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A Biharmonic Equation in $\mathcal{R}^4$ Involving Nonlinearities with Subcritical Exponential Growth

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Abstract

In this paper we consider a biharmonic equation of the form $\Delta^2 u + V(x)u = f(u)$ in the whole four-dimensional space $\mathcal{R}^4$. Assuming that the potential V satisfies some symmetry conditions and is bounded away from zero and that the nonlinearity f is odd and has subcritical exponential growth (in the sense of an Adams' type inequality), we prove a multiplicity result. More precisely we prove the existence of infinitely many nonradial sign-changing solutions and infinitely many radial solutions in $H^2(\mathcal{R}^4)$. The main difficulty is the lack of compactness due to the unboundedness of the domain $\mathcal{R}^4$ and in this respect the symmetries of the problem play an important role.

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1 Introduction

Recently, due to applications of higher order elliptic equations to conformal geometry, there has been considerable interest in the Paneitz operator which enjoys the property of conformal invariance. In $\mathcal{R}^4$, the Paneitz operator is the biharmonic operator Δ^2 where Δ is the Laplacian in $\mathcal{R}^4$. The study of problems involving powers of the Laplacian started with [17], [16], [23], [24] and we refer the reader to the paper [6] and the references therein for various results on the polyharmonic operator.

This paper is concerned with the multiplicity of solutions for biharmonic problems of the form

$$\begin{cases} \Delta^2 u + V(|x|)u = f(u) & \text{in } \mathcal{R}^4 \\ u \in H^2(\mathcal{R}^4) \end{cases} \tag{1.1}$$

where the condition $u \in H^2(\mathcal{R}^4)$ expresses explicitly that the biharmonic equation is to be satisfied in the weak sense. We will consider the case when the nonlinear term f exhibits a subcritical exponential growth. We recall that equations of the form

$$\Delta^2 u + V(x)u = f(x, u) \quad \text{in } \mathcal{R}^n$$

with $n \geq 5$ and involving nonlinearities with polynomial growth, have been studied in [10], [5] and [4].

In dimension 2 the maximal growth which can be treated variationally, in the Sobolev space H^1, is given by the *Trudinger-Moser inequality* (see [20] and [26]) which says that

$$\sup_{u \in S} \int_\Omega (e^{\alpha u^2} - 1)\, dx \begin{cases} < +\infty & \text{if } \alpha \leq 4\pi \\ = +\infty & \text{if } \alpha > 4\pi \end{cases}$$

where, denoting by $\|\cdot\|_2$ the standard L^2 norm, $S := \{u \in H_0^1(\Omega) \mid \|\nabla u\|_2^2 \leq 1\}$ if $\Omega \subset \mathcal{R}^2$ is a bounded domain, whereas $S := \{u \in H_0^1(\Omega) \mid \|u\|_2^2 + \|\nabla u\|_2^2 \leq 1\}$ if $\Omega \subseteq \mathcal{R}^2$ is an unbounded domain. The subcritical and critical exponential growth in second-order elliptic problems has been extensively investigated in recent years, starting with the works [2] and [3] concerning bounded domains in $\mathcal{R}^2$ and [9] concerning the whole space $\mathcal{R}^2$. Among the subsequent works, we can mention in chronological order [12], [11], [8], [15], [14], [21] and [13].

The natural space for a variational treatment of problem (1.1) is the Sobolev space H^2 and $\mathcal{R}^4$ is the limiting case for the corresponding Sobolev embeddings. Indeed, for $\mathcal{R}^4$ the critical growth is given by an Adams' type inequality (see [27]) which says that

$$\sup_{u \in H^2(\mathcal{R}^4),\, \|u\|_{H^2} \leq 1} \int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)\, dx \begin{cases} < +\infty & \text{for } \alpha \leq 32\pi^2\,, \\ = +\infty & \text{for } \alpha > 32\pi^2\,, \end{cases}$$

where $\|u\|_{H^2} := (\|\Delta u\|_2^2 + 2\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}$. In view of this inequality we say that a nonlinearity f has *subcritical exponential growth* if

$$\lim_{|s| \to +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \qquad \forall \alpha > 0\,.$$

In order to obtain the existence of infinitely many nonradial sign-changing and radial solutions for the biharmonic problem (1.1), we make the following assumptions on the potential V and the nonlinearity f:

(V_1) $V \in C(\mathcal{R}^4, \mathcal{R})$ is bounded from below by a positive constant V_0,

$$V(x) \geq V_0 > 0 \qquad \forall x \in \mathcal{R}^4\,;$$

(V_2) V is spherically symmetric with respect to $x \in \mathcal{R}^4$,

$$V(x) = V(|x|) \qquad \forall x \in \mathcal{R}^4 \ ;$$

(f_1) $f \in C(\mathcal{R}, \mathcal{R})$ has subcritical exponential growth;

(f_2) $f(s) = o(|s|)$ as $|s| \to 0$;

(f_3) f is odd.

We can notice that, as a consequence of assumption (f_3), nonzero solutions of (1.1) occour in antipodal pairs, namely if u is a solution of (1.1) then $-u$ is a solution of (1.1) too.

Furthermore, setting $F(s) := \int_0^s f(t)\,dt$, we will assume that:

(F_1) $\exists \mu > 2$ such that

$$\mu F(s) \le s f(s) \qquad \forall s \in \mathcal{R} \ ;$$

(F_2) $\exists \overline{s} > 0$ such that $\inf\limits_{|s| \ge \overline{s}} F(s) > 0$.

