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Multiple Solutions for Nonlinear Neumann Problems with Asymmetric Reaction, via Morse Theory

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Abstract

We consider a nonlinear Neumann problem driven by the p-Laplacian and with a reaction which exhibits an asymmetric behaviour near $+\infty$ and near $-\infty$. Namely, it is $(p-1)$-superlinear near $+\infty$ (but need not satisfy the Ambrosetti-Rabinowitz condition) and it is $(p-1)$-linear near $-\infty$. Combining variational methods with Morse theory, we show that the problem has at least three nontrivial smooth solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) = f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Δ_p denotes the p-Laplacian differential operator, defined by

$$\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u) \qquad \forall u \in W^{1,p}(\Omega),$$

with $p \in (1, +\infty)$. Also $f(z, \zeta)$ is a Carathéodory function and $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. The aim of this paper is to prove a "three solutions theorem" for problem (1.1), when the reaction term $f(z, \cdot)$ exhibits an asymmetric behaviour as we approach $+\infty$ and $-\infty$ respectively. More precisely, we assume that for almost all $z \in \Omega$, $f(z, \cdot)$ grows in a $(p-1)$-superlinear fashion near $+\infty$ and it is $(p-1)$-linear near $-\infty$.

In the past such Neumann problems have been studied only in the context of semilinear equations (i.e., $p = 2$). We mention the works of de Figueiredo-Ruf [8], Villegas [27], Perera [24], Dong [10], which deal with ordinary differential equations (i.e., $N = 1$) and Arcoya-Villegas [2], which considers partial differential equations. From the above works, only Perera [24] proves a multiplicity result, while the other works contain only existence theorems. To the best of our knowledge, no such existence and multiplicity results exist for the Neumann p-Laplacian (ordinary or partial). We should mention that p-Laplacian Dirichlet problems with an asymmetric reaction, were studied using the Fŭcik spectrum of the negative Dirichlet p-Laplacian (problems with jumping nonlinearities). We mention the works of Dancer-Perera [7], de Paiva [9], Tanaka [25], Zhang-Li-Liu-Feng [29] and the references therein. The use of the Fŭcik spectrum, dictates a $(p-1)$-linear growth in both directions and requires that the limits $\lim\limits_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta}$ exist. Therefore these works do not address the case of nonlinearities with fundamentally different behaviour at $+\infty$ and at $-\infty$, as we do here.

There are three more important works with asymptotically linear equations ("Amann-Zenhder type problems") that we should mention. These are the works of Arcoya-Orsina [1], Cingolani-Degiovanni [5] and Drábek-Robinson [11]. In Arcoya-Orsina [1] the authors deal with a Neumann problem which is more general than the p-Laplacian and has the form

$$-\operatorname{div}\,(a(z, u)\|\nabla u\|^{p-2}\nabla u)$$

where $a(z, u)$ is a measurable function, $a(z, \cdot) \in C^1(\mathbb{R})$, $0 < \alpha \leqslant a(z, \zeta) \leqslant \beta$, $\left|a'_\zeta(z, \zeta)\right| \leqslant \gamma$ and $a'_\zeta(z, \zeta)\zeta \geqslant 0$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$. Using Landesman-Lazer type conditions, they establish the existence of a solution (see Theorem 1.3). Cingolani-Degiovanni [5] deal with a Dirichlet problem where the differential operator is of the form

$$-\Delta_p u - \mu\Delta u, \quad \mu \geqslant 0,\ p > 2$$

and the reaction term is $\lambda|u|^{p-2}u + g(u)$, with $\lambda \in \mathbb{R} \setminus \sigma(-\Delta_p)$ and $g \in C^1(\mathbb{R})$. Using an interesting combination of variational techniques with Morse theoretic methods, they prove an existence theorem (see Theorem 4.2). Finally Drábek-Robinson [11] consider the Dirichlet problem

$$\begin{cases} -\Delta_p u(z) = \lambda|u|^{p-2}u - f(z, u) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ \lambda \in \mathbb{R},\ p > 1, \end{cases}$$

with $f(z, \zeta)$ a Carathéodory function which is $L^{p'}(\Omega)$-bounded ($\frac{1}{p}+\frac{1}{p'}=1$). Using Landesman-Lazer type condition, they prove existence of solutions for every $\lambda \in \mathbb{R}$ (see Theorem 1). In the process of proof they generate a new sequence of variational inequalities for the p-Laplacian (see Theorem 5).

Finally we mention two recent papers related to this work. In Gasiński-Papageorgiou [17], a Dirichlet problem with the p-Laplacian and combined nonlinearities is considered, namely with a singular term, a concave (i.e., $(p-1)$-sublinear) term and a Carathéodory perturbation. In Gasiński-Papageorgiou [16], a nonlinear anisotropic Neumann problem with a p-superlinear reaction is studied.

Our approach combines variational methods based on the critical point theory with Morse theory (critical groups). Truncation techniques are also employed, in order to make possible the use of these theories. In the next section, for the convenience of the reader, we briefly review main mathematical tools used in this paper.

2 Mathematical background

In the analysis of problem (1.1), we will use the Banach space

$$C^1_n(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and the Sobolev space

$$W^{1,p}_n(\Omega) = \overline{C^1_n(\overline{\Omega})}^{\|\cdot\|},$$

where $\|\cdot\|$ stands for the usual norm of Sobolev space $W^{1,p}(\Omega)$. Note that $C^1_n(\overline{\Omega})$ is an ordered Banach space with order cone

$$C_+ = \{u \in C^1_n(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Consider a Carathéodory nonlinearity $f_0 \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ with subcritical growth in $\zeta \in \mathbb{R}$, i.e.,

$$\left| f_0(z, \zeta) \right| \leqslant a_0(z) + c_0 |\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $r \in (1, p^*)$, where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } \quad p < N, \\ +\infty & \text{if } \quad p \geqslant N. \end{cases}$$

We introduce the C^1-functional $\varphi_0 \colon W^{1,p}_n(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F_0(z, u(z))\, dz \quad \forall u \in W^{1,p}_n(\Omega),$$

where

$$F_0(z, \zeta) = \int_0^\zeta f_0(z, s)\, ds.$$

In Iannizzotto-Papageorgiou [19] we can find the following result.

Proposition 2.1 *If $u_0 \in W^{1,p}_n(\Omega)$ is a local $C^1_n(\overline{\Omega})$-minimizer of φ_0, then $u_0 \in C^1_n(\overline{\Omega})$ and it is a local $W^{1,p}_n(\Omega)$-minimizer of φ_0.*

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. A point $x_0 \in X$ is a *critical point of* φ, if $\varphi'(x_0) = 0$. A value $c \in \mathbb{R}$ is a *critical value of* φ, if there is a critical point $x_0 \in X$, such that $\varphi(x_0) = c$. For a given $c \in \mathbb{R}$, we define the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leqslant c\} \quad \text{(the sublevel set of } \varphi \text{ at } c) \\ \dot{\varphi}^c &= \{x \in X : \varphi(x) < c\} \quad \text{(the strict sublevel set of } \varphi \text{ at } c) \\ K^\varphi &= \{x \in X : \varphi'(x) = 0\} \quad \text{(the critical set of } \varphi) \\ K^\varphi_c &= \{x \in K^\varphi : \varphi(x) = c\} \quad \text{(the critical set of } \varphi \text{ at level } c). \end{aligned}$$

The next compactness-type condition is crucial in critical point theory. So, for a given $\varphi \in C^1(X)$, we say that φ satisfies the *Cerami condition at level* $c \in \mathbb{R}$ (the *C_c-condition* for short), if every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0 \text{ in } X^*,$$

has a strongly convergent subsequence.

We say that φ satisfies the *C-condition*, if it satisfies the C_c-condition at every level $c \in \mathbb{R}$. This condition is more general than the usual Palais-Smale condition, but it was shown by Bartolo-Benci-Fortunato [3] that it suffices to establish the minimax theory of the critical values of functions $\varphi \in C^1(X)$. In particular, we have the following theorem, known in the literature as the "mountain pass theorem"

Theorem 2.1 *If $\varphi \in C^1(X)$, $x_0, x_1 \in X$, $r > 0$, $\|x_0 - x_1\| > r$,*

$$\max\{\varphi(x_0),\ \varphi(x_1)\} \leqslant \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_0,$$

and φ satisfies the C_c-condition with

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geqslant \eta_0$ and $K^\varphi_c \neq \emptyset$. Moreover, if $c = \eta_0$, then there exists $x \in K^\varphi_c$, such that $\|x - x_0\| = r$.

Next let us recall some basic definitions and facts from Morse theory. If (Y_1, Y_2) is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$, then for every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. Then for $\varphi \in C^1(X)$, the critical groups of φ at an isolated critical point x_0 with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{x_0\}) \quad \forall k \geqslant 0,$$

where U is a neighbourhood of x_0, such that $K^\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology, implies that the above definition is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K^\varphi) > -\infty$. Let

$$c \;<\; \inf \varphi(K^\varphi).$$

The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) \;=\; H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The deformation theorem implies that the above definition is independent of the particular choice of $c < \inf \varphi(K^\varphi)$. In fact, if $\eta < \inf \varphi(K^\varphi)$, then

$$C_k(\varphi, \infty) \;=\; H_k(X, \dot{\varphi}^\eta) \quad \forall k \geqslant 0.$$

To see this, let $c < \eta < \inf \varphi(K^\varphi)$. Then φ^c is a strong deformation retract of $\dot{\varphi}^\eta$ (see Granas-Dugundji [18, p. 407]). Hence

$$H_k(X, \varphi^c) \;=\; H_k(X, \dot{\varphi}^\eta) \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi, \infty) \;=\; H_k(X, \dot{\varphi}^\eta) \quad \forall k \geqslant 0.$$

Suppose that K^φ is finite. We define

$$M(t, x) \;=\; \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \forall x \in K^\varphi$$

and

$$P(t, \infty) \;=\; \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k.$$

Then the Morse relation says

$$\sum_{x \in K^\varphi} M(t, x) \;=\; P(t, \infty) + (1 + t) Q(t), \tag{2.2}$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients.

