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Symmetrization for Neumann Anisotropic Problems and Related Questions

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Abstract

In this paper we prove some comparison results for Neumann elliptic problems whose model involves the anisotropic Laplacian

$$\Delta_H u = \operatorname{div}\left(H(Du)H_\xi(Du)\right),$$

where H is a positively homogeneous convex function. Finally, we find a Poincaré inequality in the anisotropic setting.

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1 Introduction

Let Ω be a bounded connected open set of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary. The classical relative isoperimetric inequality states that

$$C_N = \inf_{E\subset\Omega} \frac{P(E;\Omega)}{(\min\{|E|, |\Omega\setminus E|\})^{1-\frac{1}{N}}} > 0 \tag{1.1}$$

(see, for example, [14],[22],[10]). Here $|E|$ is the Lebesgue measure of E, and $P(E;\Omega)$ is the usual relative perimeter in Ω. As matter of fact, it is known that, when Ω is a ball of $\mathbb{R}^N$ (see [22], [10]) or, when $N = 2$ and Ω is convex (see [11]), the infimum in (1.1) is actually a minimum and the minimizers can be characterized.

It is well-known that the inequality (1.1) is a basic tool to prove some comparison results for elliptic problems with Neumann boundary condition, and that it is related to some Poincaré inequalities. For example, in [20] the authors consider the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$. They prove a comparison result between the solutions of (1.2) and the solutions of a "symmetrized" problem which is not of the same type of (1.2). More precisely, they compare the positive and negative parts of u with the solutions of two suitable Dirichlet problems involving C_N, and defined in two balls having measure $|\Omega|/2$. Similar results are contained in [3], where a zero order term is also considered.

Furthermore, in [9] the eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is considered, and a reverse Hölder inequality for eigenfunctions u of (1.3) corresponding to the first nonzero eigenvalue μ_1, is proved; namely

$$||u||_{L^p} \leq C||u||_{L^1}. \tag{1.4}$$

Inequality (1.4) is a consequence of a comparison result between an eigenfunction of (1.3) and a solution v of a suitable "symmetrized" Dirichlet problem involving the best constant C_N of (1.1).

In this paper, using the so-called convex symmetrization (see [1]), we show that analogous comparison results hold when we replace in (1.2) and (1.3) the Laplace operator with an "anisotropic" Laplacian, that is

$$\Delta_H u = \operatorname{div}(H(Du)H_\xi(Du)),$$

where H is an arbitrary norm on $\mathbb{R}^N$ (see Section 2 for the precise assumptions on H).

In this setting, a key role is played by the anisotropic relative isoperimetric inequality:

$$K_N = \inf_{E\subset\Omega} \frac{P_H(E;\Omega)}{(\min\{|E|, |\Omega\setminus E|\})^{1-\frac{1}{N}}} > 0, \tag{1.5}$$

where $P_H(E;\Omega)$ denotes the relative perimeter of E in Ω with respect to H. When E has sufficiently smooth boundary, $P_H(E;\Omega)$ is given by

$$P_H(E;\Omega) = \int_{\partial E\cap\Omega} H(n_E)\, d\mathcal{H}^{N-1},$$

where n_E denotes the unit outer normal to E (see Section 2 for the details).

We stress that when $N = 2$, the inequality (1.5) is proved in [13], where, under the additional assumption of convexity of Ω, the authors prove that K_N is attained, and the minimizers are characterized.

Finally, it is well-known that the classical Poincaré inequality in $BV(\Omega)$, namely

$$\|u - (u)_\Omega\|_{L^q(\Omega)} \le c_q \int_\Omega |Du|, \tag{1.6}$$

with $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$, is related to the relative isoperimetric inequality (1.1). In particular, in [12] it is proved that the best constant in (1.6) is proportional to C_N. Moreover, when Ω is a ball the author computes explicitly the best constant. Here we prove (1.6) in the anisotropic setting, showing that the best constant is proportional to K_N. Moreover, by means of some examples, we describe some differences with the Euclidean case.

The paper is organized as follows. In Section 2 we give the precise definition of anisotropic perimeter, and recall some basic facts about rearrangements and convex symmetrization. In Sections 3 and 4 we prove the comparison results. Finally, in Section 5 the anisotropic Poincaré inequality is studied.

2 Notation and preliminaries

Let $N \ge 2$, and $H : \mathbb{R}^N \to [0,+\infty[$ be a $C^2(\mathbb{R}^N \setminus \{0\})$ function such that $H^2(\xi)$ is strictly convex and

$$H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^N,\ \forall t \in \mathbb{R}. \tag{2.1}$$

Moreover, suppose that there exist two positive constants $\alpha \le \beta$ such that

$$\alpha|\xi| \le H(\xi) \le \beta|\xi|, \quad \forall \xi \in \mathbb{R}^N. \tag{2.2}$$

The polar function $H^o\colon \mathbb{R}^N \to [0,+\infty[$ of H is

$$H^o(v) = \sup_{\xi\neq 0} \frac{\xi\cdot v}{H(\xi)}.$$

It is easy to verify that also H^o is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$H(v) = \sup_{\xi\neq 0} \frac{\xi\cdot v}{H^o(\xi)}.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^N\colon H^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. We put $\kappa_N = |\mathcal{W}|$ and, for $r > 0$, denote with $\mathcal{W}_r$ the set $r\mathcal{W} = \{r\xi,\ \xi \in \mathcal{W}\}$. The following properties of H and H^o hold true (see for example [6]):

$$H(\nabla H^o(\xi)) = H^o(\nabla H(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\},$$
$$H^o(\xi)\nabla H(\nabla H^o(\xi)) = H(\xi)\nabla H^o(\nabla H(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Let Ω be a bounded open set of $\mathbb{R}^N$. The total variation of a function $u \in BV(\Omega)$ with respect to H is (see [4]):

$$\int_\Omega |Du|_H = \sup\left\{\int_\Omega u \operatorname{div}\sigma dx\colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ H^o(\sigma) \le 1\right\}.$$

This yields the following definition of anisotropic relative perimeter of $F \subset \mathbb{R}^N$ in Ω:

$$P_H(F;\Omega) = \int_\Omega |D\chi_F|_H = \sup\left\{\int_F \operatorname{div}\sigma dx\colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ H^o(\sigma) \le 1\right\}.$$

The equality

$$P_H(F;\Omega) = \int_{\Omega\cap\partial^* F} H(n_F) d\mathcal{H}^{N-1}$$

holds, where $\partial^* F$ is the reduced boundary of F and n_F is the outer normal to F (see [4]).