Remark 1.1 (F_1) and (F_2) implies the Ambrosetti-Rabinowitz condition, namely

$$(A - R) \quad \exists \mu > 2 \quad \text{such that} \quad 0 < \mu F(s) \le s f(s) \quad \forall s \ge \overline{s} \ .$$

As we will see during the proof, we need the stronger condition (F_1) to obtain the Palais-Smale condition.

We can now state our main result:

Theorem 1.1 *Assume that* (V_1), (V_2), (f_1), (f_2), (f_3), (F_1) *and* (F_2) *hold. Then there exists an unbounded sequence* $\{\pm u_k\}_{k \in \mathbb{N}}$ *of sign-changing solutions of (1.1) which are not radial. There also exists an unbounded sequence* $\{\pm u_k\}_{k \in \mathbb{N}}$ *of radial solutions of (1.1).*

Here and below the unboundedness of the sequences $\{u_k\}_{k \in \mathbb{N}}$ of solutions has to be understood as follows:

$$\int_{\mathcal{R}^4} \left[(\Delta u_k)^2 + V(|x|) u_k^2 \right] dx \to +\infty \quad \text{as } k \to +\infty \ .$$

We point out that, since problem (1.1) is invariant under rotations, it is natural to look for radially symmetric solutions. Therefore it seems to be more interesting the multiplicity of nonradial solutions of (1.1). Concerning the unbounded sequence $\{u_k\}_{k \in \mathbb{N}}$ of sign-changing solutions of (1.1) which are *not radial*, we can notice that the *orbit of* u_k

$$\mathcal{O}(4) * u_k := \{ u \circ g \ : \ g \in \mathcal{O}(4) \} \subseteq H^2(\mathcal{R}^4)$$

is diffeomorphic to the quotient space $\mathcal{O}(4)/\mathcal{Z}(u_k)$, where

$$\mathcal{Z}(u_k) := \left\{ g \in \mathcal{O}(4) \ : \ u_k(gx) = u_k(x) \ \forall x \in \mathcal{R}^4 \right\} \subseteq \mathcal{O}(4)$$

is the *isotropy group of* u_k. Since u_k is not radial, it is possible that for some $g \in \mathcal{O}(4)$

$$u(gx) = u(x) \quad \forall x \in \mathcal{R}^4 \ ,$$

but this cannot happen for all $g \in \mathcal{O}(4)$, namely $\mathcal{Z}(u_k) \subsetneq \mathcal{O}(4)$. Therefore

$$\dim \mathcal{O}(4) * u_k = \dim \mathcal{O}(4)/\mathcal{Z}(u_k) \geq 1$$

and to each u_k corresponds a nontrivial orbit of solutions to (1.1). Furthermore the orbits $\mathcal{O}(4) * u_{k_1}$ and $\mathcal{O}(4) * u_{k_2}$ with $k_1 \neq k_2$ and k_1, k_2 sufficiently large are disjoint because

$$\int_{\mathcal{R}^4} \left[(\Delta\,(u_k \circ g)\,)^2 + V(|x|)(u_k \circ g)^2 \right] dx =$$
$$= \int_{\mathcal{R}^4} \left[(\Delta u_k)^2 + V(|x|) u_k^2 \right] dx \to +\infty \quad \text{as } k \to +\infty \,.$$

It will be clear during the proof that it is possible to obtain an unbounded sequence of nonradial sign-changing solutions of (1.1) without requiring the potential V to be spherically symmetric with respect to $x \in \mathcal{R}^4$. In fact we may replace the assumption (V_2) on the potential V with the following weaker assumptions:

(V_2') V is spherically symmetric with respect to $x_1,\ x_2 \in \mathcal{R}^2$,

$$V(x) = V(|x_1|,\ |x_2|) \qquad \forall x = (x_1,\ x_2) \in \mathcal{R}^2 \times \mathcal{R}^2 \;;$$

(V_2'') $V(|x_1|,\ |x_2|) = V(|x_2|,\ |x_1|) \quad \forall x_1,\ x_2 \in \mathcal{R}^2$.

Theorem 1.2 *Assume that* (V_1), (V_2'), (V_2''), (f_1), (f_2), (f_3), (F_1) *and* (F_2) *hold. Then there exists an unbounded sequence* $\{\pm u_k\}_{k \in \mathbb{N}}$ *of sign-changing solutions of*

$$\begin{cases} \Delta^2 u + V(|x_1|,\ |x_2|)u = f(u) & \text{in } \mathcal{R}^4 \\ u \in H^2(\mathcal{R}^4) \end{cases} \tag{1.2}$$

which are not radial.

Furthermore, requiring only the potential V to be spherically symmetric with respect to $x_1,\ x_2 \in \mathcal{R}^2$, it is possible to obtain an unbounded sequence of solutions of (1.2).

Theorem 1.3 *Assume that* (V_1), (V_2'), (f_1), (f_2), (f_3), (F_1) *and* (F_2) *hold. Then (1.2) possesses an unbounded sequence* $\{\pm u_k\}_{k \in \mathbb{N}}$ *of solutions.*

To prove these theorems we will follow a variational approach. Let X be the subspace of $H^2(\mathcal{R}^4)$ defined as

$$X := \left\{ u \in H^2(\mathcal{R}^4) \;\middle|\; \int_{\mathcal{R}^4} \left[(\Delta u)^2 + V(x)u^2 \right] dx < +\infty \right\} .$$

By (V_1), it follows that X is a Hilbert space endowed with the scalar product

$$\langle u,\ v \rangle := \int_{\mathcal{R}^4} \Delta u \Delta v\, dx + \int_{\mathcal{R}^4} V(x) uv\, dx \qquad u,\ v \in X$$

to which corresponds the norm $\|u\| := \sqrt{\langle u,\ u \rangle}$. Applying an interpolation inequality, it is easy to see that the embedding $X \hookrightarrow H^2(\mathcal{R}^4)$ is continuous.

