} \leq \int_{B_1} |\nabla u|^2 \quad \text{for } u \in H^1_0(B_1), \tag{7.6}$$

where the constant is sharp. Therefore, (7.5) implies that the operator \mathbf{H}^* is semi-bounded (in say metric of $H^{-1}(B_1)$), whence the necessary properties. For compact embedding of the corresponding spaces, see Maz’ja [45, p. 65, etc.].

It follows that (7.5) holds in the supercritical Joseph–Lundgren range

$$p \geq p_{JL} = 1 + \frac{4}{N-4-2\sqrt{N-1}} \quad \text{for } N \geq 11, \tag{7.7}$$

which, by obvious reasons, coincides with that for (6.22) in [37].

7.3 Inner Region I

Thus, we assume that (7.5) holds and $\sigma(\hat{\mathbf{B}}^*) = \{\hat{\lambda}_k\}$ is discrete, with some eigenfunctions $\{\hat{\psi}_\beta^*, |\beta| = k\}$. Furthermore, it is also convenient to assume that the spectrum is (at least partially) *real*. To justify such an assumption for this non-self-adjoint operator, we rewrite (7.4) in the form

$$\hat{\mathbf{B}}^* = \mathbf{B}^* - c\Delta(\frac{1}{|y|^2} I) - \frac{1}{2(p-1)} I, \quad \text{where } c = pD \tag{7.8}$$

and \mathbf{B}^* is the standard adjoint operator (6.25) with the real spectrum shown in Lemma 6.1. Actually, this means that \mathbf{B}^* admits a natural self-adjoint representation in the space $l^2_{\rho^*}$ of sequences, where it is also sectorial, [21]. Therefore, we claim that the real spectrum of (7.8) can be obtained by branching-perturbation theory (see Kato [39]) from the spectrum

$$\{\lambda_\beta = -\frac{k}{4} - \frac{1}{2(p-1)}, k = |\beta| \geq 0\}$$

of $\mathbf{B}^* - \frac{1}{2(p-1)} I$ at $c = 0$. Indeed, this is proved by the following result.

Proposition 7.2 *The operators (7.8)*

$$\hat{\mathbf{B}}^* = \mathbf{B}^* - c\Delta\left(\frac{1}{|y|^2} I\right) - \frac{1}{2(p-1)} I, \quad \text{where } c = pD,$$

converge to the operator

$$\mathbf{B}^* - \frac{1}{2(p-1)} I,$$

as $c \rightarrow 0$, in the generalized sense of Kato.

Proof. Indeed, for each $u \in W^{2,2}(B_1)$ we have that

$$\left\| \hat{\mathbf{B}}^* u - \left(\mathbf{B}^* - \frac{1}{2(p-1)} I \right) u \right\|_{L^2(B_1)} \leq c \left\| \Delta\left(\frac{1}{|y|^2} I\right) u \right\|_{L^2(B_1)}.$$

Thanks to Hardy’s inequality (7.5), we arrive at

$$\left\| \hat{\mathbf{B}}^* u - \left(\mathbf{B}^* - \frac{1}{2(p-1)} I \right) u \right\|_{L^2(B_1)} \leq cK \|u\|_{W^{2,2}(B_1)},$$

with $K > 0$, a positive constant. Therefore, for any $\varepsilon > 0$, there exists c_0 such that

$$\left\| \hat{\mathbf{B}}^* u - \left(\mathbf{B}^* - \frac{1}{2(p-1)} I \right) u \right\|_{L^2(B_1)} \leq \varepsilon \|u\|_{W^{2,2}(B_1)},$$

for all $c \in (0, c_0)$ and $u \in W^{2,2}(B_1)$.

This shows the convergence of the graphs of the operator $\hat{\mathbf{B}}^*$ to the graph of the operator \mathbf{B}^* and, hence, the previous claim is proved.

Next, the branch must be extended to $c = pD$, which is also a difficult mathematical problem; see [27, § 6] for some extra details, which are not necessary here in such a formal blow-up analysis.

