

## A Variational Approach to a Nonlocal Elliptic Problem with Sign-Changing Nonlinearity

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### Abstract

In this paper we are concerned with the nonlocal elliptic problem

$$\begin{cases} -\Delta u &= f(u) \left[ \int_{\Omega} g(u) \right]^p & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbf{R}^N$  is a bounded smooth domain,  $f, g : \mathbf{R} \rightarrow \mathbf{R}$  are given functions and  $p$  is a fixed real number. We use variational methods to prove multiplicity results.

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# 1 Introduction

In this paper we deal with a class of elliptic nonlocal problems whose prototype is

$$\begin{cases} -\Delta u = f(u) \left[ \int_{\Omega} g(u) \right]^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where  $\Omega \subset \mathbb{R}^N, N \geq 1$ , is a bounded smooth domain,  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are given functions and  $p$  is a fixed real number.

Nonlocal problems have been intensively studied since their first appearance in the work of Kirchhoff [25] who studied a wave equation which is a generalization of the D’Alembert equation. On this subject the reader may also consult Carrier [7] and Lions [29].

However, non-local problems are not restricted to mechanical motivations as in the aforementioned works. They also appear in a wide variety of applications as population dynamics (Chipot [11], [12], [10], [13], [14], [15]), Ohmic heating (Freitas and Grinfeld [20] and Lacey [28]), the formation of shear bands in materials (Olmstead [30] and Bebernes [3]), heat transfer in thermistors (Fowler [19]), combustion theory (Pao [31]), the electric ballast resistor (Chafee [9]), microwave heating of ceramic materials (Bose-Kriegsmann [4] and Kriegsmann [27]).

In particular, the present work was motivated by Gomes-Sanchez [22] who studied a variational counterpart of problem (1.1), namely

$$\begin{cases} -\Delta u = f(u) \left[ \int_{\Omega} F(u) \right]^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a given function, with a sort of exponential growth, and  $F(t) = \int_0^t f(s)ds$ ,

Problems like (1.2),

$$\begin{cases} -\Delta u = \delta \frac{e^u}{\int_{\Omega} e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and

$$\begin{cases} -\Delta u = \alpha \frac{e^{-u}}{\left(\int_{\Omega} e^{-u}\right)^p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

were previously studied by several authors as Bebernes-Lacey [2] and Caglioti-Lions-Marchiori-Pulvirenti [6] and appear in problems related with the theory of gravitational equilibrium of polytropic stars, the fully turbulent behavior of a real flow, among others.

In particular, problem (1.4) was studied by Gogny-Lions [21], when  $p = 1$ , by showing that there is solution for all values  $\alpha > 0$ . For general  $p > 0$  problem (1.4) was studied by Carrillo [8] where the author extends to a general bounded smooth domain of  $\mathbb{R}^N$  results previously obtained by [2] for a ball of  $\mathbb{R}^N$ . In this last work, and in the others mentioned above, local nonlinearity is decreasing and strictly positive and the authors are able to prove results on existence, uniqueness and multiplicity, depending on the ranges of  $p$  and  $\alpha$ .

Here, we attack cases in which  $f$  changes sign and the area of the bumps of the graph of  $f$  plays a key role. Multiplicity of solutions are obtained by exploring these geometric properties. At least to our knowledge, these types of nonlinearities have not been previously studied in the context of nonlocal problems.

These sort of problems, where area conditions are explored in order to obtain multiplicity results, were previously considered in local situations, that is,  $p = 0$ , by some authors. Let us cite some of them.

At least to our knowledge, the first authors to consider the influence of the areas of the bumps on the multiplicity of positive solutions were Brown-Budin [5] by using a combination of variational techniques and the sub and supersolution method. For autonomous ordinary differential equations these authors obtain more detailed results via simple quadrature arguments.

Later Hess [24] studied this sort of problem using variational methods combined with the Leray-Schauder degree and improves the results previously obtained in [5].

After that de Figueiredo [18] attacks this class of problems by using solely variational methods to obtain multiplicity results. Beside this the author treats the question of boundary layer formation.

Related to this subject the reader may also consult, for example, Clément & Sweers [16], Dancer & Schmitt [17] and Sweers [32].

Here we extend for the problem (1.2) some of the results obtained by these authors.

In Section 2 we study the case in which  $f(0) \geq 0$  and  $f$  is positive near zero. We prove the existence of a positive solution of (1.2) for any bounded domain  $\Omega$ . Then, when  $f$  changes sign and  $\Omega$  contains a large ball  $B_R$  but it is “near” of  $B_R$ , in a sense that will be clarified later, there exists a second positive solution of (1.2).

In Section 3 we attack the problem in which  $f(0) = 0$  and  $f$  is negative near zero. We show the existence of at least two positive solutions when  $\Omega$  is large, using the Mountain Pass Theorem.