A map $A\colon X \longrightarrow X^*$ is said to be of type $(S)_+$, if for every sequence $\{x_n\}_{n \geqslant 1} \subseteq X$, such that

$$x_n \;\longrightarrow\; x \text{ weakly in } X \quad \text{and} \quad \limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \;\leqslant\; 0,$$

one has that

$$x_n \;\longrightarrow\; x \quad \text{in } X.$$

Consider the map $A\colon W_n^{1,p}(\Omega) \longrightarrow W_n^{1,p}(\Omega)^*$, defined by

$$\langle A(u), y \rangle \;=\; \int_\Omega \|\nabla u\|^{p-2} (\nabla u,\ \nabla y)_{\mathbb{R}^N}\, dz \quad \forall u, y \in W_n^{1,p}(\Omega). \tag{2.3}$$

The next result is a particular case of Proposition 3.1 in Gasiński-Papageorgiou [15].

Proposition 2.2 *The map $A\colon W_n^{1,p}(\Omega) \longrightarrow W_n^{1,p}(\Omega)^*$, defined by (2.3) is bounded, continuous, monotone (hence maximal monotone too) and of type $(S)_+$.*

We have the following lemma from Iannizzotto-Papageorgiou [19].

Lemma 2.1 *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant 0$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $\xi_0 > 0$, such that*

$$\|\nabla u\|_p^p - \int_\Omega \vartheta |u|^p \, dz \;\geqslant\; \xi_0 \|u\|^p \quad \forall u \in W_n^{1,p}(\Omega).$$

Finally throughout this work, for every $r \in \mathbb{R}$, $r^\pm = \max\{\pm r, 0\}$. By $\|\cdot\|$ we denote the norms both for $W_n^{1,p}(\Omega)$ and $\mathbb{R}^N$ (no confusion is possible, since it is always clear from the context, which one we mean).

In the next section we formulate and prove the "three solutions theorem".

3 Existence of three solutions

The hypotheses on the nonlinearity $f(z,\zeta)$ are the following:

$\underline{H(f)}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable;

(ii) for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous, $f(z,0) = 0$;

(iii) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in (p, p^*)$, such that

$$\left|f(z,\zeta)\right| \;\leqslant\; a(z) + c|\zeta|^{r-1} \qquad \text{for almost all } z \in \Omega \text{ and all } \zeta \in \mathbb{R};$$

(iv) if $F(z,\zeta) = \int_0^\zeta f(z,s)\,ds$, then

$$\lim_{\zeta \to +\infty} \frac{F(z,\zeta)}{\zeta^p} \;=\; +\infty,$$

uniformly for almost all $z \in \Omega$ and there exists $\tau \in (\frac{r-p}{p}N, p^*)$, such that

$$\liminf_{\zeta \to +\infty} \frac{f(z,\zeta)\zeta - pF(z,\zeta)}{\zeta^\tau} \;>\; 0, \tag{3.4}$$

uniformly for almost all $z \in \Omega$;

(v) there exist functions $\xi, \vartheta \in L^\infty(\Omega)$, such that

$$\xi(z) \;\leqslant\; \vartheta(z) \;\leqslant\; 0 \text{ for almost all } z \in \Omega,$$

the last inequality is strict on a set of positive measure and

$$\xi(z) \;\leqslant\; \liminf_{\zeta \to -\infty} \frac{pF(z,\zeta)}{|\zeta|^p} \;\leqslant\; \limsup_{\zeta \to -\infty} \frac{pF(z,\zeta)}{|\zeta|^p} \;\leqslant\; \vartheta(z),$$

uniformly for almost all $z \in \Omega$ and

$$\limsup_{\zeta \to -\infty} \big(pF(z,\zeta) - f(z,\zeta)\zeta\big) \;<\; +\infty,$$

uniformly for almost all $z \in \Omega$;

(vi) there exist two constants $c_0 < 0 < \delta_0$, such that

$$F(z,\zeta) \leqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta_0$$

and

$$\int_\Omega F(z,c_0)\,dz > 0$$

and for all $R > 0$, there exists $\xi_R > 0$, such that for almost all $z \in \Omega$, the function

$$\zeta \longmapsto f(z,\zeta) + \xi_R|\zeta|^{p-2}\zeta \quad \text{is nondecreasing on } [-R,R].$$

Remark 3.1 Hypothesis $H(f)(iv)$ implies that for almost all $z \in \Omega$, the primitive $F(z,\cdot)$ grows p-superlinearly near $+\infty$. However, we do not use the common in such cases Ambrosetti-Rabinowitz condition. We recall that the Ambrosetti-Rabinowitz condition on the positive semiaxis says that there exist $q > p$ and $M > 0$, such that

$$0 < qF(z,\zeta) \leqslant f(z,\zeta)\zeta \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant M. \tag{3.5}$$

Integrating (3.5), we obtain

$$c_1\zeta^q \leqslant F(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant M, \tag{3.6}$$

with some $c_1 > 0$. In particular then, $F(z,\cdot)$ satisfies the much weaker condition

$$\lim_{\zeta\to+\infty} \frac{F(z,\zeta)}{\zeta^p} = +\infty, \tag{3.7}$$

uniformly for almost all $z \in \Omega$.

Here we employ (3.7) and another asymptotic at $+\infty$ condition, namely (3.4). Note that (3.4) is weaker than both (3.5) and (3.6). Conditions similar to (3.4), were used in the past by Costa-Magalhães [6] (partial differential equations) and Fei [13] (Hamiltonian systems). Hypothesis $H(f)(v)$ dictates a p-linear growth for $F(z,\cdot)$ near $-\infty$ for almost all $z \in \Omega$. Evidently this is the case, if for almost all $z \in \Omega$, the function $f(z,\cdot)$ is $(p-1)$-linear near $-\infty$.

Example 3.1 The following function $f(\zeta)$ satisfies hypotheses $H(f)$ (for the sake of simplicity we drop the z-dependence).

$$f(\zeta) = \begin{cases} \vartheta(|\zeta|^{p-2}\zeta + 2^{p-1}) + 1 & \text{if } \zeta < -2, \\ -2\zeta - 3 & \text{if } \zeta \in [-2,-1), \\ -|\zeta|^{p-2}\zeta + 2|\zeta|^{d-2}\zeta & \text{if } \zeta \in [-1,1], \\ \zeta^{p-1}\left(\ln\zeta + \frac{1}{p}\right) - \frac{1}{p} + 1 & \text{if } 1 < \zeta, \end{cases}$$

with $-2^{1-p} < \vartheta < 0$ and $1 < p < d < 2p$. Then hypothesis $H(f)(iii)$ is satisfied with any $r > p$. So taking $r \in (p, \min\{p^*, p + \frac{p^2}{N}\})$, we have that $\frac{r-p}{p}N < p$ and we can check that (3.4) in hypothesis $H(f)(iv)$ holds for any $\tau \in (\frac{r-p}{p}N, p]$. As for $H(f)(v)$, we have

$$\lim_{\zeta\to-\infty} \frac{pF(\zeta)}{|\zeta|^p} = \vartheta < 0 \quad \text{and} \quad \lim_{\zeta\to-\infty} (pF(\zeta) - f(\zeta)\zeta) = -\infty.$$

Finally, for $H(f)(vi)$, we have that $F(\zeta) \leqslant 0$ for all $|\zeta| \leqslant \delta_0 = (\frac{d}{2p})^{\frac{1}{d-p}}$; for $c_0 = -1$ we have $F(c_0) = \frac{2}{d} - \frac{1}{p} > 0$ and for any $R > 0$, $\xi_R = 2$.