Thus, we fix a certain exponentially decaying pattern in *Inner Region I*:

$$Y(y, \tau) = Ce^{\lambda\beta\tau} \hat{\psi}_\beta^*(y) + \dots \quad \text{as } \tau \rightarrow +\infty \quad (\hat{\lambda}_\beta < 0). \tag{7.9}$$

If there exists $\hat{\lambda} = 0 \in \sigma(\hat{\mathbf{B}}^*)$, the expansion will correspond to a centre subspace one. Note that (7.9) includes all the non-radial linearized blow-up patterns.

7.4 Matching with Inner Region II close to the origin

In order to match (7.9) with a smooth bounded flow close to $y = 0$, which we call *Inner Region II*, one needs the behaviour of the eigenfunction $\hat{\psi}_\beta^*(y)$ as $y \rightarrow 0$. To get this, without loss of generality, we assume the radial geometry. Then, the principal operator in the eigenvalue problem

$$\mathbf{H}^* \hat{\psi}^* + \dots = \lambda \hat{\psi}^* \quad \text{as } y \rightarrow 0 \tag{7.10}$$

yields the following characteristic polynomial

$$\hat{\psi}^*(y) = |y|^\gamma + \dots \implies H_c(\gamma) = (\gamma - 2)(\gamma - 3)[\gamma^2 + (N - 2)\gamma + c] = 0. \tag{7.11}$$

Obviously, the roots $\gamma = 2$ or 3 are not suitable, so that we have

$$\gamma^2 + (N - 2)\gamma + c \implies \gamma_\pm = -\frac{N-2}{2} \pm \sqrt{\frac{(N-2)^2}{4} - c}, \tag{7.12}$$

which makes sense in the subcritical range $c = pD < \frac{(N-2)^2}{4}$.

Consider the most interesting critical case

$$c \equiv pD = c_H = \frac{(N-2)^2}{4}. \tag{7.13}$$

Then, there exists the double root $\gamma_{1,2} = -\frac{N-2}{2} < 0$, which generates two L^2 -behaviours:

$$\hat{\psi}_1^*(y) = |y|^{-\frac{N-2}{2}} \ln |y|(1 + o(1)) \quad \text{and} \quad \hat{\psi}_2^*(y) = |y|^{-\frac{N-2}{2}}(1 + o(1)) \quad \text{as } y \rightarrow 0. \tag{7.14}$$

Note that H_0^1 -approximations of $\hat{\psi}_2^*$ establish that c_H is the best constant in (7.5). Thus, in L^2 in the radial (ODE) setting, the deficiency indexes of \mathbf{H}^* are (2, 2), and the straightforward conclusion on the discreteness of the spectrum of Friedrich’s extension of \mathbf{H}^* follows, [47, p. 90]. Note that this leads to the so-called *principal solution* with the minimally possible growth at the singular point.

Overall, this gives the following behaviour of the proper eigenfunctions at the origin:

$$\hat{\psi}_\beta^*(y) = -v_\beta |y|^{-\frac{N-2}{2}} + \dots \quad \text{as } y \rightarrow 0 \quad (v_\beta > 0 \text{ are normalization constants}). \tag{7.15}$$

This allows detection of the rate of blow-up of such patterns by estimating the maximal value of the expansion near the origin:

$$v_\beta(y, \tau) = C_* |y|^{-\frac{2}{p-1}} - v_\beta C e^{\lambda_\beta \tau} |y|^{-\frac{N-2}{2}} + \dots \quad \text{as } y \rightarrow 0 \quad \text{and} \quad \tau \rightarrow +\infty, \tag{7.16}$$

where we observe the natural condition of matching:

$$v_\beta C > 0. \tag{7.17}$$

Calculating the absolute maximum in y of the function on the right-hand side of (7.17) (this is a standard and justified trick in some R–D problems; see e.g., [12]) yields an exponential divergence:

$$\|v_\beta(\cdot, \tau)\|_\infty = d_\beta e^{\rho_\beta \tau} + \dots, \quad \text{where} \quad \rho_\beta = \frac{4|\hat{\lambda}_\beta|}{(N-2)(p-p_s)} > 0 \quad (p > p_s), \tag{7.18}$$

and $d_\beta > 0$ are some constants. Depending on the spectrum $\{\hat{\lambda}_\beta < 0\}$, (7.18) can determine a countable set of various Type II blow-up asymptotics.

