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On Radial Solutions of the Schrödinger Type Equation

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Abstract

We establish compact embeddings of the radial Sobolev space $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ into weighted Lebesgue spaces $L^q_w(\mathbb{R}^N)$ under various assumptions on the weight function w. We use these, along with a variety of variational techniques, to prove the existence of nontrivial nonnegative solutions for a class of nonlinear Schrödinger type equations.

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1 Introduction

In recent years there has been considerable interest in nonlinear elliptic equations involving singular potentials of Hardy type. In particular, the existence of positive solutions of these equations both in $\mathbb{R}^N$ and on bounded domains has been extensively investigated. We refer to the papers [5], [12] and [9] in which bibliographical references can be found. The aim of this paper is to study the existence of nontrivial and nonnegative solutions of the Schrödinger type equation

$$-\Delta_p u + \lambda |u|^{p-2}u = |x|^{-\alpha}|u|^{q-2}u + h(|x|)f(|x|, u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian, $1 < p < N$, $0 < \alpha < p$ and $p < q < N$. We also consider the case $1 < q < p < N$. The assumptions on the nonlinearity f and the weight function h will be formulated later. The existence of solutions of equation (1.1) has been studied in [12], but the techniques developed in the current paper mean that we can establish the existence of solutions of (1.1) in a number of cases not covered by those authors. We also mention that a variety of embeddings theorems for weighted radial Sobolev spaces into weighted Lebesgue spaces have been established in the paper [19]. However, the embedding results presented in this paper are new, and are not covered by those in [19], nor do they follow from those authors' methods. In the recent papers [7] and [8] the existence of solutions for (1.1) has been established in the special case $p = 2$ and $f \equiv 0$ with the coefficient $|x|^{-\alpha}$ replaced by a radial function behaving as $|x|^{-\alpha}$ close to 0 and infinity. Equations of this type arise in the study of standing wave solutions of a nonlinear time depending Schrödinger equation (see [6]).

By $H^{1,p}(\mathbb{R}^N)$, $1 \le p < \infty$, we denote the standard Sobolev space equipped with norm

$$\|u\|^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx.$$

We denote by $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ the subspace of radial functions. It is known that $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^s(\mathbb{R}^N)$ for $p < s < p^*$, where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. However, this is not true for $s = p$ and $s = p^*$ (see [3]).

Throughout this paper, in a given Banach space X, we denote strong convergence by " $\to$ " and weak convergence by " $\rightharpoonup$ ". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p < \infty$, are denoted by $\|\cdot\|_{L^p}$.

The paper is organized as follows. In Section 2 we prove embedding theorems for $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ into weighted Lebesgue spaces. Sections 3, 4, 5 and 6 are devoted to the existence results for equation (1.1) under various assumptions on the weight function h. In particular, in Section 3 we consider equation (1.1) in the case that the nonlinearity f depends only on u. This is extended in Section 4 to the case in which the nonlinearity f depends on x and u. We point out that the weight function h in these two sections is singular at 0 and remains bounded for large $|x|$. In Section 5 we also allow the weight function to be unbounded for large x. In Section 6 we consider the concave-convex case, that is, $1 < q < p < N$. In this situation we prove the existence of at least two distinct solutions.

2 Compact embeddings

As in the paper [12] we introduce the following class of weight functions for a fixed $\beta > 0$:

$$W_\beta = \{w \in L^\infty(\mathbb{R}^N \setminus B(0,R)) \text{ for every } R > 0, w \ge 0 \text{ on } \mathbb{R}^N, \lim_{|x|\to 0} |x|^\beta w(x) = 0\}.$$

For $w \in W_\beta$ we denote by $L^s_w(\mathbb{R}^N)$, $1 \le s < \infty$, the weighted Lebesgue space

$$L^s_w(\mathbb{R}^N) = \{u \in L^1_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^s w(x)\, dx < \infty\}$$

equipped with norm

$$\|u\|_{s,w} = \left(\int_{\mathbb{R}^N} |u(x)|^s w(x)\, dx\right)^{\frac{1}{s}}.$$

If $w(x) = |x|^{-\alpha}$, we use the notation $L^p_\alpha(\mathbb{R}^N)$.

Lemma 2.1 *Let* $0 < \beta < p$, $1 < p < N$ *and* $w \in W_\beta$. *Then the embedding of* $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ *into* $L^s_w(\mathbb{R}^N)$ *is compact for* $p < s \le p^*_\beta := \frac{p(N-\beta)}{N-p}$.

Proof. For $p < s < p^*_\beta$ this follows from [12]. This can also be deduced from Theorem 1 in [19]. We sketch the proof of this case in order to show how the constant p^*_β has been obtained. Let $\{u_n\} \subset H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ be a bounded sequence. After passing to a suitable subsequence we may assume that $u_n \rightharpoonup u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and $u_n \to u$ in $L^s(\mathbb{R}^N)$. Given $\epsilon > 0$ we choose $\delta > 0$ such that $w(x) \le \epsilon |x|^{-\beta}$ for $0 < |x| < \delta$. We then write

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n - u|^s w(x)\,dx &\le \epsilon \int_{B(0,\delta)} |u_n - u|^{s-\beta} \frac{|u_n - u|^\beta}{|x|^\beta}\,dx \\ &+ \|w\|_{L^\infty(\mathbb{R}^N\setminus B(0,\delta))} \int_{\mathbb{R}^N\setminus B(0,\delta)} |u_n - u|^s\,dx.\end{aligned}$$

Since the embedding of $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$ is compact, the second integral on the right-hand side of this inequality converges to 0 as $n \to \infty$. The first integral, denoted by I, can be estimated in the following way:

$$I \le \left(\int_{B(0,\delta)} \frac{|u_n - u|^p}{|x|^p}\,dx\right)^{\frac{\beta}{p}} \left(\int_{B(0,\delta)} |u_n - u|^{\frac{(s-\beta)}{(p-\beta)}p}\,dx\right)^{\frac{p-\beta}{p}}.$$

Since $s < p^*_\beta$ one can easily check that $\frac{(s-\beta)p}{p-\beta} < p^* := \frac{Np}{N-p}$ (the Sobolev critical exponent). Therefore applying the Hölder, Hardy and Sobolev inequalities we can show that I is bounded independently of n and the result follows in this case. If $s = p^*_\beta$ we split the corresponding integral in a similar way:

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n - u|^{p^*_\beta} w(x)\,dx &\le \epsilon \int_{B(0,\delta)} |u_n - u|^{p^*_\beta} |x|^{-\beta}\,dx \\ &+ \|w\|_{L^\infty(\mathbb{R}^N\setminus B(0,\delta))} \int_{\mathbb{R}^N\setminus B(0,\delta)} |u_n - u|^{p^*_\beta}\,dx = \epsilon I^1_n + I^2_n.\end{aligned}$$

Since $\beta > 0$, we have $p^*_\beta < p^*$ and consequently $I^2_n \to 0$ as $n \to \infty$. We now use the Hölder inequality to estimate I^1_n:

$$\begin{aligned}I^1_n &= \int_{B(0,\delta)} |u_n - u|^{p^*_\beta - \beta} |u_n - u|^\beta |x|^{-\beta}\,dx \\ &\le \left(\int_{B(0,\delta)} |u_n - u|^p |x|^{-p}\,dx\right)^{\frac{\beta}{p}} \left(\int_{B(0,\delta)} |u_n - u|^{p^*}\,dx\right)^{\frac{p-\beta}{p}}.\end{aligned}$$

The Hardy and Sobolev inequalities yield the boundedness of I^1_n which completes the proof. ■

Inspection of the proof of Lemma 2.1 shows that in the case $p < s < p^*_\beta$ the assumption $\lim_{|x|\to 0} |x|^\beta w(x) = 0$ can be replaced by $\limsup_{|x|\to 0} |x|^\beta w(x) < \infty$.

The case $s = p$ will be treated later in Lemma 2.3. We now consider the case $\beta = 0$. This case is well known. For the sake of completeness we provide a proof of this result in Lemma 2.2 below.

We need the following estimate for functions in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ (see [11], [12], [14], [17] and [18]):

$$|u(x)| \leq \begin{cases} (\frac{p-1}{N-p})^{\frac{1}{p'}} \omega_N^{-\frac{1}{p}} \|Du\|_p |x|^{-\frac{N-p}{p}} & \text{if } 0 < |x| \leq R \\ (\max(p-1,1))^{\frac{1}{p}} \omega_N^{-\frac{1}{p}} \|u\|_{H^{1,p}} |x|^{-\frac{N-1}{p}} & \text{if } |x| > R \end{cases} \tag{2.1}$$

Lemma 2.2 *Let $w \in W_\beta$ with $\beta = 0$. Then $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^{p^*}_w(\mathbb{R}^N)$.*

Proof. If $\beta = 0$, then $p^*_\beta = p^*$. By the definition of W_β, $\lim_{|x|\to 0} w(x) = 0$. Given $\epsilon > 0$ we choose $\delta > 0$ so that $w(x) \leq \epsilon$ for $x \in B(0,\delta)$. Let $\{u_n\}$ be a sequence as in Lemma 2.1. By (2.1) we have

$$|u_n - u|^{p^*} \leq C|x|^{-\frac{(N-1)N}{N-p}} \quad \text{for } |x| > \delta. \tag{2.2}$$

We now have

$$\int_{\mathbb{R}^N} |u_n - u|^{p^*} w\,dx \leq \epsilon \int_{B(0,\delta)} |u_n - u|^{p^*}\,dx + \int_{\mathbb{R}^N\setminus B(0,\delta)} |u_n - u|^{p^*} w\,dx = \epsilon I^1_n + I^2_n. \tag{2.3}$$

Since $|x|^{-\frac{(N-1)N}{N-p}} \in L^1(\mathbb{R}^N \setminus B(0,\delta))$ and $w \in L^\infty(\mathbb{R}^N \setminus B(0,\delta))$, by (2.2) and the Lebesgue dominated convergence theorem we deduce that $\lim_{n\to\infty} I^2_n = 0$. Since $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^N)$ the result follow from (2.3). ■

To obtain a compact embedding in the case $s = p$ we change assumptions on w.

Lemma 2.3 *Suppose that $w(x) = w(|x|)$ is measurable , $w \geq 0, \not\equiv 0$ and*

(i) $\int_\delta^\infty w(s)\,ds < \infty$ *and*

(ii) $\int_0^\delta w(s)s^{p-1}\,ds < \infty$ *for some $\delta > 0$.*

Then $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^p_w(\mathbb{R}^N)$.

Proof. Let $\{u_n\}$ be a sequence as in Lemma 2.1. Then

$$\int_{\mathbb{R}^N} |u_n - u|^p w\,dx \leq \int_{B(0,\delta)} |u_n - u|^p w\,dx + \int_{\mathbb{R}^N\setminus B(0,\delta)} |u_n - u|^p w\,dx = I^1_n + I^2_n.$$

By (2.1) we have

$$|u_n - u|^p w \leq C|x|^{-(N-p)} w \quad \text{for } |x| < \delta.$$

Since

$$\int_{B(0,\delta)} |x|^{-(N-p)} w(|x|)\,dx = \omega_N \int_0^\delta w(s)s^{p-1}\,ds < \infty,$$

by the Lebesgue dominated convergence theorem $\lim_{n\to\infty} I^1_n = 0$. Similarly for $|x| > \delta$

$$|u_n - u|^p w \leq C|x|^{-(N-1)} w$$

and

$$\int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^{-(N-1)} w(|x|)\,dx = \omega_N \int_\delta^\infty w(s)\,ds < \infty,$$

so $\lim_{n\to\infty} I^2_n = 0$. ■

Remark 2.1 Condition (ii) of Lemma 2.3 is obviously satisfied if

$$\lim_{|x|\to 0} |x|^{\beta} w(|x|) = 0 \text{ with } 0 \le \beta < p.$$

Remark 2.2 Lemma 2.3 continues to hold with (i) replaced by

(i') $w \in L^{\infty}_{\text{loc}}(\mathbb{R}^N - \{0\})$ and $\lim_{|x|\to\infty} w(|x|) = 0$.

In this situation the integral I_n^2 can be estimated in the following way

$$I_n^2 = \int_{\delta \le |x| \le R} |u_n - u|^p \, dx + \epsilon \int_{|x|\ge R} |u_n - u|^p \, dx,$$

where $R > \delta$ is chosen so that $w(|x|) \le \epsilon$ for $|x| \ge R$. Since $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ the first integral tends to 0 as $n \to \infty$.

Remark 2.3 Lemma 2.3 is related to the second assertion of Theorem 1 in [19], where (i) is replaced by $w(x) = O(|x|^b)$ with $b < 0$ for large $|x|$, and (ii) is replaced by $w(x) = O(|x|^{-\beta})$ as $|x| \to 0$, with $\beta < b$.