The anisotropic relative perimeter of a set F in Ω is finite if and only if the usual Euclidean perimeter

$$P(F;\Omega) = \sup\left\{\int_F \operatorname{div}\sigma dx\colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ |\sigma| \le 1\right\}$$

is finite. Indeed, by properties (2.1) and (2.2) we have that

$$\frac{1}{\beta}|\xi| \le H^o(\xi) \le \frac{1}{\alpha}|\xi|,$$

and then

$$\alpha P(E;\Omega) \le P_H(E;\Omega) \le \beta P(E;\Omega).$$

We recall that if $u \in W^{1,1}(\Omega)$, then (see [4])

$$\int_\Omega |Du|_H = \int_\Omega H(Du)dx.$$

Reasoning as in [13], from the well-known relative isoperimetric inequality (see for instance [10],[14],[22]) and the properties of H we have the following result:

Theorem 2.1 *Let Ω be an bounded connected open set of $\mathbb{R}^N$, with Lipschitz boundary. Then a relative isoperimetric inequality with respect to the anisotropic perimeter holds. Namely, there exists a constant $K_N > 0$ such that*

$$K_N = \inf_{E\subset\Omega} \frac{P_H(E;\Omega)}{(\min\{|E|, |\Omega \setminus E|\})^{1-\frac{1}{N}}}. \tag{2.3}$$

Finally, we recall some basic definitions on rearrangements. Let Ω be a bounded open set of $\mathbb{R}^N$, and $u : \Omega \to \mathbb{R}$ be a measurable function, and denote with $|\Omega|$ the Lebesgue measure of Ω.

The distribution function of u is the map $\mu_u : \mathbb{R} \to [0, \infty[$ defined by

$$\mu_u(t) = |\{x \in \Omega : u(x) > t\}|.$$

Such function is decreasing and right continuous.

The decreasing rearrangement of u is the map $u^* : [0, \infty[\to \mathbb{R}$ defined by

$$u^*(s) := \sup\{t \in \mathbb{R} : \mu_u(t) > s\}.$$

The function u^* is the generalized inverse of μ_u.

Following [1], the convex symmetrization of u is the function $u^\star(x)$, $x \in \Omega^\star$ defined by:

$$u^\star(x) = u^*(\kappa_N H^o(x)^N),$$

where $\Omega^\star$ is a set homothetic to the Wulff shape having the same measure of Ω.

In addition to the above rearrangements, it is useful to consider the increasing rearrangement of u, that is the function

$$u_*(s) = u^*(|\Omega| - s), \quad s \in [0, |\Omega|].$$

Moreover, for $x \in \Omega^\star$, we put

$$u_\star(x) = u_*(\kappa_N H^o(x)^N).$$

Finally, we recall a property that will be useful in the next sections.

Lemma 2.1 *Let $f_1(s)$, $f_2(s)$ be measurable, nonnegative functions such that*

$$\int_0^r f_1(s)ds \le \int_0^r f_2(s)ds, \quad \forall r \in [0, \delta].$$

If $g \ge 0$ is a decreasing function then:

$$\int_0^r f_1(s)g(s)ds \le \int_0^r f_2(s)g(s)ds, \quad \forall r \in [0, \delta].$$

3 Comparison results

We consider the following homogeneous Neumann boundary value problem:

$$\begin{cases} -\mathrm{div}\,(a(x, u, Du)) + c(x)u = f & \text{in } \Omega, \\ a(x, u, Du) \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \ge 2$, with Lipschitz boundary, n is the outer normal to $\partial\Omega$, $a(x, s, \xi) \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function verifying

$$a(x, s, \xi) \cdot \xi \ge (H(\xi))^2, \quad \forall \xi \in \mathbb{R}^N, \tag{3.2}$$

and $c(x) \in L^\infty(\Omega)$, with $c(x) \ge 0$, $c(x) \not\equiv 0$ in Ω. Moreover, $f \in L^p(\Omega)$, with $p = \frac{2N}{N+2}$ if $N \ge 3$, or $p > 1$ if $N = 2$.

The following result allows us to compare a solution of (3.1) with the solution of a problem that has some simmetry properties but which is not of the same type of (3.1).

Theorem 3.1 *Let $u \in W^{1,2}(\Omega)$ be a solution to* (3.1)*, and consider the following problem:*

$$\begin{cases} -\gamma\Delta_H \overline{v} + c_\star \overline{v} = f^\star & \text{in } \mathcal{W}_\lambda \\ -\gamma\Delta_H \underline{v} + c^\star \underline{v} = f_\star & \text{in } \mathcal{W}_\lambda \\ \overline{v} - \underline{v} = 0 & \text{on } \partial\mathcal{W}_\lambda \\ \Big[H(D\overline{v})H_\xi(D\overline{v}) + H(D\underline{v})H_\xi(D\underline{v})\Big]\cdot n = 0 & \text{on } \partial\mathcal{W}_\lambda, \end{cases} \tag{3.3}$$