Let us specify more clearly the necessary matching procedure. In a standard manner, we return to the original rescaled equation (6.24) and perform the rescaling in Region II according to (7.18):

$$v(y, \tau) = e^{\rho_\beta \tau} w(\xi, s), \quad \xi = e^{\mu_\beta \tau} y, \quad \mu_\beta = \frac{(p-1)\rho_\beta}{2}, \quad s = \frac{1}{(p-1)\rho_\beta} e^{(p-1)\rho_\beta \tau}. \tag{7.19}$$

Then w solves the following exponentially perturbed uniformly parabolic equation:

$$w_s = -\Delta^2 w - \Delta(|w|^{p-1} w) - \frac{1}{(p-1)\rho_\beta} \frac{1}{s} \left[\left(\frac{1}{4} + \mu_\beta \right) \xi \cdot \nabla w + \left(\frac{1}{2(p-1)} + \rho_\beta \right) w \right]. \tag{7.20}$$

As above, we arrive at a stabilization problem as $s \rightarrow +\infty$ to a bounded stationary solution, which is widely used in blow-up applications (see examples in [32]). In general, once the uniform boundedness of the orbit $\{w(s), s > 0\}$ is established (an open problem), the passage to the limit in (7.20) as $s \rightarrow +\infty$ is a standard issue of asymptotic parabolic theory, even for the present higher-order case. Recall that the limit equation

$$w_s = -\Delta^2 w - \Delta(|w|^{p-1} w)$$

is a gradient system in H^{-1} ; cf. (4.2), (4.3).

Our blow-up patterns correspond to the stabilization uniformly on compact subsets:

$$w(\xi, s) \rightarrow W(\xi), \quad s \rightarrow +\infty, \quad \text{where } \Delta W + |W|^{p-1}W = 0, \quad \xi \in \mathbb{R}^N, \quad W(0) = d_\beta, \quad (7.21)$$

for all admissible $|\beta| = 0, 1, 2, \dots$. As is well-known, for $p \in (1, p_S)$, the stationary problem in (7.21) does not admit nontrivial nonnegative solutions, while for any $p \geq p_S$, such solutions always exist. This is not different from the analysis in [37] for (6.22), so we can omit some details.

The supercritical case $p > p_{JL}$ is analyzed similarly, with some natural changes in asymptotics of eigenfunctions and in equations such as (7.20); see [24, § 5].

Let us comment on an extended semigroup for $t > T$. Since according to our construction, this Type II blow-up leaves less singular final time profile $u(x, T^-)$ than the SSS (see (7.17)), the similarity Type I blow-up via (6.1) are expected to be extensible for $t > T$, so that this blow-up is expected to be incomplete. However, this does not guarantee uniqueness of such an extension at all, which is always a hard problem. Moreover, sometimes, for special kinds of singularities for nonlinear PDEs the uniqueness problem is not solvable (a so-called *principal non-uniqueness*; see an example in [25]).

7.5 On related non-radial blow-up patterns

These can be predicted in a couple of ways. Firstly, one can start with a non-radial SSS solving the elliptic equation in (6.19), if such solutions exist. Secondly, under the condition (7.5), a non-radial eigenfunction $\psi_\beta^*(y)$ (e.g., corresponding to an “angular” logarithmic blow-up swirl obtained by angular separation of variables, see [24, § 3]) of $\hat{\mathbf{B}}^*$ can be taken into account. Then the matching will assume using non-radial entire solutions of (7.9), which deserves further study.