Let $\varphi\colon W_n^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (1.1), defined by

$$\varphi(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F(z,u(z))\,dz \quad \forall u \in W_n^{1,p}(\Omega).$$

Evidently $\varphi \in C^1(W_n^{1,p}(\Omega))$. As we mentioned our approach also uses suitable truncation techniques. For this reason we introduce the following truncations and modifications of the nonlinearity $f(z,\cdot)$. For $\varepsilon \in (0,1)$, we define:

$$f_+^\varepsilon(z,\zeta) \;=\; \begin{cases} 0 & \text{if} \quad \zeta \leqslant 0, \\ \varepsilon\zeta^{p-1} + f(z,\zeta) & \text{if} \quad \zeta > 0, \end{cases}$$

$$f_-(z,\zeta) \;=\; \begin{cases} f(z,\zeta) & \text{if} \quad \zeta < 0, \\ 0 & \text{if} \quad \zeta \geqslant 0. \end{cases}$$

Both are Carathéodory functions. We set

$$F_+^\varepsilon(z,\zeta) \;=\; \int_0^\zeta f_+^\varepsilon(z,s)\,ds \quad \text{and} \quad F_-(z,\zeta) \;=\; \int_0^\zeta f_-(z,s)\,ds$$

and we introduce the C^1-functionals $\varphi_+^\varepsilon, \varphi_-\colon W_n^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\varphi_+^\varepsilon(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_\Omega F_+^\varepsilon(z,u(z))\,dz \quad \forall u \in W_n^{1,p}(\Omega),$$

$$\varphi_-(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F_-(z,u(z))\,dz \quad \forall u \in W_n^{1,p}(\Omega).$$

Proposition 3.1 *If hypotheses $H(f)$ hold, then φ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$, such that

$$\left|\varphi(u_n)\right| \;\leqslant\; M_1 \quad \forall n \geqslant 1, \tag{3.8}$$

for some $M_1 > 0$ and

$$(1 + \|u_n\|)\varphi'(u_n) \;\longrightarrow\; 0 \quad \text{in } W_n^{1,p}(\Omega)^*. \tag{3.9}$$

If

$$N(u)(\cdot) \;=\; f(\cdot, u(\cdot)) \quad \forall u \in W_n^{1,p}(\Omega),$$

then from (3.9), we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega N(u_n)h\,dz\right| \;\leqslant\; \frac{\varepsilon_n}{1 + \|u_n\|}\|h\| \quad \forall h \in W_n^{1,p}(\Omega), \tag{3.10}$$

with $\varepsilon_n \searrow 0$, where A is defined by (2.3).

Claim. The sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is bounded.

We proceed by contradiction. So, suppose that the Claim is not true. Passing to a subsequence if necessary, we may assume that

$$\|u_n\| \;\longrightarrow\; +\infty.$$

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so, passing to another subsequence if necessary, we may assume that

$$y_n \longrightarrow y \qquad \text{weakly in } W_n^{1,p}(\Omega), \tag{3.11}$$

$$y_n \longrightarrow y \qquad \text{in } L^r(\Omega). \tag{3.12}$$

We divide (3.10) by $\|u_n\|^{p-1}$ and exploiting the $(p-1)$-homogeneity of A, we obtain

$$\left|\langle A(y_n), h\rangle - \int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} h\, dz\right| \leqslant \frac{\varepsilon_n}{(1+\|u_n\|)\|u_n\|^{p-1}}\|h\| \quad \forall h \in W_n^{1,p}(\Omega). \tag{3.13}$$

In (3.13) we choose $h = y^+ \in W_n^{1,p}(\Omega)$. Then

$$\int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} y^+\, dz \leqslant \varepsilon_n + \langle A(y_n), y^+\rangle \leqslant \varepsilon_n + c_1\|y^+\|, \tag{3.14}$$

for some $c_1 > 0$ (see (3.11)).

Suppose that $y^+ \neq 0$. Then from (3.14), we obtain

$$\limsup_{n\to+\infty} \int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz \leqslant c_1. \tag{3.15}$$

On the other hand, by virtue of hypothesis $H(f)(iv)$, we can find $\beta > 0$ and $M_2 > 0$, such that

$$pF(z,\zeta) + \beta\zeta^\tau \leqslant f(z,\zeta)\zeta \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant M_2,$$

so

$$\frac{pF(z,\zeta)}{\zeta} + \beta\zeta^{\tau-1} \leqslant f(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant M_2. \tag{3.16}$$

Therefore, using hypothesis $H(f)(iii)$ and (3.16), we have

$$\begin{aligned}
\int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz &= \int_{\{u_n \geqslant M_2\}} \frac{f(z,u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz + \int_{\{u_n < M_2\}} \frac{f(z,u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz \\
&\geqslant \int_{\{u_n \geqslant M_2\}} \frac{f(z,u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz - c_2 \\
&\geqslant \int_{\{u_n \geqslant M_2\}} \frac{pF(z,u_n)}{u_n^p} \frac{u_n^{p-1}}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|} dz - c_2 \\
&= \int_{\{u_n \geqslant M_2\}} \frac{pF(z,u_n)}{u_n^p} y_n^{p-1} \frac{y^+}{\|y^+\|} dz - c_2,
\end{aligned} \tag{3.17}$$

for some $c_2 > 0$. Note that

$$u_n(z) \longrightarrow +\infty \quad \text{for almost all } z \in \{y > 0\}.$$

So, if in (3.17) we pass to the limit as $n \to +\infty$ and we use Fatou's lemma and hypothesis $H(f)(iv)$, we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} \frac{y^+}{\|y^+\|}\, dz \;=\; +\infty. \tag{3.18}$$

Comparing (3.15) and (3.18) we reach a contradiction, which means that $y^+ = 0$ and so $y \leqslant 0$.

In (3.10), we choose $h = u_n \in W_n^{1,p}(\Omega)$. Then

$$-\|\nabla u_n\|_p^p + \int_\Omega f(z, u_n) u_n\, dz \;\leqslant\; \varepsilon_n \quad \forall n \geqslant 1. \tag{3.19}$$

On the other hand, from (3.8), we have

$$\|\nabla u_n\|_p^p - \int_\Omega pF(z, u_n)\, dz \;\leqslant\; pM_1 \quad \forall n \geqslant 1. \tag{3.20}$$

Adding (3.19) and (3.20), we obtain

$$\int_\Omega (f(z, u_n) u_n - pF(z, u_n))\, dz \;\leqslant\; M_3 \quad \forall n \geqslant 1, \tag{3.21}$$

for some $M_3 > 0$. From hypotheses $H(f)(iii)$ and (iv), we have

$$\beta\zeta^\tau - c_3 \;\leqslant\; f(z,\zeta)\zeta - pF(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0, \tag{3.22}$$

for some $c_3 > 0$. Using (3.22) in (3.21), we obtain

$$\int_{\{u_n \leqslant 0\}} (f(z, u_n) u_n - pF(z, u_n))\, dz + \beta \int_\Omega (u_n^+)^\tau\, dz \;\leqslant\; M_4 \quad \forall n \geqslant 1, \tag{3.23}$$

for some $M_4 > 0$. Hypotheses $H(f)(iii)$ and (v) imply that there exists $M_5 > 0$, such that

$$f(z,\zeta)\zeta - pF(z,\zeta) \;\geqslant\; -M_5 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0. \tag{3.24}$$

Using (3.24) in (3.23), we have

$$\int_\Omega (u_n^+)^\tau\, dz \;\leqslant\; M_6 \quad \forall n \geqslant 1, \tag{3.25}$$

for some $M_6 > 0$, so the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq L^\tau(\Omega)$ is bounded.

In (3.10), we choose $h = u_n^+ \in W_n^{1,p}(\Omega)$ and we have

$$\|\nabla u_n^+\|_p^p - \int_\Omega f(z, u_n^+) u_n^+\, dz \;\leqslant\; \varepsilon_n \quad \forall n \geqslant 1. \tag{3.26}$$

From hypothesis $H(f)(iii)$, for almost all $z \in \Omega$ and all $n \geqslant 1$, we have

$$f(z, u_n^+(z)) u_n^+(z) \;\leqslant\; c_4(|u_n^+(z)| + |u_n^+(z)|^r), \tag{3.27}$$

for some $c_4 > 0$. Using (3.27) in (3.26), we obtain

$$\|\nabla u_n^+\|_p^p \;\leqslant\; \varepsilon_n + c_5(\|u_n^+\|_p + \|u_n^+\|_r^r) \quad \forall n \geqslant 1, \tag{3.28}$$

for some $c_5 > 0$.

From hypothesis $H(f)(iv)$, it is clear that we can always assume that $\tau \leqslant r < p^*$. So, we can find $t \in [0,1)$, such that $\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}$. Invoking the interpolation theorem (see e.g., Gasiński-Papageorgiou [14, p. 905]), we have

$$\|u_n^+\|_r \leqslant \|u_n^+\|_\tau^{1-t} \|u_n^+\|_{p^*}^t \quad \forall n \geqslant 1,$$

so, if $N \neq p$, using also (3.25), we obtain

$$\|u_n^+\|_r^r \leqslant M_7 \|u_n^+\|^{tr} \quad \forall n \geqslant 1, \tag{3.29}$$

for some $M_7 > 0$. If $N = p$, in the above argument replace p^* by $q > r$ large.

Using (3.29) in (3.28), we have

$$\|\nabla u_n^+\|_p^p \leqslant c_6(1 + \|u_n^+\| + \|u_n^+\|^{tr}) \quad \forall n \geqslant 1, \tag{3.30}$$

for some $c_6 > 0$. Also, from (3.25), we have

$$\|u_n^+\|_\tau^p \leqslant M_8 \quad \forall n \geqslant 1, \tag{3.31}$$

for some $M_8 > 0$. From (3.30) and (3.31), it follows that

$$\|u_n^+\|_\tau^p + \|\nabla u_n^+\|_p^p \leqslant M_9 + c_6(\|u_n^+\| + \|u_n\|^{tr}) \quad \forall n \geqslant 1, \tag{3.32}$$

for $M_9 = M_8 + c_6 > 0$. Recall that $\|\cdot\|_\tau^p + \|\nabla \cdot\|_p^p$ is an equivalent norm on $W_n^{1,p}(\Omega)$ (see e.g., Gasiński-Papageorgiou [14, p. 227]). Moreover, the hypothesis on τ (see $H(f)(iv)$), implies that $tr < p$. So, from (3.32), it follows that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is bounded.

