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Equivalence of Some Multi-Valued Elliptic Variational Inequalities and Variational-Hemivariational Inequalities

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Abstract

First, we prove existence and comparison results for multi-valued elliptic variational inequalities involving Clarke's generalized gradient of some locally Lipschitz functions as multi-valued term. Only by applying the definition of Clarke's gradient it is well known that any solution of such a multi-valued elliptic variational inequality is also a solution of a corresponding variational-hemivariational inequality. The reverse is known to be true if the locally Lipschitz functions are regular in the sense of Clarke. Without imposing this kind of regularity the equivalence of the two problems under consideration is not clear at all. The main goal of this paper is to show that the equivalence still holds true without any additional regularity, which will fill a gap in the literature. Existence and comparison results for both multi-valued variational inequalities and variational-hemivariational inequalities are the main tools in the proof of the equivalence of these problems.

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1 Introduction

Let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, with $1 < p < \infty$, denote the usual Sobolev spaces, where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with Lipschitz boundary $\partial\Omega$. We denote by V^* and V_0^* the dual spaces corresponding to V and V_0, respectively, and by $\langle \cdot, \cdot \rangle$ the duality pairing. Let K be a closed, convex subset of V, and let $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be functions that are only supposed to be measurable in their first and locally Lipschitz continuous with respect to their second argument.

Let q denote the Hölder conjugate to p, i.e., q satisfies $1/p + 1/q = 1$. We deal with the following two variational problems:

Multi-valued elliptic variational inequality: Find $u \in K$, $\eta \in L^q(\Omega)$, and $\xi \in L^q(\partial\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial j_1(x, u(x)), \text{ a.e. } x \in \Omega, \quad \xi(x) \in \partial j_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \\ \langle Au - h, v - u\rangle + \displaystyle\int_\Omega \eta\,(v - u)\,dx + \int_{\partial\Omega} \xi\,(\gamma v - \gamma u)\,d\sigma \geq 0, \ \forall\, v \in K. \end{cases} \tag{1.1}$$

where

1. the multi-valued functions $s \mapsto \partial j_k(x, s)$ are given by Clarke's generalized gradient of the locally Lipschitz functions $s \mapsto j_k(x, s)$, $k = 1, 2$, defined by
$$\partial j_k(x, s) := \{\zeta \in \mathbb{R} : j_k^o(x, s; r) \geq \zeta r, \ \forall\, r \in \mathbb{R}\}$$
for a.a. $x \in \Omega$ in case $k = 1$, and for a.a. $x \in \partial\Omega$ in case $k = 2$, with $j_k^o(x, s; r)$ denoting the generalized directional derivative of $s \mapsto j_k(x, s)$ at s in the direction r given by
$$j_k^o(x, s; r) = \limsup_{y \to s,\ t \downarrow 0} \frac{j_k(x, y + t\,r) - j_k(x, y)}{t},$$
(cf., e.g., [6, Chap. 2]),
2. A is a second-order quasilinear elliptic differential operator of the form
$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \qquad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right),$$
3. $\gamma : V \to L^p(\partial\Omega)$ denotes the trace operator which is known to be linear and compact from V into $L^p(\partial\Omega)$,
4. $h \in V^*$.

Variational-hemivariational inequality: Find $u \in K$ such that

$$\langle Au - h, v - u\rangle + \int_\Omega j_1^o(x, u; v - u)\,dx + \int_{\partial\Omega} j_2^o(x, \gamma u; \gamma v - \gamma u)\,d\sigma \geq 0, \ \forall\, v \in K. \tag{1.2}$$

Only by applying the definition of Clarke's generalized gradient $s \mapsto \partial j_k(x, s)$ given above (see 1.) one readily observes that any solution of the multi-valued variational inequality (1.1) must be a solution of the variational-hemivariational inequality (1.2). The reverse is well known to be true if the locally Lipschitz functions $s \mapsto j_k(x, s)$ are assumed to be regular in the sense of Clarke, see [6, Chap. 2.3]. For example, if the functions $s \mapsto j_k(x, s)$ are smooth (i.e., differentiable) or convex then they are regular in the sense of Clarke.

The main goal of this paper is to show that the reverse still holds true without imposing any additional regularity on the functions $s \mapsto j_k(x, s)$. This result on the equivalence of problems (1.1) and (1.2) will fill a gap in the literature where both problems are treated separately. The main tools in the proof are existence and comparison results for both problems (1.1) and (1.2) based on an appropriate notion of sub-supersolution which are of interest in their own.