References

- [1] O.V. Admaev and V.V. Pukhnachev, *Self-similar solutions of the equation $u_t + \Delta^2 u + \Delta(u^2) = 0$* , Preprint No. 3, Comp. Centre, Krasnoyarsk, 1997.
- [2] P. Álvarez-Caudevilla and V.A. Galaktionov, *Countable families of solutions of a semilinear fourth-order elliptic equation of Cahn–Hilliard type in \mathbb{R}^N* , in preparation.
- [3] M. Berger, *Nonlinearity and Functional Analysis*, Acad. Press, New York, 1977.
- [4] A.J. Bernoff and A.L. Bertozzi, *Singularities in a modified Kuramoto–Sivashinsky equation describing interface motion for the phase transition*, Phys. D. **85** (1995), 375–404.
- [5] T. Boggio, *Sull’equilibrio delle piastre elastiche incastrate*, Rend. Acc. Lincei **10** (1901), 197–205.
- [6] T. Boggio, *Sulle funzioni di Green d’ordine m* , Rend. Circ. Mat. Palermo **20** (1905), 97–135.
- [7] H. Brezis, *Analyse fonctionnelle. Collection Mathématiques Appliquées pour la Maîtrise*. [Collection of Applied Mathematics for the Masters Degree]. Masson, Paris, 1983. *Théorie et applications*. [Theory and applications],
- [8] K.J. Brown and T.-F. Wu, *A fibering map approach to a semilinear elliptic boundary value problem*, Electronic J. Differ. Equat. **69** (2007), 1–9.

- [9] K.J. Brown and T.-F. Wu, *A fibering map approach to potential operator equation and its applications*, Differ. Int. Equat. **22** (2009), 1097–1114.
- [10] C.J. Budd, V.A. Galaktionov, and J.F. Williams, *Self-similar blow-up in higher-order semilinear parabolic equations*, SIAM J. Appl. Math. **64** (2004), 1775–1809.
- [11] D.C. Clark, *A variant of Lusternik–Schnirelman theory*, Indiana Univ. Math. J. **22** (1972), 65–74.
- [12] J.W. Dold, V.A. Galaktionov, A.A. Lacey, and J.L. Vazquez, *Rate of approach to a singular steady state in quasilinear reaction-diffusion equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1998), 663–687.
- [13] P. Drábek and S.I. Pohozaev, *Positive solutions for the p -Laplacian: application of the fibering method*, Proc. Roy. Soc. Edinburgh **127A** (1997), 703–726.
- [14] Yu.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, and S.I. Pohozaev, *Global solutions of higher-order semilinear parabolic equations in the supercritical range*, Adv. Differ. Equat. **9** (2004), 1009–1038.
- [15] S.D. Eidelman, *Parabolic Systems*, North-Holland Publ. Comp., Amsterdam/London, 1969.
- [16] U. Elias, *Eigenvalue problems for the equation $Ly + p(x)y = 0$* , J. Differ. Equat. **29** (1978), 28–57.
- [17] J.D. Evans, V.A. Galaktionov, and J.F. Williams, *Blow-up and global asymptotics of the limit unstable Cahn–Hilliard equation*, SIAM J. Math. Anal. **38** (2006), 64–102.
- [18] S. Filippas, M.A. Herrero, and J.J.L. Velázquez, *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, Proc. R. Soc. London A **456** (2000), 2957–2982.
- [19] A. Friedman, *Partial Differential Equations*, Robert E. Krieger Publ. Comp., Malabar, 1983.
- [20] J. Funada, *Nonlinear Diffusion Equation for interfacial waves generating by the Marangoni effect*, Notes Res. Inst. Math. Sci. Kyoto Univ. **510** (1984), 64–102.
- [21] V.A. Galaktionov, *Sturmian nodal set analysis for higher-order parabolic equations and applications*, Adv. Differ. Equat. **12** (2007), 669–720.
- [22] V.A. Galaktionov, *On blow-up space jets for the Navier–Stokes equations in \mathbb{R}^3 with convergence to Euler equations*, J. Math. Phys. **49** (2008), 113101.
- [23] V.A. Galaktionov, *On blow-up “twistors” for the Navier–Stokes equations in \mathbb{R}^3 : a view from reaction-diffusion theory*, arXiv:0901.4286.
- [24] V.A. Galaktionov, *Five types of blow-up in a semilinear fourth-order reaction-diffusion equation: an analytic-numerical approach*, Nonlinearity **22** (2009), 1695–1741 (arXiv:0901.4307).
- [25] V.A. Galaktionov, *Shock waves and compactons for fifth-order nonlinear dispersion equations*, Europ. J. Appl. Math. **21** (2010), 1–50 (an earlier preprint in arXiv:0911.4446).
- [26] V.A. Galaktionov, *Incomplete self-similar blow-up in a semilinear fourth-order reaction-diffusion equation*, Stud. Appl. Math. **124** (2010), 347–381. (arXiv:0902.1090).
- [27] V.A. Galaktionov and I.V. Kamotski, *On nonexistence of Baras–Goldstein type for higher-order parabolic equations with singular potentials*, Trans. Amer. Math. Soc. **362** (2010), 4117–4136 (arXiv:0901.4171).
- [28] V.A. Galaktionov, E. Mitidieri, and S.I. Pohozaev, *On global solutions and blow-up for Kuramoto–Sivashinsky–type models and well-posed Burnett equations*, Nonlinear Anal. **70** (2009), 2930–2952 (arXiv:0902.0257).