The solutions of (1.1) are critical points of the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathcal{R}^4} F(u)\, dx \qquad \forall u \in X$$

which is well defined and differentiable on X, namely $I \in C^1(X, \mathcal{R})$. The difficulty in working in this variational framework is the lack of compactness, infact I fails to satisfy the Palais-Smale condition in X. However, to gain compactness, we shall exploit the symmetries of the problem imposing the invariance with respect to a group G acting on X.

Let X_G be the space of fixed points in X with respect to the action of the group G:

$$X_G := \left\{u \in X \mid u(gx) = u(x) \quad \forall g \in G \text{ and a.e. } x \in \mathcal{R}^4\right\} \subseteq X\,.$$

We will prove the following result:

Proposition 1.1 *Let G be a group acting on X via orthogonal maps such that:*

(G_1) $I : X \to \mathcal{R}$ *is G-invariant;*

(G_2) X_G *is compactly embedded in* $L^p(\mathcal{R}^4)$ *for any* $p \in (4, +\infty)$*;*

(G_3) *dim* $X_G = +\infty$.

Then I has an unbounded sequence of critical points lying on X_G.

To prove Proposition 1.1, we will show that the problem reduces to the study of the multiplicity of critical points of the restriction $I|_{X_G}$ which behaves like a mountain pass and satisfies the Palais-Smale condition. More precisely $I|_{X_G}$ satisfies the assumptions of a generalized mountain pass theorem due to A. Ambrosetti and P. H. Rabinowitz [25] which gives the multiplicity of critical points.

This paper is organized as follows. In Section 2, we will introduce some preliminary results and, in Section 3, we will prove Proposition 1.1. Finally, in Section 4, we will show the existence of a group G, which satisfies the assumptions of Proposition 1.1, following an approach introduced by T. Bartsch and M. Willem in [7] (see also [6]). This approach allows to obtain additional informations on the nodal structure of the critical points. The existence of such a group together with Proposition 1.1 will allow us to conclude that the main theorem, Theorem 1.1, holds. We will also explain how to adapt these arguments to prove Theorem 1.2 and Theorem 1.3.

2 Some preliminary results

Let $\Omega \subset \mathcal{R}^4$ be a bounded domain in $\mathcal{R}^4$. The well known Adams' inequality [1] says that:

Theorem 2.1 ([1], Theorem 3) *There exists a constant* $C > 0$ *such that*

$$\sup_{u \in H_0^2(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{\alpha u^2}\, dx < C|\Omega| \tag{2.1}$$

if $\alpha \le 32\pi^2$*, while if* $\alpha > 32\pi^2$ *the supremum in (2.1) is infinite.*

Here, as we are looking for entire solutions of problem (1.1), we need an extension of this result to the whole space $\mathcal{R}^4$. More precisely we will use the following:

Theorem 2.2 ([27]) *There exists a constant* $C > 0$ *such that*

$$\sup_{u\in H^2(\mathcal{R}^4),\, \|u\|_{H^2}\leq 1} \int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)\, dx < +\infty \tag{2.2}$$

for all $\alpha \leq 32\pi^2$*, where* $\|u\|_{H^2} := (\|\Delta u\|_2^2 + 2\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}$*. This inequality is sharp, in the sense that if* $\alpha > 32\pi^2$ *then the supremum in (2.2) is infinite.*

In the rest of this preliminary section we will introduce some results that will be useful to establish the mountain pass structure of the functional associated to (1.1).

Lemma 2.1 ([14], Lemma 2.2) *Let* $\alpha > 0$ *and* $r > 1$*. Then for any* $\beta > r$ *there exists a constant* $C(\beta) > 0$ *such that*

$$\left(e^{\alpha s^2} - 1\right)^r \leq C(\beta)\left(e^{\alpha\beta s^2} - 1\right) \qquad \forall s \in \mathcal{R}\,.$$

For a proof of Lemma 2.1, the reader is referred to the proof of Lemma 2.2 in [14].

Remark 2.1 As a consequence of Lemma 2.1 and Hlder's inequality, it is easy to see that if $\alpha > 0$ and $q \geq 1$ then the function $|u|^q(e^{\alpha u^2} - 1)$ belongs to $L^1(\mathcal{R}^4)$ for all $u \in H^2(\mathcal{R}^4)$.