2 Hypotheses and main result

We assume the following hypotheses of Leray–Lions type on the coefficient functions a_i, $i = 1, ..., N$, of the operator A:

(A1) Each $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x, \zeta)$ is measurable in $x \in \Omega$ for all $\zeta \in \mathbb{R}^N$, and continuous in ζ for a.a. $x \in \Omega$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ such that

$$|a_i(x, \zeta)| \le k_0(x) + c_0\, |\zeta|^{p-1}\,,$$

for a.a. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^N$.

(A2) For a.a. $x \in \Omega$, and for all $\zeta, \zeta' \in \mathbb{R}^N$ with $\zeta \neq \zeta'$ the following monotonicity holds:

$$\sum_{i=1}^{N} (a_i(x, \zeta) - a_i(x, \zeta'))(\zeta_i - \zeta'_i) > 0.$$

(A3) There is some constant $\nu > 0$ such that for a.a. $x \in \Omega$ and for all $\zeta \in \mathbb{R}^N$ the inequality

$$\sum_{i=1}^{N} a_i(x, \zeta)\zeta_i \ge \nu|\zeta|^p - k_1(x)$$

is satisfied for some function $k_1 \in L^1(\Omega)$.

A particular case of the operator A satisfying (A1)–(A3) is the negative p-Laplacian, i.e., $A = -\Delta_p$, which is obtained if

$$a_i(x, \zeta) = |\zeta|^{p-2}\zeta_i, \;\; i = 1, \ldots, N, \; \zeta = (\zeta_1, \ldots, \zeta_N).$$

In view of (A1), (A2) the operator A defined by

$$\langle Au, \varphi\rangle := \int_\Omega \sum_{i=1}^{N} a_i(x, \nabla u)\frac{\partial \varphi}{\partial x_i}\, dx, \qquad \forall\, \varphi \in V_0$$

is known to provide a continuous, bounded, and monotone (resp. strictly monotone) mapping from V (resp. V_0) into V_0^*.

As for the functions $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ we assume the following:

(H1) (i) $x \mapsto j_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto j_1(x, s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.

(ii) There exist a constant $c > 0$ and a function $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ the growth condition

$$|\eta| \le k_\Omega(x) + c|s|^{p-1}, \quad \forall\, \eta \in \partial j_1(x,s)$$

is fulfilled.

(H2) (i) $x \mapsto j_2(x,s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$, and $s \mapsto j_2(x,s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \partial\Omega$.

(ii) There exist a constant $c > 0$ and a function $k_{\partial\Omega} \in L^q_+(\partial\Omega)$ such that for a.e. $x \in \partial\Omega$ and for all $s \in \mathbb{R}$ the growth condition

$$|\xi| \le k_{\partial\Omega}(x) + c|s|^{p-1}, \quad \forall\, \xi \in \partial j_2(x,s)$$

is fulfilled.

The main result of this paper is the following theorem about the equivalence of problems (1.1) and (1.2).

Theorem 2.1 *Let hypotheses (A1)–(A3) and (H1)–(H2) be satisfied, and assume the following lattice property of the closed convex subset $K \subseteq V$:*

$$K \wedge K \subseteq K, \quad K \vee K \subseteq K. \tag{2.1}$$

Then u is a solution of the multi-valued variational inequality (1.1) if and only if u is a solution of the variational-hemivariational inequality (1.2).

As an important tool to achieve our goal we develop in the next section an existence and comparison result for (1.1), and shortly recall corresponding results for (1.2) which have been recently obtained by the author in [3, 5].

Remark 2.1 The lattice condition (2.1) on the closed, convex subset $K \subseteq V$ is fulfilled for a number of important models in applied sciences, see e.g. [2, p. 216]. Moreover, Theorem 2.1 remains true also in case that the operator A is replaced by a more general Leray–Lions operator in the form:

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,\nabla u(x)) + a_0(x,u,\nabla u(x)).$$

3 Multi-valued variational inequalities

The main goal of this section is to establish the method of sub-supersolution for the multi-valued variational inequality (1.1). Our presentation is motivated by results recently obtained in [1, 2, 3, 4, 5]. It should be noted that the results we are going to present here only require a local growth condition of the generalized gradients $s \mapsto \partial j_k(x,s)$ with respect to an ordered pair of sub-supersolutions whose definition will be given next. In particular, we should emphasize that unlike in earlier works by the author, such as in [2], the following one-sided growth on Clarke's generalized gradients $s \mapsto \partial j_k(x,s)$ in the form: $\exists\, c_k \ge 0$ such that

$$\zeta_1 \le \zeta_2 + c_k(s_2 - s_1)^{p-1},\ \forall\, \zeta_i \in \partial j_k(x,s_i),\ \forall\, s_1, s_2 \text{ with } s_1 < s_2, \tag{3.1}$$

can completely be dropped in our treatment.

