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Stability and the Growth of the Number of Closed Characteristics on Compact Convex Hypersurfaces

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Abstract

In this paper, we study relationship between the stability and the growth of the number of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$, and also the stability of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$ under some pinching conditions.

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1 Introduction and main results

Let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$, bounding a strictly convex compact set C with non-empty interior, and with a non-vanishing Gaussian curvature. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose C contains the origin. For $x \in \Sigma$, let $N_\Sigma(x)$ be the outward normal unit vector at x of Σ. We consider the problem of finding $\tau > 0$ and a C^1 curve $x : [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\begin{cases} \dot{x}(t) = JN_\Sigma(x(t)), & x(t) \in \Sigma, \forall\, t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{1.1}$$

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where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

I_n is the identity matrix in $\mathbf{R}^n$. A solution (τ, x) of the above problem is called a *closed characteristic* on Σ. Two closed characteristics (τ, x) and (σ, y) are geometrically distinct, if $x(\mathbf{R}) \neq y(\mathbf{R})$. We denote by $\mathcal{J}(\Sigma)$ and $\tilde{\mathcal{J}}(\Sigma)$ the set of all closed characteristics (τ, x) on Σ with τ being the minimal period of x and the set of all geometrically distinct ones respectively.

Let $j : \mathbf{R}^{2n} \to \mathbf{R}$ be the gauge function of Σ, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian $H_\alpha : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) := j(x)^\alpha.$$

Then $H_\alpha \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$\begin{cases} \dot{x}(t) = JH'_\alpha(x(t)), \ H_\alpha(x(t)) = 1, \ \forall\, t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{1.2}$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions (τ, x) of the problem (1.2), where τ is the minimal period of x and by $\tilde{\mathcal{J}}(\Sigma, \alpha)$ the set of all geometrically distinct solutions of (1.2). Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are in one to one correspondence with each other, similarly for $\tilde{\mathcal{J}}(\Sigma)$ and $\tilde{\mathcal{J}}(\Sigma, \alpha)$. Let $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$. We call the fundamental solution $\gamma_x : [0, \tau] \to Sp(2n)$ with $\gamma_x(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{y}(t) = JH''_\alpha(x(t))y(t), \ \forall\, t \in \mathbf{R}.$$

the *associated symplectic path* of (τ, x). The eigenvalue of $\gamma_x(\tau)$ are called *Floquet multipliers* of (τ, x). By Proposition 1.6.13 of [6], the Floquet multipliers with their multiplicities of $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ do not depend on the particular choice of the Hamiltonian function in (1.2). As usual (τ, x) is *hyperbolic*, if 1 is a double Floquet multiplier of it and all the other Floquet multipliers are not on the unit circle $\mathbf{U}$ in the complex plane $\mathbf{C}$. It is *elliptic* if all the Floquet multipliers of (τ, x), i.e., all the eigenvalues of $\gamma_x(\tau)$, are on the unit circle. We shall say that it is *strictly elliptic* if all the eigenvalues $\neq 1$ are Krein-definite. It is well known that these concepts are independent of the choice of $\alpha > 1$.

For any $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$, the m-th iteration x^m of x is defined by

$$x^m(t) = x(t - j\tau), \ for\ j\tau \leq t \leq (j+1)\tau, \ 0 \leq j \leq m - 1.$$

This is simply x itself viewed as an $m\tau$-periodic function. An effective way to distinguish different elements in $\mathcal{J}(\Sigma, \alpha)$ is to compare the Maslov-type index sequences of their iterations. For periodic solutions of any periodic Hamiltonian system, this index theory was defined by C. Conley, E. Zehnder, and Y. Long (cf. [11]). This index theory assigns to the iteration sequence x^m of each solution $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ a sequence of pairs of integers $(i(x^m), \nu(x^m))_{m \in \mathbf{N}}$ through the associated symplectic path γ_x of x.