Lemma 2.2 *Let* $\alpha > 0$ *and* $q \geq 2$*. If* $M > 0$ *and* $\alpha M^2 < 32\pi^2$ *then there exists a constant* $C(\alpha, q, M) > 0$ *such that*

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)|u|^q\, dx \leq C(\alpha, q, M)\|u\|_{H^2}^q \tag{2.3}$$

holds for any $u \in H^2(\mathcal{R}^4)$ *with* $\|u\|_{H^2} \leq M$*.*

Proof. As $\alpha M^2 < 32\pi^2$, there exists $r \in \mathcal{R}$ such that $1 < r < \frac{32\pi^2}{\alpha M^2}$. Furthermore there exists $\beta \in \mathcal{R}$ such that

$$1 < r < \beta \leq \frac{32\pi^2}{\alpha M^2}$$

and in particular $\alpha\beta M^2 \leq 32\pi^2$. Let $u \in H^2(\mathcal{R}^4)$ be such that $\|u\|_{H^2} \leq M$, then by Lemma 2.1 it follows that

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^r\, dx \leq C(\alpha, M) \int_{\mathcal{R}^4} (e^{\alpha\beta u^2} - 1)\, dx \leq C(\alpha, M) \int_{\mathcal{R}^4} (e^{\alpha\beta M^2 \tilde{u}^2} - 1)\, dx$$

where $\tilde{u} := \frac{u}{\|u\|_{H^2}}$. Therefore

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^r\, dx \leq C(\alpha, M) \tag{2.4}$$

as a consequence of the Adams' type inequality (2.2).

Now, applying Hölder's inequality with $\frac{1}{r} + \frac{1}{r'} = 1$ and (2.4), we get

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)|u|^q \, dx \leq \|u\|^q_{qr'} \left(\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^r \, dx \right) \leq C(\alpha, M)\|u\|^q_{qr'}$$

and (2.3) follows easily as $qr' \geq 2$ and the Sobolev embedding theorem states that $H^2(\mathcal{R}^4)$ is continuously embedded in $L^p(\mathcal{R}^4)$ for any $p \in [2, +\infty)$.

Lemma 2.2 is a generalization of Lemma 2.4 in [14] for second order derivatives. To prove that Palais-Smale condition holds we will use the following version of Lemma 2.2.

Lemma 2.3 *Let $\alpha > 0$, $r > 1$ and $q \geq 2$. If $M > 0$ and $\alpha r M^2 < 32\pi^2$ then there exists a constant $C(\alpha, r, q, M) > 0$ such that*

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^{\frac{r}{2}} |u|^{\frac{q}{2}} \, dx \leq C(\alpha, r, q, M) \, \|u\|^{\frac{q}{2}}_{H^2} \tag{2.5}$$

holds for any $u \in H^2(\mathcal{R}^4)$ with $\|u\|_{H^2} \leq M$.

Proof. As $\alpha r M^2 < 32\pi^2$ there exists $\beta > r$ such that $\alpha\beta M^2 \leq 32\pi^2$. Let $u \in H^2(\mathcal{R}^4)$ be such that $\|u\|_{H^2} \leq M$. As in the proof of Lemma 2.2, applying Lemma 2.1 and the Adams' type inequality (2.2), we get

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^r \, dx \leq C(\alpha, r, M) \, .$$

Now

$$\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^{\frac{r}{2}} |u|^{\frac{q}{2}} \, dx \leq \left(\int_{\mathcal{R}^4} (e^{\alpha u^2} - 1)^r \, dx \right)^{\frac{1}{2}} \|u\|^{\frac{q}{2}}_q \leq \sqrt{C(\alpha, r, M)} \, \|u\|^{\frac{q}{2}}_q$$

and this ends the proof, in fact (2.5) follows by the Sobolev embedding theorem.

3 Mountain pass structure and Palais-Smale condition

We consider the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathcal{R}^4} F(u) \, dx \qquad \forall u \in X \, .$$

We can notice that, as a consequence of (f_1) and (f_2), fixed $\alpha > 0$ and $q > 0$, we have the existence of two constants c_1, $c_2 > 0$ such that

$$|f(s)| \leq c_1|s| + c_2|s|^q (e^{\alpha s^2} - 1) \qquad \forall s \in \mathcal{R} \, ;$$

therefore, from $(A - R)$, it follows the existence of two constants $\overline{c}_1$, $\overline{c}_2 > 0$ such that:

$$|F(s)| \leq \overline{c}_1|s|^2 + \overline{c}_2|s|^{q+1}(e^{\alpha s^2} - 1) \qquad \forall s \in \mathcal{R} \, .$$

This together with Remark 2.1 implies that the functional I is well defined on X. Using standard arguments, it is easy to see that $I \in C^1(X, \mathcal{R})$,

$$I'(u)v = \langle u, v\rangle - \int_{\mathcal{R}^4} f(u)v\,dx \qquad \forall u, v \in X$$

and, as already mentioned in the Introduction, the critical points of I are solutions of problem (1.1).

The main aim of this Section is the proof Proposition 1.1. Thus let G be a group acting on X via orthogonal maps satisfying (G_1), (G_2) and (G_3). Firsly we can notice that, as a consequence of the principle of symmetric criticality [22], any critical point of the restriction $I|_{X_G}$ is a critical point of I too. Therefore the proof of Proposition 1.1 reduces to show that $I|_{X_G}$ has an unbounded sequence of critical points. To do this we apply the following generalized mountain pass theorem.

Theorem 3.1 ([25], Theorem 9.12) *Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space over $\mathcal{R}$ and let $I \in C^1(E, \mathcal{R})$ be an even functional such that $I(0) = 0$. We assume that:*

- (I_1) *$\exists\rho, \gamma > 0$ such that $I|_{B_\rho\setminus\{0\}} > 0$ and $I|_{\partial B_\rho} \geq \gamma > 0$ where*

$$B_\rho := \{u \in E \mid \|u\| \leq \rho\} \subset E\ ;$$

- (I_2) *for any finite dimensional subspace $\tilde{E} \subset E$ the set $\{u \in \tilde{E} \mid I(u) \geq 0\}$ is bounded;*
- (I_3) *the Palais-Smale condition holds.*

Then I possesses an unbounded sequence of critical values $c_k \to +\infty$ as $k \to +\infty$.