where $\Delta_H u = \operatorname{div}(H(Du)H_\xi(Du))$, $\gamma = \frac{K_N^2}{N^2\kappa_N^{2/N}}$, with $\kappa_N = |\mathcal{W}|$, and λ is such that $|\mathcal{W}_\lambda| = |\Omega|/2$, namely $\lambda = (\frac{|\Omega|}{2\kappa_N})^{1/N}$. If the couple $(\overline{v}, \underline{v})$, with $\overline{v}, \underline{v} \in W^{1,2}(\mathcal{W}_\lambda)$, is the solution of (3.3)*, defining*

$$v(s) = \begin{cases} \overline{v}^*(s) & \text{if } s \in [0, |\Omega|/2], \\ \underline{v}_*(|\Omega| - s) & \text{if } s \in]|\Omega|/2, |\Omega|], \end{cases}$$

we have

$$\int_0^s u^*(t)\,dt \le \int_0^s v^*(t)\,dt, \quad s \in [0, |\Omega|].$$

Proof. Let $u \in W^{1,2}(\Omega)$ be a solution to (3.1). Using the test function

$$\varphi(x) = \begin{cases} 0 & u(x) < t, \\ u(x) - t & t \le u(x) < t + h, \\ h & t + h \le u(x), \end{cases}$$

in (3.1), by (3.2) and letting $h \to 0$ it follows that

$$-\frac{d}{dt}\int_{\{u>t\}} H(Du)^2\,dx + \int_{\{u>t\}} cu\,dx \le \int_{\{u>t\}} f\,dx. \tag{3.4}$$

On the other hand, using the Fleming-Rishel formula for the anisotropic perimeter (see [4]),

$$\int_{\{u>t\}} H(Du)\,dx = \int_t^{+\infty} P_H(\{u > s\}; \Omega)\,ds,$$

we get

$$-\frac{d}{dt}\int_{\{u>t\}} H(Du)dx = P_H(\{u > t\}; \Omega)$$

for almost every $t > 0$. Hence, by the Cauchy-Schwarz inequality we get from (3.4) that

$$P_H^2(\{u > t\}; \Omega) \le -\mu'(t)\int_{\{u>t\}} (f - cu)\,dx. \tag{3.5}$$

Now we want to estimate the left-hand side of (3.5) with the relative isoperimetric inequality (2.3). To this aim, we need to distinguish the cases when $\{u > t\}$ has measure less or greater than $|\Omega|/2$.

Hence, denoted by $t_0 = \inf\{t\colon \mu(t) \le |\Omega|/2\}$, by (2.3), and using the Hardy-Littlewood inequality and the properties of rearrangements, from (3.5) we get

$$K_N^2 \frac{\mu(t)^{2-\frac{2}{N}}}{-\mu'(t)} \le \int_0^{\mu(t)} (f^* - c_* u^*)\, ds, \text{ for a.e. } t \ge t_0,$$
$$K_N^2 \frac{(|\Omega| - \mu(t))^{2-\frac{2}{N}}}{-\mu'(t)} \le \int_0^{\mu(t)} (f^* - c_* u^*)\, ds, \text{ for a.e. } t \le t_0.$$

Making the change of variable $s = \mu(t)$ (see [2] for details), we get

$$q(s)(-u^*(s))' \le \int_0^s (f^* - c_* u^*)\, ds, \quad \text{for a.e. } s \in [0, |\Omega|], \tag{3.6}$$

where $q(s)$ is the function

$$q(s) = \begin{cases} K_N^2 s^{2-\frac{2}{N}}, & \text{if } 0 \le s \le |\Omega|/2, \\ K_N^2 (|\Omega| - s)^{2-\frac{2}{N}}, & \text{if } |\Omega|/2 \le s \le |\Omega|. \end{cases}$$

Observe that

$$\int_0^{|\Omega|} c_* u^*\, ds \le \int_0^{|\Omega|} f^*\, ds. \tag{3.7}$$

Now we reason exactly as in [3], comparing u^* with the solution of the problem

$$\begin{cases} -q(s)v'(s) + \displaystyle\int_0^s c_* v\, ds = \int_0^s f^* ds, & s \in [0, |\Omega|] \\ \displaystyle\int_0^{|\Omega|} c_* v\, ds = \int_0^{|\Omega|} f^*\, ds. \end{cases} \tag{3.8}$$

By a contradiction argument, it is easy to show that the solution $v(s)$ is decreasing in $[0, |\Omega|]$ (see [3]). Moreover, the couple $(\overline{v}, \underline{v})$, with

$$\overline{v}(x) = v(\kappa_N H^o(x)^N), \quad \underline{v}(x) = v(|\Omega| - \kappa_N H^o(x)^N), \quad x \in \mathcal{W}_\lambda,$$

is the unique solution of the problem (3.3) (see Lemma 3.1).

Now we consider the function $w = u - v$, and by (3.6), (3.7) and (3.8), we get

$$\begin{cases} -q(s)w'(s) + \displaystyle\int_0^s c_* w\, ds \le 0 & s \in [0, |\Omega|], \\ \displaystyle\int_0^{|\Omega|} c_* w\, ds \le 0. \end{cases} \tag{3.9}$$

Setting

$$W(s) = \int_0^s c_* w\, dt, \quad s \in]0, |\Omega|],$$

we claim that $W(s) \le 0$ for any s.

To this aim, we first observe that (3.9) can be written as follows:

$$\begin{cases} -w'(s) + q^{-1}(s) \displaystyle\int_0^s c_* w \, ds \leq 0, & s \in [0, |\Omega|], \\ \displaystyle\int_0^{|\Omega|} c_* w \, ds \leq 0. \end{cases} \tag{3.10}$$

Multiplying both sides of (3.10) by W^+, and integrating in $[0, |\Omega|]$ we obtain that

$$\int_{\{W>0\}} c_* w^2 \, ds + \int_{\{W>0\}} q^{-1} W^2 \, ds \leq 0.$$

Hence $W^+ \equiv 0$ in $[0, |\Omega|]$, and finally by Lemma 2.1 we get the thesis.