We now consider weight functions that are unbounded at infinity.

Lemma 2.4 *Let* $0 \le \beta < \frac{N(p-1)}{N-1}$ *and* $\frac{Np}{N-1} < q \le p^*_\beta$. *Suppose that* $w \ge 0$ *on* $\mathbb{R}^N$ *and* $w \in L^{\infty}_{\text{loc}}(\mathbb{R}^N - \{0\})$*, and moreover* w *satisfies the following two conditions*

(a) $\lim_{|x|\to 0} |x|^{\beta} w(x) = 0$ *and*

(b) $w(x) \le C|x|^r$ *for* $|x| \ge R$ *(for large* R*) with* $0 < r < \frac{N-1}{p} q - N$.

Then $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ *is compactly embedded into* $L^q_w(\mathbb{R}^N)$.

Proof. First, we point out that the inequality $0 \le \beta < \frac{N(p-1)}{N-1}$ guarantees that $\frac{Np}{N-1} < p^*_\beta$; so it makes sense to consider the exponent q satisfying $\frac{Np}{N-1} < q \le p^*_\beta$. Let $\{u_n\}$ be a sequence as in Lemma 2.1. Given $\epsilon > 0$ we choose $\delta > 0$ and $R > 0$ such that $w(x) \le \epsilon |x|^{-\beta}$ for $|x| \le \delta$ and $R^{N+r-\frac{N-1}{p}q} \le \epsilon$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n - u|^q w \, dx &\le \epsilon \int_{|x|<\delta} |x|^{-\beta} |u_n - u|^q \, dx + \int_{\delta < |x| < R} |u_n - u|^q w \, dx \\ &+ \int_{|x|\ge R} |u_n - u|^q w \, dx = \epsilon I_n^1 + I_n^2 + I_n^3. \end{aligned}$$

As in Lemma 2.1 we can show that I_n^1 is bounded. It is clear that $\lim_{n\to\infty} I_n^2 = 0$. Using (2.1) we now estimate I_n^3:

$$\begin{aligned} I_n^3 &\le C \int_{|x|\ge R} |x|^{r-\frac{N-1}{p}q} \, dx = C\omega_N \int_R^{\infty} s^{N-1+r-\frac{N-1}{p}q} \, ds \\ &\le \frac{\epsilon C \omega_N}{\frac{N-1}{p} q - N - r}. \end{aligned}$$

Hence $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^q w\,dx=0$. ■

It is clear from the proof of the above lemma that in the case $\frac{Np}{p-1}<q<p^*_\beta$ assumption (a) can be replaced by $\limsup_{|x|\to 0}|x|^\beta w(x)<\infty$.

Remark 2.4 If $\beta=0$, then $p^*_\beta=p^*$. Taking $q=p^*$ in Lemma 2.4 we obtain an improvement of Lemma 2.2. In this case we can allow w to satisfy the inequality $w(x)\le C|x|^r$ with $0\le r<N\frac{N-1}{N-p}-N=N\frac{p-1}{N-p}$.

Remark 2.5 If $p<q<p^*_\alpha$ and $0\le\alpha<p$, then $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^q_\alpha(\mathbb{R}^N)$. If $0<\alpha<p$ this continues to hold for $q=p$.

We now establish a compact embedding of $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^q_\beta(\mathbb{R}^N)$ with $q<p$. In general, the space $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ cannot be embedded into $L^q(\mathbb{R}^N)$ with $q<p$. For example the function $u(x):=\frac{1}{1+|x|^2}$ for $x\in\mathbb{R}^4$ belongs to $H^{1,2+\delta}_{\mathrm{rad}}(\mathbb{R}^4)$ for every $\delta>0$ but $u\notin L^2(\mathbb{R}^4)$. On the other hand $u\in L^2_\beta(\mathbb{R}^4)$ for every $0<\beta<4$. In Lemma 2.5 we formulate conditions on q,p, with $q<p$, and β allowing us to embed $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^q_\beta(\mathbb{R}^N)$.

Lemma 2.5 *Suppose that* $1<q<p$ *and* $N-q-q\frac{N-1}{p}<\kappa<\frac{N(p-q)}{p}$. *Then* $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ *is compactly embedded into* $L^q_{q+\kappa}(\mathbb{R}^N)$.

Proof. First we notice that inequality $N-q-q\frac{N-1}{p}<\frac{N(p-q)}{p}$ holds for $q<p$. Hence the exponent κ is well defined. Let $\{u_n\}\subset H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ be a sequence as in the proof of Lemma 2.1. Then for $\delta>0$ we have

$$\int_{\mathbb{R}^N}\frac{|u_n-u|^q}{|x|^{q+\kappa}}\,dx=\int_{B(0,\delta)}\frac{|u_n-u|^q}{|x|^{q+\kappa}}\,dx+\int_{\mathbb{R}^N\setminus B(0,\delta)}\frac{|u_n-u|^q}{|x|^{q+\kappa}}\,dx:=I_{1n}+I_{2n}.$$

By Hölder's inequality we have the following estimate for I_{1n}:

$$\begin{aligned}I_{1n}&\le\left(\int_{B(0,\delta)}\frac{|u_n-u|^p}{|x|^p}\,dx\right)^{\frac{q}{p}}\left(\int_{B(0,\delta)}\frac{dx}{|x|^{\kappa\frac{p}{p-q}}}\,dx\right)^{\frac{p-q}{p}}\\&\le\left(\int_{\mathbb{R}^N}\frac{|u_n-u|^p}{|x|^p}\,dx\right)^{\frac{q}{p}}\left(\omega_N\int_0^\delta r^{N-1-\frac{p\kappa}{p-q}}\,dr\right)^{\frac{p-q}{p}}.\end{aligned}$$

By the Hardy inequality the integrals $\int_{\mathbb{R}^N}\frac{|u_n-u|^p}{|x|^p}\,dx$ are bounded independently of n. Therefore, we can choose δ small enough so that $I_{1n}\le\epsilon$. To estimate I_{2n} we use inequality (2.1) and observe that $N-q-\kappa-\frac{N-1}{p}q<0$, to obtain

$$I_{2n}\le C\int_\delta^\infty r^{N-1-q-\kappa-\frac{(N-1)}{p}q}\,dr<\infty,$$

where $C>0$ is a constant independent of n. It follows from the Lebesgue dominated convergence theorem that $I_{2n}\to 0$ as $n\to\infty$. ■

We now observe that if $q<\frac{Np}{N+p-1}$, then $N-q-q\frac{N-1}{p}>0$ while for $p>q\ge\frac{Np}{N+p-1}$ we have $N-q-q\frac{N-1}{p}\le 0$. Hence κ is allowed to be negative. However, we always have $\kappa+q>N-q\frac{N-1}{p}>0$ since $q<p$.

