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Existence and Uniqueness of Solutions for a Class of p-Laplace Equations on a Ball

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Abstract

For a class of equations generalizing the model case

$$\Delta_p u - a(r)u^{p-1} + b(r)u^q = 0 \ \text{ in } B, \quad u = 0 \ \text{ on } \partial B,$$

where B is the unit ball in R^n, $n \geq 1$, $r = |x|$, $p, q > 1$, and Δ_p denotes the p-Laplace operator, we give conditions for the existence and uniqueness of positive solution. In case $n = 1$, we give a more general result.

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1 Introduction

We study the existence and uniqueness of positive solutions of the Dirichlet problem

$$\Delta_p u + f(r, u) = 0 \ \text{ in } B, \quad u = 0 \ \text{ on } \partial B, \tag{1.1}$$

where B is the unit ball in R^n, $n \geq 1$, $r = |x|$, and Δ_p denotes the p-Laplace operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 1 .$$

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We shall assume that $f_r(r,u) \le 0$ for all $r \in [0,1]$, and $u > 0$, which implies (in case $1 < p \le 2$) that positive solutions are radially symmetric, i.e., $u = u(r)$, and the equation (1.1) becomes

$$\left(u'|u'|^{p-2}\right)' + \frac{n-1}{r}u'|u'|^{p-2} + f(r,u) = 0, \quad 0 < r < 1, \quad u'(0) = u(1) = 0\,.$$

In case $p = 2$, exact multiplicity results were given in P. Korman, Y. Li and T. Ouyang [11], T. Ouyang and J. Shi [12], and P. Korman [10], and other papers of the same authors. Uniqueness results, in case $p = 2$, were studied by many authors, beginning with B. Gidas, W.-M. Ni and L. Nirenberg [7], for $f(u) = u^q$, see the references in [11], [12]. When $p \neq 0$, there are a number of technical difficulties, some of which were overcome only recently, through the efforts of a number of people. In particular, F. Brock [2] and L. Damascelli and F. Pacella [4] have extended the symmetry results of B. Gidas, W.-M. Ni and L. Nirenberg [7]. J. Serrin and H. Zou [16] have proved a Liouville type result, which implies a priori estimates of B. Gidas and J. Spruck type [8], provided that the point of maximum of solution is bounded away from the boundary, as was observed in A. Aftalion and F. Pacella [1], and J. Fleckinger and W. Reichel [5]. A. Aftalion and F. Pacella [1] pointed out the appropriate function spaces, in order to apply the implicit function theorem. In fact, A. Aftalion and F. Pacella [1] have proved uniqueness of solution to (1.1), for $f(r,u)$ similar to ours. In this paper, we provide a simpler proof of the crucial step in [1], involving the non-degeneracy of solutions. Moreover, we clarify the proper conditions, and provide the existence part for the model examples.

Recall that solution of (1.1) is called non-degenerate if the corresponding linearized problem (the problem (2.13) below) has only the trivial solution. Non-degeneracy results were usually proved by using the method of test functions, see P. Korman, Y. Li and T. Ouyang [11], T. Ouyang and J. Shi [12], and the references in those papers. A variation on that approach, involving maximum principle, was used in A. Aftalion and F. Pacella [1]. Here we use a simpler method, based on some identities of M. Tang, and his method [17]. We have already used this approach, see [9], in case $p = 2$ (M. Tang's results are for $p = 2$ case). In this paper, we extend M. Tang's [17] identities for $p \neq 2$ case, allowing a considerable streamlining of the proof of non-degeneracy. These new identities are likely to be useful for other problems.

Our main example is $f(r,u) = -a(r)u^{p-1} + b(r)u^q$, with $p,q > 1$, and $p - 1 < q$. This function is negative for small u, and positive for u large. We give conditions for existence and uniqueness of positive solution in this case. We show that in one space dimension, and for $f = f(u)$, the same result remains true, if $f(u)$ is arbitrarily modified in the region where it is negative. This surprising result was proved, in case $p = 2$, by R. Schaaf [15]. The uniqueness part was proved (for any $p > 1$) by J. Cheng [3]. Both of these works were using time maps, while we are using a more flexible bifurcation approach.

2 Existence and uniqueness for a class of non-autonomous problems

We consider positive solutions of the problem

$$\Delta_p u + f(|x|,u) = 0, \quad \text{for } |x| < 1, \quad u = 0 \text{ on } |x| = 1. \tag{2.1}$$

We assume that $f(r,u) \in C^1([0,1] \times \bar{R}_+)$ is a continuously differentiable function (with $r = |x|$), and

$$f(r,0) = 0, \quad \text{and} \quad f_r(r,u) \leq 0, \qquad \text{for all } r \in [0,1], \text{ and } u > 0\,. \tag{2.2}$$

If $1 < p \leq 2$, then in view of B. Gidas, W.-M. Ni and L. Nirenberg [7] and L. Damascelli and F. Pacella [4], positive solutions of (2.1) are radially symmetric, and hence they satisfy

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) + f(r,u(r)) = 0, \quad 0 < r < 1, \quad u'(0) = u(1) = 0\,. \tag{2.3}$$

where we denote $\varphi(t) = t|t|^{p-2}$. In case $p > 2$, we restrict our attention to the radial solutions of (2.1), i.e., we again consider (2.3). Following B. Franchi et al [6], we consider *classical* solutions of (2.3), i.e., we assume that $u \in C^1[0,1]$ and $\varphi(u') \in C^1(0,1]$. As a consequence, if $u'(r_1) = 0$ for some $r_1 \in (0,1)$, we may define by continuity $\varphi(u'(r_1)) = 0$.

