*Advanced Nonlinear Studies* **11** (2011), 633–651

# On the Persistence of Decay Properties for the $b-$Family of Equations

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Received 20 July 2010
*Communicated by Susanna Terracini*

**Abstract**

The $b-$family of equations governs evolution of shallow water waves with the horizontal velocities in different depths related to the parameter $b$. Using the method of characteristics, we obtain the results of the exponential decay properties of the strong solutions for the $b-$family of equations. In particular, we show that a non-trivial strong solution with compact initial value can not be compactly supported at any later time.

*MSC:*. 35Q53; 35Q60.
*Key words*. $b$-family of equations; compact support; exponential decay; method of characteristics.

# 1 Introduction

Recently, Holm and Staley [27] studied the following family of evolutionary $1+1$ PDEs that describe the balance between convection and stretching for small viscosity in the dynamics of $1D$ nonlinear

*The author acknowledges the support by a China NSF Grant No. 10871097, National Basic Research Program of China (973 Program) No. 2007CB814800 and Qing Lan Project of Jiangsu Province.

waves in fluids:

$$\partial_t m + u\partial_x m + bm\partial_x u = 0, \quad with\ u = G * m, \tag{1.1}$$

where the fluid velocity $u(t, x)$ is defined on the real line, vanishing at spatial infinity, and $u = G * m$ denotes the convolution

$$u(x) = \int_{\mathbb{R}} G(x - y)m(y)dy,$$

which relates velocity $u$ to momentum density $m$ by integration against kernel $G(x)$ over the real line. The family of equations (1.1) are characterized by the kernel $G(x)$ and the real dimensionless constant $b$, which is the ratio of the stretching to convective transport. The parameter $b$ is also the number of covariant dimensions associated with the momentum density $m$. The function $G(x)$ will determine the traveling wave shape and length scale for equation (1.1), while the constant $b$ will provide a balance or bifurcation parameter of behavior of the nonlinear solution. Here the kernel $G(x)$ is chosen to be Green's function for the Helmholtz operator on the line, that is, $G(x) = \frac{1}{2}e^{-|x|}$. This means $m = u - \partial_x^2 u$. In this case, we can change (1.1) into the following form:

$$\partial_t u - \partial_t\partial_x^2 u + (b + 1)u\partial_x u = b\partial_x u\partial_x^2 u + u\partial_x^3 u, \quad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

It is the so-called $b-$family of equations. Using the method of asymptotic integrability by Degasperis and Procesi [14], Eq. (1.1) is completely integrable when $b = 2$ and $b = 3$, cf. [4, 15, 16]. And these are the only values for which one has integrability, cf. [28].

When $b = 2$, Eq. (1.2) becomes the Camassa-Holm(CH) equation of the form

$$\partial_t u - \partial_t\partial_x^2 u + 3u\partial_x u = 2\partial_x u\partial_x^2 u + u\partial_x^3 u, \quad t > 0,\ x \in \mathbb{R}. \tag{1.3}$$

The Camassa-Holm equation was first derived by Camassa and Holm [4] as a model for shallow water waves(also we can refer it to Fokas and Fuchassteiner [17]). The alternative physical derivations were provided in [13, 29, 30]. Again $u(x, t)$ stands for the fluid velocity at the time $t$ in the spatial $x$ direction [4, 15]. The equation is completely integrable [4, 5], describing permanent and breaking waves [7, 8, 38]. Here a breaking wave means that the solution remains bounded while its slope becomes unbounded cf. [39], and that after breaking the solution can be continued uniquely either as a global conservative weak solution or as a global dissipative weak solution (see [1] ). The Camassa-Holm equation possesses the peakon [10] solutions of the form

$$u(x, t) = ce^{-|x-ct|}, \quad c > 0. \tag{1.4}$$

When $b = 3$, Eq. (1.2) becomes the Degasperis-Procesi (DP) equation of the form

$$\partial_t u - \partial_t\partial_x^2 u + 4u\partial_x u = 3\partial_x u\partial_x^2 u + u\partial_x^3 u, \quad t > 0,\ x \in \mathbb{R}. \tag{1.5}$$

The DP equation is also in dimensionless space-time variables $(x, t)$ an approximation to the incompressible Euler equations for shallow water [13, 15, 26, 30]. Degasperis, Holm and Hone [16] proved the formal integrability of the Degasperis-Procesi equation by constructing a Lax pair. They also showed [16] that the DP equation has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and that it admits exact peakon solutions which are analogous to the CH equation peakons. Lundmark [33] showed that the Degasperis-Procesi equation has not only peaked solitons (1.4), but also shock peakons of the form

$$u(x,t) = -\frac{1}{t+k} sgn(x) e^{-|x|}, \quad k > 0.$$

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH shallow water equation.

Above all, the peaked traveling wave solutions of CH equation and DP equation replicate a feature that is characteristic for the waves of great height–waves of largest amplitude that are exact solutions of the governing equations for water waves cf. [5, 9, 37].

For all $b \neq 0$, the $b-$family of equations (1.2) has at least three conservation laws defined in the following [16],

$$\begin{aligned} E_1 &= \int_{\mathbb{R}} m dx, \\ E_2 &= \int_{\mathbb{R}} m^{1/b} dx, \\ E_3 &= \int_{\mathbb{R}} m^{-1/b} \left( \frac{(\partial_x m)^2}{b^2 m^2} + 1 \right) dx, \end{aligned}$$

where $m = u - u_{xx}$. There are three useful conservation laws corresponding to the CH equation (1.3),

$$\begin{aligned} F_1 &= \int_{\mathbb{R}} m dx, \\ F_2 &= \int_{\mathbb{R}} (u^2 + (\partial_x u)^2) dx, \\ F_3 &= \int_{\mathbb{R}} (u^3 + u(\partial_x u)^2) dx, \end{aligned}$$

and the corresponding conservation laws of the DP equation (1.5) are

$$\begin{aligned} F_1 &= \int_{\mathbb{R}} m dx, \\ F_2 &= \int_{\mathbb{R}} u^3 dx, \\ F_3 &= \int_{\mathbb{R}} m v dx, \end{aligned}$$

where $v = (4 - \partial_x^2)^{-1} u$.