Lemma 3.1 *There exists a unique couple* $(\overline{v}, \underline{v})$ *with* $\overline{v}, \underline{v} \in W^{1,2}(\mathcal{W}_\lambda)$ *which solves* (3.3).

Proof. The existence of a solution of (3.3) follows by looking for functions $\overline{v}(x) = \overline{\varphi}(k_N H^o(x)^N)$, $\underline{v}(x) = \underline{\varphi}(|\Omega| - \kappa_N H^o(x)^N)$, $x \in \mathcal{W}_\lambda$. Hence, (3.3) becomes a system of ODEs, which is solvable. Now we prove that the solution is unique. Let $(\overline{v}_1, \underline{v}_1)$ and $(\overline{v}_2, \underline{v}_2)$ be two solutions of (3.3), and consider $\overline{w} = \overline{v}_1 - \overline{v}_2$ and $\underline{w} = \underline{v}_1 - \underline{v}_2$. Using $\overline{w}$ as test function in the first equation of (3.3), solved by $\overline{v}_1$ and $\overline{v}_2$ respectively, and subtracting we get

$$\gamma \int_{\mathcal{W}_\lambda} [H(D\overline{v}_1) H_\xi(D\overline{v}_1) - H(D\overline{v}_2) H_\xi(D\overline{v}_2)] \cdot D\overline{w} \, dx + \\ - \gamma \int_{\partial \mathcal{W}_\lambda} \overline{w} [H(D\overline{v}_1) H_\xi(D\overline{v}_1) - H(D\overline{v}_2) H_\xi(D\overline{v}_2)] \cdot n \, d\sigma + \int_{\mathcal{W}_\lambda} c_\star \overline{w}^2 \, dx = 0. \tag{3.11}$$

Similarly, using $\underline{w}$ in the second equation of (3.3), we get

$$\gamma \int_{\mathcal{W}_\lambda} [H(D\underline{v}_1) H_\xi(D\underline{v}_1) - H(D\underline{v}_2) H_\xi(D\underline{v}_2)] \cdot D\underline{w} \, dx + \\ - \gamma \int_{\partial \mathcal{W}_\lambda} \underline{w} [H(D\underline{v}_1) H_\xi(D\underline{v}_1) - H(D\underline{v}_2) H_\xi(D\underline{v}_2)] \cdot n \, d\sigma + \int_{\mathcal{W}_\lambda} c^\star \underline{w}^2 \, dx = 0. \tag{3.12}$$

By summing (3.11) and (3.12), and recalling the boundary conditions in (3.3) we obtain that

$$\gamma \int_{\mathcal{W}_\lambda} [H(D\overline{v}_1) H_\xi(D\overline{v}_1) - H(D\overline{v}_2) H_\xi(D\overline{v}_2)] \cdot D\overline{w} \, dx + \\ + \gamma \int_{\mathcal{W}_\lambda} [H(D\underline{v}_1) H_\xi(D\underline{v}_1) - H(D\underline{v}_2) H_\xi(D\underline{v}_2)] \cdot D\underline{w} \, dx + \\ + \int_{\mathcal{W}_\lambda} c_\star \overline{w}^2 \, dx + \int_{\mathcal{W}_\lambda} c^\star \underline{w}^2 \, dx = 0.$$

Every term in the above equality is nonnegative. As regards the first two integrals, this is due to the strict convexity of H^2. Hence, each term is equal to zero. Being $c \not\equiv 0$, this implies that $\overline{w} = \underline{w} = 0$. We point out that, using the above arguments, and reasoning as in [20], it is possible to prove the following result, analogous to the comparison result obtained in [20],[21]:

Theorem 3.2 *Let $u \in W^{1,2}(\Omega)$ be a solution of* (3.1)*, and define $w = u - t_0$, where $t_0 = \inf\{t \in \mathbb{R}\colon \mu_u(t) \leq |\Omega|/2\}$. For any $x \in \mathcal{W}_\lambda$, where λ is defined in* (3.1)*, we have that:*

$$(w^+)^\star(x) \leq \overline{v}(x),$$

$$(w^-)^\star(x) \leq \underline{v}(x),$$

where $\overline{v}$ and $\underline{v}$ are respectively the solutions to the Dirichlet problems

$$\begin{cases} -\gamma\Delta_H \overline{v} = (f^+)^\star & \text{in } \mathcal{W}_\lambda \\ \overline{v} = 0 & \text{on } \partial\mathcal{W}_\lambda, \end{cases}$$

and

$$\begin{cases} -\gamma\Delta_H \underline{v} = (f_-)^\star & \text{in } \mathcal{W}_\lambda \\ \underline{v} = 0 & \text{on } \partial\mathcal{W}_\lambda, \end{cases}$$

where $\gamma = \frac{K_N^2}{N^2\kappa_N^{2/N}}$, where w^+, f^+ and w^-, f^- denote, respectively, the positive and negative part of w and f.

4 Reverse Hölder inequality for eigenfunctions of a Neumann problem

Let us consider the following eigenvalue problem with Neumann boundary condition, associated to the anisotropic Laplacian:

$$\begin{cases} -\Delta_H u = \mu u & \text{in } \Omega \\ H(Du)H_\xi(Du)\cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where n is the usual outer Euclidean normal to $\partial\Omega$. When H is the Euclidean norm, (4.1) becomes the homogeneous Neumann eigenvalue problem for the Laplacian.

In this section we obtain comparison results for eigenfunctions corresponding to the first nonzero eigenvalue of the problem (4.1). We need the following result:

Theorem 4.1 *There exists the first positive eigenvalue $\mu_1(\Omega)$ of* (4.1) *with the following characterization:*

$$\mu_1(\Omega) = \inf\left\{\frac{\int_\Omega H(Du)^2\,dx}{\int_\Omega u^2\,dx},\ u \in W^{1,2}(\Omega),\ \int_\Omega u\,dx = 0,\ u \not\equiv 0\right\}.$$

Moreover, any eigenfunction u corresponding to $\mu_1(\Omega)$ belongs to $C^{1,\alpha}(\Omega)$. In addition, if $H^2(\xi) \in C^{2,1}(\mathbb{R}^N)$, the nodal set of u, namely the set $\{x \in \Omega\colon u(x) = 0\}$, has zero Lebesgue measure.