The following lemma gives a known condition for the Hopf's lemma to hold, see J.L. Vázquez [18], or A. Aftalion and F. Pacella [1].

Lemma 2.1 *Assume that $f(r,u) \in C^1([0,1] \times \bar{R}_+)$ satisfies $f(r,0) = 0$, for all $r \in [0,1]$, and either f is non-negative, or for some $c_0 > 0$*

$$-f(r,s) < c_0 s^{p-1}, \quad \textit{for small } s > 0, \textit{ and } r \textit{ near } 1\,. \tag{2.4}$$

If $u(r)$ is a positive solution of (2.3), then

$$u'(1) < 0\,. \tag{2.5}$$

The next lemma extends the Proposition 1.2.6 in B. Franchi et al [6], which considered the case of $f = f(u)$, i.e., independent of r. Define $F(r,u) = \int_0^u f(r,t)\,dt$.

Lemma 2.2 *Assume that*

$$F_r(r,u) \leq 0, \qquad \textit{for all } r \in [0,1], \textit{ and } u > 0\,; \tag{2.6}$$

$$\textit{for each } r_0 \in [0,1)\textit{, the function } f(r_0,u) \textit{ is either positive for all } u > 0, \tag{2.7}$$
or it changes sign once from negative, for small u, to positive.

Then any positive solution of (2.3) satisfies

$$u'(r) < 0 \qquad \textit{for all } r \in (0,1)\,, \tag{2.8}$$

$$f(0,u(0)) > 0\,. \tag{2.9}$$

Proof. We show first that $u(r)$ has no local minimums. Indeed, let $r_0 \in [0,1]$ be a point of local minimum, i.e., $u'(r_0) = 0$, $u''(r_0) \geq 0$. We claim that

$$f(r_0,u(r_0)) \leq 0\,. \tag{2.10}$$

If $p \geq 2$, this follows by evaluating the equation (2.3) at r_0. In general, we can find a sequence $\{r_n\}$ tending to r_0 from the right, at which $u'(r_n) > 0$, $u''(r_n) \geq 0$. Evaluating the equation (2.3) at r_n, we have $f(r_n, u(r_n)) < 0$, and taking the limit we conclude (2.10). In case f is positive, (2.10) implies a contradiction. In the other case, (2.10) implies that $F(r_0, u(r_0)) < 0$, and hence the "energy" $E(r) \equiv \frac{p-1}{p}|u'(r)|^p + F(r, u(r))$ satisfies $E(r_0) < 0$. We have

$$E'(r) = -\frac{n-1}{r}\varphi(u')u' + F_r(r,u) \leq 0\,,$$

and so the energy is non-increasing. But $E(1) = \frac{p-1}{p}|u'(1)|^p \geq 0$, a contradiction. We conclude that $u'(r) \leq 0$ for all r. To get the strict inequality, assume that $u'(r_1) = 0$ at some $r_1 \in (0,1)$. As discussed above, we have $\varphi(u'(r_1)) = 0$, and we also have $\frac{d}{dr}\varphi(u'(r_1)) = 0$, since r_1 is not an extremum. From the equation (2.3), $f(r_1, u(r_1)) = 0$, implying that $E(r_1) < 0$, which results in the same contradiction as before. Finally, we conclude (2.9), by using the same argument one more time, since the opposite inequality would imply $E(0) < 0$. ◊

We assume that $f(r,u)$ satisfies the following additional conditions, which are similar to the ones in [1]

$$uf_u(r,u) - (p-1)f(r,u) > 0 \quad \text{for all } r \in [0,1], \text{ and } u > 0\,; \tag{2.11}$$

For any positive solution of (2.3)

$$\alpha(r) = \frac{pf(r,u(r)) + rf_r(r,u(r))}{u(r)f_u(r,u(r)) - (p-1)f(r,u(r))} \tag{2.12}$$

is a non-increasing function of r, for $r \in (0,1)$.

We shall need to consider the linearized problem corresponding to (2.3) (here $w = w(r)$)

$$(\varphi'(u'(r))w'(r))' + \frac{n-1}{r}\varphi'(u'(r))w'(r) + f_u(r,u(r))w(r) = 0, \quad 0 < r < 1, \tag{2.13}$$
$$w'(0) = w(1) = 0\,.$$

We will show that under the above conditions any positive solution of (2.3) is non-degenerate, i.e., the problem (2.13) admits only the trivial solution. The following technical lemma we proved in [9]. We include its proof for completeness.

Lemma 2.3 *Let $u(r)$ be a positive solution of (2.3), and assume that the function $f(r,u)$ satisfies the conditions (2.2) and (2.11). Then the function $f(r,u(r))$ can change sign at most once on* $(0,1)$.