It is also shown in [16] that all these equations in the peakon $b-$family have not only the peakon solitons (1.4) but also multipeakon solutions

$$u(x,t) = \sum_{k=1}^{N} p_k(t) e^{-|x - q_k(t)|}.$$

For an arbitrary constant $b$, $p_k$ and $q_k$ are not canonical variables but satisfy the dynamical system

$$\dot{p_k} = -(b-1)\frac{\partial G_N}{\partial q_k}, \quad \dot{q_k} = \frac{\partial G_N}{\partial p_k},$$

where the generating function $G_N$ is given by

$$G_N = \frac{1}{2}\sum_{j,\,k=1}^{N} p_k p_j\, e^{-|q_j - q_k|}.$$

More recently, A. Himonas, G. Misolek, G. Ponce and Y. Zhou [25] studied the strong solution of the Camassa-Holm equation, which is initially decaying exponentially together with its spacial derivative, must be identically equal to zero if it also decays exponentially at a later time; D. Henry [21, 22, 23, 24] put such kind of conclusion into the Degasperis-Procesi equation and $b-$family of equations with the method of [25]. Meanwhile, A. Constantin [6] showed that the classical solution of the Camassa-Holm equation will have compact support if its initial data has this property; and G. Mustafa [35] showed the same property in the Degasperis-Procesi equation.

In this paper, we try to get some similar properties mentioned in [6, 21, 22, 23, 24, 25, 35], for the following equation, called the $b-$family of the equations with dispersion:

$$\partial_t m + u\partial_x m + bm\partial_x u + \kappa\partial_x u - \gamma\partial_x^3 u = 0, \tag{1.6}$$

the derivation of the equation will be shown in Section 2.

We have found it is convenient to rewrite equation (1.6) in the following form

$$\partial_t u - \partial_t\partial_x^2 u + (b+1)u\partial_x u + \kappa\partial_x u = b\partial_x u\partial_x^2 u + u\partial_x^3 u + \gamma\partial_x^3 u, \quad t>0,\ x\in\mathbb{R}. \tag{1.7}$$

Note that $G(x) = \frac{1}{2}e^{-|x|},\ x\in\mathbb{R}$. Then $(1-\partial_x^2)^{-1}f = G*f$, *and* $G*(u-\partial_x^2 u) = u$. So we also can change (1.7) into the conservation law form as follows:

$$\partial_t u + u\partial_x u + \partial_x G*\left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2\right)$$

$$+\partial_x G*((\kappa-\gamma)u) + \gamma\partial_x u = 0, \quad t>0,\ x\in\mathbb{R}. \tag{1.8}$$

One of the main purposes of this paper is to establish the finite propagation speed property for the $b-$family of equations (1.6). Due to the nonlocal nature of the conservation law form (1.8) of the equation, it is not a priori clear that a localized initial data $m(x,0)$ (that is, of compact support) will not spread out instantly to the whole spatial domain. Below we will give the theorems and will prove them in Section 3.

**Theorem 1.1** *Assume that* $m_0 : R\to R$ *is a smooth function with compact support. Let* $0\le b\le 3$ *and* $\kappa=\gamma$. *If the solution* $m(x,t)$ *of equation (1.6) has* $[0,T)$ *as maximal interval of existence in the future with initial data* $m_0$, *then for any* $t\in[0,T)$ *the smooth function* $x\to m(\cdot,t)$ *has compact support.*

We established that the solutions to the $b-$family of equations (1.6) has finite propagation speed. Since (1.6) is equivalent to (1.7), the natural question arises whether this property is captured by the time evolution of function $u$. Theorem 1.2 tells us that it is not the case.

**Theorem 1.2** *Assume that $u_0 : R \to R$ is a smooth function with compact support. Suppose $\kappa = \gamma$. Let $0 \le b \le 3$ and $T > 0$ be the maximal existence time of the smooth solution $u(x,t)$ to (1.7) with initial data $u_0$. If at every $t \in [0,T)$ the smooth function $x \to u(x,t)$ has compact support, then $u$ is identically zero.*

Another main objective of the paper is trying to formulate decay conditions on a solution, at two distinct times, which guarantees that $u \equiv 0$. Following the idea in [25], we get the main theorem:

**Theorem 1.3** *Assume that for some $T > 0$ and $s > 3/2$,*

$$u \in C([0,T] : H^s(\mathbb{R}))$$

*is a strong solution of equation (1.8). Let $0 \le b \le 3$ and $\kappa = \gamma$. If $u_0(x) = u(x,0)$ satisfies the property that for some $\alpha \in (1/2, 1)$,*

$$|u_0| \sim o\left(e^{-x}\right) \quad and \quad |\partial_x u_0| \sim O\left(e^{-\alpha x}\right) \quad as \quad x \to \infty, \tag{1.9}$$

*and there exists $t_1 \in (0,T]$ such that*

$$|u(x,t_1)| \sim o\left(e^{-x}\right) \quad as \quad x \to \infty, \tag{1.10}$$

*then $u \equiv 0$.*

The remainder of this paper is organized as follows. In Section 2, we briefly recall the derivation of the $b-$family of equations. The proofs of Theorem 1.1 and Theorem 1.2 are shown in Section 3. Section 4 is devoted to establishing the proof of Theorem 1.3. Then we conclude another theorem from Theorem 1.3 and Lemma 4.1. At last, we establish that there are no smooth solitary waves of the $b-$family of equations, in Section 5.

## 2 Derivation of the *b*-family of equations

The $b-$family of equations governs evolution of shallow water waves with the horizontal velocities in the different depth related to the parameter $b$. It is shown by A. Constantin and D. Lannes [13], that the CH ($b = 2$) and DP ($b = 3$) can be derived directly from the shallow water waves. In general, we can get the $b-$family equations in a similar way from shallow water waves. To see this, suppose that the water flow is incompressible, irrotational and inviscid. Then by [13], we know that the water wave equations for one-dimensional surfaces read, in nondimensionalized form

$$\begin{cases} \mu\partial_x^2\psi + \partial_z\psi^2 = 0, & in \quad \Omega_t, \\ \partial_z\psi = 0, & at \quad z = -1, \\ \partial_t\xi - \frac{1}{\mu}(-\mu\partial_x\xi\partial_x\psi + \partial_z\psi) = 0, & at \quad z = \epsilon\xi, \\ \partial_t\psi + \frac{\epsilon}{2}(\partial_x\psi)^2 + \frac{\epsilon}{2\mu}(\partial_z\psi)^2 = 0, & at \quad z = \epsilon\xi. \end{cases} \tag{2.1}$$

Here $x \to \epsilon\xi(t,x)$ means the free surface, $\Omega_t = \{(x,z);\ -1 < z < \epsilon\xi(t,x)\}$ means the fluid domain, while $\psi(t,\cdot)$ represents the velocity potential associated to the flow and $\epsilon = \frac{a}{h}, \mu = \frac{h^2}{\lambda^2}$, where $h$ is the mean depth, $a$ is the typical amplitude, and $\lambda$ is the typical wavelength of the waves.