Next, in (3.13), we choose $h = y_n - y \in W_n^{1,p}(\Omega)$ and so

$$\left| \langle A(y_n), y_n - y \rangle - \int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} (y_n - y)\, dz \right| \leqslant \frac{\varepsilon_n}{1 + \|u_n\|} \frac{\|y_n - y\|}{\|u_n\|^{p-1}}. \tag{3.33}$$

Note that due to hypotheses $H(f)(iii)$ and (v) and from the boundedness of the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$, the sequence $\{\frac{N(u_n)}{\|u_n\|^{p-1}}\} \subseteq L^{p'}(\Omega)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) is also bounded. So

$$\int_\Omega \frac{N(u_n)}{\|u_n\|^{p-1}} (y_n - y)\, dz \longrightarrow 0,$$

thus

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y \rangle = 0$$

and, by the $(S)_+$ property of A (see Proposition 2.2), we have

$$y_n \longrightarrow y \quad \text{in } W_n^{1,p}(\Omega)$$

and so

$$\|y\| = 1. \tag{3.34}$$

From (3.13) and the boundedness of the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$, we see that

$$\left| \langle A(-y_n^-), h \rangle - \int_\Omega \frac{N(-u_n^-)}{\|u_n^-\|^{p-1}} h\, dz \right| \leqslant \varepsilon_n' \|h\| \quad \forall h \in W_n^{1,p}(\Omega), \tag{3.35}$$

with $\varepsilon_n' \searrow 0$. Note that

$$-u_n^-(z) \longrightarrow -\infty \quad \text{for almost all } z \in \{y < 0\}.$$

Using hypothesis $H(f)(v)$ and reasoning as in the proof of Proposition 5 of Motreanu-Motreanu-Papageorgiou [21], we show that

$$g_n = \frac{N(-u_n^-)}{\|u_n^-\|^{p-1}} \longrightarrow g = \eta|y|^{p-2}y \quad \text{weakly in } L^{p'}(\Omega), \tag{3.36}$$

where $\eta \in L^\infty(\Omega)$, $\xi(z) \leqslant \eta(z) \leqslant \vartheta(z)$ for almost all $z \in \Omega$. So, if in (3.35) we pass to the limit as $n \to +\infty$ and we use (3.36), we obtain

$$\langle A(y), h\rangle = \int_\Omega \eta|y|^{p-2}yh\,dz \quad \forall h \in W_n^{1,p}(\Omega),$$

so

$$A(y) = \eta|y|^{p-2}y,$$

and thus

$$\begin{cases} -\Delta_p y(z) = \eta(z)\big|y(z)\big|^{p-2}y(z) \text{ in } \Omega, \\ \dfrac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.37}$$

(see Motreanu-Papageorgiou [22]).

Recall that $y \leqslant 0$ and $\eta \neq 0$. So, from (3.37), it follows that $y = 0$, a contradiction to (3.34). Hence the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is bounded and this proves the Claim.

Because of the Claim, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_n^{1,p}(\Omega), \tag{3.38}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{3.39}$$

In (3.10) we choose $h = u_n - u \in W_n^{1,p}(\Omega)$. Passing to the limit as $n \to +\infty$ and using (3.39), we obtain

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0,$$

and, by the $(S)_+$ property of A (see Proposition 2.2 and (3.38), we have

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p}(\Omega).$$

This proves that φ satisfies the C-condition.

Next we show that φ_+^ε also satisfies the C-condition.

Proposition 3.2 *If hypotheses $H(f)$ hold, then φ_+^ε satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ be a sequence such that

$$\big|\varphi_+^\varepsilon(u_n)\big| \leqslant M_{10} \quad \forall n \geqslant 1, \tag{3.40}$$

for some $M_{10} > 0$ and

$$(\|1 + \|u_n\|)(\varphi_+^\varepsilon)'(u_n) \longrightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)^*. \tag{3.41}$$

From (3.41), we have

$$\begin{aligned} &\left|\langle A(u_n), h\rangle + \varepsilon \int_\Omega |u_n|^{p-2} u_n h\, dz - \int_\Omega f_+^\varepsilon(z, u_n) h\, dz\right| \\ \leqslant\ & \frac{\varepsilon_n}{1 + \|u_n\|}\|h\| \quad \forall h \in W_n^{1,p}(\Omega), \end{aligned} \tag{3.42}$$

with $\varepsilon_n \searrow 0$.

In (3.42) we choose $h = -u_n^- \in W_n^{1,p}(\Omega)$. Then

$$\|\nabla u_n^-\|_p^p + \varepsilon\|u_n^-\|_p^p \ \leqslant\ \varepsilon_n \quad \forall n \geqslant 1,$$

so

$$u_n^- \ \longrightarrow\ 0 \quad \text{in } W_n^{1,p}(\Omega). \tag{3.43}$$

Next in (3.42), we choose $h = u_n^+ \in W_n^{1,p}(\Omega)$. Then

$$-\|\nabla u_n^+\|_p^p - \varepsilon\|u_n^+\|_p^p + \int_\Omega f_+^\varepsilon(z, u_n^+) u_n^+\, dz \ \leqslant\ \varepsilon_n \quad \forall n \geqslant 1. \tag{3.44}$$

Moreover, from (3.40) and (3.43), it follows that

$$\|\nabla u_n^+\|_p^p + \varepsilon\|u_n^+\|_p^p - \int_\Omega pF_+^\varepsilon(z, u_n^+)\, dz \ \leqslant\ M_{11} \quad \forall n \geqslant 1, \tag{3.45}$$

for some $M_{11} > 0$. We add (3.44) and (3.45). Then

$$\int_\Omega \left(f_+^\varepsilon(z, u_n^+) u_n^+ - pF_+^\varepsilon(z, u_n^+)\right) dz \ \leqslant\ M_{12} \quad \forall n \geqslant 1,$$

for some $M_{12} > 0$, so

$$\int_\Omega \left(f(z, u_n^+) u_n^+ - pF(z, u_n^+)\right) dz \ \leqslant\ M_{12} \quad \forall n \geqslant 1. \tag{3.46}$$

From (3.22), we have

$$\beta(\zeta^+)^\tau - c_3 \ \leqslant\ f(z, \zeta^+)\zeta^+ - pF(z, \zeta^+) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.47}$$

Using (3.47) in (3.46), as in the proof of Proposition 3.1, we infer that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq L^\tau(\Omega)$ is bounded.

Hypothesis $H(f)(iii)$ implies that there exists $\widehat{a} \in L^\infty(\Omega)_+$ and $\widehat{c} > 0$, such that

$$\left|f(z, \zeta)\zeta\right| \ \leqslant\ \widehat{a}(z) + \widehat{c}|\zeta|^r \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.48}$$

If in (3.42) we choose $h = u_n^+ \in W_n^{1,p}(\Omega)$, then

$$\|\nabla u_n^+\|_p^p + \varepsilon\|u_n^+\|_p^p \ \leqslant\ \varepsilon_n + \int_\Omega f_+^\varepsilon(z, u_n^+) u_n^+\, dz \quad \forall n \geqslant 1,$$

so, using (3.48), we have

$$\|\nabla u_n^+\|_p^p \ \leqslant\ c_7(1 + \|u_n^+\|_r^r) \quad \forall n \geqslant 1, \tag{3.49}$$

for some $c_7 > 0$ and there is $t \in [0,1)$, such that $\frac{1}{r} = \frac{1-t}{r} + \frac{t}{p^*}$ (again, if $N = p$, we replace p^* by $q > r$ large). From (3.49), as in the proof of Proposition 3.1, via the interpolation inequality and the boundedness of the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq L^\tau(\Omega)$, we infer that

$$\|\nabla u_n^+\|_p^p \leqslant c_8(1 + \|u_n^+\|_{p^*}^{tr}) \quad \forall n \geqslant 1, \tag{3.50}$$

for some $c_8 > 0$.

Suppose that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is not bounded. Passing to a suitable subsequence if necessary, we may assume that

$$\|u_n^+\| \longrightarrow +\infty.$$

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so, passing to another subsequence if necessary, we may assume that

$$y_n \longrightarrow y \qquad \text{weakly in } W_n^{1,p}(\Omega), \tag{3.51}$$

$$y_n \longrightarrow y \qquad \text{in } L^p(\Omega). \tag{3.52}$$

From (3.50), we have

$$\|\nabla y_n\|_p^p \leqslant c_8\Big(\frac{1}{\|u_n^+\|^p} + \frac{1}{\|u_n^+\|^{p-tr}}\|y_n^+\|_{p^*}^{tr}\Big) \quad \forall n \geqslant 1. \tag{3.53}$$

Recall that $tr < p$ (see hypothesis $H(f)(iv)$). So, if in (3.53) we pass to the limit as $n \to +\infty$ and we use (3.52), we obtain

$$\|\nabla y\|_p^p \leqslant 0,$$

so

$$y \equiv \xi \in \mathbb{R}_+$$

(note that $y \geqslant 0$).