- [29] V.A. Galaktionov, E. Mitidieri, and S.I. Pohozaev, *Variational approach to complicated similarity solutions of higher-order nonlinear evolution equations of parabolic, hyperbolic, and nonlinear dispersion types*, In: Sobolev Spaces in Mathematics. II, Appl. Anal. and Part. Differ. Equat., Series: Int. Math. Ser., Vol. **9**, V. Maz'ya Ed., Springer, New York, 2009 (an extended version in arXiv:0902.1425).
- [30] V.A. Galaktionov, E. Mitidieri, and S.I. Pohozaev, *Variational approach to complicated similarity solutions of higher-order nonlinear PDEs. II*, Nonl. Anal.: RWA **12** (2011), 2435–2466 (arXiv:1103.2643).
- [31] V.A. Galaktionov and J.L. Vazquez, *The problem of blow-up in nonlinear parabolic equations*, Discr. Cont. Dyn. Syst. **8** (2002), 399–433.
- [32] V.A. Galaktionov and J.L. Vazquez, *A Stability Technique for Evolution Partial Differential Equations. A Dynamical Systems Approach*, Birkhäuser, Boston/Berlin, 2004.
- [33] V.A. Galaktionov and J.F. Williams, *On very singular similarity solutions of a higher-order semilinear parabolic equation*, Nonlinearity **17** (2004), 1075–1099.
- [34] H.-C. Grunau and G. Sweers, *The maximum principle and positive principal eigenfunctions for polyharmonic equations*, Reaction Diffusion Systems (Trieste, 1995), Lecture Notes in Pure and Appl. Math. **194**, Dekker, New York, (1998), 163–182.
- [35] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, RI, 1988.
- [36] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differ. Geom., Vol. II (Cambridge, MA, 1993), pp. 7–136, Int. Press, Cambridge, MA, 1995.
- [37] M.A. Herrero and J.J.L. Velázquez, *Blow-up of solutions of supercritical semilinear parabolic equations*, C.R. Acad. Sci. Paris Sér. I Math. **319** (1994), 141–145.
- [38] N.H. Ibragimov, *On the group classification of differential equations of second order*, Soviet Math. Dokl. **9** (1968), 1365–1369.
- [39] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin/New York, 1976.
- [40] M.A. Krasnosel'skii, *Vector fields which are symmetric with respect to a subspace*, Dokl. Acad. Nauk Ukrain. SSR, No. 1 (1951).
- [41] M.A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford/Paris, 1964.
- [42] M.A. Krasnosel'skii and P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin/Tokyo, 1984.
- [43] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
- [44] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, In: Contributions to Analysis, Acad. Press, New York, 1974, pp. 245–272.
- [45] V. Maz'ja, *Sobolev Spaces*, Springer-Verlag, Berlin/Tokyo, 1985.
- [46] N. Mizoguchi, *Rate of Type II blowup for a semilinear heat equation*, Math. Ann., **339** (2007), 839–877.
- [47] M.A. Naimark, *Linear Differential Operators, Part 1*, Frederick Ungar Publ. Co., New York, 1967.
- [48] Y. Naito and T. Suzuki, *Existence of type II blow-up solutions for a semilinear heat equation with critical nonlinearity*, J. Differ. Equat. **232** (2007), 176–211.
- [49] A. Novick-Cohen, *Blow up and growth in the directional solidification of dilute binary alloys*, Appl. Anal. **47** (1992), 241–257.