Proof. Let $\xi \in (0,1)$ be such that $f(\xi, u(\xi)) = 0$. We claim that $f(r,u(r)) > 0$ for all $r \in [0,\xi)$. Indeed, by (2.11) we conclude that $f_u(\xi, u(\xi)) > 0$, and in general $f_u(r,u(r)) > 0$, so long as $f(r,u(r)) > 0$, and hence

$$\frac{d}{dr}f(r,u(r)) = f_r(r,u(r)) + f_u(r,u(r))u'(r) < 0,$$

and the claim follows. So that, if the function $f(r,u(r))$ is positive near $r = 1$, it is positive for all $r \in [0,1)$. If, on the other hand, $f(r,u(r))$ is negative near $r = 1$, it will change sign exactly once on $[0,1)$ (since it cannot stay negative for all r, by the maximum principle). ◊

This lemma implies that either $f(r,u(r)) > 0$ for all $r \in [0,1)$, or else there is a $r_2 \in (0,1)$, so that $f(r,u(r)) > 0$ on $[0,r_2)$ and $f(r,u(r)) < 0$ on $(r_2,1)$. We put these cases together, we defining $r_2 = 1$ in the first case. I.e., the last lemma implies that

$$\text{there is a } r_2 \in (0,1], \text{ so that } f(r,u(r)) > 0 \text{ on } [0,r_2) \tag{2.14}$$
$$\text{and, in case } r_2 < 1, \text{ we have: } f(r,u(r)) < 0 \text{ on } (r_2,1)\,.$$

We shall consider the following two functions, depending on the solutions of (2.3) and (2.13):

$$\xi(r) = r^{n-1}\left[(p-1)\varphi(u'(r))w(r) - \varphi'(u'(r))u(r)w'(r)\right]\;; \tag{2.15}$$

$$T(r) = r^{n}\left[(p-1)\varphi(u'(r))w'(r) + f(r,u(r))w(r)\right] + (n-p)r^{n-1}\varphi(u'(r))w(r)\,. \tag{2.16}$$

In case $p = 2$, these functions were introduced by M. Tang [17]. The following crucial lemma is proved by a direct computation.

Lemma 2.4 *For any solutions of (2.3) and (2.13), we have*

$$\xi'(r) = r^{n-1}w(r)\left[u(r)f_u(r,u(r)) - (p-1)f(r,u(r))\right]\;; \tag{2.17}$$

$$T'(r) = r^{n-1}w(r)\left[pf(r,u(r)) + rf_r(r,u(r))\right]\,. \tag{2.18}$$

We shall need the following functions, depending on the solutions of (2.3), see [12], [17], [1], and [9].

$$Q(r) = r^{n}\left[(p-1)\varphi\left(u'(r)\right)u'(r) + u(r)f(r,u(r))\right] + (n-p)r^{n-1}\varphi\left(u'(r)\right)u(r); \tag{2.19}$$

$$P(r) = r^{n}\left[(p-1)\varphi\left(u'(r)\right)u'(r) + pF(r,u(r))\right] + (n-p)r^{n-1}\varphi\left(u'(r)\right)u(r)\,, \tag{2.20}$$

where we again denote $F(r,u) = \int_0^u f(r,t)\,dt$. Observe that

$$P(0) = 0, \quad \text{and} \quad P(1) = \varphi\left(u'(1)\right)u'(1) > 0\,, \tag{2.21}$$

provided (2.4) holds, and

$$\begin{aligned} P'(r) &= r^{n-1}\left[npF(r,u(r)) - (n-p)u(r)f(r,u(r)) + prF_r(r,u(r))\right] \\ &\equiv r^{n-1}I(r). \end{aligned} \tag{2.22}$$

We shall need the following lemma.

Lemma 2.5 *Let $u(r)$ be a positive solution of (2.3), and assume that the function $f(r,u)$ satisfies the condition (2.2). Then any solution of the linearized problem (2.13), $w(r)$, cannot vanish in the region where $f(r,u(r)) < 0$ (i.e., on $(r_2,1)$, see (2.14)).*

Proof. Assume that the contrary is true, and let τ denote the largest root of $w(r)$ in the region where $f(r,u(r)) < 0$. We may assume that $w(r) > 0$ on $(\tau,1)$. Then integrating the formula (2.18) over $(\tau,1)$, we have

$$(p-1)\varphi\left(u'(1)\right)w'(1) - \tau^n(p-1)\varphi\left(u'(\tau)\right)w'(\tau) = \int_\tau^1 \left(pf(r,u(r)) + rf_r(r,u(r))\right)wr^{n-1}\,dr.$$

We have a contradiction, since the left hand side is non-negative, while the integral on the right is negative. ◊

Next, we present the crucial non-degeneracy result.