For functions w, z and sets W and Z of functions defined on Ω or $\partial\Omega$ we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$.

Our basic notion of sub- and supersolution of the multi-valued variational inequality (1.1) reads as follows, see [2].

Definition 3.1 *A function* $\underline{u} \in V$ *is called a* **subsolution** *of (1.1) if there is an* $\underline{\eta} \in L^q(\Omega)$ *and a* $\underline{\xi} \in L^q(\partial\Omega)$ *satisfying*

$$
\begin{aligned}
&(i)\ \underline{u} \vee K \subseteq K,\\
&(ii)\ \underline{\eta}(x) \in \partial j_1(x, \underline{u}(x)),\ \text{a.e. } x \in \Omega,\ \underline{\xi}(x) \in \partial j_2(x, \gamma\underline{u}(x)),\ \text{a.e. } x \in \partial\Omega,\\
&(iii)\ \langle A\underline{u} - h, v - \underline{u}\rangle + \int_\Omega \underline{\eta}\,(v - \underline{u})\,dx + \int_{\partial\Omega} \underline{\xi}\,(\gamma v - \gamma\underline{u})\,d\sigma \ge 0,\\
&\quad \text{for all } v \in \underline{u} \wedge K.
\end{aligned}
$$

Definition 3.2 *A function* $\overline{u} \in V$ *is called a* **supersolution** *of (1.1) if there is an* $\overline{\eta} \in L^q(\Omega)$ *and a* $\overline{\xi} \in L^q(\partial\Omega)$ *satisfying*

$$
\begin{aligned}
&(i)\ \overline{u} \wedge K \subseteq K,\\
&(ii)\ \overline{\eta}(x) \in \partial j_1(x, \overline{u}(x)),\ \text{a.e. } x \in \Omega,\ \overline{\xi}(x) \in \partial j_2(x, \gamma\overline{u}(x)),\ \text{a.e. } x \in \partial\Omega,\\
&(iii)\ \langle A\overline{u} - h, v - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(v - \overline{u})\,dx + \int_{\partial\Omega} \overline{\xi}\,(\gamma v - \gamma\overline{u})\,d\sigma \ge 0,\\
&\quad \text{for all } v \in \overline{u} \vee K.
\end{aligned}
$$

Remark 3.1 Note that the notions for sub- and supersolution defined in Definition 3.1 and Definition 3.2 have a symmetric structure. One obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}$ in Definition 3.1 by $\overline{u}$, and interchanging $\vee$ by $\wedge$. Furthermore, the lattice condition (2.1) readily implies that any solution of the multi-valued variational inequality (1.1) is both a subsolution and a supersolution for (1.1).

To justify that Definitions 3.1 and 3.2 are in fact natural extensions of the usual notions of sub-supersolutions for elliptic boundary value problems let us discuss the following special case.

Example 3.1 If $K = V$, $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and if j_k, $k = 1, 2$, are the primitives of f_k given by

$$
j_k(x, s) := \int_0^s f_k(x, t)\,dt, \tag{3.2}
$$

then Clarke's generalized gradient ∂j_k is a singleton, i.e.,

$$
\partial j_k(x, s) = \{f_k(x, s)\},
$$

and (1.1) reduces to the following quasilinear elliptic BVP

$$
\langle Au - h, v\rangle + \int_\Omega f_1(x, u)\,v\,dx + \int_{\partial\Omega} f_2(x, \gamma u)\,\gamma v\,d\sigma = 0, \quad \forall\, v \in V, \tag{3.3}
$$

which is the formulation of the weak solution of the BVP

$$Au + f_1(x,u) = h \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu} + f_2(x,u) = 0 \ \text{ on } \ \partial\Omega, \tag{3.4}$$

where $\partial/\partial\nu$ denotes the outward pointing conormal derivative associated with A. If $\underline{u} \in V$ is a subsolution according to Definition 3.1, then the first condition (i) is trivially satisfied. The second condition (ii) of Definition 3.1 means that

$$\underline{\eta}(x) = f_1(x, \underline{u}(x)), \text{ a.e. } x \in \Omega, \ \underline{\xi}(x) = f_2(x, \gamma\underline{u}(x)), \text{ a.e. } x \in \partial\Omega.$$