[50] A. Novick-Cohen, *The Cahn-Hilliard equation: mathematical and modeling perspectives*, Adv. Math. Sci. Appl. **8** (1998), 965-985.

[51] A. Novick-Cohen, The Cahn-Hilliard equation, Handbook of Differential Equations: Evolutionary Equations, Vol. IV, 201–228, Handb. Differ. Equat., Elsevier/North-Holland, Amsterdam, 2008.

[52] A. Novick-Cohen and M. Grinfeld, *Counting the stationary states of the Sivashinsky equation*, Nonlinear Anal. TMA **24**, 4, (1995), 875 – 881.

[53] A. Novick-Cohen and L.A. Segel, *Nonlinear aspects of the Cahn-Hilliard equation*, Physica D **10** (1984), 277-298.

[54] S.I. Pohozaev, *On an approach to nonlinear equations*, Soviet Math. Dokl. **20** (1979), 912–916.

[55] S.I. Pohozaev, *The fibering method in nonlinear variational problems*, Pitman Research Notes in Math., Vol. **365**, Pitman, 1997, pp. 35–88.

[56] P. Rabinowitz, *Variational methods for nonlinear eigenvalue problems*, In: Eigenvalue of Nonlinear Problems, Edizioni Cremonese, Rome, 1974, pp. 141–195.

[57] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin/New York, 1995.

[58] T.P. Witelski, A.J. Bernoff, and A.L. Bertozzi, *Blow-up and dissipation in a critical-case unstable thin film equation*, Europ. J. Appl. Math. **15** (2004), 223–256.

Appendix a: towards existence of similarity blow-up profiles of (6.2)

Without loss of generality, we consider the ODE (6.2) for $N = 1$ (similar ideas apply to the radial case for any $N \geq 1$ and $p < p_S$):

$$\begin{cases} -f^{(4)} - \frac{1}{4}yf' - \frac{1}{2(p-1)}f - (|f|^{p-1}f)'' = 0 & \text{in } \mathbb{R}_+, \\ f'(0) = f'''(0) = 0, \end{cases} \tag{A.1}$$

where we have put two symmetry boundary conditions at the origin. The strategy of proving existence of a solution of (A.1) is to use the 2D asymptotic bundle (6.4) to “shoot” two symmetry conditions at $y = 0$. This shooting is well defined:

Claim 1. *For any $A, C \in \mathbb{R}$, the function $f = f(y; A, C)$ is well defined for all $y \in [0, \infty)$.*

Indeed, this follows from the obvious fact that the principal and leading operators,

$$f^{(4)} = -(|f|^{p-1}f)'' + \dots \implies f'' = -|f|^{p-1}f + \dots$$

do not allow finite- y blow-up.

One can see that, to match two boundary conditions in (A.1), specific “oscillatory” properties of solutions $f(y; A, C)$ are necessary. Then a “min-max”-like procedure can be applied, when we first, for a fixed parameter A , change C in such a manner to get existence of

$$C^+(A) = \inf \{C \in \mathbb{R} : f'(0; A, C) > 0\}. \tag{A.2}$$

Existence of such an $C^+(A)$ is guaranteed at least for $A \gg 1$ by necessary oscillatory properties of solutions. We next start to decrease A in such a manner to guarantee that, for some A^+ , we obtain the necessary solution:

$$f'''(0; A^+, C^+(A^+)) = 0 \implies \exists f(y) = f(y; A^+, C^+(A^+)), \tag{A.3}$$

but strong oscillatory properties are again required.