Proof. The characterization of $\mu_1(\Omega)$ follows by standard arguments of Calculus of Variations. Moreover, by [19, Theorem 5.2, Chapter 4], $u \in W^{2,2}(\Omega')$, for any open set $\Omega' \Subset \Omega$. Hence, by [19, Theorem 6.1, Chapter 4], $u \in C^{1,\alpha}(\Omega)$.

The last part of the thesis follows applying Theorem (1.7) in [17]. To do that, any eigenfunction u has to satisfy a uniform "doubling condition", namely for any $x_0 \in u^{-1}\{0\}$ it is possible to find a positive constant C such that, for any $\rho > 0$ sufficiently small,

$$\int_{B_{2\rho}(x_0)} u^2\,dx \leq C\int_{B_\rho(x_0)} u^2\,dx, \tag{4.2}$$

and C does not depend on ρ and x_0.

Being $H^2 \in C^{2,1}(\mathbb{R}^N)$, we can use the results of [15] (see also [18] and [16]), obtaining that any eigenfunction u satisfies (4.2).

In order to apply the above Theorem, from now on we suppose that

$$H^2(\xi) \in C^{2,1}(\mathbb{R}^N).$$

Let $\mu_1(\Omega)$ and u_1 be, respectively, the first nonzero eigenvalue and a corresponding eigenfunction of the problem (4.1). Let us consider

$$U(s) = \int_0^s u_1^*(t)\, dt, \quad s \in [0, |\Omega|],$$

where u_1^* is the decreasing rearrangement of u_1. Obviously, $U(0) = 0$, and $U(|\Omega|) = 0$, since u_1 has null mean value. Moreover, being u_1^* decreasing, $U(s)$ is concave in $[0, |\Omega|]$ and hence it is positive. By Theorem 4.1, being $|\{x \in \Omega\colon u = 0\}| = 0$, there exists a unique $\tilde{s} \in]0, |\Omega|[$ such that $u_1^*(\tilde{s}) = 0$. Finally, using the same arguments of Theorem 3.1, we have that $U(s)$ solves the following unidimensional problem

$$\begin{cases} -q(s)U''(s) \le \mu_1(\Omega)U & \text{in }]0, |\Omega|[, \\ U(0) = U(\Omega) = 0, \end{cases} \tag{4.3}$$

where

$$q(s) = \begin{cases} K_N^2 s^{2-\frac{2}{N}} & \text{if } 0 \le s \le |\Omega|/2, \\ K_N^2 (|\Omega| - s)^{2-\frac{2}{N}} & \text{if } |\Omega|/2 \le s \le |\Omega|. \end{cases}$$

Let $L > 0$ be such that $\mu_1(\Omega) = \lambda_1(0, L)$, where $\lambda_1(0, L)$ is the first eigenvalue of the problem

$$\begin{cases} -q(s)V''(s) = \lambda V & \text{in }]0, L[, \\ V(0) = V'(L) = 0. \end{cases} \tag{4.4}$$

Remark 4.1 We stress that the first eigenvalue $\lambda_1(0, L)$ of (4.4) can be found as the minimum of the problem

$$\min \left\{ \frac{\int_0^L \varphi'(\sigma)^2\, d\sigma}{\int_0^L q(\sigma)^{-1}\varphi(\sigma)^2\, d\sigma},\ \varphi \in H^1(0, L),\ \varphi(0) = 0, \varphi \not\equiv 0 \right\}.$$

Moreover, it is simple and monotone decreasing with respect to the inclusion of intervals, and the corresponding eigenfunction assumes just one sign (see, for instance, [23], [9]).

Remark 4.2 We observe that if such L exists, it holds necessarily that

$$L \le \min\{\tilde{s}, |\Omega| - \tilde{s}, |\Omega|/2\}, \tag{4.5}$$

where $\tilde{s}$ is defined as above (see [9]). Obviously, (4.5) holds if $L \le \tilde{s}$ and $L \le |\Omega| - \tilde{s}$. Reasoning by contradiction, we suppose first that $L > \tilde{s}$. Multiplying by U and integrating over $(0, \tilde{s})$ the equation in (4.3), we obtain that

$$\lambda_1(0, \tilde{s}) \le \frac{\int_0^{\tilde{s}} U'(\sigma)^2\, d\sigma}{\int_0^{\tilde{s}} q(\sigma)^{-1} U(\sigma)^2\, d\sigma} \le \mu_1(\Omega) = \lambda_1(0, L),$$

and this contradicts the strict monotonicity of $\lambda_1(0, \cdot)$.

The same argument applied to the eigenfunction $w = -u_1$ of (4.1) proves that $L \le |\Omega| - \tilde{s}$.

Remark 4.3 We show that a positive number L such that $\mu_1(\Omega) = \lambda_1(0, L)$ exists. Indeed, setting $v(s) = V'(s)$, with $s \in]0, L[$, since $L \le |\Omega|/2$, the function $v^\star(x) = v(\kappa_N H^o(x)^N)$, $x \in \mathcal{W}_\delta$ solves the problem

$$\begin{cases} -\gamma\Delta_H w = \mu_1(\Omega) w & \text{in } \mathcal{W}_\delta, \\ w = 0 & \text{on } \partial\mathcal{W}_\delta, \end{cases} \tag{4.6}$$

where $\gamma = \left(\frac{K_N}{N\kappa_N^{1/N}}\right)^2$ and $|\mathcal{W}_\delta| = L$. Writing $s = \kappa_N H^o(x)^N$, and recalling the properties of H, a straightforward computation gives that

$$\Delta_H v^\star(x) = N\kappa_N^{2/N} s^{2-2/N} \left(N v''(s) + 2(N-1)\frac{v'(s)}{s} \right).$$

Being $v(s) = V'(s)$ and $L \le |\Omega|/2$, by (4.4) we obtain that $v^\star$ solves (4.6).