Theorem 2.1 *Let $u(r)$ be a positive solution of (2.3). Assume that the conditions (2.2), (2.4), (2.6), (2.7), (2.11) and (2.12) hold. In the case $p < n$, assume additionally the following two conditions:*

$$u(r)f(r,u(r)) - pF(r,u(r)) > 0 \quad \textit{for } r \in (0, r_2); \tag{2.23}$$

$$\textit{the function } I(r) = npF(r,u(r)) - (n-p)u(r)f(r,u(r)) + prF_r(r,u(r)), \tag{2.24}$$

defined in (2.22), satisfies any one of the following three conditions:
(i) $I(r) > 0$ *on* $(0, r_2)$,
(ii) $I(r) < 0$ *on* $(0, 1)$,
(iii) $I(r) > 0$ *on* $(0, r_0)$ *and* $I(r) < 0$ *on* $(r_0, 1)$, *for some* $r_0 \in (0, 1)$.
Then $u(r)$ is a non-degenerate solution, i.e., the corresponding linearized problem (2.13) admits only the trivial solution.

Proof. With r_2 as defined by (2.14), we claim that

$$Q(r) > 0 \ \text{ on } [0, r_2). \tag{2.25}$$

In the case $p \geq n$, this follows immediately from the definition of $Q(r)$ in (2.19). In the other case, $p < n$, observe that $P(r) > 0$ on $[0, r_2)$ by (2.21) and (2.22). We write

$$Q(r) = P(r) + r^n \left[u(r)f(r,u(r)) - pF(r,u(r))\right],$$

and hence $Q(r) > 0$ on $[0, r_2)$ by our condition (2.23).

We now define the function $O(r) = \gamma\xi(r) - T(r)$, which in view of (2.17) and (2.18) satisfies

$$O'(r) = \left[u(r)f_u(r,u(r)) - (p-1)f(r,u(r))\right] w(r) r^{n-1} \left[\gamma - \alpha(r)\right], \tag{2.26}$$

with $\alpha(r)$ as defined by (2.12), and γ is a constant, to be selected. We may assume that $w(0) > 0$. We claim that the function $w(r)$ cannot have any roots inside $(0, 1)$. Assuming otherwise, let τ_1 be the smallest root of $w(r)$, i.e., $w(r) > 0$ on $[0, \tau_1)$. Let $\tau_2 \in (\tau_1, 1]$ denote the second root of $w(r)$. We now fix $\gamma = \alpha(\tau_1)$. By the monotonicity of $\alpha(r)$, the function $\gamma - \alpha(r)$ is non-positive on $[0, \tau_1)$ and non-negative on (τ_1, τ_2). Hence, $O'(r) \leq 0$ (at $r = \tau_1$, both $w(r)$ and $\gamma - \alpha(r)$ change sign). Since $O(0) = 0$, we conclude that

$$O(r) \leq 0 \ \text{ for all } r \in [0, \tau_2]. \tag{2.27}$$

There are two possibilities for the second root.

Case (i) $\tau_2 = 1$, i.e., $w(r) < 0$ on $(\tau_1, 1)$. From (2.27) we have $O(1) \leq 0$. On the other hand, $O(1) = -T(1) = -(p-1)\varphi(u'(1))\, w'(1) > 0$, by (2.5), a contradiction.

Case (ii) $\tau_2 < 1$. Observe that $f(\tau_2, u(\tau_2)) > 0$. Indeed, assuming otherwise, we conclude by Lemma 2.3 that $f(r, u(r)) \leq 0$ on $(\tau_2, 1)$. But this contradicts Lemma 2.5. Applying Lemma 2.3 again, we conclude that $f(r, u(r)) > 0$ over $[0, \tau_2)$. It follows that $\tau_2 < r_2$, i.e., by (2.25), $Q(r) > 0$ on $[0, \tau_2)$.

Since $\xi(\tau_1) > 0$, while $\xi(\tau_2) < 0$, we can find a point $t \in (\tau_1, \tau_2)$, such that $\xi(t) = 0$, i.e.,

$$\frac{u(t)}{w(t)} = (p-1)\frac{\varphi(u'(t))}{\varphi'(u'(t))\, w'(t)} = \frac{u'(t)}{w'(t)}. \tag{2.28}$$

Since $t \in (0, r_2)$, we have

$$Q(t) > 0. \tag{2.29}$$

In view of (2.27),

$$T(t) = -O(t) \geq 0.$$

On the other hand, using (2.28) and (2.29),

$$T(t) = \left[t^n \left((p-1)\varphi(u')\, w' \frac{u}{w} + f(t,u)u\right) + (n-p)t^{n-1}\varphi(u')\, u\right] \frac{w}{u} = Q(t)\frac{w(t)}{u(t)} < 0,$$

giving us a contradiction.

It follows that $w(r)$ cannot have any roots, i.e., we may assume that $w(r) > 0$ on $[0, 1)$. But that is impossible, as can be seen by integrating (2.17) over $(0, 1)$. Hence $w \equiv 0$. ◊

Our main example is the problem (here $r = |x|$)

$$\varphi(u'(r))' + \tfrac{n-1}{r}\varphi(u'(r)) - a(r)u^{p-1} + b(r)u^q = 0, \quad \text{for } r \in (0,1), \tag{2.30}$$
$$u = 0 \ \text{ for } r = 1.$$

We consider the sub-critical case

$$\min(1, p-1) < q < \frac{np - n + p}{n - p}, \tag{2.31}$$

and assume that the functions $a(r), b(r) \in C^1[0, 1]$ satisfy

$$a(r) > 0, \ \ b(r) > 0, \ \ a'(r) > 0, \ \ b'(r) < 0 \quad \text{for } r \in (0,1). \tag{2.32}$$

We define the functions

$$A(r) \equiv pa(r) + ra'(r),$$
$$B(r) \equiv \left(\frac{np}{q+1} - (n-p)\right) b(r) + \frac{prb'(r)}{q+1}.$$

Observe that $\frac{np}{q+1} - (n-p) > 0$ for subcritical q, i.e., when (2.31) holds.