Define the vertically averaged horizontal component of the velocity by

$$u(t,x) = \frac{1}{1+\epsilon\xi}\int_{-1}^{\epsilon\xi} \partial_x\psi(t,x,z)dz.$$

When it is in the shallow water conditions, the scaling should be $\mu \ll 1, \epsilon = O(1)$. We now consider the so-called Camassa-Holm scaling as follows: $\mu \ll 1, \epsilon = O(\mu^{\frac{1}{2}})$. With this scaling, one still has $\mu \ll 1$. The dimensionless parameter is larger here than in the long wave scaling, and the nonlinear effects are therefore stronger and it is possible that a stronger nonlinearity could allow the appearance of breaking waves.

Let us define the horizontal velocity $u^\theta(\theta \in [0,1])$ at the level line $\theta$ of the fluid domain.

$$v \equiv u^\theta(x) = \partial_x\psi|_{z=(1+\epsilon\xi)\theta-1}.$$

Let $p \in \mathbb{R},\ \lambda = \frac{1}{2}(\theta^2 - \frac{1}{3})$ and $\theta \in [0,1]$. Assume

$$\alpha = p + \lambda, \quad \beta = p - \frac{1}{6} + \lambda, \quad \eta = -\frac{2}{3}p - \frac{1}{6} - \frac{3}{2}\lambda, \quad \delta = -\frac{9}{2}p - \frac{23}{24} - \frac{3}{2}\lambda.$$

Under the Camassa-Holm scaling, one should have the following class of equations for $v \equiv u^\theta(\theta \in [0,1])$,

$$\partial_t v + \partial_x v + \frac{3}{2}\epsilon v\partial_x v + \mu(\alpha\partial_x^3 v + \beta\partial_t\partial_x^2 v) = \epsilon\mu(\eta v\partial_x^3 v + \delta\partial_x v\partial_x^2 v), \tag{2.2}$$

where $O(\epsilon^4, \mu^2)$ terms have been discarded.

Meanwhile the averaged horizontal velocity $u$ and the free surface $\xi$ satisfy

$$u = u^\theta + \mu\lambda\partial_x^2 u^\theta + 2\mu\epsilon u^\theta\partial_x^2 u^\theta,$$

$$\xi = u + \frac{\epsilon}{4}u^2 + \mu\frac{1}{6}\partial_t\partial_x u - \epsilon\mu(\frac{1}{6}u\partial_x^2 u + \frac{5}{48}(\partial_x u)^2).$$

In order to get the $b-$family of equations, we should rescale and shift the dependent variable, then apply a Galilean transformation to Eq. (2.2). If the following conditions hold

$$\beta < 0, \quad \alpha \neq \beta, \quad \beta = -\frac{2}{3}(b+1)\eta, \quad \delta = b\eta,$$

where

$$p = -\frac{10-29b}{72b}, \quad \lambda = -\frac{5}{18b} - \frac{1}{9} - p, \quad \theta^2 = \frac{11b-10}{12b},$$

then we are able to obtain the equation of the form follows:

$$\partial_t u - \partial_t\partial_x^2 u + (b+1)u\partial_x u + \kappa\partial_x u = b\partial_x u\partial_x^2 u + u\partial_x^3 u, \quad t > 0,\ x \in \mathbb{R}. \tag{2.3}$$

On the other hand, the solution $u^\theta$ of (2.2) is transformed to the solution $u$ of (2.3) by

$$\tilde{u}(t,x) = \frac{1}{a}u^\theta(\frac{x}{b_0} + \frac{\nu}{c}t,\ \frac{t}{c}),$$

with

$$a = \frac{2}{3\epsilon\kappa}(b+1)(1-\nu),\quad b_0^2 = -\frac{1}{\beta\mu},\quad \nu = \frac{\alpha}{\beta},\quad \text{and}\quad c = \frac{b_0}{\kappa}(1-\nu).$$

We now see two special values of $b$.

(1) The Camassa-Holm equation ($b = 2$)

$$\partial_t u - \partial_t\partial_x^2 u + 3u\partial_x u + \kappa\partial_x u = 2\partial_x u\partial_x^2 u + u\partial_x^3 u,$$

this equation holds if it has the following conditions:

$$\beta < 0,\quad \alpha \neq \beta,\quad \beta = -2\eta,\quad \delta = 2\eta,$$

where $p = -\frac{1}{3}, \theta^2 = \frac{1}{2}$.

(2) The Degasperis-Procesi equation ($b = 3$)

$$\partial_t u - \partial_t\partial_x^2 u + 4u\partial_x u + \kappa\partial_x u = 3\partial_x u\partial_x^2 u + u\partial_x^3 u$$

can also be derived if it meets the following conditions:

$$\beta < 0,\quad \alpha \neq \beta,\quad \beta = -\frac{8}{3}\eta,\quad \delta = 3\eta,$$

where $p = -\frac{77}{216}, \theta^2 = \frac{23}{36}$.

It is obvious that Eq. (2.3) is reversible but not Galilean invariant. With $w = u + c_0$, one has the equivalent form of the of (2.3) which is the $b-$family of equations (1.7)

$$\partial_t w - \partial_t\partial_x^2 w + (b+1)w\partial_x w + (\kappa - (b+1)c_0)\partial_x w = b\partial_x w\partial_x^2 w + w\partial_x^3 w - c_0\partial_x^3 w,$$

where $t > 0, x \in \mathbb{R}$. In particular, when $c_0 = \frac{\kappa}{b}$, we have

$$\partial_t w - \partial_t\partial_x^2 w + (b+1)w\partial_x w - c_0\partial_x w = b\partial_x w\partial_x^2 w + w\partial_x^3 w - c_0\partial_x^3 w,\quad t > 0,\ x \in \mathbb{R}.$$

# 3 Proof of theorem 1.1 and theorem 1.2

In this section, we will give the proofs of Theorem 1.1 and Theorem 1.2. Using the method of characteristics, we prove Theorem 1.1 first.