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## 2 $f(0) \geq 0$ and $f > 0$ near 0

Here we will study problem (1.2) by establishing a multiplicity result related with the number of zeroes of  $f$  and the areas of the bumps of the graph of  $f$ . More precisely, we will suppose that  $f : [0, +\infty) \rightarrow \mathbf{R}$  is a continuous function of class  $C^1$  in  $(0, +\infty)$  and satisfies

$$(f_1) \quad \liminf_{t \rightarrow 0^+} \frac{f(t)}{t^\alpha} > 0,$$

where  $\alpha, p$  are positive numbers satisfying

$$(\alpha p)_1 \quad 1 > \alpha + \alpha p + p,$$

$$(f_{21}) \quad \left\{ \begin{array}{l} \text{There is a number } 0 < \theta_1 \text{ such that} \\ f(\theta_1) = 0 \text{ and } f(t) > 0 \text{ if } 0 < t < \theta_1. \end{array} \right.$$

$$(f_{22}) \quad \left\{ \begin{array}{l} \text{There are numbers } \theta_2, \theta_3 \text{ with } 0 < \theta_1 < \theta_2 < \theta_3 \\ \text{such that } f(\theta_2) = f(\theta_3) = 0 \text{ and} \\ f(t) < 0 \text{ if } \theta_1 < t < \theta_2; f(t) > 0 \text{ if } \theta_2 < t < \theta_3, \end{array} \right.$$

$$(F_1) \quad F(\theta_2) = \int_0^{\theta_2} f(s)ds > 0,$$

$$(F_2) \quad F(\theta_3) > F(\theta_1).$$

We point out that condition  $(F_1)$ , in particular, implies that  $F(t) > 0$  for all  $0 < t < \theta_3$  (in this way the term

$$\left[ \int_{\Omega} F(u) \right]^p$$

makes sense if  $u(x) \in (0, \theta_3)$  for every  $x \in \Omega$ ), while condition  $(F_2)$  tells us that the area of the bump of the graph of  $f$  between  $(\theta_1, \theta_2)$  is less than the area of the bump of the graph of  $f$  between  $(\theta_2, \theta_3)$ .

We will suppose that  $\Omega$  contains and it is near, in a sense that will be clarified later, a ball  $B_R = \{x \in \mathbf{R}^N; |x| < R\}$  with  $R$  sufficiently large.

Note that the assumption  $(\alpha p)_1$  implies that  $0 < \alpha < 1$  and assuming  $(f_1)$  and if  $f(0) = 0$ , then  $f$  is not differentiable at 0. Indeed,  $f'_+(0) = +\infty$  and so  $(f_1)$  is a kind of sublinearity at 0.

In order to state the main result of this section, we need define for  $R > 0$  the map

$$G(R) := C_N - \frac{K_N^{p+1}}{p+1} R^{Np+1} \left( [F(\theta_3)]^{p+1} \left( 1 - \frac{1}{R} \right)^{p+1} - [F(\theta_1)]^{p+1} \right),$$

where  $K_N$  is such that  $|B_R| = K_N R^N$  and  $C_N = (\theta_3^2/2)h_N$  where  $h_N$  is such that  $|C_R| \leq h_N R^{N-1}$  being

$$C_R = \{x \in \mathbf{R}^N; R - 1 < |x| < R\}.$$

Then, thanks to  $(F_2)$ , it is clear that there exists  $R_0 > 0$  such that for  $R > R_0$  we get

$$G(R) < 0. \tag{2.1}$$

The main result of this section is as follows:

**Theorem 2.1** *Let  $f \in C^0[0, \infty) \cap C^1(0, \infty)$  be a function satisfying  $(f_1), (f_{21})$  and  $(\alpha p)_1$ . Then problem (1.2) possesses a positive solution  $u_1$  satisfying  $0 < |u_1|_\infty < \theta_1$ , for any bounded, smooth domain  $\Omega$ . If, in addition,  $f$  satisfies  $(f_{22}), (F_1)$  and  $(F_2)$ , then problem (1.2) possesses a second solution  $u_2$  satisfying*

$$0 < |u_1|_\infty < \theta_1 < \theta_2 < |u_2|_\infty < \theta_3 \tag{2.2}$$

for every bounded domain  $\Omega$  which contains a ball  $B_R$  with  $R > R_0$  and

$$|\Omega|^{p+1} - |B_R|^{p+1} < -\frac{G(R)(p+1)}{[F(\theta_1)]^{p+1}}.$$

A standard bootstrap argument shows that such solutions are classical.