Theorem 2.2 *In addition to the conditions (2.31) and (2.32), assume that the function $A(r)$ is positive and non-decreasing, while the function $B(r)$ is positive on $(0, 1)$. Assume also that the functions $\frac{rb'(r)}{b(r)}$ and $rb'(r)$ are non-increasing on $(0, 1)$. Then any positive solution of the problem (2.30) is non-degenerate.*

Proof. We shall verify the conditions of the Theorem 2.1. We have

$$uf - pF = b(r)u^{q+1}(1 - \frac{p}{q+1}) > 0 \quad \text{for all } r \in [0,1),$$

verifying (2.23). Compute

$$(q - (p-1))\alpha(r) = -\frac{A(r)}{b(r)(u(r))^{q-p+1}} + p + \frac{rb'(r)}{b(r)}.$$

In view of our assumptions, $\alpha(r)$ is a non-increasing function, for all $r \in [0, 1)$, verifying the condition (2.12). Compute

$$I(r) = -A(r)u^p + B(r)u^{q+1} = B(r)u^p \left[-\frac{A(r)}{B(r)} + u^{q-p+1} \right].$$

Since by our conditions, $B(r)$ is non-increasing, it follows that the quantity in the square bracket is a non-increasing function, which is negative near $r = 1$. Hence, either $I(r)$ is negative over $(0, 1)$, or it changes sign exactly once, from positive to negative thus verifying either part (ii), or part (iii), of the condition (2.24). Hence, the Theorem 2.1 applies, implying that any positive solution of the problem (2.30) is non-degenerate. ◊

Our second example is the problem

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) + b(r)u^q = 0, \quad r \in (0, 1), \ \ u = 0 \ \text{ for } r = 1. \tag{2.33}$$

By a similar proof, we establish the following result.

Theorem 2.3 *Let q satisfy (2.31), and assume that $b(r)$ satisfies*

$$b(r) > 0, \quad b'(r) < 0 \quad \textit{for } r \in (0, 1),$$

$$\textit{the functions } \tfrac{rb'(r)}{b(r)} \textit{ and } rb'(r) \textit{ are non-increasing on } (0, 1).$$

Then any positive solution of the problem (2.30) is non-degenerate.

We consider next an autonomous problem ($f = f(u)$)

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) + \lambda f(u) = 0, \quad r \in (0, 1), \ \ u = 0 \ \text{ for } r = 1, \tag{2.34}$$

depending on a positive parameter λ. This problem is scaling invariant, which makes it easy to prove the following lemma, see B. Franchi, E. Lanconelli and J. Serrin [6]. (Recall that the condition (2.7) implies that $u'(r) < 0$, so that $u(0)$ gives the maximum value of solution.)

Lemma 2.6 *Assume that the condition (2.7) holds. Then the maximum value of the positive solution of (2.34), $u(0) = \alpha$, uniquely identifies the solution pair $(\lambda, u(r))$ (i.e., there is at most one λ, with at most one solution $u(r)$, so that $u(0) = \alpha$).*

We now turn to the a priori estimates. Several people have already observed that the recent Liouville type result of J. Serrin and H. Zou [16], implies a priori estimates of B. Gidas and J. Spruck type [8], provided that the point of maximum of solution is bounded away from the boundary, see [1], [5]. In our case, positive solutions take their maximum at the origin, so that we have the following lemma, whose standard proof we sketch for completeness.

Lemma 2.7 *For the problem (1.1) assume that the conditions of Lemma 2.2 hold, and*

$$\lim_{u \to \infty} \frac{f(r, u)}{u^q} = b(r),$$

with $\min(1, p-1) < q < \frac{np-n+p}{n-p}$, *and* $b(0) > 0$. *Then there exists a constant c, such any positive solution of the problem (1.1) satisfies*

$$|u|_{L^\infty} < c.$$

Proof. As we have already mentioned, the maximum of any positive solution occurs at $r = 0$, by the Lemma 2.2. If there is a sequence of unbounded solutions, their rescaling tends to a non-trivial solution of

$$\Delta_p u + b(0)u^q = 0 \text{ in } R^n,$$

which is impossible by the result of J. Serrin and H. Zou [16], see [1] and [5] for more details. ◊

We can now prove existence and uniqueness results for model equations.