The existence of at least one closed characteristic was first established by P. Rabinowitz [13] (for star-shaped hypersurfaces) and A. Weinstein [16] independently in 1978. In [5] of I. Ekeland in 1986 and [10] of Y. Long in 1998, for any $\Sigma \in \mathcal{H}(2n)$ the existence of infinitely many closed characteristics on Σ was proved if all the closed characteristics are hyperbolic. In [7] of 1984, N. Hingston proved that the number $N(l)$ of closed geodesics of length $\leq l$ for a Riemannian metric on a sphere satisfies $\liminf_{l\to\infty} N(l)\frac{\log l}{l} > 0$ provided that all closed geodesics are hyperbolic. In [8] of 1993, N. Hingston proved that the number $N(l)$ of closed geodesics of length $\leq l$ for any Riemannian metric on the two-sphere satisfies $\liminf_{l\to\infty} N(l)\frac{\log l}{l} > 0$. Motivated by these results, we prove the following result in this paper:

Theorem 1.1 *On every compact C^2 hypersurface in $\mathbf{R}^{2n}$ bounding a convex set with non-empty interior, if all the closed characteristics are hyperbolic, then there exist infinitely many closed characteristics. Moreover, let $N(l)$ be the number of closed characteristics of minimal actions $\leq l$, then $N(l)$ grows at least like the prime numbers. That is,* $\liminf_{l\to\infty} N(l)\frac{\log l}{l} > 0$.

Here the action of a closed characteristic (τ, x) is defined by (cf. P. 190 of [6]) $A(\tau, x) = \frac{1}{2}\int_0^\tau (Jx \cdot \dot{x})dt$. Note that $A(\tau, x)$ is a geometric quantity depending only on how many times one runs around the closed characteristic. $A(\tau, x)$ is called the *minimal action* of the closed characteristic (τ, x) if τ is the minimal period of x.

As in Definition 5.1.6 of [6], a C^2 hypersurface Σ bounding a compact convex set C, containing 0 in its interior is (r, R)-*pinched*, with $0 < r \leq R$, if:

$$|y|^2 R^{-2} \leq \frac{1}{2}(H_2''(x)y, y) \leq |y|^2 r^{-2}, \ \forall\, x \in \Sigma.$$

On the stability problem, I. Ekeland proved in P. 413 of [5] the existence of at least one strictly elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ is (r, R)-pinched with $\frac{R}{r} < \sqrt{2}$. Comparing with it, we prove a slightly stronger result by using a different method in the following:

Theorem 1.2 *Assume Σ is (r, R)-pinched with $\frac{R}{r} \leq \sqrt{2}$. Then any closed characteristic with Ekeland index zero is elliptic. It implies that Σ always carries an elliptic closed characteristic under this pinching condition.*

Remark 1.1 Note that in this theorem, when $\frac{R}{r} = \sqrt{2}$, we can only get the existence of one elliptic closed characteristic from the proof in Section 3, because -1 may be a Floquet multiplier of the closed characteristics with Ekeland index zero and of Krein-indefinite, this is the difference from the result of I. Ekeland.

In this article, let $\mathbf{N}, \mathbf{N}_0, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbf{R}^{2n}$. Denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the standard L^2-inner product and L^2-norm. For a set A, we denote by $\#A$ the number of elements in A. We define the function $E(a) = \max\{k \in \mathbf{Z} \mid k < a\}$ as (60) in P. 31 of [6].

2 Proof of Theorem 1.1

To solve the given energy problem, we consider the following fixed period problem

$$\dot{z}(t) = JH_\alpha'(z(t)), \forall\, t \in \mathbf{R}, \ z(1) = z(0). \tag{2.1}$$

Define the dual function H_α^* of H_α by $H_\alpha^*(x) = \sup\{y \cdot x - H(y) \mid y \in \mathbf{R}^{2n}\}$. For $1 < \alpha < 2$, define

$$E_\alpha = \{u \in L^{\frac{\alpha}{\alpha-1}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^{2n}) \mid \int_0^1 u\,\mathrm{d}t = 0\},$$

with the usual $L^{\frac{\alpha}{\alpha-1}}$ norm. The Clarke-Ekeland dual action functional $f_\alpha : E_\alpha \to \mathbf{R}$ is defined by

$$f_\alpha(u) = \int_0^1 \{\frac{1}{2} Ju \cdot \Pi u + H_\alpha^*(-Ju)\}\,\mathrm{d}t$$

where Πu is defined by $\frac{d}{dt}\Pi u = u$ and $\int_0^1 \Pi u\,\mathrm{d}t = 0$. Then $f_\alpha \in C^2(E_\alpha, \mathbf{R})$.