It is worthwhile to say that even in case  $\Omega = B_R$  we may not infer that the solutions obtained are radial. We recall that in [23] the function  $f$  should be of class  $C^1[0, +\infty)$  in order to obtain radial symmetry. As it is well known that, if we consider the problem

$$\begin{cases} -\Delta u = u^\alpha & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $0 < \alpha < 1$ , then nonradial solution may appear.

We will start by establishing a basic and simple lemma which will play a key role in the proof of theorem 2.1.

**Lemma 2.1** *Let  $f \in C^0[0, \infty) \cap C^1(0, \infty)$  be a function such that  $f(\theta) \leq 0$ , for some  $\theta > 0$ , and  $f$  satisfies  $\int_0^\theta f > 0$ . Then problem (1.2) does not possess a weak positive solution  $u$  with  $|u|_\infty = \theta$ .*

*Proof.* First of all we have to point out that, due to the regularity of  $f$ , any weak solution is classical. Suppose, on the contrary, that  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is a positive solution of (1.2), that is,

$$\begin{cases} -\Delta u = f(u) \left[ \int_\Omega F(u) \right]^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $|u|_\infty = \theta$ . Consequently,  $0 < u(x) \leq \theta$  for all  $x \in \Omega$  and  $|\{x \in \Omega; u(x) < \theta\}| > 0$ . Since  $f \in C^1$  we are able to find  $M > 0$  such that  $f(t) + Mt \geq 0$  is increasing on  $[0, \theta]$ . Since

$$\begin{cases} -\Delta u + M \left[ \int_\Omega F(u) \right]^p u = \left[ \int_\Omega F(u) \right]^p [f(u) + Mu] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta \theta + M \left[ \int_\Omega F(u) \right]^p \theta \geq \left[ \int_\Omega F(u) \right]^p [f(\theta) + M\theta] & \text{in } \Omega, \\ \theta > 0 & \text{on } \partial\Omega, \end{cases}$$

we obtain

$$\begin{cases} -\Delta(\theta - u) + M \left[ \int_{\Omega} F(u) \right]^p (\theta - u) = \\ \left[ \int_{\Omega} F(u) \right]^p [f(\theta) + M\theta - (f(u) + Mu)] \stackrel{\neq}{\geq} 0 \text{ in } \Omega, \\ \theta - u > 0 \text{ on } \partial\Omega. \end{cases}$$

By the maximum principle,  $\theta - u(x) > 0$  on  $\bar{\Omega}$  and so  $|u|_{\infty} < \theta$  which contradicts our assumption. This proves the lemma.

*Proof of the Theorem 2.1.* First of all let us consider the problem

$$\begin{cases} -\Delta u = f_1(u) \left[ \int_{\Omega} F_1(u) \right]^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where  $f_1 : \mathbf{R} \rightarrow \mathbf{R}$  is the Hölder-continuous function defined by

$$f_1(t) = \begin{cases} f(0) & \text{if } t \leq 0, \\ f(t) & \text{if } 0 < t \leq \theta_1, \\ 0 & \text{if } t > \theta_1, \end{cases}$$

where  $F_1(t) = \int_0^t f_1(s)ds$ .

We will find a solution of the problem (2.3) as a critical point of the functional  $J_1 : H_0^1(\Omega) \rightarrow \mathbf{R}$  given by

$$J_1(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \left[ \int_{\Omega} F_1(u) \right]^{p+1}, \quad u \in H_0^1(\Omega), \tag{2.4}$$

where in  $H_0^1(\Omega)$  we are considering the usual norm  $\|u\| = \left( \int_{\Omega} |\nabla u|^2 \right)^{\frac{1}{2}}$ .

In view of the definition of  $f_1$ , we have  $\int_{\Omega} F_1(u) \geq 0$  for all  $u \in H_0^1(\Omega)$  and so

$$\left[ \int_{\Omega} F_1(u) \right]^p \text{ and } \left[ \int_{\Omega} F_1(u) \right]^{p+1}$$

are well defined. Also,  $\int_{\Omega} F_1(u) \leq F_1(\theta_1)|\Omega|$ , which implies that

$$J_1(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} [F_1(\theta_1)|\Omega|]^{p+1}$$

and so  $J_1$  is bounded from below. In addition to this, it is a standard matter to show that  $J_1$  is weakly lower semicontinuous and belongs to  $C^1(H_0^1(\Omega), \mathbf{R})$  with

$$\langle J_1'(u), \varphi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi - \left[ \int_{\Omega} F_1(u) \right]^p \int_{\Omega} f_1(u)\varphi, \quad \forall u, \varphi \in H_0^1(\Omega). \tag{2.5}$$

Hence  $J_1$  attains a minimum at  $u_1 \in H_0^1(\Omega)$  which is a weak solution of (2.3).