If $\mu_1(\Omega)$ is the first eigenvalue of (4.6), by a result contained in [7], $\mu_1(\Omega)$ is simple and (4.6) has a positive solution symmetric with respect the H^o norm. Hence, looking for a solution $w(x) = \tilde{w}(r)$, with $r = H^o(x)$, the equation in (4.6) becomes

$$\tilde{w}'' + \frac{N-1}{r}\tilde{w}' + \frac{\mu_1(\Omega)}{\gamma}\tilde{w} = 0, \quad r \in]0, \delta[,$$

the standard Bessel equation. Then, denoting by $j_{p,k}$ the k-th positive zero of the Bessel function J_p, we have that

$$w(x) = v^\star(x) = H^o(x)^{1-\frac{N}{2}} J_{N/2-1}\left(\sqrt{\frac{\mu_1(\Omega)}{\gamma}} H^o(x) \right).$$

Since the first eigenvalue of the Dirichlet $-\Delta_H$ in $\mathcal{W}_\delta$ is $\delta^{-2} j^2_{\frac{N}{2}-1,1}$, from the identity $\gamma^{-1}\mu_1(\Omega) = \delta^{-2} j^2_{\frac{N}{2}-1,1}$ we deduce that

$$L = |\mathcal{W}_\delta| = \kappa_N \delta^N = \kappa_N \left(\frac{j_{\frac{N}{2}-1,1} K_N}{N\kappa_N^{1/N}} \right)^N \mu_1(\Omega)^{-N/2}.$$

We observe that the one-dimensional problems (4.3) and (4.4) are the same (up to some constants) of the corresponding one-dimensional problems (3.4) and (3.5) contained in [9]. Hence we can repeat line by line the arguments contained in [9].

We first consider the case of equality in (4.5), obtaining the following main result.

Proposition 4.1 *Suppose that* $L = \min\{\tilde{s}, |\Omega| - \tilde{s}, |\Omega|/2\}$. *Then:*

(a) if $L = \tilde{s}$, *then* U *and* V *are proportional,*

(b) if $L = |\Omega| - \tilde{s}$, *then* $\int_0^s (-u_1)^*(t)$ *and* V *are proportional,*

(c) if $L = |\Omega|/2$, *then* $\tilde{s} = |\Omega|/2$, U *and* V *are proportional and* $(u_1)^* = (-u_1)^*$.

Hence from now on we can assume that

$$L < \min\{\tilde{s}, |\Omega| - \tilde{s}, |\Omega|/2\}.$$

We extend the function V, defined in (4.4), in $[0, |\Omega|]$, in order to have a function symmetric with respect to $|\Omega|/2$, still denoted by V:

$$V(s) = \begin{cases} V(s) & \text{in } [0, L], \\ V(L) & \text{in }]L, |\Omega| - L], \\ V(|\Omega| - s) & \text{in }]|\Omega| - L, |\Omega|]. \end{cases}$$

Using the notation introduced above, we can obtain also in our context the result contained in [9]:

Theorem 4.2 *Let $\mu_1(\Omega)$ be the first positive eigenvalue of problem* (4.1)*, and let u_1 be a corresponding eigenfunction. If U and V are defined as above, and normalized such that*

$$\max_{s\in[0,|\Omega|]} U(s) = \max_{s\in[0,|\Omega|]} V(s),$$

the following comparison results hold:

1. *for any $s \in [0, |\Omega|]$,*
$$U(s) \le V(s),$$
and, for any $p \ge 1$,
$$\int_0^{|\Omega|} |u_1^*(t)|^p \, dt \le \int_0^{|\Omega|} |v(t)|^p \, dt.$$
2. *for any $p \ge 1$, there exists a constant $C = C(N, p, \mu_1, \gamma, H)$ such that the following reverse Hölder inequality holds:*
$$||u_1||_{L^p(\Omega)} \le C||u_1||_{L^1(\Omega)},$$
where $C = ||v||_{L^p(\Omega)}/||v||_{L^1(\Omega)}$.

5 Poincaré inequality

In this section we study a Poincaré inequality related to the anisotropic total variation of a BV function u. To do that, let us consider, for any measurable set $\Omega \subset R^N$ with positive finite measure, $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$. Moreover, we define the value

$$I(\Omega; q) = \sup\{R_q(E) \colon E \subset \Omega, 0 < |E| < |\Omega|\},$$

where

$$R_q(E;\Omega) = \frac{\left[|\Omega \setminus E| \cdot |E| \cdot (|\Omega \setminus E|^{q-1} + |E|^{q-1})\right]^{1/q}}{|\Omega| \cdot P_H(E;\Omega)}.$$

We get the following result:

Theorem 5.1 *Let $N \geq 2$ and $1 \leq q \leq N/(N-1)$. Suppose that Ω is a bounded connected open set of $\mathbb{R}^N$ with Lipschitz boundary. Then $I(\Omega; q) < +\infty$ and*

$$\|u - (u)_\Omega\|_{L^q(\Omega)} \leq I(\Omega; q) \int_\Omega |Du|_H, \tag{5.1}$$

for any $u \in BV(\Omega)$. Moreover, the constant $I(\Omega; q)$ is sharp.

Proof. We follow the proof contained in [12]. Using the relative anisotropic inequality (2.3), we obtain that

$$I\Big(\Omega; \frac{N}{N-1}\Big) \leq \frac{C_1}{K_N}$$

for some constant $C_1 = C_1(N, \Omega)$. Hence, $I(\Omega, q)$ is finite for any $1 \leq q \leq \frac{N}{N-1}$.