*Proof.* We associate the solution $m$ with initial data $m(x,0) = m_0(x)$ to the family $\{q(\cdot,t)\}_{t\in[0,T)}$ of smooth diffeomorphisms [2] of line defined by [20, 35],

$$\begin{cases} \frac{dq}{dt} = u(q,t) + \gamma, & t \in [0,T), \\ q(0) = x. \end{cases} \tag{3.1}$$

Using (1.6) with $\kappa = \gamma$ and (4.15), it is easy to calculate as follows:

$$\begin{aligned}\frac{d}{dt}\left(m(q,t)(\partial_x q)^b\right) &= (\partial_t m + \partial_x m \partial_t q)(\partial_x q)^b + bm(t,q)(\partial_x q)^{b-1}\partial_t\partial_x q \\ &= (-u\partial_x m - bm\partial_x u - \gamma\partial_x m + (u+\gamma)\partial_x m)(\partial_x q)^b \\ &\quad + bm(t,q)(\partial_x q)^{b-1}\partial_x q \partial_x u \\ &= (-u\partial_x m - bm\partial_x u - \gamma\partial_x m + bm\partial_x u + \partial_x m(u+\gamma))(\partial_x q)^b \\ &= 0.\end{aligned}$$

So we get the identity:

$$m(q(x,t),t)\cdot(\partial_x q(x,t))^b = m(x,0), \quad x \in \mathbb{R},\ t \in [0,T). \tag{3.2}$$

On the other hand, from (3.1) we infer that

$$\partial_x q(x,t) = \exp\left(\int_0^t \partial_x u(q(x,s),s)ds\right), \quad x \in \mathbb{R},\ t \in [0,T), \tag{3.3}$$

If $m_0$ is supported in the compact interval $[a,b]$, since $q_x(x,t) > 0$ on $\mathbb{R}\times[0,T)$ in (3.3), we deduce from (3.2) that $m(x,t)$ has its support in the interval $[q(a,t), q(b,t)]$. This completes the proof of Theorem 1.1.

Let us concerning the powerful invariance property (3.2) after the proof, in the particular $\kappa = \gamma = 0$ and $b = 2$. This identity is a re-expression of Noether's law (see [11]), in the context of the fact that CH is a geodesic equation on the diffeomorphsim group cf. [2, 31]. However, the only other context when a geometric interpretation of this type is available is for $\kappa$ is different from 0, $\gamma = 0$ and $b = 2$, when one obtains a re-expression of geodesic flow on the Bott-Virasoro group cf [12, 34].

Now, we use the Paley-Wiener Theorem and simple calculations to prove the Theorem 1.2 as follows:

*Proof.* For $(x,t) \in \mathbb{R}\times[0,T)$, let $m(x,t) = u - u_{xx}$. Clearly $m(x,0)$ has compact support since $u_0$ does. From Theorem 1.1 we deduce that for every $t \in [0,T)$ the smooth function $x \to m(x,t)$ is also of compact support. Recall that by the Paley-Wiener theorem [18, 36], an entire analytic function $g(\xi)$ of the complex variable $\xi = \eta + i\zeta$ (with $\eta, \zeta \in \mathbb{R}$) is the Fourier transform $F_f$ of a smooth function $f : \mathbb{R} \to \mathbb{R}$ with compact support in the ball $x \in \mathbb{R} : |x| \le a$, where $F_f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx$, for $\xi \in \mathbb{C}$, if and only if for every integer $n \ge 0$, there is a constant $c_n > 0$ so that

$$|g(\xi)| \le \frac{c_n e^{a|\xi|}}{(1+|\xi|)^n}, \quad \xi \in \mathbb{C}. \tag{3.4}$$

Notice that

$$F_m(\xi) = (1+\xi^2)F_u(\xi), \quad \xi \in \mathbb{C},\ t \in [0,T). \tag{3.5}$$

By assumption $u(\cdot,t)$ has compact support at every $t \in [0,T)$. Hence the Paley-Wiener theorem ensures that $F_m$ and $F_u$ are entire functions.
Then (3.5) forces

$$F_m(i) = F_m(-i) = 0 \tag{3.6}$$

at any fixed $t \in [0, T)$, or

$$\int_{\mathbb{R}} e^{x} m(x,t)dx = \int_{\mathbb{R}} e^{-x} m(x,t)dx = 0. \tag{3.7}$$

However, we infer from (1.6) with $\kappa = \gamma$ and for all $t \in [0, T)$,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} e^{x} m(x,t)dx &= \int_{\mathbb{R}} e^{x}\partial_t m dx \\
&= -b\int_{\mathbb{R}} e^{x} m\partial_x u dx - \int_{\mathbb{R}} e^{x} u\partial_x m dx - \gamma\int_{\mathbb{R}} e^{x}\partial_x m dx \\
&= -b\int_{\mathbb{R}} e^{x} m\partial_x u dx + \int_{\mathbb{R}} e^{x} m\partial_x u dx + \int_{\mathbb{R}} e^{x} mu dx + \gamma\int_{\mathbb{R}} e^{x} m dx \\
&= \int_{\mathbb{R}} e^{x} mu dx + (1-b)\int_{\mathbb{R}} e^{x} m\partial_x u dx \\
&= (2-b)\int_{\mathbb{R}} e^{x} u\partial_x u dx - (1-b)\int_{\mathbb{R}} e^{x}\partial_x u\partial_x^2 u dx \\
&\quad + \int_{\mathbb{R}} e^{x} u^2 dx + \int_{\mathbb{R}} e^{x}(\partial_x u)^2 dx \\
&= \frac{b}{2}\int_{\mathbb{R}} e^{x} u^2 dx + \frac{3-b}{2}\int_{\mathbb{R}} e^{x}(\partial_x u)^2 dx.
\end{aligned}$$

Hence for all $t \in [0, T)$

$$\frac{d}{dt}\int_{\mathbb{R}} e^{x} m(x,t)dx = \frac{b}{2}\int_{\mathbb{R}} e^{x} u^2 dx + \frac{3-b}{2}\int_{\mathbb{R}} e^{x}(\partial_x u)^2 dx. \tag{3.8}$$

But (3.7) and (3.8) can hold simultaneously if and only if $u \equiv 0$ since $0 \le b \le 3$. The proof is complete.

# 4 Proof of Theorem 1.3 and other results

In this section, we prove the main result that if the initial data and the derivative of the initial data are exponential decay, with any later time exponential decay, then the solution is identically zero. In order to prove it, we will show a lemma for whatever value of $\kappa$ and $\gamma$ first.