3 Existence of a solution in the case $f(|x|, u) = f(u)$

In this section we establish the existence of a solution of equation (1.1) in the case where the nonlinearity depends only on u, that is,

$$-\Delta_p u + \lambda |u|^{p-2} u = |x|^{-\alpha} |u|^{q-2} u + h(|x|) f(u) \text{ in } \mathbb{R}^N, \tag{3.1}$$

where $0 < \alpha < p < q \le p_\alpha^*$ and $\lambda > 0$ is a parameter. It is assumed that $f : \mathbb{R}_0^+ \to \mathbb{R}$ and $h : \mathbb{R}^+ \to \mathbb{R}$ are continuous. Here we use notations $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$. Since we are interested in positive solutions, f is extended by 0 for $x \in (-\infty, 0)$. It follows from the assumption (A_2) below that $f(0) = 0$.

Throughout this section it is assumed that:

(A_1) $h \in W_\beta \cap L^1(\mathbb{R}^N \setminus B(0, \delta))$ for some $\delta > 0$ and $0 < \beta < p$;

(A_2) There exist constants $a > 0$, $b > 0$, $s \in (p, p_\beta^*]$ such that

$$|f(u)| \le a u^{p-1} + b u^{s-1} \quad \text{for } u \ge 0;$$

(A_3) $\lim_{u \to 0+} u^{-p} F(u) = 0$, where $F(u) = \int_0^u f(s)\, ds$;

(A_4) There exists $\frac{1}{q} < \sigma < \frac{1}{p}$ such that

$$\sigma u f(u) - F(u) \ge g(u) \quad \text{for } u \ge 0,$$

where $g(u)$ is a continuous and bounded function on $[0, \infty)$;

(A_5) $F(u) > 0$ for $u \in (0, \infty)$.

As an example of a function f satisfying $(A_2), \ldots, (A_5)$ we consider $f(u) = a e^{-u} u^{r-1} + b u^{s-1}$ for $u \ge 0$ and $f(u) = 0$ for $u < 0$, where $p < r < s \le p_\beta^*$ and $a > 0$ and $b > 0$ are constants. It is easy to see that f satisfies (A_2), (A_3) and (A_5). To verify (A_4) we take $\max(\frac{1}{q}, \frac{1}{s}) < \sigma < \frac{1}{p}$. Then

$$\begin{aligned} \sigma u f(u) - F(u) &= \sigma(a u^r e^{-u} + b u^s) - a \int_0^u e^{-t} t^{r-1}\, dt - \frac{b}{s} u^s \\ &\ge \sigma a u^r e^{-u} - \frac{a}{r} e^{-u} u^r - \frac{a}{r} \int_0^u e^{-t} t^r\, dt := g(u). \end{aligned}$$

It is clear that $g(u)$ is a bounded function on $[0, \infty)$. We point out that g may change sign. In particular, if $\sigma < \frac{1}{r}$, then g is negative. Hence condition (A_4) is more general than the Ambrosetti - Rabinowitz condition [1].

A solution will be obtained as a critical point of mountain-pass type of a functional J defined by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|Du|^p + \lambda |u|^p)\, dx - \frac{1}{q} \int_{\mathbb{R}^N} |x|^{-\alpha} |u|^q\, dx - \int_{\mathbb{R}^N} h(|x|) F(u)\, dx.$$

It is easy to check that J is a C^1-functional.

Theorem 3.1 *Let $0 < \alpha < p < q \leq p^*_\alpha$. We assume that (A_1), ...,(A_5) hold. Then equation (3.1) has a nontrivial nonnegative solution.*

Proof. We begin by checking that J has a mountain-pass geometry. We have

$$J(u) \geq \frac{\min(1,\lambda)}{p}\|u\|^p - \frac{1}{q}\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^q\,dx - \int_{\mathbb{R}^N} h(|x|)F(u)\,du.$$

It follows from (A_2) and (A_3) that for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$F(u) \leq \epsilon|u|^p + C_\epsilon|u|^s \text{ for } u \in \mathbb{R}.$$

Assumption (A_1) implies that $h \in L^1(\mathbb{R}^N)$. Indeed, we have

$$\int_0^\delta h(s)\,ds \leq C\int_0^\delta s^{N-\beta-1}\,dx = C\frac{\delta^{N-\beta}}{N-\beta} < \infty$$

for some constant $C > 0$. So conditions (i) and (ii) of Lemma 2.3 are satisfied. Then by Lemmas 2.1 and 2.3 we have

$$\int_{\mathbb{R}^N} h(|x|)F(u)\,dx \leq C(\epsilon\|u\|^p + \|u\|^s) \text{ for } u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N).$$

Since $0 < \alpha < p$, we have by Remark 2.5 that

$$\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^q\,dx \leq C\|u\|^q \quad \text{for all } u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\,,$$

for some constant $C > 0$. Hence

$$J(u) \geq \frac{\min(1,\lambda)}{p}\|u\|^p - \frac{C}{q}\|u\|^q - C(\epsilon\|u\|^p + \|u\|^s).$$

We now choose $\epsilon > 0$ so that $\frac{\min(1,\lambda)}{p} - C\epsilon > 0$. Since $q, s > p$, we can find constants $\rho > 0$ and $\kappa > 0$ such that

$$J(u) \geq \kappa \text{ for all } u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \text{ with } \|u\| = \rho.$$

Let $\phi \in C^1(\mathbb{R}^N)$ be a radial function with compact support, $\phi \geq 0$ and $\phi \not\equiv 0$. Then by (A_5) we have

$$J(t\phi) \leq \frac{t^p}{p}\int_{\mathbb{R}^N}(|\nabla\phi|^p + \lambda|\phi|^p)\,dx - \frac{t^q}{q}\int_{\mathbb{R}^N}|x|^{-\alpha}|\phi|^q\,dx < 0$$

for $t > 0$ sufficiently large. Letting $\psi = t\phi$, with $t > 0$ large, we have $\|\psi\| > \rho$ and $J(\psi) < 0$. We now check that J satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. Let $\{u_n\} \subset H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ be a sequence satisfying