If  $f(0) > 0$  the solution  $u_1 \neq 0$ . In case  $f(0) = 0$ , we should show that  $u_1 \neq 0$  in  $\Omega$ . For this it is enough to show that  $J_1(u_1) < 0$ . Let  $\varphi_1 > 0$  be an eigenfunction of  $(-\Delta, H_0^1(\Omega))$  associated to the first eigenvalue  $\lambda_1$ . Thus, taking  $\epsilon > 0$  such that  $0 < \epsilon\varphi_1(x) < \theta_1$ , we obtain (because  $F(\epsilon\varphi_1) = F_1(\epsilon\varphi_1)$ )

$$J_1(\epsilon\varphi_1) = \frac{1}{2}\epsilon^2\|\varphi_1\|^2 - \frac{1}{p+1} \left[ \int_{\Omega} F(\epsilon\varphi_1) \right]^{p+1},$$

where  $F(t) = \int_0^t f(s)ds$ .

We now use condition  $(f_1)$  to obtain  $0 < t_0 < \theta_1$  and  $\gamma > 0$  such that

$$\frac{f(t)}{t^\alpha} \geq \gamma \text{ if } 0 < t < t_0.$$

Then we choose  $\epsilon > 0$  such that  $0 < \epsilon\varphi_1(x) < t_0$  to obtain

$$\begin{aligned} J_1(\epsilon\varphi_1) &= \frac{1}{2}\epsilon^2\|\varphi_1\|^2 - \frac{1}{p+1} \left[ \int_{\Omega} F(\epsilon\varphi_1) \right]^{p+1} \leq \\ &\frac{1}{2}\epsilon^2\|\varphi_1\|^2 - \frac{1}{p+1} \left( \frac{\gamma}{\alpha+1} \right)^{p+1} \epsilon^{(\alpha+1)(p+1)} \left[ \int_{\Omega} \varphi_1^{\alpha+1} \right]^{p+1}. \end{aligned}$$

Consequently,

$$J_1(\epsilon\varphi_1) \leq \epsilon^{(\alpha+1)(p+1)} \left[ \frac{1}{2}\epsilon^{2-(\alpha+1)(p+1)}\|\varphi_1\|^2 - \frac{1}{p+1} \left( \frac{\gamma}{\alpha+1} \right)^{p+1} \left[ \int_{\Omega} \varphi_1^{\alpha+1} \right]^{p+1} \right]$$

and, by  $(\alpha p)_1$  we have that  $2 - (\alpha + 1)(p + 1) > 0$ , and so for  $\epsilon > 0$  small enough, we have  $J_1(\epsilon\varphi_1) < 0$ .

In this way, we have shown that  $u_1 \neq 0$ . Furthermore,

$$-\Delta u_1 = f_1(u_1) \left[ \int_{\Omega} F_1(u_1) \right]^p \geq 0$$

implies that  $u_1 \geq 0$  and because  $u_1 \neq 0$  we conclude that  $u_1 > 0$  in  $\Omega$ .

By Lemma 2.1,  $u_1 < \theta_1$  in  $\Omega$  and so  $0 < u_1(x) < \theta_1$  for all  $x \in \Omega$ . Consequently,  $u_1$  is a solution of the former problem (1.2). Moreover, for a future use, we remark that if  $u \in H_0^1(\Omega)$ ,  $0 < u(x) < \theta_1$ , one obtains

$$J_1(u) > -\frac{1}{p+1} [F(\theta_1)]^{p+1} |\Omega|^{p+1}. \tag{2.6}$$

Indeed, if  $u \in H_0^1(\Omega)$ ,  $0 < u(x) < \theta_1$  we have  $F(u(x)) < F(\theta_1)$  and so

$$\int_{\Omega} F(u) < F(\theta_1)|\Omega|.$$

Thus, using that

$$J_1(u) > -\frac{1}{p+1} \left[ \int_{\Omega} F(u) \right]^{p+1},$$

we deduce (2.6). In what follows, we will show the existence of a second solution by using a device motivated by Klaasen-Mitidieri [26].

For this we should consider the truncation  $f_2 : \mathbf{R} \rightarrow \mathbf{R}$  given by

$$f_2(t) = \begin{cases} f(0) & \text{if } t < 0; \\ f(t) & \text{if } 0 \leq t \leq \theta_3; \\ 0 & \text{if } t > \theta_3, \end{cases}$$

and the corresponding problem

$$\begin{cases} -\Delta u = f_2(u) \left[ \int_{\Omega} F_2(u) \right]^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

with  $F_2(t) = \int_0^t f_2(s) ds$ .