Since $K = V$, any $v \in \underline{u} \wedge V$ has the form $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in V$, where $w^+ = \max\{w, 0\}$, condition (iii) becomes

$$\begin{aligned} \langle A\underline{u} - h, -(\underline{u} - \varphi)^+\rangle \quad &+ \quad \int_\Omega f_1(\cdot, \underline{u})\,(-(\underline{u} - \varphi)^+)\,dx \\ &+ \quad \int_{\partial\Omega} f_2(\cdot, \gamma\underline{u})\,(-(\gamma\underline{u} - \gamma\varphi)^+)\,d\sigma \geq 0, \end{aligned} \tag{3.5}$$

for all $\varphi \in V$. Since $\underline{u} \in V$, we have

$$M = \{(\underline{u} - \varphi)^+ : \varphi \in V\} = V \cap L^p_+(\Omega),$$

where $L^p_+(\Omega)$ is the positive cone of $L^p(\Omega)$, and thus we obtain from inequality (3.5)

$$\langle A\underline{u} - h, \chi\rangle + \int_\Omega f_1(x, \underline{u})\,\chi\,dx + \int_{\partial\Omega} f_2(x, \gamma\underline{u})\,\gamma\chi\,d\sigma \leq 0, \quad \forall\, \chi \in V \cap L^p_+(\Omega, \tag{3.6}$$

which is nothing but the usual notion of a (weak) subsolution for the BVP (3.4). Similarly, one verifies that $\overline{u} \in V$ which is a supersolution according to Definition 3.2 is equivalent with the usual supersolution of the BVP (3.4).

In what follows we assume the existence of an ordered pair $\underline{u}$, $\overline{u}$ of sub-supersolutions of the multi-valued variational inequality (1.1) satisfying $\underline{u} \leq \overline{u}$. With respect to this ordered pair we impose the following hypotheses on the nonlinearities j_k, $k = 1, 2$.

(j1) $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto j_1(x,s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.

(ii) There exists a function $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [\underline{u}(x), \overline{u}(x)]$ the growth condition

$$|\eta| \leq k_\Omega(x), \quad \forall\, \eta \in \partial j_1(x,s)$$

is fulfilled.

(j2) $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $x \mapsto j_2(x,s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$, and $s \mapsto j_2(x,s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \partial\Omega$.

(ii) There exists a function $k_{\partial\Omega} \in L^q_+(\partial\Omega)$ such that for a.e. $x \in \partial\Omega$ and for all $s \in [\gamma\underline{u}(x), \gamma\overline{u}(x)]$ the growth condition

$$|\xi| \le k_{\partial\Omega}(x), \quad \forall\, \xi \in \partial j_2(x, s)$$

is fulfilled.

Remark 3.2 We note that by the growth condition (ii) of (j1) and (j2), only a local L^q-boundedness condition on Clarke's generalized gradient ∂j_k is assumed, which is trivially satisfied, in particular, if j_k satisfy (H1)–(H2).

Our goal is to prove the existence of solutions of the multi-valued variational inequality (1.1) within the ordered interval $[\underline{u}, \overline{u}]$ of the given sub- and supersolution of (1.1). The proof of our main existence and comparison result strongly relies on an appropriate modification of the functions j_k outside the interval $[\underline{u}, \overline{u}]$ formed by the given sub- and supersolutions. Let $(\underline{u}, \underline{\eta}, \underline{\xi}) \in V \times L^q(\Omega) \times L^q(\partial\Omega)$ and $(\overline{u}, \overline{\eta}, \overline{\xi}) \in V \times L^q(\Omega) \times L^q(\partial\Omega)$ satisfy the conditions of Definition 3.1 and Definition 3.2, respectively, with $\underline{u} \le \overline{u}$. Then we define the following modifications $\tilde{j}_k$ of the given j_k:

$$\tilde{j}_1(x, s) = \begin{cases} j_1(x, \underline{u}(x)) + \underline{\eta}(x)(s - \underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ j_1(x, s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ j_1(x, \overline{u}(x)) + \overline{\eta}(x)(s - \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{3.7}$$