**Lemma 4.1** *Assume that for some $T > 0$ and $s > 3/2$,*

$$u \in C([0, T] : H^s(\mathbb{R}))$$

*is a strong solution of Eq. (1.8) and that $u_0(x) = u(x, 0)$ satisfies, for some $\theta \in (0, 1)$,*

$$|u_0|,\ |\partial_x u_0| \sim O\left(e^{-\theta x}\right) \quad as \quad x \to \infty. \tag{4.1}$$

*Then*

$$|u(x,t)|,\ |\partial_x u(x,t)| \sim O\left(e^{-\theta x}\right) \quad as \quad x \to \infty, \tag{4.2}$$

*uniformly in the time interval $[0, T]$.*

*Proof.* We introduce the following notations:

$$F(u) = \frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2, \tag{4.3}$$
$$H(u) = (\kappa - \gamma)u, \tag{4.4}$$
$$M = \sup_{t\in[0,T]} \|u(t)\|_{H^s}, \tag{4.5}$$

and the weight

$$\varphi_N(x) = \begin{cases} 1, & x \leq 0, \\ e^{\theta x}, & x \in (0, N), \\ e^{\theta N}, & x \geq N, \end{cases} \tag{4.6}$$

where $N \in \mathbb{Z}^+$. Observe that for all $N$, we have

$$0 \leq \varphi_N'(x) = \frac{d\varphi_N(x)}{dx} \leq \theta\varphi_N(x) \quad a.e \quad x \in \mathbb{R}. \tag{4.7}$$

Using (4.3), we have

$$|F| \leq \left|\frac{bu^2}{2}\right| + \left|\frac{3-b}{2}(\partial_x u)^2\right| \leq \alpha^2(|u|^2 + |\partial_x u|^2) \leq \alpha^2(|u| + |\partial_x u|)^2, \tag{4.8}$$

where $\alpha^2 = max\,\{\frac{b}{2}, \frac{3-b}{2}\}$.

Using the notation (4.3), (4.4) and (4.6), from Eq. (1.8) we obtain

$$\partial_t u\,\varphi_N + u\partial_x u\,\varphi_N + \gamma\partial_x u\,\varphi_N + \varphi_N\partial_x G * F(u) + \varphi_N\partial_x G * H(u) = 0. \tag{4.9}$$

A simple calculation shows that there exists $c_0 > 0$, depending only on $\theta \in (0, 1)$, (see (4.1) and (4.6)), such that for any $N \in \mathbb{Z}^+$,

$$\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}dy \leq 2c_0 = \frac{2-\theta}{1-\theta}. \tag{4.10}$$

Thus, for any appropriate function $f$, one sees that

$$\begin{aligned} \left|\varphi_N\partial_x G * f^2(x)\right| &= \left|\frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} sgn(y-x)e^{-|x-y|}f^2(y)dy\right| \\ &\leq \frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}\varphi_N(y)f^2(y)dy \\ &\leq \frac{1}{2}\left(\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}dy\right)\|\varphi_N f\|_\infty\|f\|_\infty \\ &\leq c_0\|\varphi_N f\|_\infty\|f\|_\infty, \end{aligned} \tag{4.11}$$

$$|\varphi_N\partial_x G * f(x)| \quad \leq \quad c_0\|\varphi_N f\|_\infty. \tag{4.12}$$

Since $\partial_x^2 G = G - \delta$, the arguments in (4.11) and (4.12) also show that

$$
\begin{aligned}
|\varphi_N \partial_x^2 G * f^2(x)| &\leq c_0 \|\varphi_N f\|_\infty \|f\|_\infty, &(4.13)\\
|\varphi_N \partial_x^2 G * f(x)| &\leq c_0 \|\varphi_N f\|_\infty. &(4.14)
\end{aligned}
$$

Now we will consider the smooth diffeomorphisms of line introduced in the third part as follows:

$$
\begin{cases}
\frac{dq}{dt} = u(q,t) + \gamma, \quad t \in [0,T),\\
q(0) = x.
\end{cases}
$$

The Sobolev embedding could ensure the uniform bound of $\partial_x u(\eta, s)$ for $(\eta, s) \in \mathbb{R} \times [0,t]$ with $t \in [0,T)$, from (3.3), we get for every $t \in [0,T)$ a constant $A(t) > 0$ such that

$$
\exp(-A(t)) < \partial_x q < \exp(A(t)), \quad x \in \mathbb{R}. \tag{4.15}
$$

By the above preparations, firstly let $V = u(q(x,t),t)\varphi_N(q(x,t))$. Combining (4.9) and (3.1), we have

$$
\begin{aligned}
\frac{dV}{dt} &= \partial_t u\, \varphi_N + \partial_x u\, q_t \varphi_N + u q_t \varphi_N'\\
&= \partial_t u\, \varphi_N + (u+\gamma)\partial_x u\, \varphi_N + (u+\gamma) u \varphi_N'\\
&= u^2 \varphi_N' - \varphi_N \partial_x G * F(u) + \gamma u \varphi_N' - \varphi_N \partial_x G * H(u). &(4.16)
\end{aligned}
$$

We now insert (4.3), (4.4), (4.7), (4.8), (4.11) and (4.12) into (4.16) to deduce that

$$
\begin{aligned}
\left|\frac{dV}{dt}\right| &\leq |u^2 \varphi_N'| + |\varphi_N \partial_x G * F(u)| + |\gamma u \varphi_N'| + |\varphi_N \partial_x G * H(u)|\\
&\leq \|u\|_\infty |u\varphi_N| + c_0 \alpha^2 (\|u\|_\infty + \|\partial_x u\|_\infty)(\|u\varphi_N\|_\infty + \|\partial_x u\, \varphi_N\|_\infty)\\
&\quad + |\gamma||u\varphi_N| + |\kappa - \gamma| c_0 \|u\varphi_N\|_\infty\\
&\leq (M + |\gamma| + |\kappa-\gamma|c_0)\|u\varphi_N\|_\infty + c_0\alpha^2 M(\|u\varphi_N\|_\infty + \|\partial_x u\, \varphi_N\|_\infty)\\
&\leq (c_0\alpha^2 M + M + |\gamma| + |\kappa-\gamma|c_0)(\|u\varphi_N\|_\infty + \|\partial_x u\, \varphi_N\|_\infty). &(4.17)
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\frac{dV}{dt} &\geq -(c_0\alpha^2 M + M + |\gamma| + |\kappa-\gamma|c_0)(\|u\varphi_N\|_\infty + \|\partial_x u\, \varphi_N\|_\infty),\\
\frac{dV}{dt} &\leq (c_0\alpha^2 M + M + |\gamma| + |\kappa-\gamma|c_0)(\|u\varphi_N\|_\infty + \|\partial_x u\, \varphi_N\|_\infty).
\end{aligned}
$$