As mentioned above $I \in C^1(X, \mathcal{R})$, moreover I is even and $I(0) = 0$. We have to show that the functional $I|_{X_G}$ satysfies the remaining hypotheses of Theorem 3.1 for $E = X_G$ which is infinite dimensional by assumption (G_3).

Lemma 3.1 *Assume (f_1), (f_2) and $(A - R)$. Then $I|_{X_G}$ satisfies (I_1).*

Proof. As a consequence of (f_1), (f_2) and $(A - R)$, fixed $\alpha > 0$ and $q > 1$, we have that for any $0 < \varepsilon < 1$ there exists a constant $C(\alpha, q, \varepsilon) > 0$ such that

$$|F(s)| \leq \varepsilon|s|^2 + C(\alpha, q, \varepsilon)|s|^q(e^{\alpha s^2} - 1) \qquad \forall s \in \mathcal{R}\,. \tag{3.1}$$

Thus in particular if we fix $\alpha = 1$ and $q > 3$ then for any $0 < \varepsilon < 1$ we have

$$\int_{\mathcal{R}^4} F(u)\,dx \leq \varepsilon\|u\|_2^2 + C(q, \varepsilon)\int_{\mathcal{R}^4} |u|^q(e^{u^2} - 1)\,dx \qquad \forall u \in X_G\,.$$

Now, recalling that the embedding $X_G \hookrightarrow H^2(\mathcal{R}^4)$ is continuous, namely there exists a constant $\overline{C} > 0$ such that

$$\|u\|_{H^2} \leq \overline{C}\|u\| \qquad \forall u \in X_G\,,$$

we have that if $\|u\| \leq \frac{1}{\overline{C}}$ then $\|u\|_{H^2} \leq 1$. Applying Lemma 2.2

$$\int_{\mathcal{R}^4} F(u)\,dx \leq \varepsilon\|u\|_2^2 + C_1(q, \varepsilon)\|u\|^q \qquad \forall u \in X_G,\ \|u\| \leq \frac{1}{\overline{C}}$$

and without loss of generality we may assume that $[C_1(q,\varepsilon)]^{\frac{1}{q-2}} > \overline{C}$.

So for any $u \in X_G$ with $\|u\| \le \frac{1}{\overline{C}}$ we have

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_2^2 - C_1(q,\varepsilon)\|u\|^q \ge \left(\frac{1}{2} - \frac{\varepsilon}{V_0}\right)\|u\|^2 - C_1(q,\varepsilon)\|u\|^q = \\ &= \|u\|^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\varepsilon)\|u\|^{q-2}\right) . \end{aligned} \tag{3.2}$$

Now we choose $0 < \varepsilon < 1$ as follows

$$0 < \varepsilon < \min\left\{1, \left(\frac{1}{2} - \frac{1}{2^{q-2}}\right)V_0\right\}$$

and we set

$$\rho := \frac{1}{2[C_1(q,\varepsilon)]^{\frac{1}{q-2}}} .$$

Since $\rho < \frac{1}{\overline{C}}$, (3.2) holds for any $u \in X_G$ with $\|u\| = \rho$ and we have

$$I(u) \ge \rho^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\varepsilon)\rho^{q-2}\right) = \rho^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{2^{q-2}}\right) \quad \forall u \in X_G,\ \|u\| = \rho.$$

Setting

$$\gamma := \rho^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{2^{q-2}}\right) > 0 ,$$

we get

$$I(u) \ge \gamma \qquad \forall u \in X_G,\ \|u\| = \rho .$$

In conclusion, if $\rho_1 \le \rho$ then applying (3.2) we have that for any $u \in X_G$ with $\|u\| = \rho_1$

$$I(u) \ge \rho_1^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\varepsilon)\rho_1^{q-2}\right) \ge \rho_1^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\varepsilon)\rho^{q-2}\right) = \rho_1^2\frac{\gamma}{\rho^2} > 0$$

and this means that

$$I(u) > 0 \qquad \forall u \in X_G \setminus \{0\}, \|u\| \le \rho .$$

Lemma 3.2 *Assume (f_2) and $(A-R)$. Then $I|_{X_G}$ satisfies (I_2).*

Proof. As a consequence of (f_2) and $(A-R)$ there exist C_1, $C_2 > 0$ such that

$$F(s) \ge C_1|s|^\mu - C_2|s|^2 \qquad \forall s \in \mathcal{R} .$$

Therefore for any $u \in X_G$ we have that

$$I(u) \le \frac{1}{2}\|u\|^2 + C_2\|u\|_2^2 - C_1\|u\|_\mu^\mu \le \left(\frac{1}{2} + \frac{C_2}{V_0}\right)\|u\|^2 - C_1\|u\|_\mu^\mu . \tag{3.3}$$

Let $\tilde{E}$ be a finite dimensional subspace of X_G. Since all norms in $\tilde{E}$ are equivalent, there exists a constant $C > 0$ such that for any $u \in \tilde{E}$ we have $\|u\|_\mu \geq C\|u\|$. Thus

$$I(u) \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right)\|u\|^2 - \tilde{C}_1\|u\|^\mu \qquad \forall u \in \tilde{E}$$

and in particular for any $u \in \tilde{E}$ with $\|u\| = R$

$$I(u) \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right)R^2 - \tilde{C}_1 R^\mu \ .$$

This means that for $R > 0$ sufficiently large

$$I(u) < 0 \qquad \forall u \in \tilde{E},\ \|u\| > R$$

and the set $\{u \in \tilde{E} \mid I(u) \geq 0\}$ is bounded.