and similarly

$$\tilde{j}_2(x, s) = \begin{cases} j_2(x, \gamma\underline{u}(x)) + \underline{\xi}(x)(s - \gamma\underline{u}(x)) & \text{if} \quad s < \gamma\underline{u}(x), \\ j_2(x, s) & \text{if} \quad \gamma\underline{u}(x) \le s \le \gamma\overline{u}(x), \\ j_2(x, \gamma\overline{u}(x)) + \overline{\xi}(x)(s - \gamma\overline{u}(x)) & \text{if} \quad s > \gamma\overline{u}(x). \end{cases} \tag{3.8}$$

The following two lemmas list essential properties of the modified functions $\tilde{j}_k$.

Lemma 3.1 *Let hypotheses (j1) be satisfied. Then the function $\tilde{j}_1$ has the following properties:*

(P1) $\tilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies*

(i) $x \mapsto \tilde{j}_1(x, s)$ *is measurable in* Ω *for all* $s \in \mathbb{R}$, *and* $s \mapsto \tilde{j}_1(x, s)$ *is Lipschitz continuous in* $\mathbb{R}$ *for a.e.* $x \in \Omega$.

(ii) Let $\partial\tilde{j}_1$ *denote Clarke's generalized gradient of* $s \mapsto \tilde{j}_1(x, s)$. *Then for a.e.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$ *the growth*

$$|\eta| \le k_{\Omega}(x), \quad \forall\, \eta \in \partial\tilde{j}_1(x, s)$$

is fulfilled.

(iii) Clarke's generalized gradient of $s \mapsto \tilde{j}_1(x, s)$ *is given by*

$$\partial\tilde{j}_1(x, s) = \begin{cases} \underline{\eta}(x) & \text{if} \quad s < \underline{u}(x), \\ \partial\tilde{j}_1(x, \underline{u}(x)) & \text{if} \quad s = \underline{u}(x), \\ \partial j_1(x, s) & \text{if} \quad \underline{u}(x) < s < \overline{u}(x), \\ \partial\tilde{j}_1(x, \overline{u}(x)) & \text{if} \quad s = \overline{u}(x), \\ \overline{\eta}(x) & \text{if} \quad s > \overline{u}(x), \end{cases} \tag{3.9}$$

where $\partial\tilde{j}_1(x, \underline{u}(x)) \subseteq \partial j_1(x, \underline{u}(x))$ *and* $\partial\tilde{j}_1(x, \overline{u}(x)) \subseteq \partial j_1(x, \overline{u}(x))$ *hold true.*

Proof. The proof follows immediately from the definition (3.7) of $\tilde{j}_1$, and using the assumptions (j1) on j_1 as well as from the fact that Clarke's generalized gradient $\partial j_1(x, s)$ is a convex set. □

Lemma 3.2 *Let hypotheses (j2) be satisfied. Then the function $\tilde{j}_2$ has the following properties:*

(P2) $\tilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies*

- *(i)* $x \mapsto \tilde{j}_2(x, s)$ *is measurable in* $\partial\Omega$ *for all* $s \in \mathbb{R}$*, and* $s \mapsto \tilde{j}_2(x, s)$ *is Lipschitz continuous in* $\mathbb{R}$ *for a.e.* $x \in \partial\Omega$*.*
- *(ii) Let* $\partial\tilde{j}_2$ *denote Clarke's generalized gradient of* $s \mapsto \tilde{j}_2(x, s)$*. Then for a.e.* $x \in \partial\Omega$ *and for all* $s \in \mathbb{R}$ *the growth*
$$|\xi| \le k_{\partial\Omega}(x), \quad \forall\, \xi \in \partial\tilde{j}_2(x, s)$$
is fulfilled.
- *(iii) Clarke's generalized gradient of* $s \mapsto \tilde{j}_2(x, s)$ *is given by*
$$\partial\tilde{j}_2(x, s) = \begin{cases} \underline{\xi}(x) & \text{if} \quad s < \gamma\underline{u}(x), \\ \partial\tilde{j}_2(x, \gamma\underline{u}(x)) & \text{if} \quad s = \gamma\underline{u}(x), \\ \partial j_2(x, s) & \text{if} \quad \gamma\underline{u}(x) < s < \gamma\overline{u}(x), \\ \partial\tilde{j}_2(x, \gamma\overline{u}(x)) & \text{if} \quad s = \gamma\overline{u}(x), \\ \overline{\xi}(x) & \text{if} \quad s > \gamma\overline{u}(x), \end{cases} \tag{3.10}$$
where $\partial\tilde{j}_2(x, \gamma\underline{u}(x)) \subseteq \partial j_2(x, \gamma\underline{u}(x))$ *and* $\partial\tilde{j}_2(x, \gamma\overline{u}(x)) \subseteq \partial j_2(x, \gamma\overline{u}(x))$ *hold true.*