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ in } H^{-1,p'}_{\mathrm{rad}}(\mathbb{R}^N),\ p' = \frac{p}{p-1}. \tag{3.2}$$

First we show that $\{u_n\}$ is bounded in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Let σ be as in assumption (A_4). Then

$$\begin{aligned} J(u_n) &- \sigma\langle J'(u_n), u_n\rangle \geq (\frac{1}{p} - \sigma)\min(1,\lambda)\|u_n\|^p \\ &+ (\sigma - \frac{1}{q})\|u\|^q_{q,\alpha} + \int_{\mathbb{R}^N}(\sigma u_n f(u_n) - F(u_n))h(|x|)\,dx \\ &\geq (\frac{1}{p} - \sigma)\min(1,\lambda)\|u_n\|^p + \int_{\mathbb{R}^N} h(|x|)g(u_n)\,dx. \end{aligned} \tag{3.3}$$

Since $h \in L^1(\mathbb{R}^N)$, the last integral is bounded independently of n. Therefore inequality (3.3) yields the existence of constant $C_1 > 0$, independent of n, such that

$$C_1(1 + \|u_n\|) \geq (\frac{1}{p} - \sigma) \min(1, \lambda)\|u_n\|^p.$$

This inequality shows that $\{u_n\}$ is bounded in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$. We can assume that $u_n \rightharpoonup u$ in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$, $u_n \to u$ in $L^p_h(\mathbb{R}^N)$, $L^s_h(\mathbb{R}^N)$ and $L^q_\alpha(\mathbb{R}^N)$. To show that $u_n \to u$ in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$ we write

$$\begin{aligned} \int_{\mathbb{R}^N} \langle |\nabla u_n|^{p-2}\nabla u_n &- |\nabla u|^{p-2}\nabla u, u_n - u\rangle\, dx \qquad (3.4)\\ &+ \lambda \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, dx\\ &= \int_{\mathbb{R}^N} |x|^{-\alpha}(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\, dx\\ &+ \int_{\mathbb{R}^N} h(|x|)(f(u_n) - f(u))(u_n - u)\, dx. \end{aligned}$$

Both integrals on the right-hand side converge to 0 as $n \to \infty$. (For the last one we use Lemma 2.5 from [12] and the Lebesgue convergence theorem.) To the left-hand side of (3.4) we apply the following inequality (see [10], [16]): for all $\xi, \zeta \in \mathbb{R}^N$ we have

$$|\xi - \zeta|^p \leq \begin{cases} c(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) & \text{for } p \geq 2,\\ c\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle^{\frac{p}{2}}(|\xi|^p + |\zeta|^p)^{\frac{2-p}{2}} & \text{for } 1 < p < 2. \end{cases} \qquad (3.5)$$

This yields $u_n \to u$ in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$. The fact that u can be taken nonnegative follows from Theorem 10 in [2]. ∎

Remark 3.1 Assumption (A_4) was used to show that a $(PS)_c$ sequence is bounded. Inspection of the proof of Theorem 3.1 shows that this theorem continues to hold if in assumption (A_4) the boundedness of g is replaced by

(A_4') $\lim_{u\to\infty} \frac{g(u)}{u^{r_1}} = c$ where $c \in \mathbb{R}$ and $1 < r_1 < p$.

In this situation inequality (3.3) takes the form

$$\begin{aligned} J(u_n) &- \sigma\langle J'(u_n), u_n\rangle \geq (\frac{1}{p} - \sigma) \min(1, \lambda)\|u_n\|^p - \int_{\mathbb{R}^N} h(|x|)g(u_n)\, dx\\ &\geq (\frac{1}{p} - \sigma) \min(1, \lambda)\|u_n\|^p - \int_{|u_n|\leq R} h(|x|)g(u_n)\, dx\\ &- (|c| + 1) \int_{|u_n|\geq R} h(|x|)|u_n|^{r_1}\, dx, \end{aligned}$$

where $R > 0$ is chosen so that $|g(u)| \leq (|c| + 1)|u|^{r_1}$ for $|u| \geq R$. The first integral on the right hand side of this inequality is bounded independently of n because $h(|x|) \in L^1(\mathbb{R}^N)$. We now estimate the integral over $\{|u_n| \geq R\}$. To estimate the second integral we fix $R_1 > 0$ and split the integration

$$\int_{|u_n|\geq R} h(|x|)|u_n)|^{r_1}\,dx \leq \int_{B(0,R_1)} h(|x|)|u_n|^{r_1}\,dx + \int_{\mathbb{R}^N\setminus B(0,R_1)} h(|x|)|u_n|^{r_1}\,dx = J_{n,1} + J_{n,2}.$$

Since $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N \setminus B(0,R_1))$ for each $R_1 > 0$, by Young's inequality, we have for small $\epsilon > 0$

$$J_{n,2} \leq \epsilon \int_{\mathbb{R}^N} |u_n|^p\,dx + C(\epsilon) \int_{\mathbb{R}^N\setminus B(0,R_1)} h(|x|)^{\frac{p}{p-r_1}}\,dx,$$

where $C(\epsilon) > 0$ is a constant. To estimate $J_{n,1}$ we observe that inequality $N - \frac{N-p}{p} r_1 > \beta$ yields $p\frac{N-\beta}{N-p} > p > r_1$. So by (2.1) we get

$$J_{n,1} \leq C\|u_n\|^{r_1} \int_0^{R_1} s^{N-1-\beta-\frac{N-p}{p}r_1}\,ds = C_1\|u_n\|^{r_1}.$$

Obviously this allows us to show that $\{u_n\}$ is bounded in $H^{1,p}_{\rm rad}(\mathbb{R}^N)$.

We now give an example of a nonlinearity satisfying conditions (A_1), (A_2), (A_3), (A'_4) and (A_5):

$$f(u) = u^{s-1} + \frac{u^{r-1}}{(1+u)^d} \quad \text{for } u \in [0,\infty),$$

where $q < s < r \leq p^*_\alpha$ and $r_1 := r - d$ with $0 < r - d < p$. Indeed, we have

$$f(u) \leq u^{p-1} + u^{r-1} \quad \text{for } u \geq 0.$$

Thus condition (A_2) is satisfied. Conditions (A_3) and (A_5) are obviously satisfied. Finally, let $\frac{1}{s} < \sigma < \frac{1}{p}$. Then

$$\sigma f(u)u - F(u) \geq -\frac{d}{r}\int_0^u \frac{t^r}{(1+t)^{d+1}}\,dt \geq Cu^{r-d},$$

where $C < 0$ and (A_4) holds.