Lemma 3.3 *Assume* (f_1), (f_2) *and* (F_1). *Then* $I|_{X_G}$ *satisfies* (I_3).

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_G$ be a Palais-Smale sequence, that is $|I(u_n)| \leq C_1\ \forall n \in \mathbb{N}$ and $I'(u_n) \to 0$ as $n \to +\infty$.

Firstly we prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G. For any $u, v \in X_G$

$$I'(u)v = \langle u, v\rangle - \int_{\mathcal{R}^4} f(u)v\,dx$$

therefore, for any $n \in \mathbb{N}$

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \int_{\mathcal{R}^4} \left(F(u_n) - \frac{1}{\mu} f(u_n)u_n\right) dx \ .$$

As a consequence of (F_1) we have that

$$\int_{\mathcal{R}^4} \left(F(u_n) - \frac{1}{\mu} f(u_n)u_n\right) dx \leq 0$$

and so we obtain

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \qquad \forall n \in \mathbb{N} \ . \tag{3.4}$$

On the other hand

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \leq |I(u_n)| + \frac{1}{\mu}|I'(u_n)u_n| \leq C_1 + \frac{C_2}{\mu}\|u_n\| \qquad \forall n \in \mathbb{N} \ . \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$0 \leq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \leq C_1 + \frac{C_2}{\mu}\|u_n\| \qquad \forall n \in \mathbb{N}$$

which means that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G.

It remains to prove that $\{u_n\}_{n\in\mathbb{N}}$ converges up to subsequences. Since, by assumption, (G_2) holds and since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G, we have that $u_n \to u$ in $L^p(\mathcal{R}^4)$ for any $p \in (4, +\infty)$. Here and below, up to the end of the proof, convergence has to be understood up to the passage to a subsequence.

Fix $n \in \mathbb{N}$. Since

$$[I'(u_n) - I'(u)](u - u_n) = \|u - u_n\|^2 - \int_{\mathcal{R}^4} [f(u_n) - f(u)](u - u_n)\, dx\ ,$$

then

$$\|u - u_n\|^2 = [I'(u_n) - I'(u)](u - u_n) + \int_{\mathcal{R}^4} [f(u_n) - f(u)](u - u_n)\, dx\ .$$

As $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence we have that

$$[I'(u_n) - I'(u)](u - u_n) \to 0$$

when $n \to +\infty$. If we show that for any $0 < \varepsilon < 1$ there exist constants $C_3 > 0$ and $C_4(\varepsilon) > 0$ such that

$$E_n := \int_{\mathcal{R}^4} [f(u_n) - f(u)](u - u_n)\, dx \leq C_3\varepsilon + C_4(\varepsilon)\|u - u_n\|_p \qquad \forall n \in \mathbb{N} \tag{3.6}$$

for some $p > 4$, then it follows that $\|u - u_n\|^2 \to 0$ as $n \to +\infty$ that is what we wanted to prove.

Therefore to end the proof we have to show that (3.6) holds. At this aim we can notice that as a consequence of (f_1) and (f_2), fixed $\alpha > 0$ and $q > 0$, for any $0 < \varepsilon < 1$ there exists a constant $C(\alpha,\, q,\, \varepsilon) > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C(\alpha,\, q,\, \varepsilon)\, |s|^q (e^{\alpha s^2 - 1}) \qquad \forall s \in \mathcal{R}\ .$$

Let $\alpha > 0$ and $q > 0$ to be choosen during the proof. Then for any $0 < \varepsilon < 1$ we have

$$\begin{aligned}
E_n &\leq \int_{\mathcal{R}^4} \Big[|f(u_n)| + |f(u)|\Big]\, |u - u_n|\, dx \leq \\
&\leq \int_{\mathcal{R}^4} \Big[\varepsilon(|u_n| + |u|) + \\
&\qquad + C(\alpha,\, q,\, \varepsilon)\Big(|u_n|^q (e^{\alpha u_n^2} - 1) + |u|^q (e^{\alpha u^2} - 1)\Big)\Big]\, |u - u_n|\, dx = \\
&= \varepsilon\, E_{1,n} + C(\alpha,\, q,\, \varepsilon)\, E_{2,n}
\end{aligned}$$

where we have set

$$\begin{aligned}
E_{1,n} &:= \int_{\mathcal{R}^4} \Big(|u_n| + |u|\Big)\, |u - u_n|\, dx \qquad \forall n \in \mathbb{N}\ , \\
E_{2,n} &:= \int_{\mathcal{R}^4} \Big(|u_n|^q (e^{\alpha u_n^2} - 1) + |u|^q (e^{\alpha u^2} - 1)\Big)\, |u - u_n|\, dx \qquad \forall n \in \mathbb{N}\ .
\end{aligned}$$

We estimate $E_{1,n}$ as follows:

$$E_{1,n} \leq \int_{\mathcal{R}^4} (|u_n|^2 + |u|^2)\, dx \leq 2(\|u_n\|_2^2 + \|u\|_2^2) \leq \frac{2}{V_0}(\|u_n\|^2 + \|u\|^2) \leq C_3 \quad \forall n \in \mathbb{N}.$$