Integrating $\frac{dV}{dt}$ in the $t-$variable, we get

$$
|V| \leq (c_0\alpha^2 M + M + |\gamma| + |\kappa-\gamma|c_0)\int_0^t (\|V\|_\infty + \|\partial_x u\, \varphi_N\|_\infty) d\tau + |V(q,0)|, \tag{4.18}
$$

or equivalently

$$
\|V\|_\infty \leq (c_0\alpha^2 M + M + |\gamma| + |\kappa-\gamma|c_0)\int_0^t (\|V\|_\infty + \|\partial_x u\, \varphi_N\|_\infty) d\tau + \|V(q,0)\|_\infty \tag{4.19}
$$

Next, using the notation (4.3), (4.4), differentiating (1.8)in the $x-$variable, and multiplying $\varphi_N$ on both sides produces the equation:

$$\partial_x(\partial_t u)\,\varphi_N + (\partial_x u)^2\varphi_N + u\partial_x^2 u\,\varphi_N + \gamma\partial_x^2 u\,\varphi_N + \varphi_N\partial_x^2 G * F(u) + \varphi_N\partial_x^2 G * H(u) = 0. \tag{4.20}$$

Let $W = \partial_x u(q(x,t),t)\varphi_N(q(x,t))$. Combining (3.1), (4.20), we have

$$\begin{aligned}\frac{dW}{dt} &= \partial_x(\partial_t u)\varphi_N + \partial_x^2 u q_t\varphi_N + \partial_x u q_t\varphi_N' \\ &= \partial_x(\partial_t u)\varphi_N + (u+\gamma)\partial_x^2 u\varphi_N + (u+\gamma)\partial_x u\varphi_N' \\ &= u\partial_x u\varphi_N' + \gamma\partial_x u\varphi_N' - \varphi_N\partial_x^2 G * F(u) - \varphi_N\partial_x^2 G * H(u).\end{aligned} \tag{4.21}$$

With the same argument of $\frac{dV}{dt}$, we have

$$\left|\frac{dW}{dt}\right| \leq (c_0\alpha^2 M + 2M + |\gamma| + |\kappa - \gamma|c_0)(\|u\varphi_N\|_\infty + \|\partial_x u\varphi_N\|_\infty). \tag{4.22}$$

So we can reach the following inequality

$$\|W\|_\infty \leq (c_0\alpha^2 M + 2M + |\gamma| + |\kappa-\gamma|c_0)\int_0^t (\|W\|_\infty + \|u\varphi_N\|_\infty)d\tau + \|W(q,0)\|_\infty. \tag{4.23}$$

Thus, choosing a constant $\tilde{C}_0 = \tilde{C}_0(M,T,b,\theta,\kappa,\gamma) \geq 2(c_0\alpha^2 M + 2M + |\gamma| + |\kappa-\gamma|c_0)$ and combining (4.19) and (4.23) yield

$$\|W\|_\infty + \|V\|_\infty \leq \tilde{C}_0\int_0^t (\|W\|_\infty + \|V\|_\infty)d\tau + (\|W(q,0)\|_\infty + \|V(q,0)\|_\infty)\,. \tag{4.24}$$

Hence for any $N \in \mathbb{Z}_+$ and any $t \in [0,T]$, choosing $\tilde{C} = e^{\tilde{C}_0 t}$, it follows from Gronwall's inequality that

$$\begin{aligned}\|u\varphi_N\|_\infty + \|\partial_x u\varphi_N\|_\infty &\leq (\|u\varphi_N|_{t=0}\|_\infty + \|\partial_x u\varphi_N|_{t=0}\|_\infty)e^{\tilde{C}_0 t} \\ &\leq \tilde{C}(\|u_0\, max\,\{1, e^{\theta x}\}\|_\infty \\ &\quad +\|\partial_x u_0\, max\,\{1, e^{\theta x}\}\|_\infty).\end{aligned} \tag{4.25}$$

Now we deduce from (4.15) that the function $q(\cdot,\ t)$ is strictly increasing on $\mathbb{R}$ with

$$lim_{x\to\pm\infty} q(x,t) = \pm\infty$$

as long as $t \in [0,T)$ [32]. So we can deduce (4.25) into the following inequality:

$$\begin{aligned}\|u(x,t)\varphi_N(x)\|_\infty + \|\partial_x u(x,t)\varphi_N(x)\|_\infty &= \|u(q(x,t),t)\varphi_N(q(x,t))\|_\infty \\ &\quad +\|\partial_x u(q(x,t),t)\varphi_N(q(x,t))\|_\infty \\ &\leq \tilde{C}\|u_0\, max\,\{1, e^{\theta x}\}\|_\infty \\ &\quad +\tilde{C}\|\partial_x u_0\, max\,\{1, e^{\theta x}\}\|_\infty.\end{aligned} \tag{4.26}$$

Finally, taking the limit as $N$ goes to infinity in (4.26), we find that for any $t \in [0,T]$

$$|e^{\theta x}u(x,t)| + |e^{\theta x}\partial_x u(x,t)| \leq \tilde{C}(\|u_0\, max\,\{1, e^{\theta x}\}\|_\infty + \|\partial_x u_0\, max\,\{1, e^{\theta x}\}\|_\infty), \tag{4.27}$$

which completes the proof of Lemma 4.1.

By Lemma 4.1, we can prove Theorem 1.3 as follows:

*Proof.* Integrating Eq. (1.8) with $\kappa = \gamma$ over the time interval $[0, t_1]$, we get

$$u(x,t_1) - u(x,0) + \int_0^{t_1} u\partial_x u(x,\tau)d\tau + \int_0^{t_1} \gamma u_x + \int_0^{t_1} \partial_x G * \left(\frac{bu^2}{2} + \frac{3-b}{2}(\partial_x u)^2\right)(x,\tau)d\tau = 0. \tag{4.28}$$

By hypothesis (1.9) and (1.10), we have

$$u(x,t_1) - u(x,0) \sim o(e^{-x}) \quad as \quad x \to \infty. \tag{4.29}$$

By (1.9), (1.10) and Lemma 4.1, it deduced that for all $\alpha \in (1/2, 1)$,

$$\int_0^{t_1} u\partial_x u(x,\tau)d\tau \sim O(e^{-2\alpha x}) \quad as \quad x \to \infty, \tag{4.30}$$