Similar to the way we obtained the first solution, we find a minimizer  $u_2 \in H_0^1(\Omega)$  to the functional

$$J_2(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \left[ \int_{\Omega} F_2(u) \right]^{p+1}, \quad u \in H_0^1(\Omega).$$

Such a minimizer is a weak solution (which is a classical one) of the problem (2.7) satisfying  $0 < u_2(x) < \theta_3$ . We have to point out that  $u_2$  is positive because  $f_2(t) + Mt$  is positive for some  $M > 0$  and for all  $t \in \mathbf{R}$ . So,  $u_2$  is a solution of (1.2).

However, we can not affirm, up to now, that  $u_1 \neq u_2$ . For this, we consider the function  $u_R \in H_0^1(\Omega)$  defined by

$$u_R(x) = \begin{cases} \theta_3 & \text{if } |x| \leq R-1, \\ (R-|x|)\theta_3 & \text{if } R-1 < |x| < R, \\ 0 & \text{if } x \in \Omega \setminus B_R \end{cases}$$

and set

$$C_R = \{x \in \mathbf{R}^N; R-1 < |x| < R\}.$$

Thus

$$\begin{aligned} J_2(u_R) &= \frac{1}{2} \int_{C_R} |\nabla u_R|^2 - \frac{1}{p+1} \left[ \int_{B_{R-1}} F(\theta_3) + \int_{C_R} F(u_R) \right]^{p+1} \\ J_2(u_R) &= \frac{1}{2} \int_{C_R} |\nabla u_R|^2 - \frac{1}{p+1} \left[ F(\theta_3)|B_{R-1}| + \int_{C_R} F(u_R) \right]^{p+1} \\ J_2(u_R) &= \frac{\theta_3^2}{2} |C_R| - \frac{1}{p+1} \left[ F(\theta_3)|B_{R-1}| + \int_{C_R} F(u_R) \right]^{p+1}. \end{aligned}$$

Using that  $\int_{C_R} F(u_R) \geq 0$  we have

$$\left[ F(\theta_3)|B_{R-1}| + \int_{C_R} F(u_R) \right]^{p+1} \geq (F(\theta_3)|B_{R-1}|)^{p+1}$$

we get

$$J_2(u_R) \leq \frac{\theta_3^2}{2} h_N R^{N-1} - \frac{1}{p+1} [F(\theta_3)]^{p+1} K_N^{p+1} (R-1)^{N(p+1)},$$

that is,

$$J_2(u_R) \leq C_N R^{(N-1)} - K_N^{p+1} \frac{[F(\theta_3)]^{p+1}}{p+1} (R-1)^{N(p+1)}.$$

We would like to show that

$$C_N R^{(N-1)} - K_N^{p+1} \frac{[F(\theta_3)]^{p+1}}{p+1} (R-1)^{N(p+1)} < -\frac{1}{p+1} [F(\theta_1)]^{p+1} |\Omega|^{p+1}. \tag{2.8}$$

Before continuing, we remember that  $N(p+1) - (N-1) = Np+1$  and, by virtue of  $(F_2)$ ,  $[F(\theta_3)]^{p+1} > [F(\theta_1)]^{p+1}$ .

It is worthy to remark that all the calculations made up to now are valid for all  $B_R \subset \Omega, R > 1$ .

Adding on both sides of (2.8) the term

$$\frac{[F(\theta_1)]^{p+1}}{p+1} |B_R|^{p+1},$$

and taking into account that  $|B_R| = K_N R^N$ , (2.8) is equivalent to

$$\frac{1}{p+1} (F(\theta_1))^{p+1} \left[ |\Omega|^{p+1} - |B_R|^{p+1} \right] < -G(R).$$

Then, by (2.6) and (2.8)

$$J_2(u_2) \leq J_2(u_R) < -\frac{1}{p+1} (F(\theta_1))^{p+1} |\Omega|^{p+1} < J_1(u_1) = J_2(u_1).$$

This shows that  $u_1 \neq u_2$  and the proof of the theorem is over.

**Remark 2.1** We should emphasize that, in the previous theorem, if  $f(0) > 0$  we may suppose that  $f$  is of  $C^1$ -class on  $[0, \infty)$ . So, under this assumption and if  $\Omega = B_R$ , the obtained solutions are radial and decreasing with respect to  $r = |x|$ .

**Remark 2.2** As we have shown,  $u_1$  is a minimum of  $J_1$  and  $u_2$  is a minimum of  $J_2$ . We conjecture that  $u_1$  is also a minimum for  $J_2$ . If this is the case, an application of the Mountain Pass Theorem would lead us to a third solution  $u_3$ . However, we were not able to prove this. We also conjecture that  $u_1$  and  $u_2$  are ordered, i.e.,  $0 < u_1(x) < u_2(x)$  for all  $x \in \Omega$ .