To estimate $E_{2,n}$ we apply Hölder's inequality with $\frac{4}{5}+\frac{1}{5}=1$ obtaining that

$$\begin{aligned} E_{2,n} &\leq \left[\left(\int_{\mathcal{R}^4} |u_n|^{\frac{5}{4}q} (e^{\alpha u_n^2}-1)^{\frac{5}{4}}\, dx \right)^{\frac{4}{5}} + \right. \\ &\quad \left. + \left(\int_{\mathcal{R}^4} |u|^{\frac{5}{4}q} (e^{\alpha u^2}-1)^{\frac{5}{4}}\, dx \right)^{\frac{4}{5}} \right] \|u-u_n\|_5 = \\ &= \left[(E_{4,n})^{\frac{4}{5}} + (E_{5,n})^{\frac{4}{5}} \right] \|u-u_n\|_5 \end{aligned}$$

where we have set

$$E_{4,n} := \int_{\mathcal{R}^4} |u_n|^{\frac{5}{4}q} (e^{\alpha u_n^2}-1)^{\frac{5}{4}}\, dx \qquad E_{5,n} := \int_{\mathcal{R}^4} |u|^{\frac{5}{4}q} (e^{\alpha u^2}-1)^{\frac{5}{4}}\, dx \quad \forall n \in \mathbb{N}.$$

Now, it suffices to prove that $E_{4,n}$ and $E_{5,n}$ are bounded by a constant independent of n to conclude that (3.6) holds with $p = 5$. As $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X there exists a constant $M > 0$ such that $\|u_n\|_{H^2} \leq M$ $\forall n \in \mathbb{N}$ and $\|u\|_{H^2} \leq M$. Thus, choosing $\alpha < \frac{64\pi^2}{5M^2}$ and $q \geq \frac{4}{5}$, we can apply Lemma 2.3 obtaining the desired estimate for $E_{4,n}$ and $E_{5,n}$. This completes the proof of Lemma 3.3.

In conclusion $I|_{X_G}$ satisfies the assumptions of Theorem 3.1 and possesses a sequence $\{c_k\}_{k\in\mathbb{N}}$ of critical values such that $c_k \to +\infty$ as $k \to +\infty$. The associated sequence of critical points $\{u_k\}_{k\in\mathbb{N}}$ lies in X_G and is unbounded. Infact, reasoning as in (3.3), we get

$$c_k = I(u_k) \leq \left(\frac{1}{2} + \frac{C_2}{V_0} \right) \|u_k\|^2$$

from which it follows that $\|u_k\| \to +\infty$ as $k \to +\infty$.

4 Exploiting symmetries

We have to construct a group acting on X which satisfies the assumptions of Proposition 1.1, namely a subgroup $G \subseteq \mathcal{O}(4)$ acting on X and satisfying (G_1), (G_2) and (G_3). As already mentioned, at this aim we will follow an idea of T. Bartsch and M. Willem ([7], see also [6]).

Let H be the subgroup of $\mathcal{O}(4)$ defined as

$$H := \mathcal{O}(2) \times \mathcal{O}(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ : \ a,\, b \in \mathcal{O}(2) \right\} \subset \mathcal{O}(4)$$

and consider

$$\tau := \begin{pmatrix} 0 & i_2 \\ i_2 & 0 \end{pmatrix} \in \mathcal{O}(4)$$

where i_2 denotes the identity matrix in $\mathcal{R}^2$. We can notice that $\tau^{-1} = \tau$ and τ is in the normalizer of H in $\mathcal{O}(4)$, namely $\tau H = H\tau$. We define $G :=< H \cup \{\tau\} >$, an element $g \in G$ can be written uniquely in the form

$$g = h \quad \text{or} \quad g = h\tau$$

with $h \in H$. We consider the action of G on X defined by

$$\begin{aligned} h * u(x) &:= u(h^{-1}x) && \text{for a.e. } x \in \mathcal{R}^4,\ \forall h \in H\ , \\ h\tau * u(x) &:= -u(\tau h^{-1}x) && \text{for a.e. } x \in \mathcal{R}^4,\ \forall h \in H \end{aligned}$$

for any $u \in X$. It is easy to see that this indeed defines an action of G on X, namely $i_4 * g = g$ and $(g_1 g_2) * u = g_1 * (g_2 * u)$ for $g_1, g_2 \in G$, $u \in X$, and that this action is continuous.

Remark 4.1 A special case of the action of G over X is the following:

$$\tau * u(x) = -u(\tau x) \qquad \text{for a.e. } x \in \mathcal{R}^4\ .$$

So in particular if $u \in X_G$ and $x \in \mathcal{R}^4$ with $\tau x = hx$ for some $h \in H$ then

$$-u(x) = u(\tau x) = u(\tau h^{-1}\tau x) = u(h^{-1}x) = u(x)$$

and $u(x) = 0$. Therefore any $u \in X_G$ must necessarily be zero on the set

$$\{x \in \mathcal{R}^4 \mid \tau x = hx \ \text{ for some } h \in H\}\ .$$

Since I is even according to (f_3), I is G-invariant. In fact if we assume that (V_2) holds then the potential V is spherically symmetric and in particular V is G-invariant

$$V(gx) = V(|gx|) = V(|x|) = V(x) \qquad \forall g \in G \subset \mathcal{O}(4),\ \forall x \in \mathcal{R}^4\ .$$