If $\xi = 0$, then

$$\nabla y_n \longrightarrow 0 \quad \text{in } L^p(\Omega; \mathbb{R}^N)$$

and so

$$y_n \longrightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)$$

(see (3.52)), a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

If $\xi > 0$, then

$$u_n^+(z) \longrightarrow +\infty \quad \text{for almost all } z \in \Omega.$$

Hence by virtue of hypothesis $H(f)(iv)$ and Fatou's lemma, we have

$$\lim_{n\to+\infty} \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz = \lim_{n\to+\infty} \int_\Omega \frac{F(z,u_n^+)}{(u_n^+)^p} y_n^p\,dz = +\infty. \tag{3.54}$$

From (3.40) and (3.43), it follows that

$$\Big|\frac{1}{p}\|\nabla u_n^+\|_p^p - \int_\Omega F(z,u_n^+)\,dz\Big| \leqslant M_{13} \quad \forall n \geqslant 1,$$

for some $M_{13} > 0$, so

$$\int_\Omega F(z, u_n^+)\, dz \leqslant M_{13} + \frac{1}{p}\|\nabla u_n^+\|_p^p,$$

thus

$$\int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \leqslant \frac{M_{13}}{\|u_n^+\|^p} + \frac{1}{p}\|\nabla y_n^+\|_p^p$$

and so

$$\limsup_{n\to+\infty} \int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \leqslant 0. \tag{3.55}$$

Comparing (3.54) and (3.55) we reach a contradiction. This shows that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is bounded and so also the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$ is bounded (see (3.43)). Hence, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{weakly in } W_n^{1,p}(\Omega),\\ u_n &\longrightarrow u \quad \text{in } L^r(\Omega). \end{aligned}$$

As before (see the proof of Proposition 3.1), using Proposition 2.2, we conclude that

$$u_n \longrightarrow u \quad \text{in } W_n^{1,p}(\Omega).$$

This proves that φ_+^ε satisfies the C-condition.

Concerning the functional φ_- we have the following result.

Proposition 3.3 *If hypotheses $H(f)$ hold, then φ_- is coercive.*

Proof. By virtue of hypotheses $H(f)(iii)$ and (v), for a given $\varepsilon > 0$, we can find $\beta_\varepsilon > 0$, such that

$$f(z, \zeta) \geqslant (\vartheta(z) + \varepsilon)|\zeta|^{p-2}\zeta - \beta_\varepsilon \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0,$$

so

$$F(z, \zeta) \leqslant \frac{1}{p}(\vartheta(z) + \varepsilon)|\zeta|^p + \beta_\varepsilon|\zeta| \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0. \tag{3.56}$$

Then, using Lemma 2.1, for every $u \in W_n^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_-(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F_-(z, u)\, dz\\ &\geqslant \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \vartheta|u|^p\, dz - \frac{\varepsilon}{p}\|u\|^p - \beta_\varepsilon\|u\|\\ &\geqslant \frac{1}{p}(\xi_0 - \varepsilon)\|u\|^p - \beta_\varepsilon\|u\|. \end{aligned} \tag{3.57}$$

Choosing $\varepsilon \in (0, \xi_0)$, from (3.57) and since $p > 1$, we infer that φ_- is coercive.

Next we compute the critical groups of φ and φ_+^ε at infinity.

Proposition 3.4 *If hypotheses $H(f)$ hold, then $C_k(\varphi, \infty) = 0$ for all $k \geqslant 0$.*

Proof. Let $\psi = \varphi|_{C^1_n(\overline{\Omega})}$. Nonlinear regularity theory (see e.g., Gasiński-Papageorgiou [14, pp. 737-738]), implies that ψ and φ have identical critical sets. Recall that the embedding $C^1_n(\overline{\Omega}) \subseteq W^{1,p}_n(\Omega)$ is dense. Invoking Proposition 16 of Palais [23], we have

$$H_k(W^{1,p}_n(\Omega), \dot{\varphi}^a) = H_k(C^1_n(\overline{\Omega}), \dot{\psi}^a) \quad \forall a \in \mathbb{R},\ k \geqslant 0. \tag{3.58}$$

Let $K \subseteq C^1_n(\overline{\Omega})$ denote the common critical set. We choose

$$a < \inf \varphi(K) = \inf \psi(K)$$

and we have

$$C_k(\varphi, \infty) = H_k(W^{1,p}_n(\Omega), \varphi^a) = H_k(W^{1,p}_n(\Omega), \dot{\varphi}^a) \quad \forall k \geqslant 0 \tag{3.59}$$

and

$$C_k(\psi, \infty) = H_k(C^1_n(\overline{\Omega}), \psi^a) = H_k(C^1_n(\overline{\Omega}), \dot{\psi}^a) \quad \forall k \geqslant 0. \tag{3.60}$$

From (3.58), (3.59) and (3.60), it follows that it suffices to show that

$$H_k(C^1_n(\overline{\Omega}), \psi^a) = 0 \quad \forall k \geqslant 0$$

and for all $a < 0$ with $|a|$ large. To this end, we introduce the following two sets

$$\begin{aligned} \partial B^C_1 &= \{u \in C^1_n(\overline{\Omega}) :\ \|u\|_{C^1_n(\overline{\Omega})} = 1\}, \\ \partial B^C_{1,+} &= \{u \in \partial B^C_1 :\ u(z) > 0 \text{ for some } z \in \Omega\}. \end{aligned}$$

Consider the homotopy $h_+ \colon [0,1] \times \partial B^C_{1,+} \longrightarrow \partial B^C_{1,+}$, defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_0}{\|(1-t)u + t\widehat{u}_0\|_{C^1_n(\overline{\Omega})}} \quad \forall (t,u) \in [0,1] \times \partial B^C_{1,+},$$

where $\widehat{u}_0$ is the L^p-normalized principal eigenfunction of the negative Neumann p-Laplacian, i.e.,

$$\widehat{u}_0(z) = \frac{1}{|\Omega|_N^{\frac{1}{p}}} \quad \forall z \in \overline{\Omega}.$$

We have

$$h_+(0,u) = u \quad \text{and} \quad h_+(1,u) = \frac{\widehat{u}_0}{\|\widehat{u}_0\|_{C^1_n(\overline{\Omega})}}.$$

This shows that the set $\partial B^C_{1,+}$ is contractible in itself.

Choose $u \in \partial B^C_{1,+}$ and let $t > 0$. Then

$$\begin{aligned} \varphi(tu) &= \frac{t^p}{p}\|\nabla u\|_p^p - \int_\Omega F(z, tu)\, dz \\ &= \frac{t^p}{p}\|\nabla u\|_p^p - \int_\Omega F(z, tu^+)\, dz - \int_\Omega F(z, -tu^-)\, dz. \end{aligned} \tag{3.61}$$

Hypotheses $H(f)(iii)$ and (iv) imply that, for a given $\beta > 0$, we can find $c_\beta > 0$, such that

$$F(z,\zeta) \geqslant \frac{\beta}{p}\zeta^p - c_\beta \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{3.62}$$

Also, hypotheses $H(f)(iii)$ and (v) imply that we can find $\beta_0 > 0$ and $c_9 > 0$, such that

$$F(z,\zeta) \geqslant -\frac{\beta_0}{p}|\zeta|^p - c_9 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0. \tag{3.63}$$

Using (3.62) and (3.63) in (3.61), we obtain

$$\begin{aligned} \varphi(tu) &\leqslant \frac{t^p}{p}\|\nabla u\|_p^p - \frac{\beta t^p}{p}\|u^+\|_p^p + \frac{\beta_0 t^p}{p}\|u^-\|_p^p + c_{10} \\ &\leqslant \frac{t^p}{p}(\|\nabla u\|_p^p - \beta\|u^+\|_p^p + \beta_0\|u\|_p^p) + c_{10}, \end{aligned} \tag{3.64}$$

for some $c_{10} > 0$. Since $u \in \partial B^c_{1,+}$, we see that $\|u^+\|_p \neq 0$. Since $\beta > 0$ is arbitrary, we can always choose it such that

$$\|\nabla u\|_p^p + \beta_0\|u\|_p^p < \beta\|u^+\|_p^p$$

and so from (3.64), we infer that

$$\varphi(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.65}$$

In addition, hypotheses $H(f)(iv)$ and (v) imply that there exist $\widehat{\beta} > 0$, $M_{14} > 0$ and $\widehat{\gamma} > 0$, such that

$$pF(z,\zeta) - f(z,\zeta)\zeta \leqslant -\widehat{\beta}\zeta^\tau \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant M_{14}, \tag{3.66}$$

$$pF(z,\zeta) - f(z,\zeta)\zeta \leqslant \widehat{\gamma} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0. \tag{3.67}$$

Let $u \in W^{1,p}_n(\Omega)$. From (3.66), (3.67) and hypothesis $H(f)(iii)$, we have

$$\begin{aligned} \int_\Omega (pF(z,u) - f(z,u)u)\,dz &= \int_{\{u\leqslant 0\}} (pF(z,u) - f(z,u)u)\,dz \\ &\quad + \int_{\{0<u\leqslant M_{14}\}} (pF(z,u) - f(z,u)u)\,dz \\ &\quad + \int_{\{M_{14}<u\}} (pF(z,u) - f(z,u)u)\,dz \\ &\leqslant c_{11} - \widehat{\beta}\int_{\{M_{14}\leqslant u\}} u^\tau\,dz, \end{aligned} \tag{3.68}$$

for some $c_{11} > 0$. We consider the embedding $i\colon C^1_n(\overline{\Omega}) \longrightarrow W^{1,p}_n(\Omega)$. We have

$$\psi = \varphi \circ i,$$

so, by the chain rule, we have

$$\psi'(u) = i^*\varphi'(u) \quad \forall u \in C^1_n(\overline{\Omega}). \tag{3.69}$$