Suppose $u \in E_\alpha/\{0\}$ is a critical point of f_α. By [6], there exists $\xi_u \in \mathbf{R}^{2n}$ such that $z_u(t) = \Pi u(t) + \xi_u$ is a 1-periodic solution of the problem (2.1). Denote the Ekeland index and nullity (cf. [4, 5, 6] for details) of f_α at u by $i^E(u)$ and $\nu^E(u)$ respectively. We denote the corresponding Maslov-type indices of z_u by $(i(z_u), \nu(z_u))$. Let $h = H_\alpha(z_u(t))$ and $1/m$ be the minimal period of z_u for some $m \in \mathbf{N}$. Define

$$x_u(t) = h^{\frac{-1}{\alpha}} z_u(h^{\frac{2-\alpha}{\alpha}} t) \quad and \quad \tau = \frac{1}{m} h^{\frac{\alpha-2}{\alpha}}. \tag{2.2}$$

Then there hold $x_u(t) \in \Sigma$ for all $t \in \mathbf{R}$ and $(\tau, x_u) \in \mathcal{J}(\Sigma, \alpha)$. Note that the period 1 of z_u corresponds to the period $m\tau$ of the solution $(m\tau, x_u^m)$ of with minimal period τ.

On the other hand, every solution $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ gives rise to a sequence $\{z_m^x\}_{m\in\mathbf{N}}$ of solutions of the problem (2.1), and a sequence $\{u_m^x\}_{m\in\mathbf{N}}$ of critical points of f_α defined by

$$z_m^x(t) = (m\tau)^{\frac{-1}{2-\alpha}} x(m\tau t) \tag{2.3}$$

$$u_m^x(t) = (m\tau)^{\frac{\alpha-1}{2-\alpha}} \dot{x}(m\tau t) \tag{2.4}$$

Lemma 2.1 *(cf. Proposition 15.2.1 of [11]) For u and $z = z_u$ defined above, there hold*

$$i(z) = i^E(u) + n \quad and \quad \nu(z) = \nu^E(u). \tag{2.5}$$

Lemma 2.2 *(cf. Proposition 15.2.2 of [11]) For $z = z_u$, $x = x_u$, τ and m defined above, there hold*

$$i(z) = i(x^m) \quad and \quad \nu(z) = \nu(x^m). \tag{2.6}$$

Lemma 2.3 *(cf. Corollary 15.1.4 of [11]) It holds that*

$$i(x) \geq n, \ \forall\, (\tau, x) \in \mathcal{J}(\Sigma, \alpha). \tag{2.7}$$

Following Section V.3 of [6], denote by "ind" the S^1-action cohomology index theory for S^1-invariant subset of E_α defined in [6]. For $[f_\alpha]_c \equiv \{u \in E_\alpha \mid f_\alpha(u) \leq c\}$ define

$$c_k = \inf\{c < 0 \mid ind([f_\alpha]_c) \geq k\}. \tag{2.8}$$

Then there hold

$$-\infty < \min f_\alpha = c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots < 0, \tag{2.9}$$

$$c_k \to 0 \ \ as\ k \to +\infty, \ all\ c_k\ are\ critical\ values\ of\ f_\alpha. \tag{2.10}$$

Lemma 2.4 *(cf. Theorem 15.3.1 of [11]) For any given $k \in \mathbf{N}$, there exists $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$ such that for u_m^x defined by* (2.4)

$$f'_\alpha(u_m^x) = 0 \quad \text{and} \quad f_\alpha(u_m^x) = c_k, \tag{2.11}$$
$$i(x^m) \leq 2k - 2 + n \leq i(x^m) + \nu(x^m) - 1. \tag{2.12}$$

Lemma 2.5 *Let $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$. Then there exist positive constants a and b such that*

$$i(x) \geq a\tau - b.$$

where a and b are independent of the choice of $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$.