Remark 3.2 If $q = p$, we consider the equation

$$-\Delta_p u + \lambda|u|^{p-2}u = \mu|x|^{-\alpha}|u|^{-p}u + h(|x|)f(u) \text{ in } \mathbb{R}^N,$$

where $\mu \in \mathbb{R}$ is a parameter. The mountain-pass structure is valid for μ small. In this case we use a compact embedding of $H^{1,p}_{\rm rad}(\mathbb{R}^N)$ into $L^p_\alpha(\mathbb{R}^N)$. The case $q < p$, that is, an equation with a concave - convex nonlinearity, will be discussed in Section 6.

Remark 3.3 If $p < s < q < p^*$, then assumption (A_5) is not needed. In this situation $F(u) = O(|u|^s)$ for large $|u|$. Hence, we can find a radial function ϕ of class C^1 with compact support such that $J(\phi) < 0$ and $\|\phi\| \geq \rho$.

Remark 3.4 Theorem 3.1 remains true if assumptions (A_1) and (A_4) are replaced by

(A_1') $h \in W_\beta$ and $\lim_{|x|\to\infty} h(|x|) = 0$,

(A_4') There exists $\frac{1}{q} < \sigma < \frac{1}{p}$ such that

$$\sigma u f(u) - F(u) \geq 0 \quad \text{for} \quad u \geq 0,$$

respectively.

4 Existence of a solution for problem (1.1)

In this section we consider equation (1.1), where the nonlinearity f depends on x and u. We assume that $0 < \alpha < p < N$, and $p < q < p^*_\alpha$. It is assumed that the function h satisfies

(H) $h \in W_\beta \cap L^1(\mathbb{R}^N \setminus B(0,\delta))$ for every $\delta > 0$, where $0 < \beta < p$.

Since

$$\int_0^\delta h(s)s^{p-1}\,ds \leq C\int_0^\delta s^{p-\beta-1}\,ds = C\frac{\delta^{p-\beta}}{p-\beta} < \infty,$$

both Lemmas 2.1, 2.3 are applicable to $h(|x|)$. Moreover, we assume that

(F_1) $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, $f(|x|, 0) = 0$. Since we are interested in positive solutions we extend f by 0 for $u < 0$.

(F_2) There exists $R_\circ > 0$ such that for every $R \geq R_\circ$ there exist constants $a_R > 0$, $b_R > 0$ and $\theta_R \in (p, p^*_\beta]$ such that

$$|f(|x|, u)| \leq a_R u^{\theta_R - 1} + b_R \text{ for } u \geq 0 \text{ and } x \in B(0, R).$$

(F_3) There exist $p < s \leq p^*_\beta$ and constants $a > 0$, $b > 0$ such that

$$f(|x|, u) \leq au^{p-1} + bu^{s-1} \text{ for } u \geq 0, \ x \in \mathbb{R}^N$$

and $\limsup_{u\to 0+} F(|x|, u)u^{-p} \leq 0$ uniformly in $x \in \mathbb{R}^N$, where $F(|x|, u) = \int_0^u f(|x|, s)\,dx$.

(F_4) There exists $\frac{1}{q} < \sigma < \frac{1}{p}$ such that

$$\sigma u f(|x|, u) - F(|x|, u) \geq 0$$

for $u \geq 0$, $x \in \mathbb{R}^N$.

(F_5) There exists $R_1 > 0$ such that $F(|x|, u) \geq 0$ for $u \geq 0$ and $x \in B(0, R_1)$.

(F_6) For every $\delta > 0$ there exist constants $M > 0$ and $R_2 > 0$ such that $|f(|x|, u)| \leq M|x|^{\frac{N-1}{p}}$ for $|x| \geq R_2$ and $0 \leq u \leq \delta$.

We now give an example of a nonlinearity f satisfying conditions $(F_1), \dots, (F_6)$. Let

$$f(|x|, u) = A(|x|)u^{s-1} + C(|x|)\frac{u^{r-1}}{(1+u)^d} \quad \text{for } u \geq 0,\ x \in \mathbb{R}^N$$

and $f(|x|, u) = 0$ for $u \leq 0$ and $x \in \mathbb{R}^N$, where $p < r, s \leq p^*_\beta$, $s, q > r$ and $d > 0$. We make the following assumptions on the functions $A(|x|)$ and $C(|x|)$:

(a) $A(|x|) \geq 0$ on $\mathbb{R}^N$, A is continuous and bounded on $\mathbb{R}^N$;

(b) $C(|x|) \leq 0$ on $\mathbb{R}^N$, C is continuous on $\mathbb{R}^N$;

(c) there exist constants $M > 0$ and $R > 0$ such that

$$|C(|x|)| \leq M|x|^{\frac{N-1}{p}} \quad \text{for } |x| \geq R; \quad \text{and}$$

(d) there exists $R_1 > 0$ such that $C(|x|) = 0$ for $x \in B(0, R_1)$.

We only check condition (F_4). We choose $\max(\frac{1}{s}, \frac{1}{q}) < \sigma < \frac{1}{r}$. Since $C(|x|) \leq 0$ on $\mathbb{R}^N$ and $\sigma < \frac{1}{r}$ we see that

$$\sigma u f(|x|, u) - F(|x|, u) \geq (\sigma - \frac{1}{r})C(|x|)\frac{u^r}{(1+u)^d} - \frac{d}{r}C(|x|)\int_0^u \frac{t^r}{(1+t)^{d+1}}\,dt \geq 0$$

and condition (F_4) follows.