Now, let $u \in BV(\Omega)$. It is not restrictive to assume that $u \geq 0$. We have that

$$u(x) = \int_0^{+\infty} \chi_{\{u>t\}}(x) dt,$$

and then

$$(u)_\Omega = \frac{1}{|\Omega|} \int_0^{+\infty} \mu_u(t) dt,$$

where μ_u is the distribution function of u. Hence, using the Minkowski inequality we get

$$\|u - (u)_\Omega\|_{L^q(\Omega)} \leq \int_0^{+\infty} \frac{1}{|\Omega|} \left[\int_\Omega (|\Omega| \chi_{\{u>t\}} - \mu_u(t))^q dx \right]^{1/q} dt =$$

$$= \int_0^{+\infty} P_H(\{u > t\}) R_q(\{u > t\}; \Omega)\, dt.$$

By the Fleming-Rishel formula for the anisotropic perimeter and the definition of R_q we have that the best constant in (5.1) is smaller than $I(\Omega, q)$. On the other hand, if we take $u = \chi_E$, where E is any measurable subset of Ω, we have

$$\frac{\|\chi_E - (\chi_E)_\Omega\|_{L^q(\Omega)}}{P_H(E; \Omega)} = R_q(E; \Omega).$$

Hence

$$\|u - (u)_\Omega\|_{L^q(\Omega)} \leq I(\Omega; q) \int_\Omega |Du|_H,$$

and $I(\Omega; q)$ is the best constant.

Remark 5.1 We observe that, in general, the constant $I(\Omega; q)$ is not attained, as shown in the following example. Let $N = 2$, $q = 2$ and Ω bounded convex open set. Then

$$R_2(E; \Omega) = \frac{1}{\sqrt{|\Omega|}} \frac{\sqrt{(|\Omega| - |E|)|E|}}{P_H(E; \Omega)}.$$

Following [13], given $0 < k \leq |\Omega|/2$ and denoted by

$$\mu(k) = \min \left\{ \frac{P_H^2(F; \Omega)}{k}, F \subset \Omega, |F| = k \right\} > 0,$$

then

$$I(\Omega;2) = \frac{1}{\sqrt{|\Omega|}} \sup_{0<k\le|\Omega|/2} \frac{\sqrt{|\Omega|-k}}{\sqrt{\mu(k)}}.$$

If $\Omega = \mathcal{W} \cap A$, where A is a convex cone centered at the origin, by [13], $\mu(k) = 4|\mathcal{W} \cap A|$ and

$$I(\mathcal{W} \cap A;2) = \frac{1}{2\sqrt{|\mathcal{W} \cap A|}}.$$

Hence the measure of any sequence maximizing $R_2(\,\cdot\,;\mathcal{W} \cap A)$ tends to zero, and $I(\mathcal{W} \cap A;2)$ is not attained.

Remark 5.2 If H is the Euclidean norm, in [12] it is proved that the best constant $I(\Omega;q)$ is achieved when Ω is a ball. More generally, this holds also when Ω is a bounded connected open set of $\mathbb{R}^N$ with C^2 boundary (see [8] and [5]). Moreover, in [12] the optimal constant is explicitly computed when Ω is a ball B_r of $\mathbb{R}^N$ of radius r, that is

$$I\Big(B_r;\frac{N}{N-1}\Big) = \frac{\omega_N^{1-\frac{1}{N}}}{2\omega_{N-1}},$$

and it is attained at a half ball of B_r. Hence, when Ω is a ball, the extremal sets of the relative isoperimetric inequality and of $R_{\frac{N}{N-1}}(\,\cdot\,;\Omega)$ are the same. This is not true, in general, when Ω is not a ball, as shown in the following simple example.

Example 5.1 We construct the set Ω in the following way. Given a square with side 2, we cut away from it, near the vertices, four pieces A^i_ε, $i = 1,\dots,4$, of measure ε (see Figure 1). We do that in such a way the obtained set Ω is C^2 and symmetric with respect to the origin. Hence $|\Omega| = 4(1-\varepsilon)$. By [11], the optimal constant of the relative isoperimetric inequality is attained on a set E whose boundary in Ω is a straight segment passing through the origin and parallel to one of the axes (see Figure 1), and

$$R_2(E;\Omega) = \frac{\sqrt{1-\varepsilon}}{2}.$$

On the other hand, denoted by F the set such that $\partial F \cap \Omega$ is the circular arc of radius $1/2$ and centered in a vertex of the square (compare Figure 1), we have that

$$R_2(F;\Omega) = \frac{\sqrt{(\pi-16\varepsilon)\left(4-\frac{\pi}{16}-3\varepsilon\right)}}{2\pi\sqrt{1-\varepsilon}} > R_2(E;\Omega)$$

as $\varepsilon > 0$ is small enough. Hence the minimizer, whose existence is guaranteed by [8], is not the same of the minimizer of the relative isoperimetric inequality.

On the other hand, in the following example we show that in the anisotropic setting it does not hold that the extremal sets in the relative isoperimetric inequality and in the Poincaré inequality are the same when Ω is some level set of H or H^o.