**Remark 2.3** As we have said before, if  $f(0) = 0$ , then  $f'_+(0) = +\infty$  which combined with the fact that  $f(t) > 0$ , for  $0 < t < \theta_1$  and  $f(\theta_1) = 0$  implies that the graph of  $f$  crosses the graph of the straight line  $\lambda_1(\Omega)t$  for some  $0 < t < \theta_1$ . It seems that this crossing produces the first solution  $u_1$ . However, the existence of positive solution fails if  $f(0) = 0$  and  $(f_1)$  does not hold. To check this, let us suppose that the function  $f, g$  satisfy  $0 \leq f(t) \leq mt, 0 \leq g(t) \leq C$ , for all  $t \geq 0$  where  $m, C$  are positive constants, and  $u$  is a positive solution of (1.1). A simple calculation in (1.1) leads us to

$$0 < \lambda_1(\Omega) \leq C|\Omega|^p$$

where  $C > 0$  is a positive constant does not depend on  $\Omega$ . This is a contradiction because  $\lambda_1(\Omega) \rightarrow +\infty$  as  $|\Omega| \rightarrow 0$ . The equation below

$$\begin{cases} -\Delta u = \sin u \left[ \int_{\Omega} g(u) \right]^{2p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

with  $0 \leq g(t) \leq C$  satisfies these conditions.

### 3 $f(0) = 0$ and $f < 0$ near 0

In this section we consider the problem

$$\begin{cases} -\Delta u = f(u) \left[ \int_{\Omega} F(u) \right]^{2p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with  $p = 1, 2, \dots$  and  $f \in C^1[0, \infty)$  satisfying

$$(f_3) \quad \begin{cases} f(0) = 0 \text{ and there are } 0 < a_1 < a_2 \\ \text{such that } f(a_1) = f(a_2) = 0, \\ f(t) < 0 \text{ if } 0 < t < a_1 \text{ and } f(t) > 0 \text{ if } a_1 < t < a_2. \end{cases}$$

$$(F_3) \quad F(a_2) > 0, \text{ where } F(t) = \int_0^t f(s)ds.$$

Observe that due to Dirichlet boundary condition,  $u$  attains small positive value and so  $\int_{\Omega} F(u)$  could take negative values, and  $[\int_{\Omega} F(u)]^p$  does not make sense for  $p > 0$ ; so we pose  $[\int_{\Omega} F(u)]^{2p}$ .

In view of condition  $(F_3)$ , there is  $0 < a_3 < a_2$  such that  $F(a_3) = 0, F(t) < 0$  if  $0 < t < a_3$  and  $F(t) > 0$  if  $a_3 < t < a_2$ .

From now on, we still denote by  $f$  the extension-truncation of  $f$  such that  $f(t) = f(0) = 0$  if  $t < 0$  and  $f(t) = 0$  if  $t > a_2$ .

Let  $J : H_0^1(\Omega) \rightarrow \mathbb{R}$  be the energy functional associated to problem (3.1) given by

$$J(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2p+1} \left[ \int_{\Omega} F(u) \right]^{2p+1} \tag{3.2}$$

where, as before,  $\|\cdot\|$  is the usual norm of  $H_0^1(\Omega)$ .

**Theorem 3.1** *Under assumptions  $(f_3) - (F_3)$  and if  $\Omega$  contains a ball  $B_R$  with  $R \gg 0$ , problem (3.1) has at least two positive solutions satisfying*

$$a_1 < |u_1|_\infty, |u_2|_\infty < a_2.$$

Furthermore, if  $\Omega = B_R$  such solutions are radial and  $\frac{\partial u_i}{\partial r} < 0$  for  $0 < r < R$  and  $i = 1, 2$ .

*Proof.* As in the previous result, we may show that  $J$  is coercive, bounded from below, weakly lower semicontinuous and of  $C^1$ -class. Hence  $J$  attains a global minimum at a certain  $u_1 \in H_0^1(\Omega)$ . Since  $f(0) = 0$  we have to show that  $u_1 \neq 0$ . For this, we consider, like before, the function

$$u_R(x) = \begin{cases} a_2 & \text{if } |x| \leq R - 1, \\ (R - |x|)a_2 & \text{if } R - 1 < |x| < R, \\ 0 & \text{if } x \in \Omega \setminus B_R. \end{cases}$$

Let us evaluate  $J(u_R)$ : Observing that  $0 < u_R(x) \leq a_2$  for all  $x \in \Omega$ , we obtain

$$J(u_R) = \frac{1}{2}a_2^2|C_R| - \frac{1}{2p+1} \left[ F(a_2)|B_{R-1}| + \int_{C_R} F(u_R) \right]^{2p+1}$$

where  $C_R = \{x \in \mathbb{R}^N; R - 1 < |x| < R\}$ . We now note that  $F(u_R(x)) \geq \min_{0 \leq t \leq a_2} F(t) = F(a_1)$  to obtain  $\int_{C_R} F(u_R(x))dx \geq F(a_1)|C_R|$  and so

$$F(a_2)|B_{R-1}| + \int_{C_R} F(u_R) \geq F(a_2)|B_{R-1}| + F(a_1)|C_R|$$

and, because the function  $t \rightarrow t^{2p+1}, t \in \mathbb{R}$  is increasing,

$$\left[ F(a_2)|B_{R-1}| + \int_{C_R} F(u_R) \right]^{2p+1} \geq [F(a_2)|B_{R-1}| + F(a_1)|C_R|]^{2p+1}.$$