Proof. Since H_α is convex, then there exist a positive constant $R = R(\Sigma, \alpha)$ such that

$$|y|^2 R^{-2} \leq \frac{1}{2}(H''_\alpha(x(t))y, y), \ \forall\, x \in \Sigma, \ y \in \mathbf{R}^{2n}. \tag{2.13}$$

By the same method of the proof of Proposition V.1.8 in [6], $i^E(x) \geq 2nE[\frac{\tau}{\pi R^2}]$, where $i^E(x)$ denotes the corresponding index defined by I. Ekeland for convex Hamiltonian system in [6]. From Lemma 2.1, we obtain $i(x) = i^E(x) + n$. Hence, $i(x) \geq 2nE[\frac{\tau}{\pi R^2}] + n \geq \frac{2n\tau}{\pi R^2} - n$. Let $a = \frac{2n}{\pi R^2}$, $b = n$, our lemma follows.

Proof of Theorem 1.1. By (21) in P. 191 of [6], we only need to prove that the number $N_1(l)$ of $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ with minimal period $\tau \leq l$ grows at least like the prime numbers if all the closed characteristics are hyperbolic.

Suppose all the closed characteristics are hyperbolic. Then the inequalities (2.12) become equalities and $\nu(x^m) = 1$; we know

$$(2\mathbf{N} - 2 + n) \subset \{i(x^m) \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \textit{ with minimal period } \tau, m \in \mathbf{N}\}. \tag{2.14}$$

Using the precise iteration formulae of the Maslov-type index theory for any symplectic path which is established by Y. Long(cf. [11]), we can get $i(x^m) = m(i(x) + 1) - 1$, $\forall$ hyperbolic (τ, x), $m \in \mathbf{N}$. Thus (2.14) is equivalent to

$$(2\mathbf{N} - 1 + n) \subset \{m(i(x) + 1) \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \textit{ with minimal period } \tau\}. \tag{2.15}$$

Define Functions

$$N_1(l) = \#\{(\tau, x) \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \textit{ with minimal period } \tau \leq l\},$$

$$N_2(l) = \#\{i(x) + 1 \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \textit{ with minimal period } \tau, \ i(x) + 1 \leq l\},$$

$$N_3(l) = \#\{p \textit{ is prime} \mid n + 1 \leq p \leq l\}, \ l \in \mathbf{R}.$$

Let $P(l)$ denote the number of prime numbers less than or equal to l. Then we have $P(l) - (n - 1) \leq N_3(l) \leq P(l)$, and

$$\lim_{l\to\infty} \frac{N_3(l)\log l}{l} = \lim_{l\to\infty} \frac{P(l)\log l}{l} = 1, \tag{2.16}$$

where in the last step we used the Prime number theorem in Chapter 9 of [9]. On the other hand, by Lemma 2.5, we get

$$N_1(l) \geq N_2(al - b + 1), \tag{2.17}$$

where a, b are positive constants dependent only on α, Σ.

We continue our study in two cases:

Case 1. n is even. In this case, let $A = \{p \text{ is prime} \mid p \geq n+1\}$. Using (2.15), we obtain

$$A \subset (2\mathbf{N} - 1 + n) \subset \{m(i(x)+1) \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \text{ with minimal period } \tau,\ m \in \mathbf{N}\}.$$

Hence, for any $p \in A$, we can find (τ, x) such that

$$p = m(i(x) + 1). \tag{2.18}$$

Since p is a prime and $2 \leq n \leq i(x)$ by Lemma 2.3, (2.18) holds only when $m = 1$, i.e., $i(x) + 1 = p$. Thus

$$A \subset \{i(x) + 1 \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \text{ with minimal period } \tau\}. \tag{2.19}$$

Then

$$N_3(l) \leq N_2(l). \tag{2.20}$$

From (2.16) and (2.20), it follows that

$$\liminf_{l\to\infty} \frac{N_2(l)\log l}{l} \geq \lim_{l\to\infty} \frac{N_3(l)\log l}{l} = 1. \tag{2.21}$$

Combining (2.17) with (2.21), we obtain

$$\begin{aligned}\liminf_{l\to\infty} \frac{N_1(l)\log l}{l} &\geq \liminf_{l\to\infty} \frac{N_2(al-b+1)\log(al-b+1)}{al-b+1}\left(\frac{al-b+1}{l}\frac{\log l}{\log(al-b+1)}\right)\\ &\geq a > 0.\end{aligned} \tag{2.22}$$

Thus our theorem holds for Case 1.