Theorem 2.4 *Assuming the conditions of Theorem 2.3, the problem (2.33) has a unique positive solution.*

Proof. Assume first that $b(r) \equiv 1$. It is known that there exists a positive solution (proved by using variational methods). To see that the solution is unique, we consider a family of problems

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) + \lambda u^q = 0, \quad r \in (0,1), \quad u = 0 \text{ for } r = 1, \tag{2.35}$$

depending on a positive parameter λ. Since by Theorem 2.3, positive solutions of (2.35) are non-degenerate, we can continue our solution at $\lambda = 1$, for both increasing and decreasing λ on a solution curve, in the Banach space X, defined on p. 382 of A. Aftalion and F. Pacella [1], and this solution curve does not admit any turns. (The space X, which is a subspace of $C^1(B)$, was originally defined in F. Pacard and T. Rivière [13]. A. Aftalion and F. Pacella [1] showed that one may apply the implicit function theorem, when working in X.) By Lemma 2.1, solutions stay positive for all λ. By scaling, we see that the maximum value $u(0,\lambda) \to 0$, as $\lambda \to \infty$, and $u(0,\lambda) \to \infty$, as $\lambda \to 0$ (see Lemma 2.7). Since this solution curve covers all possible values of $u(0,\lambda)$, it follows by Lemma 2.6 that there is only one solution curve. We conclude uniqueness of positive solution at all λ, in particular at $\lambda = 1$.

Turning to the general case, we consider a family of problems

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) + b^{\theta}(r)u^q = 0, \quad r \in (0,1), \quad u = 0 \text{ for } r = 1, \tag{2.36}$$

depending on a parameter $0 \le \theta \le 1$. It is easy to check that the conditions of Theorem 2.3 hold for all θ. Using the Theorem 2.3 and Lemma 2.7, we can continue the solutions between $\theta = 0$ and $\theta = 1$, to prove existence of positive solutions to the problem (2.33). Continuing backwards, between $\theta = 1$ and $\theta = 0$, we conclude the uniqueness of positive solution to the problem (2.33). ◊

Theorem 2.5 *Assuming the conditions of Theorem 2.2, the problem (2.30) has a unique positive solution.*

Proof. We consider a family of problems

$$\varphi(u'(r))' + \frac{n-1}{r}\varphi(u'(r)) - \theta a(r)u^{p-1} + b(r)u^q = 0, \quad r \in (0,1), \quad u = 0 \text{ for } r = 1,$$

depending on a parameter $0 \le \theta \le 1$. When $\theta = 0$, we have a unique positive solution, by the preceding result. Arguing as before, we conclude existence of positive solutions at $\theta = 1$. ◊

3 A more general result in the one-dimensional case

We give an exact description of the curve of positive solutions for the p-Laplace problem in the one-dimensional case

$$\varphi(u'(x))' + \lambda f(u(x)) = 0 \quad \text{for } -1 < x < 1, \;\; u(-1) = u(1) = 0\,, \tag{3.1}$$

where $\varphi(t) = t|t|^{p-2}$, $p > 1$, λ a positive parameter, and the function $f(u) \in C^1(\bar{R}_+)$ satisfies

$$f(u) < 0, \;\; \text{for } u \in (0, \gamma), \;\; f(u) > 0, \;\; \text{for } u > \gamma\,, \tag{3.2}$$

for some $\gamma > 0$, and

$$f'(u) - (p-1)\frac{f(u)}{u} > 0, \;\; \text{for } u > \gamma\,. \tag{3.3}$$

This problem is autonomous, so that it can be posed on any interval. Here we chose the interval $(-1, 1)$ for convenience, related to the symmetry of solutions.

The following lemma gives a known condition for the Hopf's lemma to hold, see J.L. Vázquez [18]. Moreover, in the present ODE case a completely elementary proof, using the Gronwall's lemma, can be easily given.

Lemma 3.1 *Assume that* $f(u) \in C^1(\bar{R}_+)$ *satisfies* $f(0) = 0$, *and*

$$-f(s) < c_0 s^{p-1}, \;\; \textit{for small } s > 0, \textit{ and some } c_0 > 0\,. \tag{3.4}$$

If $u(x)$ *is a positive solution of (3.1), then*

$$u'(1) < 0\,.$$

We shall need the linearized problem for (3.1)

$$(\varphi'(u'(x))w'(x))' + \lambda f'(u(x))w(x) = 0 \quad \text{for } -1 < x < 1, \;\; w(-1) = w(1) = 0\,. \tag{3.5}$$

We call a solution $u(x)$ of (3.1) *non-degenerate*, if the corresponding problem (3.5) admits only the trivial solution $w = 0$, otherwise $u(x)$ is a *degenerate* solution. We have the following precise description of the solution curve. The uniqueness part was proved previously by J. Cheng [3].

Theorem 3.1 *Assume that the conditions (3.2) and (3.3) hold, and* $\lim_{u\to\infty} \frac{f(u)}{u^q} > 0$ *for some* $q > \min(1, p-1)$. *In case* $f(0) = 0$, *we assume additionally that (3.4) holds. Then the problem (3.1) has at most one positive solution for any* λ, *while for* λ *sufficiently small, the problem (3.1) has a unique positive solution. Moreover, all positive solutions of (3.1) are non-degenerate, they lie on a unique solution curve, extending for* $0 < \lambda \le \lambda_0 \le \infty$.