Proof. The proof is similar as for Lemma 3.1. □

By means of $\tilde{j}_1$ and $\tilde{j}_2$ we introduce integral functionals $\tilde{J}_1$ and $\tilde{J}_2$ defined on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively, and given by

$$\tilde{J}_1(u) = \int_\Omega \tilde{j}_1(x, u(x))\, dx, \quad u \in L^p(\Omega),$$
$$\tilde{J}_2(v) = \int_{\partial\Omega} \tilde{j}_2(x, v(x))\, d\sigma, \quad v \in L^p(\partial\Omega).$$

Due to (P1)(ii) and (P2)(ii), and applying Lebourg's mean value theorem (see [2, Theorem 2.177]) the functionals $\tilde{J}_1 : L^p(\Omega) \to \mathbb{R}$ and $\tilde{J}_2 : L^p(\partial\Omega) \to \mathbb{R}$ are well-defined and Lipschitz continuous, so that Clarke's generalized gradients $\partial\tilde{J}_1 : L^p(\Omega) \to 2^{(L^p(\Omega))^*}$ and $\partial\tilde{J}_2 : L^p(\partial\Omega) \to 2^{(L^p(\partial\Omega))^*}$ are well-defined too. Moreover, Aubin–Clarke theorem (cf. [6, p. 83]) provides the following characterization of the generalized gradients. For $u \in L^p(\Omega)$ we have

$$\tilde{\eta} \in \partial\tilde{J}_1(u) \Longrightarrow \tilde{\eta} \in L^q(\Omega) \text{ with } \tilde{\eta}(x) \in \partial\tilde{j}_1(x, u(x)) \text{ for a.e. } x \in \Omega, \tag{3.11}$$

and similarly for $v \in L^p(\partial\Omega)$

$$\tilde{\xi} \in \partial\tilde{J}_2(v) \Longrightarrow \tilde{\xi} \in L^q(\partial\Omega) \text{ with } \tilde{\xi}(x) \in \partial\tilde{j}_2(x, v(x)) \text{ for a.e. } x \in \partial\Omega. \tag{3.12}$$

By means of Clarke's generalized gradient $\partial\tilde{J}_k$ we introduce the following multi-valued operators:

$$\Phi_1(u) := (i^* \circ \partial\tilde{J}_1 \circ i)(u), \quad \Phi_2(u) := (\gamma^* \circ \partial\tilde{J}_2 \circ \gamma)(u), \quad u \in V, \tag{3.13}$$

where $i^* : L^q(\Omega) \to V^*$ and $\gamma^* : L^q(\partial\Omega) \to V^*$ denote the adjoint operators of the embedding $i : V \hookrightarrow L^p(\Omega)$ and the trace operator $\gamma : V \to L^p(\partial\Omega)$, respectively, defined by: If $\eta \in L^q(\Omega)$ and $\xi \in L^q(\partial\Omega)$ then

$$\langle i^*\eta, \varphi\rangle = \int_\Omega \eta\,\varphi\,dx, \quad \langle \gamma^*\xi, \varphi\rangle = \int_{\partial\Omega} \xi\,\gamma\varphi\,d\sigma, \quad \forall \varphi \in V. \tag{3.14}$$

As for the existence theory for abstract equations governed by multi-valued operators we make use of the following notion of pseudomonotonicity, see, e.g., [7], or [2, Chap. 2].

Definition 3.3 *Let X be a real reflexive Banach space with dual space X^*. The operator $\mathcal{A} : X \to 2^{X^*}$ is called* **pseudomonotone** *if the following conditions hold:*

(i) The set $\mathcal{A}(u)$ is nonempty, bounded, closed and convex for all $u \in X$;

(ii) $\mathcal{A}$ is upper semicontinuous from each finite dimensional subspace of X to X^* equipped with the weak topology;

(iii) If $(u_n) \subset X$ with $u_n \rightharpoonup u$, and $u_n^* \in \mathcal{A}(u_n)$ is such that $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then to each element $v \in X$ there exists $u^*(v) \in \mathcal{A}(u)$ with

$$\liminf\langle u_n^*, u_n - v\rangle \ge \langle u^*(v), u - v\rangle.$$

The following proposition provides sufficient conditions for an operator $\mathcal{A} : X \to 2^{X^*}$ to be pseudomonotone.