Case 2. n is odd. In this case, let $A = \{2p \mid p \text{ is prime},\ 2p \geq n+1\}$. Using (2.15), we obtain

$$A \subset (2\mathbf{N} - 1 + n) \subset \{m(i(x)+1) \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \text{ with minimal period } \tau,\ m \in \mathbf{N}\}.$$

Hence, for any $2p \in A$, we can find (τ, x) such that

$$2p = m(i(x) + 1). \tag{2.23}$$

Since p is a prime and $2 \leq n \leq i(x)$ by Lemma 2.3, (2.23) holds only when $m = 1$ and $i(x) + 1 = 2p$, or $m = 2$ and $i(x) + 1 = p$. Thus, for any prime $p \geq \frac{n+1}{2}$, either $p \in B$ or $2p \in B$, where $B = \{i(x) + 1 \mid (\tau, x) \in \mathcal{J}(\Sigma, \alpha) \text{ with minimal period } \tau\}$. Then

$$N_3(l) \leq N_2(2l). \tag{2.24}$$

From (2.16) and (2.24), it follows

$$\liminf_{l\to\infty} \frac{N_2(l)\log l}{l} \ge \lim_{l\to\infty} \frac{N_3(l/2)\log(l/2)}{l/2} \frac{\log l}{2\log(l/2)} = \frac{1}{2}. \tag{2.25}$$

Combining (2.17) with (2.25), we obtain

$$\begin{aligned} \liminf_{l\to\infty} \frac{N_1(l)\log l}{l} &\ge \liminf_{l\to\infty} \frac{N_2(al-b+1)\log(al-b+1)}{al-b+1}\left(\frac{al-b+1}{l}\frac{\log l}{\log(al-b+1)}\right) \\ &\ge \frac{a}{2} > 0. \end{aligned} \tag{2.26}$$

Thus our theorem holds for Case 2.

3 Proof of Theorem 1.2

The following two lemmas will be useful in our proof of Theorem 1.2:

Lemma 3.1 *(cf. Lemma 3.2 of [15]) Let $\gamma(t), 0 \le t \le \tau$ be a symplectic path in $\mathbf{R}^{2n}$. Then we have $i(\gamma^2)-2i(\gamma) \le n$. Moreover, if $i(\gamma^2)-2i(\gamma) = n$, then $\gamma(\tau)$ can be deformed in $\Omega^0(\gamma(\tau))$(cf. Definition 1.8.5 and Theorem 1.8.10 of [11]) to M as follows:*

$$M = N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \tag{3.1}$$

with $p_- + p_0 + r = n$ and $\theta_k \in (\pi, 2\pi)$ for $1 \le k \le r$. In particular, if γ is the associated symplectic path of $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, then (τ, x) is strictly elliptic.

Note that the proof of Lemma 3.2 of [15] works also for any symplectic path γ although it was given for closed characteristics.

Lemma 3.2 *(cf. Proposition V.1.9 of [6]) Let Σ be a compact C^2 hypersurface bounding a convex set with non-empty interior. Then Σ carries at least one closed characteristic with Ekeland index zero.*

Proof of Theorem 1.2. In this section, we fix $\alpha = 2$. Let (τ, x) be a closed characteristic with Ekeland index zero, i.e., $i^E(x) = 0$. By Lemma 3.1 of [15], we obtain

$$\tau_2 = A(\tau, x) \ge \pi r^2 \tag{3.2}$$

where τ_2 is the quantity T_2 in P. 191 of [6]. On the other hand, from Proposition V.1.8 of [6], Lemmas 2.1 and 3.1, and the fact that $i^E(x) = 0$, it follows that

$$2nE[\frac{2\tau_2}{\pi R^2}] = 2nE[\frac{A(2\tau, x)}{\pi R^2}] \le i^E(x^2) = i(x^2) - n = i(x^2) - 2i(x) + n \le 2n \tag{3.3}$$

where $i^E(x^m)$ is the Ekeland index defined in [6], $i(x^m)$ is the corresponding Maslov-type index, $m \in \mathbf{N}$. We continue the proof in two steps according to the value of $\frac{R}{r}$.

Step 1. If $\frac{R}{r} < \sqrt{2}$. From (3.2), we get $1 \leq \frac{\tau_2}{\pi r^2} < \frac{2\tau_2}{\pi R^2}$, the inequalities in (3.3) become equalities. Hence, $i(x^2) - 2i(x) = n$. It follows from Lemma 3.1 that (τ, x) is strictly elliptic .