Claim 2. *There exists a subset of solutions (A.1), which are oscillatory close to the origin.*

Note that this is not straightforward: e.g., the bundle (6.4) is not oscillatory at all. Therefore, we have to find oscillatory structures of solutions, which are *not small*. To get a functional “topology” of oscillatory solutions, we perform the similarity scaling (not invariant) in (A.1),

$$f(y) = \varepsilon^{-\frac{1}{2(p-1)}} g(z), \quad z = \frac{y}{\varepsilon^{1/4}}, \quad \varepsilon > 0, \tag{A.4}$$

to get the following singularly perturbed ODE for $\varepsilon \ll 1$:

$$-\varepsilon g^{(4)} - \frac{1}{4} z g' - \frac{1}{2(p-1)} g - (|g|^{p-1} g)' = 0. \tag{A.5}$$

It follows that we can describe a set of solutions which are oscillatory about the limit profile $g_0(z)$ satisfying

$$\begin{cases} -\frac{1}{4} z g_0' - \frac{1}{2(p-1)} g_0 - (|g_0|^{p-1} g_0)'' = 0, & z > 0, \\ g_0(0) = 1, \quad g_0'(0) = 0. \end{cases} \tag{A.6}$$

One can check by maximum principle arguments that such a $g_0(z) > 0$ exists on some interval $z \in [0, z_0)$ and is strictly monotone there.

We next perform the linearization about g_0 by introducing the new fast variable Z :

$$g(z) = g_0(z) + G(Z), \quad \text{where } Z = \frac{z}{\sqrt{\varepsilon}}. \tag{A.7}$$

Substituting (A.7) into (A.5) and performing the linearization in the last term, we conclude that there exists a subset of solutions satisfying, uniformly on compacts in Z the linearized ODE

$$-G_Z^{(4)} - p(g_0^p(z) G)_Z'' + \dots = 0, \tag{A.8}$$

where we omit further linear and nonlinear terms of the order, at least, $O(\varepsilon)$. Since $g_0(z) = 1 + o(1)$ on such compacts, the linearized ODE admits further simplification:

$$-G^{(4)} - p G'' = 0. \tag{A.9}$$

It follows that there exists a 2D subset of purely oscillatory solutions about $g_0(z)$ with the typical behaviour, for $\varepsilon \ll 1$,

$$g(z) = g_0(z) + B_1 \cos\left(\frac{\sqrt{p}z}{\sqrt{\varepsilon}}\right) + B_2 \sin\left(\frac{\sqrt{p}z}{\sqrt{\varepsilon}}\right) + \dots, \quad \text{where } B_{1,2} \in \mathbb{R}. \tag{A.10}$$

We stop at this moment our analysis and refer to [17, § 4] for a similar and much more detailed study of such oscillatory solutions of the third-order ODE (A.1) for $p = 3$.

Finally, we claim that, using the oscillatory bundle (A.10), it is possible to prove existence of a finite limit (A.2) for $|A| \gg 1$. Extending this strategy further to get (A.3) is more difficult, but seems doable (in finite time).

Anyway, the above analysis clearly shows (but does not prove completely rigorously) existence of a first blow-up profile $f(y)$. What is very difficult and remains entirely open is how to catch a possible multiplicity of

solutions. It seems that the ideas of a μ -bifurcation analysis (see Section 6.2 and [10, § 4] for the RD equation (6.22)), when (A.2) is replaced by the equation

$$-f^{(4)} - \mu y f' - \frac{1}{2(p-1)} f - (|f|^{p-1} f)'' = 0 \quad \text{in } \mathbb{R}_+, \quad (\text{A.11})$$

changing μ up to the required $\mu = \frac{1}{4}$ and using the discrete spectrum of the linearized operator (cf. (6.25))

$$\mathbf{B}^*(\mu) = -\Delta^2 - \mu y \cdot \nabla, \quad (\text{A.12})$$

are not applicable for the C–H-type nonlinearities in the divergent form.

Finally, let us mention that, for $p \neq p_0 = 3$, when (A.1) reduces to the third order, we have not succeeded in obtaining similarity profiles numerically by solving the ODE. Recall, that Figure 2 was obtained by a PDE numerical modelling.