So we can get

$$\int_0^{t_1} u\partial_x u(x,\tau)d\tau \sim o(e^{-x}) \quad as \quad x \to \infty \tag{4.31}$$

and

$$\gamma \int_0^{t_1} \partial_x u(x,\tau)d\tau \sim O(e^{-\alpha x}) \quad as \quad x \to \infty.$$

We shall show that if $u \neq 0$, then the last term in (4.28) is not $O(e^{-\alpha x})$. Thus, we have

$$\int_0^{t_1} \partial_x G * \left(\frac{bu^2}{2} + \frac{3-b}{2}(\partial_x u)^2\right)(x,\tau)d\tau = \partial_x G * \int_0^{t_1} \left(\frac{bu^2}{2} + \frac{3-b}{2}(\partial_x u)^2\right)(x,\tau)d\tau \triangleq \partial_x G * \rho(x), \tag{4.32}$$

where by (1.9), (1.10) and Lemma 4.1, we have

$$0 \leq \rho(x) \sim O\,(e^{-2\alpha x}), \quad as \quad x \to \infty,$$

since $0 \leq b \leq 3$, so that $\rho(x) \sim o(e^{-x})$ as $x \to \infty$, and we get,

$$\begin{aligned} \partial_x G * \rho(x) &= -\frac{1}{2}\int_{-\infty}^{\infty} sgn(x-y)e^{-|x-y|}\rho(y)dy \\ &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\rho(y)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\rho(y)dy. \end{aligned} \tag{4.33}$$

From (4.33), it follows that

$$e^x \int_x^{\infty} e^{-y}\rho(y)dy = o(1)e^x \int_x^{\infty} e^{-2y}dy \sim o(1)e^{-x} \sim o(e^{-x}) \quad as \quad x \to \infty. \tag{4.34}$$

So we should look at the contributions of the first integral in (4.33). Suppose $u \neq 0$, then $\rho(x) \neq 0$ since $0 \leq b \leq 3$, and we get

$$\int_{-\infty}^{x} e^{y}\rho(y)dy = \int_{-\infty}^{0} e^{y}\rho(y)dy + \int_{0}^{M} e^{y}\rho(y)dy + \int_{M}^{x} e^{y}\rho(y)dy, \qquad x > M \gg 1.$$

Therefore

$$C_1 \leq \int_{-\infty}^{x} e^{y}\rho(y)dy \leq C_0 \quad for \quad x \gg 1, \tag{4.35}$$

where $C_0, C_1$ are constants. Hence, the last term in (4.33) satisfies

$$-\frac{C_0}{2}e^{-x} \leq \partial_x G * \rho(x) \leq -\frac{C_1}{2}e^{-x} \quad for \quad x \gg 1, \tag{4.36}$$

Finally, combining (4.28)-(4.32) with (4.36), we have

$$o(e^{-x}) + o(e^{-x}) + O(e^{-x}) = O(e^{-\alpha x}), \quad as \quad x \to \infty. \tag{4.37}$$

It can't be true when $u \neq 0$ for any $\gamma \neq 0$. Thus, $u \equiv 0$. This completes the proof of Theorem 1.3.

In the case when the solution $u(x,t)$ possesses further regularity and its data $u_0$ has stronger decay properties we shall give a more precise description of its behavior at infinity in the space variable.

By Theorem 1.3 and Lemma 4.1, we have the following result:

**Theorem 4.1** *Assume that for some $T > 0$ and $s > 5/2$,*

$$u \in C([0,T] : H^s(\mathbb{R}))$$

*is a strong solution of Eq. (1.8) with $\kappa = \gamma$.*

*(a) If $u_0(x) = u(x,0)$ has compact support, then for any $t \in (0,T]$,*

$$u(x,t) = \begin{cases} c_+(t)e^{-x}, & for \quad x > q(b,t), \\ c_-(t)e^{x}, & for \quad x < q(a,t). \end{cases} \tag{4.38}$$

*(b) If for some $\mu > 0$*

$$\partial_x^j u_0 \sim O\left(e^{-(1+\mu)|x|}\right) \quad as \quad |x| \to \infty \quad j = 0,1,2, \tag{4.39}$$

*then for any $t \in (0,T]$,*

$$m(x,t) = (1-\partial_x^2)u(x,t) \sim O\left(e^{-(1+\mu)|x|}\right) \quad as \quad |x| \to \infty, \tag{4.40}$$

*and*

$$\lim_{x\to\pm\infty} e^{\pm x}u(x,t) = c_{\pm}(t), \tag{4.41}$$

*where in (4.38), $c_+(\cdot)$, $c_-(\cdot)$ denote continuous non-vanishing functions, with $c_+(t) > 0$ and $c_-(t) < 0$ for $t \in (0,T]$. Furthermore $c_+(\cdot)$ is a strong increasing function, while $c_-(\cdot)$ is a strong decreasing function.*

*Proof.* Since $u(x) = \int_{\mathbb{R}} G(x-y)m(y)dy$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}m(y,t)dy \\ &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}m(y,t)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}m(y,t)dy. \end{aligned} \tag{4.42}$$

First, let us prove part (a). Using the proof of Theorem 1.2, we know that if $u_0$ has compact support in $x$ in the interval $[a,b]$, then so does $m(\cdot,t)$ in the interval $[q(a,t), q(b,t)]$ for any $t \in [0,T]$. Furthermore, defining

$$E_+(t) = \int_{q(a,t)}^{q(b,t)} e^{y}m(y,t)dy, \quad \text{and} \quad E_-(t) = \int_{q(a,t)}^{q(b,t)} e^{-y}m(y,t)dy, \tag{4.43}$$

and from the definition of $G * m = u$, we have

$$u(x,t) = \frac{1}{2}e^{-|x|} * m(x,t) = \frac{1}{2}e^{-x}E_+(t), \quad x > q(b,t), \tag{4.44}$$

and

$$u(x,t) = \frac{1}{2}e^{-|x|} * m(x,t) = \frac{1}{2}e^{x}E_-(t), \quad x < q(a,t). \tag{4.45}$$

Hence, it follows that for $x > q(b,t)$,

$$u(x,t) = -\partial_x u(x,t) = \partial_x^2 u(x,t) = \frac{1}{2}e^{-x}E_+(t) \tag{4.46}$$

and for $x < q(a,t)$,

$$u(x,t) = \partial_x u(x,t) = \partial_x^2 u(x,t) = \frac{1}{2}e^{x}E_-(t). \tag{4.47}$$