Theorem 4.1 *Assume $(F_1), \dots, (F_6)$ and (H). Then equation (1.1) has a nonnegative nontrivial solution in $H^{1,p}_{\text{rad}}(\mathbb{R}^N)$.*

Proof. We use some ideas from the paper [15]. We commence by showing that the functional J has a mountain pass structure, that is, there exist constants $\kappa > 0$ and $\rho > 0$ such that

$$J(u) \geq \kappa \quad \text{for } u \in H^{1,p}_{\text{rad}}(\mathbb{R}^N) \text{ with } \|u\| = \rho. \tag{4.1}$$

We have by Lemmas 2.1, 2.3 and (F_3)

$$\begin{aligned} J(u) \;&\geq\; \frac{\min(1,\lambda)}{p}\|u\|^p - \frac{1}{q}\int_{\mathbb{R}^N} |x|^{-\alpha}(u^+)^q\,dx - \epsilon\int_{\mathbb{R}^N} h(|x|)|u|^p\,dx \\ &-\; C_\epsilon \int_{\mathbb{R}^N} h(|x|)|u|^s\,dx \geq \frac{\min(1,\lambda)}{p}\|u\|^p - C\|u\|^q - \epsilon C\|u\|^p - CC_\epsilon\|u\|^s. \end{aligned}$$

Taking $\epsilon > 0$ and $\rho > 0$ sufficiently small, inequality (4.1) follows. We now choose $v \in H^{1,p}_{\text{rad}}(\mathbb{R}^N)$, $v \not\equiv 0$, $v \geq 0$ and with supp $v \subset B(0, R_1)$. Then by (F_5) we get for $t > 0$ large that $\|tv\| > \rho$ and

$$J(tv) \leq \frac{t^p \min(1,\lambda)}{p}\|v\|^p - \frac{t^q}{q}\int_{\mathbb{R}^N} |x|^{-\alpha}v^q\,dx < 0.$$

Letting $\tilde{v} = tv$ we put

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H^{1,p}_{\rm rad}(\mathbb{R}^N));\ \gamma(0) = 0,\ \gamma(1) = \tilde{v}\}.$$

Obviously, we have $\kappa \leq c$. The mountain - pass level c generates a sequence $\{u_n\} \subset H^{1,p}_{\rm rad}(\mathbb{R}^N)$, $j = 1, 2, \ldots$ such that

$$J(u_n) \to c_n \ \text{ and } \ J'(u_n) \to 0 \ \text{ in } \ H^{-1,p'}_{\rm rad}(\mathbb{R}^N) \ \text{as}\, n \to \infty.$$

We now show that $\{u_n\}$ is bounded in $H^{1,p}_{\rm rad}(\mathbb{R}^N)$. Using assumption (F_4) we write

$$\begin{aligned} J(u_n) \quad - \quad & \sigma\langle J'(u_n), u_n\rangle \geq (\frac{1}{p} - \sigma)\min(1, \lambda)\|u_n\|^p + (\sigma - \frac{1}{q})\|u_n\|^q_{\alpha,q} \\ + \quad & \int_{\mathbb{R}^N} (\sigma u_n f(|x|, u_n) - F(|x|, u_n))h(|x|)\,dx \\ \geq \quad & (\frac{1}{p} - \sigma)\min(1, \lambda)\|u_n\|^p + (\sigma - \frac{1}{q})\|u_n\|^q_{\alpha,q}, \end{aligned}$$

and the boundedness of $\{u_n\}$ in $H^{1,p}(\mathbb{R}^N)$ follows. We may assume that $u_n \rightharpoonup u$ in $H^{1,p}_{\rm rad}(\mathbb{R}^N)$ and $u_n \to u$ in $L^q_{|x|^{-\alpha}}(\mathbb{R}^N)$. Testing $J'(u_n) \to 0$ with u_n^- we get that $u_n^- \to 0$ in $H^{1,p}_{\rm rad}(\mathbb{R}^N)$. We now show that $u_n \to u$ in $H^{1,p}_{\rm rad}(\mathbb{R}^N)$. We distinguish two cases: (i) $2 \leq p$ and (ii) $1 < p < 2$. In the case (i) we have by (3.5) for $m > n$

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^p\,dx \quad + \quad & \lambda \int_{\mathbb{R}^N} |u_n - u_m|^p\,dx \\ \leq \quad & C\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^p \nabla u_m, \nabla u_n - \nabla u_m)\,dx \\ + \quad & C\lambda \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)\,dx \\ = \quad & C\int_{\mathbb{R}^N} |x|^{-\alpha}((u_n^+)^{q-2}u_n^+ - (u_m^+)^{q-2}u_m^+)(u_n - u_m)\,dx \\ + \quad & C\int_{\mathbb{R}^N} h(|x|)(f(|x|, u_n) - f(|x|, u_m))(u_n - u_m)\,dx \\ := \quad & I^1_{nm} + I^2_{nm}. \end{aligned}$$

Since $u_n \to u$ in $L^q_\alpha(\mathbb{R}^N)$, $I^1_{nm} \to 0$ as $n, m \to \infty$. We now estimate I^2_{nm}. We use assumptions (F_2) and (F_6) to estimate I^2_{nm}:

$$\begin{aligned} \int_{\mathbb{R}^N} h(|x|)(f(|x|, u_n) \quad - \quad & f(|x|, u_m))(u_n - u_m)\,dx \\ = \quad & \int_{\mathbb{R}^N} h(|x|)f(|x|, u_n)(u_n - u_m)\,dx \\ - \quad & \int_{\mathbb{R}^N} h(|x|)f(|x|, u_m)(u_n - u_m)\,dx := \tilde{I}^1_{nm} + \tilde{I}^2_{nm}. \end{aligned}$$

Let $\delta > 0$ be given. By (F_6) we can find $R_2 > 0$ and $M > 0$ so that $|f(|x|, u)| \leq M|x|^{\frac{N-1}{p}}$ for $|x| \geq R_2$ and $0 \leq u \leq \delta$. Using estimates (2.1) we can assume that $|u_n(x)|, |u_m(x)| \leq \delta$ for $|x| \geq R_2$ by taking

R_2 larger, if necessary. We then have by (F_2)

$$\begin{aligned} |\tilde{I}^1_{nm}| &\leq a_{R_2}\Big(\int_{|x|\leq R_2} h(|x|)|u_n - u_m|^{\theta_{R_2}}\, dx\Big)^{\frac{1}{\theta_{R_2}}}\Big(\int_{|x|\leq R_2} h(|x|)|u_n|^{\theta_{R_2}}\, dx\Big)^{\frac{\theta_{R_2}-1}{\theta_{R_2}}} \\ &+ b_{R_2}\int_{|x|\leq R_2} h(|x|)|u_n - u_m|\, dx + \int_{|x|\geq R_2} h(|x|)|f(|x|, u_n)||u_n - u_m|\, dx. \end{aligned}$$

It clear that the first two terms on the right-side converge to 0 as $n, m \to \infty$. Indeed, we apply Lemma 2.1 to the first term. The second integral can be estimated, using the Hölder inequality, in the following way

$$\int_{|x|\leq R_2} h|u_n - u_m|\, dx \leq \Big(\int_{|x|\leq R_2} h\, dx\Big)^{\frac{1}{s'}}\Big(\int_{|x|\leq R_2} h|u_n - u_m|^s\, dx\Big)^{\frac{1}{s}},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$ and $p < s \leq p^*_\beta$. Again, by Lemma 2.1 this integral tends to 0 as $n, m \to \infty$.