In case $f(0) < 0$, *we have* $\lambda_0 < \infty$, *there are no turns on the solution curve, with* $u(0, \lambda) \to \infty$ *as* $\lambda \to 0$, *and* $u'(\pm 1, \lambda_0) = 0$. *I.e., for* $\lambda > \lambda_0$ *there are no positive solutions, while for* $0 < \lambda \le \lambda_0$, *there is a unique positive non-degenerate solution.*

In case $f(0) = 0$, *we have* $\lambda_0 = \infty$, *there are no turns on the solution curve, with* $u(0, \lambda) \to \infty$ *as* $\lambda \to 0$, *and* $\lim_{\lambda\to\infty} u(x, \lambda) = 0$, *for* $x \ne 0$. *I.e., for all* $\lambda > 0$, *there is a unique positive non-singular solution.*

What is remarkable here is that essentially no restrictions are placed on $f(u)$ in the region where it is negative. We shall present the proof, after a series of lemmas, most of which hold under considerably more general conditions.

Lemma 3.2 *Assume that $f(u) \in C^1(\bar{R}_+)$. Then any positive solution of (3.1) is an even function, with $u'(x) < 0$ for $x > 0$.*

Proof. If $\xi > 0$ is a critical point of $u(x)$, i.e., $u'(\xi) = 0$, then the solutions $u(x)$ and $u(2\xi - x)$ have the same Cauchy data, and so by uniqueness for initial value problems, the graph of $u(x)$ is symmetric with respect to ξ, which is impossible. It follows that $x = 0$ is the only critical point, the point of global maximum. ◊

Lemma 3.3 *Assume that $f(u) \in C^1(\bar{R}_+)$, while $u(x)$ and $w(x)$ are any solutions of (3.1) and (3.5) respectively. Then*

$$\varphi'(u')\,(u''w - u'w') = constant, \quad for\ all\ x \in [-1, 1]\,. \tag{3.6}$$

Proof. Just differentiate the left hand side of (3.6), and use the equations (3.1) and (3.5). ◊

Lemma 3.4 *Assume that $f(u) \in C^1(\bar{R}_+)$. If the linearized problem (3.5) admits a non-trivial solution, then it does not change sign on $(-1, 1)$, i.e., we may assume that $w(x) > 0$ on $(-1, 1)$.*

Proof. Assume on the contrary that $w(x)$ vanishes on $(0, 1)$ (the case when $w(x)$ vanishes on $(-1, 0)$ is similar). Let $\xi > 0$ be the largest root of $w(x)$ on $(0, 1)$. We may assume that $w(x) > 0$ on $(\xi, 1)$, and then $w'(\xi) > 0$. Evaluating the expression in (3.6) at $x = \xi$, and at $x = 1$,

$$\varphi'(u'(\xi))u'(\xi)w'(\xi) = \varphi'(u'(1))u'(1)w'(1)\,.$$

The quantity on the left is negative, while the one on the right is non-negative, a contradiction. ◊

Given $u(x)$ and $w(x)$, solutions of (3.1) and (3.5) respectively, we again consider the following function, motivated by M. Tang [17]

$$T(x) = x\left[(p-1)\varphi(u'(x))w'(x) + \lambda f(u(x))w(x)\right] - (p-1)\varphi(u'(x))w(x)\,.$$

The following lemma is proved by a direct computation.

Lemma 3.5 *Assume that $f(u) \in C^1(\bar{R}_+)$, while $u(x)$ and $w(x)$ are any solutions of (3.1) and (3.5) respectively. Then*

$$T'(x) = p\lambda f(u(x))w(x)\,. \tag{3.7}$$

By Lemma 3.2, $u(0)$ gives the maximum value of the solution. If we now assume that $f(u)$ satisfies the condition (3.2), then it is easy to see that $f(u(0)) > 0$ (otherwise we get a contradiction, multiplying (3.1) by u, and integrating over $(-1, 1)$). So that we can find a point $x_0 \in (0, 1)$, such that

$$u(x_0) = \gamma\,.$$

(I.e., $f(u(x)) > 0$ on $(0, x_0)$, and $f(u(x)) < 0$ on $(x_0, 1)$.) Define

$$q(x) = (p-1)(1-x)\varphi(u'(x)) + \varphi'(u'(x))u(x)\,.$$

Lemma 3.6 *Assume that $f(u) \in C^1(\bar{R}_+)$ satisfies the condition (3.2). Then*

$$q(x_0) < 0\,.$$

Proof. We have $(p-1)\varphi(t) = t\varphi'(t)$, and so we can rewrite

$$q(x) = \varphi'(u'(x))\left[(1-x)u'(x) + u(x)\right]\,.$$

Since $\varphi'(t) > 0$ for all $t \neq 0$, it suffices to show that $r(x) \equiv (1-x)u'(x) + u(x) < 0$ on $[x_0, 1)$. We have $r(1) = 0$, and

$$r'(x) = (1-x)u''(x) = -(1-x)\lambda \frac{f(u(x))}{\varphi'(u'(x))} > 0, \quad \text{on } (x_0, 1)\,,$$

and the proof follows. ◊

Lemma 3.7 *Assume that $f(u) \in C^1(\bar{R}_+)$ satisfies the conditions (3.2) and (3.3), while $u(x)$ and $w(x)$ are any solutions of (3.1) and (3.5) respectively. Then*

$$(p-1)w(x_0)\varphi(u'(x_0)) - u(x_0)w'(x_0)\varphi'(u'(x_0)) > 0\,, \tag{3.8}$$

which implies, in particular,

$$w'(x_0) < 0\,. \tag{3.9}$$

Proof. By a direct computation

$$\left[(p-1)w(x)\varphi(u'(x)) - u(x)w'(x)\varphi'(u'(x))\right]' = \lambda\left[f'(u) - (p-1)\frac{f(u)}{u}\right]uw\,.$$