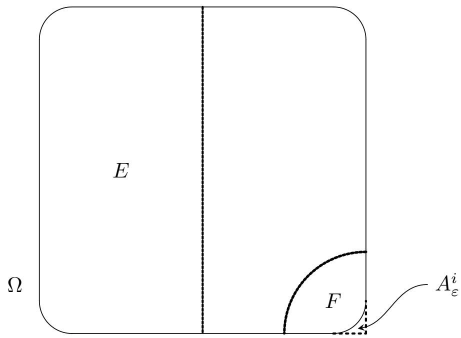

Figure 1: Example 5.1

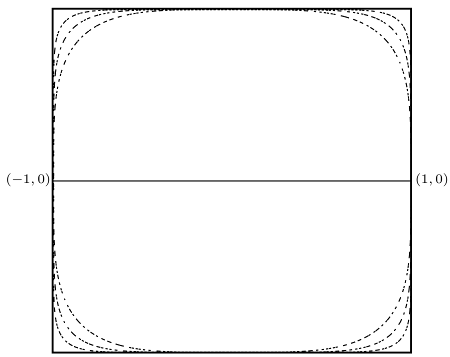

Figure 2: Example 5.2

Example 5.2 Let $N = 2$, and for $p \geq 2$, $H_p(x, y) = (|x|^p + |y|^p)^{1/p}$, $H_\infty(x, y) = \max\{|x|, |y|\}$ and $\Omega = \Omega_p = \{(x, y) \colon H_p(x, y) < 1\}$. Let us suppose, by contradiction, that for any $2 \leq p < +\infty$ $I(\Omega_p, 2)$ is attained at the extremal sets of the relative isoperimetric inequality. Following [13], we take the set whose boundary in Ω_p is the straight segment joining $(-1, 0)$ and $(1, 0)$ (see Figure 2). Hence, for any fixed set $F \subseteq \Omega_\infty$, and for any p we have

$$\frac{\sqrt{|\Omega_p|}}{4} = I(\Omega_p; 2) \geq \frac{\sqrt{|F \cap \Omega_p|(|\Omega_p| - |F \cap \Omega_p|)}}{\sqrt{|\Omega_p|} P_{H_p}(F; \Omega_p)}. \tag{5.2}$$

We can pass to the limit as $p \to +\infty$ in (5.2), obtaining

$$\frac{1}{2} \geq \frac{\sqrt{|F|(|\Omega_\infty| - |F|)}}{\sqrt{|\Omega_\infty|} P_{H_\infty}(F; \Omega_\infty)} = \frac{\sqrt{|F|(4 - |F|)}}{2 P_{H_\infty}(F; \Omega_\infty)}, \quad \text{for any } F \subseteq \Omega_\infty. \tag{5.3}$$

Consider in (5.3) F as an isosceles triangle having a vertex in one of the vertices of Ω_∞, with the equal sides on $\partial\Omega_\infty$ and measure $|F| = \varepsilon$. Being $P_{H_\infty}(F; \Omega_\infty) = \sqrt{2\varepsilon}$, as ε tends to zero the right hand side in (5.3) is $\sqrt{2}/2$, and this is absurd. Hence the above argument shows that there exists at least one p such that the extremal sets of $I(\Omega_p;2)$ and those of the relative isoperimetric inequality are not the same.

A similar example can be given if Ω is a level set of H^o.

References

[1] A. Alvino, V. Ferone, P.-L. Lions and G.Trombetti, *Convex symmetrization and applications*, Ann. Inst. Henri Poincaré **14** (1997), 275–293.

[2] A. Alvino, P.-L. Lions, S. Matarasso and G. Trombetti, *Comparison results for solutions of elliptic problems via symmetrization*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), 167–188.

[3] A. Alvino, S. Matarasso and G. Trombetti, *Elliptic boundary value problems: comparison results via symmetrization*, Ricerche Mat. **51** (2002), 341–355.

[4] M. Amar and G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 91–133.

[5] T. Bartsch and M. Willem, *Some critical minimization problems for functions of bounded variations*, J. Funct. Anal. **259** (2010), 3025–3035.

[6] G. Bellettini and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), 537–566.

[7] M. Belloni, V. Ferone and B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*, Z. Angew. Math. Phys. **54** (2003), 771–783.

[8] V. Bouchez and J. Van Schaftingen, *Extremal functions in Poincaré-Sobolev inequalities for functions of bounded variation*, in Nonlinear Elliptic Partial Differential Equations, Amer. Math. Soc., Contemporary Mathematics **240** (2011), 47–58.

[9] B. Brandolini, F. Chiacchio and C. Trombetti, *Sharp estimates for eigenfunctions of a Neumann problem*, Comm. Partial Differential Equations **34** (2009), 1317–1337.

[10] Yu.D. Burago and V.A. Zalgaller, Geometric Inequalities, Springer-Verlag, Berlin, 1988.

[11] A. Cianchi, *On relative isoperimetric inequalities in the plane*, Boll. U.M.I. (7) **3-B** (1989), 289–325.

[12] A. Cianchi, *A sharp form of Poincaré type inequalities on balls and spheres*, Z. Angew. Math. Phys. **40** (1989), 558–569.

[13] F. Della Pietra and N. Gavitone, *The relative isoperimetric inequality in the plane: the anisotropic case*, preprint del Dip. di Matematica e Applicazioni, Università degli Studi di Napoli “Federico II” (2011).

[14] H. Federer and W.H. Fleming, Normal and integral currents, *Ann. of Math.* **72** (1960), 458–520.

[15] N. Garofalo and F-H. Lin, *Monotonicity properties of variational integrals, A_p−weights, and unique continuation*, Indiana J. Math. **35** (1986), 245–268.

[16] Q. Han and F-H. Lin, *Nodal sets of solutions of elliptic differential equations*, book in preparation.

[17] R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*, J. Differential Geom., **30** (1989), 505–522.

[18] L. Hörmander, *Uniqueness theorems for second order elliptic operators*, Comm. Partial Differential Equations **8** (1983), 21–64.

[19] Ladyženskaja O.A. and Ural’ceva N.N. Linear and Quasilinear Elliptic Equations, Academic press, New York and London, 1968.

[20] C. Maderna and S. Salsa, *Symmetrization in Neumann problems*, Applicable Anal. **9** (1979), 247–256.

[21] C. Maderna and S. Salsa, *A priori bounds in nonlinear Neumann problems*, Boll. U.M.I. B **16** (1979), 1144–1153.

[22] V.G. Maz'ja, Sobolev spaces, Springer-Verlag, Berlin, 1985.

[23] J. Nečas, Le Méthodes Directes En théorie des Équations Elliptiques, Paris, Masson, 1967.