Proposition 3.1 *Let X be a real reflexive Banach space, and assume that $\mathcal{A} : X \to 2^{X^*}$ satisfies the following conditions:*

(i) For each $u \in X$ we have that $\mathcal{A}(u)$ is a nonempty, closed and convex subset of X^*;

(ii) $\mathcal{A} : X \to 2^{X^*}$ is bounded;

(iii) If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* provided one has that $u_n^* \in \mathcal{A}(u_n)$ and $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then $u^* \in \mathcal{A}(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Then the operator $\mathcal{A} : X \to 2^{X^*}$ is pseudomonotone.

As for the proof of Proposition 3.1 we refer, e.g., to [7, Chap. 2]. In the proof of our main result we make use of the following surjectivity result for multi-valued pseudomonotone mappings perturbed by maximal monotone operators in reflexive Banach spaces (cf., e.g., [7, Theorem 2.12]).

Theorem 3.1 *Let X be a real reflexive Banach space with dual space X^*, $\Phi : X \to 2^{X^*}$ a maximal monotone operator, and $v_0 \in dom\,(\Phi)$. Let $\mathcal{A} : X \to 2^{X^*}$ be a pseudomonotone operator, and assume that either $\mathcal{A}_{v_0}$ is quasi-bounded or Φ_{v_0} is strongly quasi-bounded. Assume further that $\mathcal{A} : X \to 2^{X^*}$ is v_0-coercive, i.e., there exists a real-valued function $a : \mathbb{R}_+ \to \mathbb{R}$ with $a(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in$ graph$\,(\mathcal{A})$ one has $\langle u^*, u - v_0\rangle \ge a(\|u\|_X)\|u\|_X$. Then $\mathcal{A} + \Phi$ is surjective, i.e., range$(\mathcal{A} + \Phi) = X^*$.*

Remark 3.3 The operators $\mathcal{A}_{v_0}$ and Φ_{v_0} that appear in the theorem above are defined by $\mathcal{A}_{v_0}(v) := \mathcal{A}(v_0 + v)$ and similarly for Φ_{v_0}. As for the notion of *quasi-bounded* and *strongly quasi-bounded* we refer to [7, p.51]. In particular, any bounded operator is quasi-bounded and strongly quasi-bounded as well.

The operators Φ_k, $k = 1, 2$, defined in (3.13) possess the following properties.

Lemma 3.3 *The operators $\Phi_k : V \to 2^{V^*}$, $k = 1, 2$, are bounded and pseudomonotone.*

Proof. As for $\Phi_1 : V \to 2^{V^*}$ we refer to [1, Lemma 3.1], and for $\Phi_1 : V \to 2^{V^*}$ the proof is given in [1, Lemma 3.2]. □

Let b be the cut-off function defined as follows:

$$b(x, s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x) \\ -(\underline{u}(x) - s)^{p-1} & \text{if} \quad s < \underline{u}(x). \end{cases}$$

Apparently, $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|b(x, s)| \le k_2(x) + c_1 |s|^{p-1} \tag{3.15}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $c_1 > 0$ is a constant and $k_2 \in L^q(\Omega)$. Moreover, one has the following estimate

$$\int_\Omega b(x, u(x))\, u(x)\, dx \ge c_2 \|u\|^p_{L^p(\Omega)} - c_3, \quad \forall u \in L^p(\Omega), \tag{3.16}$$

for some constants $c_2 > 0$ and $c_3 > 0$. Let B denote the Nemytskij operator associated with b, i.e.,

$$B(u)(x) = b(x, u(x)).$$

In view of (3.15) the Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ is continuous and bounded, and thus due to the compact embedding $V \hookrightarrow L^p(\Omega)$, it follows that the composed operators $\tilde{B} := i^* \circ B \circ i : V \to V^*$ is completely continuous.