Let $\langle\cdot,\cdot\rangle_c$ denote the duality brackets for the pair $(C^1_n(\overline{\Omega})^*, C^1_n(\overline{\Omega}))$. Then, using (3.69) and (3.68), we have

$$\begin{aligned}\frac{d}{dt}\psi(tu) &= \langle\psi'(tu),u\rangle_c = \langle i^*\varphi'(tu),u\rangle_c = \langle\varphi'(tu),u\rangle \\ &= t^{p-1}\|\nabla u\|_p^p - \int_\Omega f(z,tu)u\,dz \\ &= \frac{1}{t}\Big(\|\nabla(tu)\|_p^p - \int_\Omega f(z,tu)tu\,dz\Big) \\ &\leqslant \frac{1}{t}\Big(\|\nabla(tu)\|_p^p - \int_\Omega pF(z,tu)\,dz + c_{11}\Big) \\ &= \frac{1}{t}(p\varphi(tu)+c_{11}).\end{aligned}$$

By virtue of (3.65), we see that, if $t>0$ is so large that $\varphi(tu) < -\frac{c_{11}}{p}$, then

$$\frac{d}{dt}\psi(tu) < 0. \tag{3.70}$$

From Proposition 3.3, we have

$$\inf_{-C_+}\psi = \inf_{-C_+}\varphi > -c_{12},$$

for some $c_{12}>0$. Let

$$a < \min\Big\{-c_{12},\ -\frac{c_{11}}{p},\ \inf_{\overline{B}_1^c}\psi\Big\},$$

where

$$\overline{B}_1^c = \{u\in C^1_n(\overline{\Omega}) :\ \|u\|_{C^1_n(\overline{\Omega})} \leqslant 1\}.$$

Because of (3.70), we can find a unique $\lambda(u)\geqslant 1$, such that

$$\begin{cases}\psi(tu)>a & \text{if} \quad t\in[0,\lambda(u)),\\ \psi(tu)=a & \text{if} \quad t=\lambda(u),\\ \psi(tu)<a & \text{if} \quad t>\lambda(u)\end{cases}$$

and

$$\psi^a = \{tu:\ u\in\partial B^c_{1,+},\ t\geqslant\lambda(u)\}. \tag{3.71}$$

The implicit function theorem implies that $\lambda\in C(\partial B^c_{1,+};[1,+\infty))$.

Let

$$D_+ = \{tu:\ u\in\partial B^c_{1,+},\ t\geqslant 1\}.$$

Using the radial retraction, we see that $\partial B^c_{1,+}$ is a retract of D_+ and D_+ is deformable onto $\partial B^c_{1,+}$ in $C^1_n(\overline{\Omega})$. Invoking Theorem 6.5 of Dugundji [12, p. 325], we infer that $\partial B^c_{1,+}$ is a deformation retract of D_+. Hence

$$D_+ \text{ and } \partial B^c_{1,+} \text{ are homotopy equivalent.} \tag{3.72}$$

Next, consider the homotopy $\widehat{h}_+ : [0,1]\times D_+ \longrightarrow D_+$, defined by

$$\widehat{h}_+(s,tu) = \begin{cases}(1-s)tu + s\lambda(u)u & \text{if} \quad t\in[0,\lambda(u)),\\ tu & \text{if} \quad t\geqslant\lambda(u).\end{cases}$$

Note that, using (3.71), we have

$$\widehat{h}_+(0,\cdot) \;=\; id, \quad \widehat{h}_+(1,tu) \;\in\; \psi^a \quad \forall tu \in D_+$$

and

$$\widehat{h}_+(s,\cdot)|_{\psi^a} \;=\; id|_{\psi^a}.$$

So, it follows that ψ^a is a strong deformation retract of D_+, hence

$$D_+ \text{ and } \psi^a \text{ are homotopy equivalent.} \tag{3.73}$$

From (3.72) and (3.73), it follows that

$$\psi^a \text{ and } \partial B^c_{1,+} \text{ are homotopy equivalent}$$

and so

$$H_k(C^1_n(\overline{\Omega}), \psi^a) \;=\; H_k(C^1_n(\overline{\Omega}), \partial B^c_{1,+}) \quad \forall k \geqslant 0 \tag{3.74}$$

(see Granas-Dugundji [18, p. 387]). But from the first part of the proof, we know that $\partial B^c_{1,+}$ is contractible. Hence

$$H_k(C^1_n(\overline{\Omega}), \partial B^c_{1,+}) \;=\; 0 \quad \forall k \geqslant 0 \tag{3.75}$$

(see Granas-Dugundji [18, p. 389]). From (3.74) and (3.75), it follows that

$$H_k(C^1_n(\overline{\Omega}), \psi^a) \;=\; 0 \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi, \infty) \;=\; 0 \quad \forall k \geqslant 0.$$

A suitable modification of the proof of Proposition 3.4 gives the following result.

Proposition 3.5 *If hypotheses $H(f)$ hold, then $C_k(\varphi^\varepsilon_+, \infty) = 0$ for all $k \geqslant 0$.*

Proof. From hypotheses $H(f)(iii)$ and (v) (similarly as in (3.62)), we see that for any $\beta > 0$, we can find $c_\beta > 0$, such that

$$F^\varepsilon_+(z,\zeta) \;\geqslant\; \frac{\beta+\varepsilon}{p}(\zeta^+)^p - c_\beta \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.76}$$

Let

$$S_+ \;=\; \{u \in W^{1,p}_n(\Omega) : \|u\| = 1,\ u^+ \neq 0\}.$$

Then, for $u \in S_+$, we have

$$\begin{aligned} \varphi^\varepsilon_+(tu) \;&=\; \frac{t^p}{p}\|\nabla u\|^p_p + \frac{\varepsilon t^p}{p}\|u\|^p_p - \int_\Omega F^\varepsilon_+(z,tu)\,dz \\ &\leqslant\; \frac{t^p}{p}(\|\nabla u\|^p_p + \varepsilon\|u\|^p_p - (\varepsilon+\beta)\|u^+\|^p_p) + c_{13}, \end{aligned} \tag{3.77}$$

for some $c_{13} > 0$. Since $u \in S_+$, $\|u^+\|_p \neq 0$. Also, as $\beta > 0$ is arbitrary, we can choose $\beta > 0$ large enough, such that

$$\|\nabla u\|_p^p + \varepsilon \|u^-\|_p^p \;<\; \beta \|u^+\|_p^p.$$

So, from (3.77), it follows that

$$\varphi_+^\varepsilon(tu) \;\longrightarrow\; -\infty \quad \text{as } t \to +\infty. \tag{3.78}$$

Also, similarly as in (3.66), for all $u \in W_n^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (pF_+^\varepsilon(z,u) - f_+^\varepsilon(z,u)u\,dz) &= \int_{\{u>0\}} (pF(z,u) - f(z,u)u\,dz) \\
&= \int_{\{0<u<M_{14}\}} (pF(z,u) - f(z,u)u\,dz) \\
&\quad + \int_{\{M_{14}\leqslant u\}} (pF(z,u) - f(z,u)u\,dz) \\
&\leqslant c_{14} - \widehat{\beta} \int_{\{M_{14}\leqslant u\}} u^\tau\,dz,
\end{aligned} \tag{3.79}$$

for some $c_{14} > 0$. Then, using (3.79), we have

$$\begin{aligned}
\frac{d}{dt}\varphi_+^\varepsilon(tu) &= \langle (\varphi_+^\varepsilon)'(tu), u\rangle = \frac{1}{t}\Big(t^p\|\nabla u\|_p^p + \varepsilon t^p \|u\|_p^p - \int_\Omega f_+^\varepsilon(z,tu)tu\,dz\Big) \\
&\leqslant \frac{1}{t}\Big(t^p\|\nabla u\|_p^p + \varepsilon t^p \|u\|_p^p - \int_\Omega pF_+^\varepsilon(z,tu)\,dz + c_{14}\Big) \\
&= \frac{1}{t}(p\varphi_+^\varepsilon(tu) + c_{14}),
\end{aligned}$$

So, for all $t > 0$ large, such that $\varphi_+^\varepsilon(tu) < -\frac{c_{14}}{p}$, we have

$$\frac{d}{dt}\varphi_+^\varepsilon(tu) \;<\; 0.$$

Let $\eta < -\frac{c_{14}}{p}$. Then, we obtain a unique $\lambda_+(u) > 0$, $\lambda_+ \in C(S_+)$ (from the implicit function theorem), such that

$$\varphi_+^\varepsilon(\lambda_+(u)u) \;=\; \eta \quad \forall u \in S_+. \tag{3.80}$$

Let

$$E_+ \;=\; \{u \in W_n^{1,p}(\Omega) : \; u^+ \neq 0\}$$

and define

$$\widehat{\lambda}_+(u) \;=\; \frac{1}{\|u\|}\lambda_+\Big(\frac{u}{\|u\|}\Big) \quad \forall u \in E_+.$$

Evidently $\widehat{\lambda}_+ \in C(E_+)$ and from (3.80), we have that

$$\varphi_+^\varepsilon(\widehat{\lambda}_+(u)u) \;=\; \eta \quad \forall u \in E_+. \tag{3.81}$$

Moreover, if $\varphi_+^\varepsilon(u) = \eta$, then $\widehat{\lambda}_+(u) = 1$. We define

$$\overline{\lambda}_+(u) \;=\; \begin{cases} 1 & \text{if } \varphi_+^\varepsilon(u) \leqslant \eta, \\ \widehat{\lambda}_+(u) & \text{if } \varphi_+^\varepsilon(u) > \eta. \end{cases} \tag{3.82}$$