Also under the assumptions (V_2') and (V_2'') on the potential V we have that

$$V(hx) = V(|ax_1|, |bx_2|) = V(|x_1|, |x_2|) = V(x)\ ,$$

$$V(h\tau x) = V(|ax_2|, |bx_1|) = V(|x_2|, |x_1|) = V(|x_1|, |x_2|) = V(x)\ ,$$

$$\forall h = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H,\ \forall x_1,\ x_2 \in \mathcal{R}^2$$

and V is G-invariant. Therefore condition (G_1) is satified. The compactness condition (G_2) is a consequence of a result due to E. Hebey and M. Vaugon (see [18], Corollary 4) which generalize a well known result of P. L. Lions (see [19], Théorème III.1). For the convenience of the reader we report here below the part of this more general result that we will use. If $x \in \mathcal{R}^4$ then we write $x = (x_1, x_2)$ where $x_1, x_2 \in \mathcal{R}^2$ with respect to the splitting $\mathcal{R}^4 = \mathcal{R}^2 \times \mathcal{R}^2$. Let $W^{1,4}_H(\mathcal{R}^4)$ the subspace of $W^{1,4}(\mathcal{R}^4)$ consisting of all $u \in W^{1,4}(\mathcal{R}^4)$ radially symmetric with respect to $x_i \in \mathcal{R}^2$ for $i \in \{1, 2\}$

$$W^{1,4}_H(\mathcal{R}^4) := \{u \in W^{1,4}(\mathcal{R}^4) \mid h * u = u \quad \forall h \in H\}\ .$$

$W^{1,4}_H(\mathcal{R}^4)$ is nothing but the space of fixed points in $W^{1,4}(\mathcal{R}^4)$ with respect to the action of H.

Theorem 4.1 ([18], Corollary 4) *For any $p \in (4, +\infty)$ the embedding*

$$W^{1,4}_H(\mathcal{R}^4) \hookrightarrow L^p(\mathcal{R}^4)$$

is compact, i.e. $W^{1,4}_H(\mathcal{R}^4) \hookrightarrow\hookrightarrow L^p(\mathcal{R}^4)$.

Now, as a consequence of Theorem 4.1, the hypothesis (G_2) of Proposition 1.1 easily follows

$$X_G \hookrightarrow W_H^{1,4}(\mathcal{R}^4) \hookrightarrow\hookrightarrow L^p(\mathcal{R}^4) \qquad \forall p \in (4, +\infty) .$$

Obviously we have also that the G satisfies hypothesis (G_3). Therefore, from Proposition 1.1, we obtain the existence of an unbounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset X_G$ of critical points for the functional I. These critical points are not radial, infact by construction

$$u_k(x) = -u_k(\tau x) \qquad \text{for a.e. } x \in \mathcal{R}^4,\ \forall k \in \mathbb{N}$$

and, furthermore, are sign-changing (see Remark 4.1 above). This ends the proof of Theorem 1.2 and of the first part of Theorem 1.1.

We can notice that it is easy to adapt the previous arguments to obtain a proof of Theorem 1.3. In fact we can apply again Proposition 1.1 with the action of H defined by

$$h * u(x) := u(h^{-1}x) \qquad \text{for a.e. } x \in \mathcal{R}^4,\ \forall h \in H .$$

Since the potential V is spherically symmetric with respect to $x_1, x_2 \in \mathcal{R}^2$ according to (V_2'), I is H-invariant and condition (G_1) is satisfied. Furthermore, from Theorem 4.1 we have

$$X_H \hookrightarrow W_H^{1,4}(\mathcal{R}^4) \hookrightarrow\hookrightarrow L^p(\mathcal{R}^4) \qquad \forall p \in (4, +\infty)$$

and also condition (G_2) is satisfied.

To conclude the proof of Theorem 1.1, it remains to prove the existence of an unbounded sequence of critical points of I which are radial. At this aim it suffices to notice that the orthogonal group $\mathcal{O}(4)$ satisfies the assumptions of Proposition 1.1 with respect to the action defined by

$$g * u(x) := u(g^{-1}x) \qquad \text{for a.e. } x \in \mathcal{R}^4,\ \forall g \in \mathcal{O}(4) .$$

Infact, since the potential V is spherically symmetric according to (V_2), condition (G_1) is satisfied and we have the following result of P. L. Lions [19] which states that $\mathcal{O}(4)$ satisfies (G_2). Let $H^2_{\text{rad}}(\mathcal{R}^4)$ be the subspace of $H^2(\mathcal{R}^4)$ consisting of all $u \in H^2(\mathcal{R}^4)$ which are radially symmetric. $H^2_{\text{rad}}(\mathcal{R}^4)$ is nothing but the space of fixed points in $H^2(\mathcal{R}^4)$ with respect to the action of $\mathcal{O}(4)$.

Theorem 4.2 ([19], Théorème II.1) *For any $p \in (2, +\infty)$ the embedding*

$$H^2_{rad}(\mathcal{R}^4) \hookrightarrow L^p(\mathcal{R}^4)$$

is compact, i.e. $H^2_{rad}(\mathcal{R}^4) \hookrightarrow\hookrightarrow L^p(\mathcal{R}^4)$.

Therefore, applying again Proposition 1.1 with $G = \mathcal{O}(4)$ we obtain an unbounded sequence of critical points for the functional I lying on $H^2_{\text{rad}}(\mathcal{R}^4)$.

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