Then integration by parts, (4.46), (4.47) and Eq. (1.7) with $\kappa = \gamma$ yield the identities,

$$\begin{aligned} E_+(0) &= \int_{-\infty}^{\infty} e^{y}m(y,0)dy \\ &= \int_{-\infty}^{\infty} e^{y}u_0dy - \int_{-\infty}^{\infty} e^{y}\partial_y^2 u_0(y)dy \\ &= \int_{-\infty}^{\infty} e^{y}u_0dy + \int_{-\infty}^{\infty} e^{y}\partial_y u_0(y)dy = 0 \end{aligned} \tag{4.48}$$

and

$$\begin{aligned}
\frac{dE_+(t)}{dt} &= \int_{-\infty}^{\infty} e^y \frac{d}{dt}(u - \partial_y^2 u)dy \\
&= \int_{-\infty}^{\infty} e^y (b\partial_y u \partial_y^2 u + (u+\gamma)\partial_y^3 u - (b+1)u\partial_y u - \gamma\partial_y u)dy \\
&= -\int_{-\infty}^{\infty} e^y u\partial_y u dy + \int_{-\infty}^{\infty} e^y \partial_y^2 (u\partial_y u)dy \\
&\quad - \int_{-\infty}^{\infty} e^y \partial_y F(u)dy - \gamma \int_{-\infty}^{\infty} e^y \partial_y (u - \partial_y^2 u)dy \\
&= e^y(\partial_y(u\partial_y u) - u\partial_y u)|_{-\infty}^{\infty} - e^y F(u)|_{-\infty}^{\infty} \\
&\quad + \int_{-\infty}^{\infty} e^y F(u)dy + \gamma \int_{-\infty}^{\infty} e^y m dy \\
&= \int_{-\infty}^{\infty} e^y \left(\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_y u)^2\right) dy > 0.
\end{aligned} \tag{4.49}$$

Therefore, in the lifespan of the solution $u(x,t)$, $E_+(t)$ is an increasing function. Thus, from (4.48), it follows that $E_+(t) > 0$ for $t \in (0,T]$.

Similarly, it's easy to see $E_-(t)$ is decreasing with $E_-(0) = 0$, therefore $E_-(t) < 0$ for $t \in (0,T]$.

Taking $c_\pm(t) = \frac{1}{2}E_\pm(t)$, we have proved (4.38).

Next, let us consider part (b). Since $m(x,t) = (1-\partial_x^2)u(x,t)$ satisfies the equation (1.6), an argument similar to that gives in the proof of Lemma 4.1 shows that

$$\sup_{t\in[0,T]} \|m(t)e^{(1+\mu)|x|}\|_\infty \le \widetilde{c}\, \|m(0)e^{(1+\mu)|x|}\|_\infty, \tag{4.50}$$

with $\widetilde{c} = \widetilde{c}(M,T,b,\theta,\gamma)$ and that for any $\theta \in (0,1)$,

$$\partial_x^j u(t) \sim O\,(e^{-\theta|x|}), \quad as \quad |x| \to \infty \quad j = 0,1,2. \tag{4.51}$$

Thus, the definitions in (4.43) make sense with the integrals extended to the whole real line and the computations in (4.48) and (4.49) can be carried out in the same fashion. Finally, using (4.50) in (4.42), we obtain (4.41).

# 5 Nonexistence of smooth solitary waves

It is observed that the peaked solutions of the $b-$family of equations are not smooth solutions [19]. The same is true about the $b-$family of equations with a dispersive term. Actually, one can establish the following result for any traveling wave solutions of the $b-$family of equations.

**Theorem 5.1** *There is no nontrivial traveling-wave solution*

$$u \in C([0,\infty);\ H^3) \cap C^1([0,\infty);\ H^2)$$

*for Eq. (1.7) with $\kappa = \gamma$.*

*Proof.* Arguing by contradiction, assume that $w \in H^3$ and $u(t,x) = w(x-ct)$ is a strong solution of (1.7). Then we have

$$cw' - cw''' - (b+1)ww' + bw'w'' + ww''' - \gamma(w' - w''') = 0 \quad in\ L^2. \tag{5.1}$$

We find that

$$\left(cw - cw'' - \frac{b+1}{2}w^2 + \frac{b-1}{2}(w')^2 + ww'' - \gamma(w - w'')\right)' = 0 \quad in\ L^2. \tag{5.2}$$

So we have

$$cw - cw'' - \frac{b+1}{2}w^2 + \frac{b-1}{2}(w')^2 + ww'' - \gamma(w - w'') = 0 \quad in\ H^1, \tag{5.3}$$

or, what is same,

$$((c-\gamma) - w)(w - w'') - \frac{b-1}{2}(w^2 - (w')^2) = 0 \quad in\ H^1, \tag{5.4}$$

since $w \in H^3 \subset C_0^2(\mathbb{R})$. Multiplying this identity with $2w'$ yields that

$$((c-\gamma) - w)(w^2 - (w')^2)' - (b-1)w'(w^2 - (w')^2) = 0, \tag{5.5}$$

since $w \in H^3 \subset C_0^2(\mathbb{R})$ and $w \neq 0$, we have $w \neq c - \gamma$, a.e. and $w^2 \neq (w')^2$, a.e. Let $w_0 = w(\xi) = max_{x\in\mathbb{R}}\ w(x) > 0$. Then taking integration for (5.5) in $[\xi, x]$ yields that

$$\int_\xi^x \frac{d(w^2 - (w')^2)}{w^2 - (w')^2} = \int_\xi^x \frac{(b-1)dw}{c - \gamma - w}, \quad x \in \mathbb{R}. \tag{5.6}$$

This implies that

$$(w - c + \gamma)^{b-1}|w^2 - (w')^2| = w_0^2(w_0 - c + \gamma)^{b-1}, \quad x \in \mathbb{R}. \tag{5.7}$$

If we take into account $w,\ w' \to 0$ as $x \to \infty$, it is then inferred from (5.7) that

$$w_0^2(w_0 - c + \gamma)^{b-1} = 0, \quad x \in \mathbb{R}, \tag{5.8}$$

which also implies from (5.7) that

$$(w - c + \gamma)^{b-1}|w^2 - (w')^2| = 0, \quad x \in \mathbb{R}. \tag{5.9}$$

This is a contradiction to $w \neq c - \gamma$ and $w^2 \neq (w')^2$ since $w \in H^3$.

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