Evidently $\overline{\lambda}_+ \in C(E_+)$. We consider the homotopy $\overline{h}\colon [0,1] \times E_+ \longrightarrow E_+$, defined by

$$\overline{h}_+(t,u) \;=\; (1-t)u + t\overline{\lambda}_+(u)u \quad \forall (t,u) \in [0,1] \times W_n^{1,p}(\Omega).$$

Using (3.81) and (3.82), we note that

$$\overline{h}_+(0,u) \;=\; u, \quad \overline{h}_+(1,u) \;\in\; (\varphi_+^{\varepsilon})^{\eta}$$

and

$$\overline{h}_+(t,u) \;=\; u \quad \forall (t,u) \in [0,1] \times (\varphi_+^{\varepsilon})^{\eta}.$$

This shows that $(\varphi_+^{\varepsilon})^{\eta}$ is a strong deformation retract of E_+, hence

$$E_+ \text{ and } (\varphi_+^{\varepsilon})^{\eta} \text{ are homotopy equivalent,}$$

so

$$H_k(W_n^{1,p}(\Omega), E_+) \;=\; H_k(W_n^{1,p}(\Omega), (\varphi_+^{\varepsilon})^{\eta}) \quad \forall k \geqslant 0. \tag{3.83}$$

Assuming with any loss of generality that the critical set $K^{\varphi_+^{\varepsilon}}$ of φ_+^{ε} is bounded (otherwise we already have a sequence of distinct positive solutions in $\operatorname{int} C_+$ of (1.1) and so we are done) and choosing

$$\eta \;<\; \min\left\{ \inf \varphi_+^{\varepsilon}(K^{\varphi_+^{\varepsilon}}),\; -\frac{c_{14}}{p} \right\},$$

we have

$$H_k(W_n^{1,p}(\Omega), (\varphi_+^{\varepsilon})^{\eta}) \;=\; C_k(\varphi_+^{\varepsilon}, \infty) \quad \forall k \geqslant 0 \tag{3.84}$$

(see Section 2). Also, consider the homotopy $h_0\colon [0,1] \times E_+ \longrightarrow E_+$, defined by

$$h_0(t,u) \;=\; \frac{(1-t)u + t\widehat{u}_0}{\|(1-t)u + t\widehat{u}_0\|}.$$

Note that

$$h_0(1,u) \;=\; \frac{\widehat{u}_0}{\|\widehat{u}_0\|}$$

and so E_+ is contractible in itself. Hence

$$H_k(W_n^{1,p}(\Omega), E_+) \;=\; 0 \quad \forall k \geqslant 0 \tag{3.85}$$

(see Granas-Dugundji [18, p. 389]). From (3.83), (3.84) and (3.85), we conclude that

$$C_k(\varphi_+^{\varepsilon}, \infty) \;=\; 0 \quad \forall k \geqslant 0.$$

Remark 3.2 Such a result was first proved for $p = 2$ and for a nonlinearity bilaterally superlinear (i.e., in both $\mathbb{R}_+$ and $\mathbb{R}_-$) and satisfying the Ambrosetti-Rabinowitz condition, by Wang [28].

Proposition 3.6 *If hypotheses $H(f)$ hold, then $u = 0$ is a local minimizer of φ and φ_+^{ε}.*

Proof. Let $u \in C^1_n(\overline{\Omega})$ be such that $\|u\|_{C^1_n(\overline{\Omega})} \leqslant \delta_0$, with $\delta_0 > 0$ as in hypothesis $H(f)(vi)$. Then by virtue of that hypothesis, we have

$$F(z, u(z)) \leqslant 0 \quad \text{for almost all } z \in \Omega.$$

So, if $u \in C^1_n(\overline{\Omega})$ with $\|u\|_{C^1_n(\overline{\Omega})} \leqslant \delta_0$, then

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F(z, u(z))\, dz \geqslant 0 = \varphi(0),$$

so $u = 0$ is a local $C^1_n(\overline{\Omega})$-minimizer of φ.

Invoking Proposition 2.1, we conclude that $u = 0$ is a local $W^{1,p}_n(\Omega)$-minimizer of φ.

The proof for φ_+^ε is similar.

Now we are ready for the multiplicity theorem.

Theorem 3.1 *If hypotheses $H(f)$ hold, then problem (1.1) has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in C^1_n(\overline{\Omega}).$$

Proof. From Proposition 3.6, we know that $u = 0$ is a local minimizer of φ_+^ε. We may assume that it is an isolated critical point of φ_+^ε. Indeed, if this is not the case, then we can find the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_n(\Omega)$, such that

$$(\varphi_+^\varepsilon)'(u_n) = 0 \quad \forall n \geqslant 1 \tag{3.86}$$

and

$$u_n \longrightarrow 0 \quad \text{in } W^{1,p}_n(\Omega). \tag{3.87}$$

From (3.86), we have

$$A(u_n) + \varepsilon |u_n|^{p-2} u_n = N_+^\varepsilon(u_n), \tag{3.88}$$

where

$$N_+^\varepsilon(u)(\cdot) = f_+^\varepsilon(\cdot, u(\cdot)) \quad \forall u \in W^{1,p}_n(\Omega).$$

On (3.88), we act with $-u_n^- \in W^{1,p}_n(\Omega)$ and obtain

$$\|\nabla u_n^-\|_p^p + \varepsilon \|u_n^-\|_p^p = 0,$$

so $u_n^- = 0$, i.e., $u_n \geqslant 0$ for all $n \geqslant 1$. Therefore (3.88) becomes

$$A(u_n) = N(u_n),$$

where

$$N(u)(\cdot) = f(\cdot, u(\cdot)) \quad \forall u \in W^{1,p}_n(\Omega),$$

so

$$\begin{cases} -\Delta_p u_n(z) = f(z, u_n(z)) \text{ in } \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$$

(see Motreanu-Papageorgiou [22]). So, we have generated a whole sequence of distinct nontrivial solutions $\{u_n\}_{n\geqslant 1} \subseteq C_+$ (by the nonlinear regularity theory) of (1.1), which converge to $u = 0$ in $W_n^{1,p}(\Omega)$ (see (3.87)). Hence we are done.

Now, assuming that $u = 0$ is an isolated critical point of φ_+^ε and reasoning as in Motreanu-Motreanu-Papageorgiou [21, proof of Proposition 6], we can find $\varrho > 0$ small, such that

$$\varphi_+^\varepsilon(0) \;=\; 0 \;<\; \inf\{\varphi_+^\varepsilon(u):\; \|u\| = \varrho\} \;=\; \eta_\varrho^\varepsilon. \tag{3.89}$$

Moreover, hypothesis $H(f)(iv)$ implies that

$$\varphi_+^\varepsilon(c) \;\longrightarrow\; -\infty \quad \text{as } c \to +\infty,\ c \in \mathbb{R}. \tag{3.90}$$

Then (3.89) and (3.90) together with Proposition 3.2, permit the use of the mountain pass theorem (see Theorem 2.1). So, there is $u_0 \in W_n^{1,p}(\Omega)$, such that

$$\varphi_+^\varepsilon(0) \;=\; 0 \;<\; \eta_\varrho^\varepsilon \;\leqslant\; \varphi_+^\varepsilon(u_0) \tag{3.91}$$

and

$$(\varphi_+^\varepsilon)'(u_0) \;=\; 0. \tag{3.92}$$

From (3.91), we have $u_0 \neq 0$, while from (3.92), we obtain

$$A(u_0) + \varepsilon|u_0|^{p-2}u_0 \;=\; N_+^\varepsilon(u_0). \tag{3.93}$$

As before, acting on (3.93) with $-u_0^- \in W_n^{1,p}(\Omega)$, we obtain that $u_0 \geqslant 0$, $u_0 \neq 0$. Hence

$$A(u_0) \;=\; N(u_0),$$

so

$$\begin{cases} -\Delta_p u_0(z) \;=\; f(z, u_0(z)) \text{ in } \Omega, \\ \dfrac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Nonlinear regularity theory (see Gasiński-Papageorgiou [14, pp. 737-738]), implies that $u_0 \in C_+ \setminus \{0\}$. Moreover, if $R > \|u_0\|_\infty$, then by virtue of hypothesis $H(vi)$, for almost all $z \in \Omega$, we have

$$-\Delta_p u_0(z) + \xi_R u_0(z)^{p-1} \;=\; f(z, u_0(z)) + \xi_R u_0(z)^{p-1} \;\geqslant\; 0,$$

so

$$\Delta_p u_0(z) \;\leqslant\; \xi_R u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega$$

and using the nonlinear maximal principle due to Vázquez [26], we obtain

$$u_0 \;\in\; \operatorname{int} C_+.$$

From Proposition 3.3, we know that φ_- is coercive. Also, exploiting the compactness of the embedding $W_n^{1,p}(\Omega) \subseteq L^p(\Omega)$, we can easily check that φ_- is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $v_0 \in W_n^{1,p}(\Omega)$, such that