Step 2. If $\frac{R}{r} = \sqrt{2}$, when the strictly inequality in (3.2) holds, then $1 < \frac{\tau_2}{\pi r^2} = \frac{2\tau_2}{\pi R^2}$, (τ, x) is strictly elliptic from Step 1. In the following, we assume

$$\tau_2 = \pi r^2. \tag{3.4}$$

Consider the two linear Hamiltonian systems:

$$\dot{y} = 2JR^{-2}y \tag{3.5}$$

$$\dot{y} = JH_2''(x(t))y \tag{3.6}$$

and the two corresponding quadratic forms on

$$L_o^2([0,s],\mathbf{R}^{2n}) = \{v \in L^2([0,s],\mathbf{R}^{2n}) \mid \int_0^s v\,\mathrm{d}t = 0\} \tag{3.7}$$

$$Q_s^R(v,v) := \frac{1}{2}\int_0^s \{Jv \cdot \Pi v + \frac{R^2}{2}\|v\|^2\}\,\mathrm{d}t \tag{3.8}$$

$$Q_s(v,v) := \frac{1}{2}\int_0^s \{Jv \cdot \Pi v + (H_2''(x(t))^{-1}Jv, Jv)\}\,\mathrm{d}t \tag{3.9}$$

From the pinching condition, it follows that

$$Q_{2\tau_2}^R(v,v) \geq Q_{2\tau_2}(v,v),\ \forall\, v \in L_o^2([0,2\tau_2],\mathbf{R}^{2n}). \tag{3.10}$$

Denote by i_s^R and i_s the indices of Q_s^R and Q_s respectively. Denote by V the kernel of $Q_{2\tau_2}^R$. Let $u \in V$, then $Q_{2\tau_2}^R(v,u) = 0, \forall\, v \in L_o^2([0,2\tau_2],\mathbf{R}^{2n})$.
That is

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^{2\tau_2} \{(Jv,\Pi u) + \frac{R^2}{2}(v,u)\}\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{2\tau_2} \{(-v,J\Pi u) + \frac{R^2}{2}(v,u)\}\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{2\tau_2} \{(v,-J\Pi u + \frac{R^2}{2}u)\}\,\mathrm{d}t, \forall\, v \in L_o^2([0,2\tau_2],\mathbf{R}^{2n}). \end{aligned} \tag{3.11}$$

Therefore,

$$-J\Pi u(t) + \frac{R^2}{2}u(t) = c,\ \forall\, 0 \leq t \leq 2\tau_2 \tag{3.12}$$

where c is a constant. Differentiating (3.12) with respect to t yields

$$\dot{u}(t) = \frac{2}{R^2}Ju(t),\ \forall\, 0 \leq t \leq 2\tau_2. \tag{3.13}$$

By solving this equation, we obtain

$$u(t) = \exp(\frac{2}{R^2}Jt)u(0),\ \forall\, 0 \leq t \leq 2\tau_2. \tag{3.14}$$

From this we know V is a $2n$-dimensional subspace of $L_o^2([0, 2\tau_2], \mathbf{R}^{2n})$, i.e.,

$$dim V = 2n \tag{3.15}$$

Claim. *$Ju(0)$ is an eigenvector of $\gamma(\tau_2)$ for the eigenvalue -1 if $Q_{2\tau_2}(u, u) = 0$ for $u \in V$, where $\gamma(t)$ is the associated symplectic path of (τ, x).*

In fact, if $Q_{2\tau_2}(u, u) = 0$, then

$$0 = Q^R_{2\tau_2}(u, u) - Q_{2\tau_2}(u, u) = \frac{1}{2}\int_0^{2\tau_2} \{(\frac{R^2}{2}Ju - H_2''(x(t))^{-1}Ju, Ju)\}\, dt. \tag{3.16}$$

But, by the pinching condition, $\frac{R^2}{2}I_{2n} - H_2''(x(t))^{-1}$ is semi-positive definite for all $0 \le t \le 2\tau_2$, it follows from (3.14) that

$$(\frac{R^2}{2}Ju(t) - H_2''(x(t))^{-1}Ju(t), Ju(t)) = 0, \forall\, 0 \le t \le 2\tau_2. \tag{3.17}$$