Consequently,

$$J(u_R) \leq \frac{1}{2}a_2^2|C_R| - \frac{1}{2p+1} [F(a_2)|B_{R-1}| + F(a_1)|C_R|]^{2p+1}.$$

We now point out that:

- $\frac{1}{2}a_2^2|C_R|$  behaves like  $\frac{1}{2}a_2^2R^{N-1}$  at infinity;
- $F(a_2)|B_{R-1}|$  behaves like  $F(a_2)(R - 1)^N$  at infinity and recall that  $F(a_2) > 0$ ;
- $F(a_1)|C_R|$  behaves like  $F(a_1)R^{N-1}$  at infinity.

Thus  $(F(a_2)|B_{R-1}| + F(a_1)|C_R|)^{2p+1}$  behaves like  $F(a_2)^{2p+1}R^{N(2p+1)}$  at infinity and so

$$J(u_R) < 0 \text{ if } R \text{ is large enough.}$$

From now on, we fix such a large  $R$ . Consequently,  $J(u_1) \leq J(u_R) < 0$  and so  $u_1 \neq 0$ . Let  $u^-$  be the negative part of  $u = u^+ - u^-$ . Multiplying both sides of

$$\begin{cases} -\Delta u_1 = f(u_1) \left[ \int_{\Omega} F(u_1) \right]^{2p} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

by  $u_1^-$  and integrating by parts, we obtain

$$-\|u_1^-\|^2 = \left[ \int_{\Omega} F(u_1) \right]^{2p} \int_{\Omega} f(u_1) u_1^- = 0.$$

Hence,  $u_1 \geq 0$  in  $\Omega$  and reasoning as in Lemma 2.1 and in view of the maximum principle we conclude that  $0 < u_1(x) < a_2$  and  $a_1 < |u_1|_{\infty} < a_2$ .

We now claim that 0 is also a local minimum of  $J$ . Indeed, if  $t \geq 0$  we have  $f(t) \leq \alpha t$ , for some  $\alpha > 0$ , and so  $F(t) \leq \frac{\alpha}{2} t^2$  for all  $t \geq 0$ . Since  $F(t) = 0$  if  $t < 0$ , this inequality remains true for all  $t \in \mathbb{R}$ . Hence,

$$\int_{\Omega} F(u) \leq C\|u\|^2$$

from which we have

$$\left[ \int_{\Omega} F(u) \right]^{2p+1} \leq C\|u\|^{2(2p+1)}.$$

Consequently,

$$J(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^{2(2p+1)} = \|u\|^2 \left( \frac{1}{2} - C\|u\|^{4p} \right) > 0$$

if  $\|u\| = \rho > 0$ ,  $\rho$  small enough. This shows that 0 is a strict local minimum.

We now use the well known Mountain Pass Theorem due to Ambrosetti-Rabinowitz [1].

Combining the fact that

$$J(u_R) < 0, \text{ if } R \text{ is fixed and large enough,}$$

with

$$J(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^{2(2p+1)} = \|u\|^2 \left( \frac{1}{2} - C\|u\|^{4p} \right) > 0$$

we find  $\rho, \alpha$ , with  $\rho > 0$  small enough, such that  $J(u) \geq \alpha$  if  $\|u\| = \rho$  and  $J(u_R) < 0$  if  $\|u_R\| > \rho$  and so  $J$  satisfies the geometry of the Mountain Pass Theorem.

For the sake of completeness we show that  $J$  enjoys the (PS) condition. For this, let  $(u_n) \subset H_0^1(\Omega)$  such that  $|J(u_n)| \leq C$  and  $J'(u_n) \rightarrow 0$  in  $(H_0^1(\Omega))^* = H^{-1}(\Omega)$ . Since  $J$  is coercive, it follows that  $(u_n)$  is bounded in  $H_0^1(\Omega)$  and so  $u_n \rightharpoonup u$ , weakly in  $H_0^1(\Omega)$ ,  $u_n \rightarrow u$  in  $L^r(\Omega)$ ,  $1 \leq r < 2^*$ , where  $2^*$  is the critical Sobolev exponent, and  $u_n(x) \rightarrow u(x)$  a.e. in  $\Omega$ , perhaps for subsequences.