$$\varphi_-(v_0) \;=\; \inf \varphi_-. \tag{3.94}$$

From hypothesis $H(f)(vi)$, we know that there exists $c_0 < 0$, such that

$$\varphi_-(c_0) = -\int F(z, c_0)\,dz < 0,$$

so

$$\varphi_-(v_0) = \inf \varphi_- < 0 = \varphi_-(0) \tag{3.95}$$

and thus

$$v_0 \neq 0. \tag{3.96}$$

From (3.94), we have

$$\varphi_-'(v_0) = 0,$$

so

$$A(v_0) = N_-(v_0), \tag{3.97}$$

where

$$N_-(u)(\cdot) = f_-(\cdot, u(\cdot)) \quad \forall u \in W_n^{1,p}(\Omega).$$

On (3.97) we act with $v_0^+ \in W_n^{1,p}(\Omega)$ and obtain

$$\|\nabla v_0^+\|_p^p = 0,$$

so, using (3.96), we have

$$v_0^+(z) = v_0(z) \equiv \xi \quad \text{for almost all } z \in \Omega,$$

with $\xi \in \mathbb{R}_+ \setminus \{0\}$ and thus

$$\varphi_-(v_0) = \varphi_-(\xi) = 0,$$

a contradiction to (3.95). Therefore $v_0^+ = 0$ and so $v_0 \leqslant 0$. Hence (3.97) becomes

$$A(v_0) = N(v_0)$$

and so

$$\begin{cases} -\Delta_p v_0(z) = f(z, v_0(z)) \text{ in } \Omega, \\ \dfrac{\partial v_0}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Nonlinear regularity theory implies that $v_0 \in (-C_+) \setminus \{0\}$ and as before, using hypothesis $H(f)(vi)$ and the nonlinear maximum principle of Vázquez [26], we obtain

$$v_0 \in -\operatorname{int} C_+.$$

Now assume that $\{0, u_0\}$ are the only critical points of φ_+^ε. Indeed, if $u \notin \{0, u_0\}$ is another critical point of φ_+^ε, then reasoning as above, we can show that $u \in \operatorname{int} C_+$ and so it solves (1.1). Then $\{v_0, u_0, u\}$ are three nontrivial smooth solutions of (1.1) (all of constant sing) and so, we are done.

Claim. $C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geqslant 0$.

We choose

$$\mu < \varphi_+^\varepsilon(0) = 0 < \eta < \eta_\varrho^\varepsilon \leqslant \varphi_+^\varepsilon(u_0)$$

and then we consider the following triple of sets

$$(\varphi_+^\varepsilon)^\mu \subseteq (\varphi_+^\varepsilon)^\eta \subseteq W = W_n^{1,p}(\Omega).$$

For this triple of sets, we consider the corresponding long exact sequence of homology groups

$$\dots H_k(W, (\varphi_+^\varepsilon)^\mu) \xrightarrow{j_*} H_k(W, (\varphi_+^\varepsilon)^\eta) \xrightarrow{\partial_*} H_{k-1}(\varphi_+^\varepsilon)^\eta, (\varphi_+^\varepsilon)^\mu)\dots \tag{3.98}$$

where j_* is the group homomorphism induced by the inclusion $(W, (\varphi_+^\varepsilon)^\mu) \subseteq (W, (\varphi_+^\varepsilon)^\eta)$ and ∂_* is the boundary homomorphism. Using Proposition 3.5 and Proposition 3.6, for all $k \geq 0$, we have

$$H_k(W, (\varphi_+^\varepsilon)^\mu) = C_k(\varphi_+^\varepsilon, \infty) = 0, \tag{3.99}$$

$$H_k(W, (\varphi_+^\varepsilon)^\eta) = C_k(\varphi_+^\varepsilon, u_0), \tag{3.100}$$

$$H_{k-1}((\varphi_+^\varepsilon)^\eta, (\varphi_+^\varepsilon)^\mu) = C_{k-1}(\varphi_+^\varepsilon, 0) = \delta_{k-1,1}\mathbb{Z} = \delta_{k,1}\mathbb{Z}. \tag{3.101}$$

From (3.99), (3.100) and (3.101), we see that in the long exact sequence (3.98), only the tail $k = 1$ is nontrivial. So, we focus on it and using (3.98)–(3.101), we have

$$\begin{aligned} \operatorname{rank} C_1(\varphi_+^\varepsilon, u_0) &= \operatorname{rank} H_1(W, (\varphi_+^\varepsilon)^\eta) \\ &= \operatorname{rank}\ker\partial_* + \operatorname{rank}\operatorname{im}\partial_* \\ &= \operatorname{rank}\operatorname{im} j_* + \operatorname{rank}\operatorname{im}\partial_* \\ &\leqslant 0 + 1. \end{aligned} \tag{3.102}$$

On the other hand, since u_0 is a critical point of mountain pass type, we have

$$\operatorname{rank} C_1(\varphi_+^\varepsilon, u_0) \geqslant 1. \tag{3.103}$$

From (3.102) and (3.103) (and recalling that the critical groups are trivial for $k \geqslant 1$), we conclude that

$$C_k(\varphi_+^\varepsilon, u_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.104}$$

We consider the homotopy $h\colon [0,1]\colon W_n^{1,p}(\Omega) \longrightarrow W_n^{1,p}(\Omega)$, defined by

$$h(t,u) = (1-t)\varphi(u) + t\varphi_+^\varepsilon(u) \quad \forall (t,u) \in [0,1] \times W_n^{1,p}(\Omega).$$

As in the proofs of Propositions 3.1 and 3.2, we can check that for all $t \in [0,1]$, $h(t,\cdot)$ satisfies C-condition. Suppose that we could find two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_n^{1,p}(\Omega)$, such that

$$\begin{cases} t_n \longrightarrow t \quad \text{in } [0,1], \\ u_n \longrightarrow u_0 \quad \text{in } W_n^{1,p}(\Omega), \\ h'_u(t_n, u_n) = 0 \quad \forall n \geqslant 1. \end{cases} \tag{3.105}$$

From the equation in (3.105), we have

$$A(u_n) + t_n|u_n|^{p-2}u_n = (1-t_n)N(u_n) + t_n N_+^\varepsilon(u_n) \quad \forall n \geqslant 1,$$

so

$$\begin{cases} -\Delta_p u_n(z) + t_n\left|u_n(z)\right|^{p-2}u_n(z) \\ \qquad = (1-t_n)f(z,u_n(z)) + t_n f_+^\varepsilon(z,u_n(z)) \text{ in } \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0 \text{ on } \partial\Omega \end{cases} \tag{3.106}$$

(see Motreanu-Papageorgiou [22]). From (3.106) and due to hypothesis $H(f)(iii)$, we know that we can find $\overline{M} > 0$, such that

$$\|u_n\| \leqslant \overline{M} \quad \forall n \geqslant 1$$

(see e.g., Gasiński-Papageorgiou [14, p. 737]). Invoking Theorem 2 of Lieberman [20] (see also Gasiński-Papageorgiou [14, p. 738]), we can find $s \in (0,1)$ and $M_0 > 0$, such that

$$u_n \in C_n^{1,s}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_n^{1,s}(\overline{\Omega})} \leqslant M_0 \quad \forall n \geqslant 1. \tag{3.107}$$

Recalling that the embedding $C_n^{1,s}(\overline{\Omega}) \subseteq C_n^1(\overline{\Omega})$ is compact and using (3.105), we have

$$u_n \longrightarrow u_0 \quad \text{in } C_n^1(\overline{\Omega}).$$

But we know that $u_0 \in \operatorname{int} C_+$, so we can find $n_0 \geqslant 1$, such that

$$u_n \in \operatorname{int} C_+ \quad \forall n \geqslant n_0.$$

Therefore for $n \geqslant n_0$, u_n is a solution of (1.1) (see (3.106)). So, we have produced a whole sequence of distinct positive solutions of (1.1) and so we are done. Hence, we may assume that there exists $r > 0$, such that $u_0 \in \operatorname{int} C_+$ is the only critical point of $\{h(t,\cdot)\}_{t\in[0,1]}$ in $\overline{B}_r(u_0)$, with

$$\overline{B}_r(u_0) = \{u \in W_n^{1,p}(\Omega) : \|u - u_0\| \leqslant r\}.$$

Invoking the homotopy invariance property of the critical groups (see e.g., Chang [4, p. 334]), we have

$$C_k(h(0,\cdot),u_0) = C_k(h(1,\cdot),u_0) \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi,u_0) = C_k(\varphi_+^{\varepsilon},u_0) \quad \forall k \geqslant 0$$

and thus, from (3.104), we have

$$C_k(\varphi,u_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0.$$

This proves the Claim.

Recall that $v_0 \in -\operatorname{int} C_+$ is a minimizer of φ_- and $\varphi_-|_{-C_+} = \varphi|_{-C_+}$. Hence v_0 is a local $C_n^1(\overline{\Omega})$-minimizer of φ and so by virtue of Proposition 2.1, it is also a local $W_n^{1,p}(\Omega)$-minimizer of φ. Therefore

$$C_k(\varphi,v_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.108}$$

Moreover, from Propositions 3.4 and 3.6, we have

$$C_k(\varphi,\infty) = 0 \quad \forall k \geqslant 0 \tag{3.109}$$

and

$$C_k(\varphi,0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.110}$$

Suppose that $\{0,u_0,v_0\}$ are the only critical points of φ. Using the Morse relation (2.2), with $t=-1$, we obtain

$$(-1)^1 + 2(-1)^0 = 0,$$

a contradiction. This means that there is another critical point y_0 of φ, such that $y_0 \notin \{0,u_0,v_0\}$. Then y_0 is a solution of (1.1) and nonlinear regularity theory implies that $y_0 \in C_n^1(\overline{\Omega})$.

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