Let $\frac{R^2}{2}I_{2n} - H_2''(x(t))^{-1} = P^T(t)D(t)P(t)$, where $P(t)$ is an orthogonal matrix, $P^T(t)$ is its transpose, $D(t) = diag(\lambda_1(t), \lambda_2(t), \cdots, \lambda_{2n}(t))$ is a diagonal matrix, $\lambda_i(t) \ge 0$, for any $1 \le i \le 2n$, then,

$$\begin{aligned}(\frac{R^2}{2}Ju(t) - H_2''(x(t))^{-1}Ju(t), Ju(t)) &= (P^T(t)D(t)P(t)Ju(t), Ju(t)) \\ &= (D(t)P(t)Ju(t), P(t)Ju(t)).\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we obtain $(D(t)P(t)Ju(t), P(t)Ju(t)) = 0$. Let $P(t)Ju(t) = w(t) = (w_1(t), w_2(t), \cdots, w_{2n}(t))^T$. Then

$$\sum_{i=1}^{2n} \lambda_i(t)w_i^2(t) = 0. \tag{3.19}$$

But $\lambda_i(t) \ge 0,\ \forall\ 1 \le i \le 2n$. Thus $\lambda_i(t)w_i(t) = 0,\ \forall\ 1 \le i \le 2n,\ 0 \le t \le 2\tau_2$ follows from (3.19). Hence, $D(t)P(t)Ju(t) = 0, \forall\, 0 \le t \le 2\tau_2$. Then

$$\frac{R^2}{2}Ju(t) - H_2''(x(t))^{-1}Ju(t) = P^T(t)D(t)P(t)Ju(t) = 0,\ \forall\ 0 \le t \le 2\tau_2. \tag{3.20}$$

Combining (3.13) with (3.20), we obtain

$$\begin{aligned}JH_2''(x(t))Ju(t) &= JH_2''(x(t))(\frac{2}{R^2}H_2''(x(t))^{-1}Ju(t)) \\ &= -\frac{2}{R^2}u(t) = J\dot{u}(t).\end{aligned} \tag{3.21}$$

i.e., Ju is a solution of (3.6), thus

$$Ju(t) = \gamma(t)Ju(0), \tag{3.22}$$

Combining (3.14) with (3.22), we obtain

$$(\gamma(t) - \exp(\frac{2}{R^2}Jt))Ju(0) = 0. \tag{3.23}$$

Let $t = \tau_2$, then $\frac{2}{R^2}t = \pi$. From (3.23), we get

$$0 = (\gamma(\tau_2) + I_{2n})(Ju(0)).$$

Hence, the Claim holds.

Denote by ν the dimension of the eigenspace for eigenvalue -1 of $\gamma(\tau_2)$. From (3.10), (3.15) and the above claim, we know $Q_{2\tau_2}(v, v)$ is negative-definite on a $(2n - \nu)$-dimensional subspace of V, it follows that

$$2n - \nu \leq i_{2\tau_2}. \tag{3.24}$$

From Theorem 1.8.10 of [11], there is a path $f : [0, \tau_2] \to \Omega^0(\gamma(\tau_2))$(cf. Definition 1.8.5 of [11]) such that $f(0) = \gamma(\tau_2)$ and $f(\tau_2) = N_1(-1, 1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1, -1)^{\diamond q_+} \diamond M$, where M has no eigenvalue -1. Then the following holds:

$$\nu = q_- + 2q_0 + q_+. \tag{3.25}$$

Denote by $\xi = f * \gamma$ the joint path of γ and f. By the homotopy invariance of Maslov-type index, $i(\gamma^m) = i(\xi^m)$, $\forall\, m \in \mathbf{N}$. Let ξ_1 and ξ_2 be two symplectic paths starting from the identities which satisfy $\xi_1(\tau_2) = N_1(-1, 1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1, -1)^{\diamond q_+}$, $\xi_2(\tau_2) = M$ and $i(\xi) = i(\xi_1) + i(\xi_2)$. Then for any $m \in \mathbf{N}$, $i(\xi^m) = i(\xi_1^m) + i(\xi_2^m)$ (cf. Lemma 8.1.3 of [11]). Hence

$$\begin{aligned} i_{2\tau_2} &= i^E(x^2) = i(x^2) - n = i(\gamma^2) - n \\ &= i(\xi^2) - n = i(\xi_1^2) + i(\xi_2^2) - n \\ &= i_{-1}(\xi_1) + i_1(\xi_1) + i_{-1}(\xi_2) + i_1(\xi_2) - n \\ &= i_{-1}(\xi_1) - i(\xi_1) + i_{-1}(\xi_2) - i(\xi_2) + n. \end{aligned} \tag{3.26}$$