The quantity on the right is positive on $(0, x_0)$, in view of our conditions, and Lemma 3.4. Integrating over $(0, x_0)$, we conclude (3.8). ◊

We have all the pieces in place for the following crucial lemma.

Lemma 3.8 *Under the conditions (3.2) and (3.3), any positive solution of (3.1) is non-degenerate, i.e., the corresponding linearized problem (3.5) admits only the trivial solution.*

Proof. Assuming the contrary, let $w(x) > 0$ on $(-1, 1)$ be a solution of (3.5) (see Lemma 3.4). By Lemma 3.3, and since $f(u(x_0)) = 0$,

$$\varphi'(u'(1))u'(1)w'(1) = \varphi'(u'(x_0))u'(x_0)w'(x_0) = (p-1)\varphi(u'(x_0))w'(x_0)\,. \tag{3.10}$$

Integrating (3.7) over $(x_0, 1)$, we have $T(1) - T(x_0) = p\lambda \int_{x_0}^1 f(u(x))w(x)\,dx < 0$, i.e.,

$$L \equiv (p-1)\varphi(u'(1))w'(1) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + (p-1)\varphi(u'(x_0))w(x_0) < 0\,.$$

On the other hand, using (3.10), then (3.8), followed by (3.9) and Lemma 3.6, we have

$$\begin{aligned} L &= (p-1)\varphi(u'(x_0))w'(x_0) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + (p-1)\varphi(u'(x_0))w(x_0) \\ &> (p-1)\varphi(u'(x_0))w'(x_0) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + u(x_0)w'(x_0)\varphi'(u'(x_0)) \\ &= w'(x_0)q(x_0) > 0\,, \end{aligned}$$

a contradiction. ◊

This lemma implies that there are no turns on the solution curves. For a more detailed description of the bifurcation diagram, we need to establish some additional properties of solutions. Denoting $F(u) = \int_0^u f(t)\,dt$, one shows by differentiation that

$$E(x) \equiv \frac{p-1}{p}|u'(x)|^p + \lambda F(u(x)) = constant, \quad \text{for all } x \in [-1,1]. \tag{3.11}$$

It is customary to think of $E(x)$ as "energy".

We shall also need the following well-known lemma, see e.g. P. Korman [10].

Lemma 3.9 *The maximum value of the positive solution of (3.1), $u(0) = \alpha$, uniquely identifies the solution pair $(\lambda, u(x))$ (i.e., there is at most one λ, with at most one solution $u(x)$, so that $u(0) = \alpha$).*

Proof of the Theorem 3.1. We begin by showing that the problem (3.1) has solutions for some $\lambda > 0$. Observe that our condition (3.3) implies that the function $\frac{f(u)}{u^{p-1}}$ is increasing, i.e., $f(u) \to \infty$ for large u. Let $\theta > \gamma$ be the point where $\int_0^\theta f(t)\,dt = 0$. Consider the solutions of the initial value problem

$$\varphi(u')' + f(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0\,.$$

If $\alpha > \theta$, then the conservation of energy formula (3.11) implies that $\frac{p-1}{p}|u'(x)|^p + F(u(x)) = F(\alpha) > 0$, and so this solution must vanish (for the first time) at some $r > 0$. Rescaling, we get a positive solution of (3.1) at $\lambda = r^2$.

We now use the implicit function theorem to continue solutions for both decreasing and increasing λ (working, as before, in the space X from [1], or in the framework of B.P. Rynne [14]), and in both directions there are no turns by Lemma 3.8. For decreasing λ, we have $u(0,\lambda) \to \infty$, since otherwise the solution curve would have no place to go (it cannot go to zero, because $f(u)$ is negative near zero). By Lemma 2.7, $u(0,\lambda) \to \infty$ as $\lambda \to 0$. For increasing λ, $u(0,\lambda)$ is decreasing, in view of Lemma 3.9, and hence the oscillation of the solutions is decreasing. Consider first the case $f(0) = 0$. Then positive solutions continue for all λ, since $u_x(1,\lambda) < 0$, by Lemma 3.1. In case $f(0) < 0$, the opposite is true, i.e., we have $u_x(1,\lambda_0) = 0$ at some λ_0, and the solution becomes sign-changing for $\lambda > \lambda_0$, since otherwise we shall get arbitrarily large oscillations of $u(x,\lambda)$ for large λ, by integrating the equation (3.1) over any interval $(x, 1)$. ◊

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