We now use assumption (F_6) to estimate the last integral. Given $\epsilon > 0$ we choose $R_3 > R_2$ so that $\int_{|x|\geq R_3} h(|x|)\, dx \leq \epsilon$. Then

$$\begin{aligned} \int_{|x|\geq R_2} h(|x|)|f(|x|, u_m)||u_n - u_m|\, dx &\leq \int_{R_2\leq|x|\leq R_3} h(|x|)|f(|x|, u_n)||u_n - u_m|\, dx \\ &+ \int_{|x|\geq R_3} h(|x|)f(|x|, u_n)||u_n - u_m|\, dx. \end{aligned}$$

It is clear that the first integral on the right-hand side of this inequality tends to 0 as $n, m \to \infty$. The second integral can be estimated in the following way:

$$\int_{|x|\geq R_3} h(|x|)|f(|x|, u_n)||u_n - u_n|\, dx \leq C\int_{|x|\geq R_3} h(|x|)\, dx \leq C\epsilon.$$

Here we have used assumption (F_6) and estimate (2.1). In a similar way we show that $\tilde{I}^2_{nm} \to 0$ as $n, m \to \infty$. The case (ii) is treated in a similar manner. Finally, using (F_6) it is easy to show that u is a weak solution of (1.1) in a distributional sense. ■

5 Problem with a weight function tending to ∞ as $|x| \to \infty$

In this section we consider equation (3.1) with nonlinearity f depending only on u. As in Section 3 we assume that $f : \mathbb{R}^+_0 \to \mathbb{R}$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and $0 < \alpha < p$, $p < q < p^*_\alpha$.

We make the following assumptions on f and h:

(a_1) $h \in L^\infty_{\text{loc}}(\mathbb{R} - \{0\})$;

(a_2) $\lim_{|x|\to 0} |x|^\beta h(|x|) = 0$, $0 < \beta < \frac{N(p-1)}{N-1}$;

(a_3) There exist constants $a > 0$, $b > 0$ and $\frac{Np}{N-1} < s_1 < s_2 < p^*_\beta$ such that

$$|f(u)| \leq au^{s_1-1} + bs^{s_2-1} \quad \text{for} \quad u \geq 0;$$

(a_4) There exist constants $C > 0$, $R > 0$ and $0 < r < \frac{N-1}{p} s_1 - N$ such that

$$h(|x|) \leq C|x|^r \text{ for } |x| \geq R;$$

(a_5) There exist a constant $\frac{1}{q} < \sigma < \frac{1}{p}$ such that

$$\sigma u f(u) - F(u) \geq 0 \text{ for } u \geq 0 \text{ and } x \in \mathbb{R}^N.$$

Since we look for nonnegative solutions we extend f by 0 for $u \in (-\infty, 0)$.

Theorem 5.1 *Let $0 < \alpha < p$ and $p < q < p_\alpha^*$. Assume $(a_1), \ldots, (a_6)$ hold. Then there exists nonnegative and nontrivial solution of equation (3.1).*

Proof. The proof is similar to that of Theorem 3.1. We only point out that by Lemma 2.4 the space $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L_h^{s_1}(\mathbb{R}^N)$ and $L_h^{s_2}(\mathbb{R}^N)$. Moreover it follows from (a_3) that $\lim_{u \to 0+} \frac{F(u)}{u^p} = 0$. Therefore we can repeat the proof of Theorem 3.1 with some obvious modifications. ■

6 Multiple solutions

As an application of Lemma 2.5 we consider equation

$$-\Delta_p + \lambda |u|^{p-2} u = \frac{\mu}{|x|^\alpha} |u|^{q-2} u + h(|x|) f(u) \text{ in } \mathbb{R}^N, \tag{6.1}$$

where $\lambda > 0$ and $\mu > 0$ are parameters, $q < p$ and $\alpha = q + \kappa$, where κ is a constant from Lemma 2.5. Solutions will be obtained as critical points of the functional

$$J_\mu(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \lambda |u|^p \, dx) \, dx - \frac{\mu}{q} \int_{\mathbb{R}^N} \frac{|u|^q}{|x|^\alpha} \, dx - \int_{\mathbb{R}^N} h(|x|) F(u) \, dx.$$

Theorem 6.1 *Suppose that assumptions $(A_1), \ldots, (A_5)$ hold. Moreover assume that $1 < q < p$, $N - q - \frac{N-1}{p} q < \kappa < \frac{N(p-q)}{p}$ and $\alpha = q + \kappa$. Then there exists $\mu_\circ > 0$ such that equation (6.1) has at least two nontrivial nonnegative solutions for $0 < \mu < \mu_\circ$.*

Proof. Using Lemmas 2.1, 2.3 and 2.5 we obtain the following inequality

$$J_\mu(u) \geq \frac{\min(1, \lambda)}{p} \|u\|^p - C \frac{\mu}{q} \|u\|^q - C(\epsilon \|u\|^p + \|u\|^s),$$

where $C > 0$ is a constant. We choose $\epsilon > 0$ so that $\frac{\min(1,\lambda)}{p} - C\epsilon > 0$. Since $s > p$ we can find constants $\rho > 0$ and $\gamma > 0$ such that

$$(\frac{\min(1, \lambda)}{p} - C\epsilon) \|u\|^p - C \|u\|^s \geq \gamma \text{ for } u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \text{ with } \|u\| = \rho.$$

We then choose $\mu_\circ > 0$ so that $\gamma - C\mu_\circ \|u\|^q \geq \frac{\kappa}{2}$ for $0 < \mu \leq \mu_\circ$ and $\|u\| = \rho$. Let ϕ be nonzero function in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Since $q < p$ it is clear that $J_\mu(t\phi) < 0$ for $t > 0$ sufficiently small. This shows that $\inf_{\|u\| \leq \rho} J(u) < 0$. It is now routine to show that equation (6.1) has a mountain-pass solution. A second solution, which is a local minimizer of J_μ, is obtained with the aid of the Ekeland variational principle [4]. ■

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