Furthermore,

$$\langle J'(u), v \rangle = \int_{\Omega} \nabla u \cdot \nabla v - \left[ \int_{\Omega} F(u) \right]^{2p} \int_{\Omega} f(u)v, \text{ for all } u, v \in H_0^1(\Omega)$$

and so

$$\langle \nabla J(u), v \rangle = \langle u, v \rangle - \langle Tu, v \rangle \quad \text{for all } u, v \in H_0^1(\Omega)$$

where  $\langle \cdot, \cdot \rangle$  is the usual inner product in  $H_0^1(\Omega)$ ,  $\nabla J(u)$  is the gradient of  $J$  in  $u \in H_0^1(\Omega)$  and  $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is defined by

$$\langle Tu, v \rangle = \left[ \int_{\Omega} F(u) \right]^{2p} \int_{\Omega} f(u)v, \quad \text{for all } u, v \in H_0^1(\Omega)$$

and clearly is compact. Hence  $\nabla J(u) = u - Tu$  and because  $\nabla J(u_n) \rightarrow 0$  in  $H_0^1(\Omega)$  and, in view of compactness of  $T$ ,  $(T(u_n))$  is convergent in  $H_0^1(\Omega)$ , we conclude that  $u_n \rightarrow u$  in  $H_0^1(\Omega)$  up to a subsequence. This shows that  $J$  satisfies the assumptions of the Mountain Pass Theorem and so problem (3.1) possesses a second solution  $u_2$ .

To finish the proof we have to point out that both positive solutions we have found are classical and satisfy  $a_1 < |u_1|_{\infty}, |u_2|_{\infty} < a_2$ . In case we are working in a ball, both  $u_1, u_2$  are radial and  $\frac{\partial u_1}{\partial r}, \frac{\partial u_2}{\partial r} < 0$  if  $0 < r < R$ .

**Remark 3.1** Let  $u$  be a solution of problem (3.1) with  $0 < u(x) < a_2$ . Thus

$$\|u\|^2 = \left[ \int_{\Omega} F(u) \right]^{2p} \int_{\Omega} f(u)u.$$

and because

$$\|u\|^2 = \left[ \int_{\Omega} F(u) \right]^{2p} \left[ \int_{0 < u(x) \leq a_1} f(u)u + \int_{a_1 < u(x) \leq a_2} f(u)u \right] \leq \|u\|^2$$

we obtain

$$\|u\|^2 \leq \left[ \int_{\Omega} F(u) \right]^{2p} \int_{a_1 < u(x) \leq a_2} f(u)u$$

because  $f(t) < 0$  for  $0 \leq t < a_1$ . Since  $f$  is bounded, we may find a constant  $C > 0$  such that  $f(t)t \leq C(1 + t^2)$  which yields

$$\|u\|^2 \leq C \left[ \int_{\Omega} F(u) \right]^{2p} \int_{a_1 < u(x) \leq a_2} (1 + u^2).$$

Since  $\frac{u(x)}{a_1} > 1$ , we obtain

$$\|u\|^2 \leq C_1 \left[ \int_{\Omega} F(u) \right]^{2p} \int_{a_1 < u(x) \leq a_2} u^2 \leq C_2 \left[ \int_{\Omega} F(u) \right]^{2p} \int_{\Omega} u^2.$$

We now use the variational characterization of  $\lambda_1 = \lambda_1(\Omega)$ , and recalling that  $\Omega = B_R$ , to get

$$\|u\|^2 \leq C_2 [F(a_2)]^{2p} \frac{|\Omega|^{2p}}{\lambda_1(\Omega)} \|u\|^2.$$

Observing that

$$\frac{|B_R|^{2p}}{\lambda_1(R)} \rightarrow 0 \text{ as } R \rightarrow 0$$

and using that

$$0 < C_2 \leq \frac{|B_R|^{2p}}{\lambda_1(R)} \rightarrow 0 \text{ as } R \rightarrow 0,$$

we conclude that problem (3.1) does not possess solution if  $R > 0$  is small.

It follows from this remark and theorem 3.1 that there are  $0 < R_0 \leq R_1 < \infty$  such that problem (3.1) has no solution if  $R \in (0, R_0]$  and has two positive solutions if  $R \in [R_\infty, \infty)$ . Consequently, there is a range  $[R_0, R_\infty]$  for which we were not able to determine existence of solution.

**Remark 3.2** From condition  $(F_3)$ , there exists  $a_1 < a_3 < a_2$  such that  $F(a_3) = 0$ ,  $F(t) < 0$  if  $0 < t < a_3$ , and  $F(t) > 0$  if  $a_3 < t < a_2$ . We conjecture that positive solutions  $u$  of (3.1) should satisfy  $a_3 < |u|_\infty < a_2$ .

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