Here we have used the Bott-type iteration formulae(cf. Theorem 9.2.1 of [11]), and Lemma 2.1. By computations on the splitting numbers(cf. Section 9.1 of [11]),

$$i_{-1}(\xi_1) - i(\xi_1) = -q_+ - q_0. \tag{3.27}$$

From the Bott-type iteration formulae(cf. Theorem 9.2.1 of [11]) and (3.24)-(3.27), we know

$$\begin{aligned} i(\xi_2^2) - 2i(\xi_2) &= i_{-1}(\xi_2) - i(\xi_2) \\ &= i_{2\tau_2} - (i_{-1}(\xi_1) - i(\xi_1)) - n \\ &\geq 2n - \nu + q_+ + q_0 - n \\ &= n - q_+ - 2q_0 - q_- + q_+ + q_0 \\ &= n - q_- - q_0 \\ &\geq n - q_- - q_0 - q_+ = \frac{1}{2} rank(M). \end{aligned} \tag{3.28}$$

It follows from Lemma 3.1 that we have $i(\xi_2^2) - 2i(\xi_2) = \frac{1}{2}rank(M)$, $q_+ = 0$, all the eigenvalues of M are on the unit circle and Krein-definite. Therefore, (τ, x) is elliptic. Here when $q_- + q_0 > 0$, the eigenvalue -1 is Krein-indefinite. Furthermore, it follows from Lemma 3.2 that there always exists one elliptic closed characteristic. The proof is complete.

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References

[1] G. Dell'Antoio, Variational Calculus and Stability of Periodic Solutions of a Class of Hamiltonian Systems, SISSA Ref. 185/92/FM, October, 1992.

[2] G. Dell'Antoio, B. D'Onofrio, and I. Ekeland, *Les systèmes hamiltoniens convexes et pairs ne sont pas ergodiques en général*, C. R. Acad. Sci. Paris Sér. I **315** (1992), 1413-1415.

[3] B. D'Onofrio and I. Ekeland, *Hamiltonian systems with elliptic periodic orbits*, Nonlinear Anal. **14** (1990), 11-21.

[4] I. Ekeland, *Une théorie de Morse pour les systèmes hamiltoniens convexes*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 19-78.

[5] I. Ekeland, *An index theory for periodic solutions of convex Hamiltonian systems*, Proc. Symp. in Pure Math. **45** (1986), 395-423.

[6] I. Ekeland, Convexity Methods in Hamiltonian Mechanics. Springer-Verlag. Berlin, 1990.

[7] N. Hingston, *Equivariant Morse theory and closed geodesics*, J. Differential Geom. **19** (1984), 85-116.

[8] N. Hingston, *On the growth of the number of closed geodesics on the two-sphere*, Internat. Math. Res. Notices 1993, no. 9, 253-262. MR 94m: 58044.

[9] Hua Loo Keng, Introduction to Number Theory, Springer, Berlin, 1982.

[10] Y. Long, *Hyperbolic closed characteristics on compact convex smooth hypersurfaces in* $\mathbf{R}^{2n}$. J. Diff. Equa. **150** (1998), 227-249.

[11] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Mathematics. 207, Birkhäuser. Basel. 2002.

[12] Y. Long and C. Zhu, *Closed characteristics on compact convex hypersurfaces in* $\mathbf{R}^{2n}$. Ann. of Math. **155** (2002), 317-368.

[13] P. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157-184.

[14] W. Wang, X. Hu and Y. Long, *Resonance identity, stability and multiplicity of closed characteristics on compact convex hypersurfaces*. Duke Math. J. **139** (2007), 411-462.

[15] W. Wang, *Stability of closed characteristics on compact hypersurfaces in* $\mathbf{R}^{2n}$ *under a pinching condition*. Advanced Nonlinear Studies.**10** (2010), 263-272.

[16] A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. of Math. **108** (1978), 507-518.