Consider the multi-valued operator $\mathcal{A}$ defined by

$$\mathcal{A}(u) = Au + \tilde{B}(u) + \Phi_1(u) + \Phi_2(u). \tag{3.17}$$

Lemma 3.4 *The operator $\mathcal{A} : V \to 2^{V^*}$ is bounded, pseudomonotone, and v_0-coercive for $v_0 \in K$.*

Proof. Hypotheses (A1)–(A2) imply that $A : V \to V^*$ is continuous, bounded, and monotone, and thus, in particular, pseudomonotone, see, e.g., [2, Theorem 2.109]. The cut-off operator $\tilde{B} : V \to V^*$ is bounded and completely continuous. Therefore, the (single-valued) operator $A + \tilde{B} : V \to V^*$ is continuous, bounded and pseudomonotone. Due to Lemma 3.3, the multi-valued operators $\Phi_k : V \to 2^{V^*}$, $k = 1, 2$, are bounded and pseudomonotone. Since pseudomonotonicity is invariant under addition (see [7, Chap. 2]), it follows that $\mathcal{A} : V \to 2^{V^*}$ is bounded and pseudomonotone, and so it remains to show that $\mathcal{A}$ is v_0-coercive for $v_0 \in K$, i.e., we need to verify that there exists a real-valued function $a : \mathbb{R}_+ \to \mathbb{R}$ with $a(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in$ graph $(\mathcal{A})$ one has

$$\langle u^*, u - v_0\rangle \geq a(\|u\|_V)\|u\|_V. \tag{3.18}$$

To check (3.18) let $u^* \in \mathcal{A}(u)$, i.e., u^* has the form

$$u^* = Au + \tilde{B}(u) + i^*\tilde{\eta} + \gamma^*\tilde{\xi},$$

where $\tilde{\eta} \in L^q(\Omega)$ with $\tilde{\eta}(x) \in \partial\tilde{j}_1(x, u(x))$, for a.e. $x \in \Omega$, and $\tilde{\xi} \in L^q(\partial\Omega)$ with $\tilde{\xi}(x) \in \partial\tilde{j}_2(x, \gamma u(x))$, for a.e. $x \in \partial\Omega$. Hypotheses (A1), (A2), and the estimates (3.15) and (3.16), as well as the uniform boundedness of $\partial\tilde{j}_1$ and $\partial\tilde{j}_2$ in view of (P1) (ii) of Lemma 3.1 and (P2) (ii) of Lemma 3.2, respectively, allows us to estimate as follows:

$$|\langle Au + \tilde{B}(u) + i^*\tilde{\eta} + \gamma^*\tilde{\xi}, v_0\rangle| \leq c(1 + \|u\|_V^{p-1}), \quad \forall\, u \in V, \tag{3.19}$$

for some constant $c > 0$, and

$$\begin{aligned} \langle u^*, u\rangle &= \langle Au + \tilde{B}(u) + i^*\tilde{\eta} + \gamma^*\tilde{\xi}, u\rangle \\ &\geq \nu\|\nabla u\|^p_{L^p(\Omega)} - c_4 + c_2\|u\|^p_{L^p(\Omega)} - c_3 - c_5\,\|u\|_{L^p(\Omega)} - c_6\,\|\gamma u\|_{L^p(\partial\Omega)} \\ &\geq c_7\,\|u\|_V^p - c_8\,\|u\|_V - c_9. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we get the estimate

$$\langle Au + \tilde{B}(u) + i^*\tilde{\eta} + \gamma^*\tilde{\xi}, u - v_0\rangle \geq c_7\,\|u\|_V^p - c\,\|u\|_V^{p-1} - c_8\,\|u\|_V - c_{10},$$

for some positive constants c_i, which proves the v_0-coercivity. □

The method of sub-supersolution which provides an existence and comparison result for (1.1) reads as follows.

Theorem 3.2 *Let (A1)–(A3) be satisfied. Assume the existence of sub- and supersolutions* $\underline{u}$ *and* $\overline{u}$*, respectively, of the multi-valued variational inequality (1.1) with* $\underline{u} \leq \overline{u}$ *such that (j1)–(j2) are fulfilled. Then there exist solutions of (1.1) within the ordered interval* $[\underline{u}, \overline{u}]$.

Proof. Let $I_K : V \to \mathbb{R}\cup\{+\infty\}$ be the indicator function related to the given closed convex set $K \neq \emptyset$, i.e.,

$$I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K, \\ +\infty & \text{if} \quad u \notin K, \end{cases}$$

which is known to be proper, convex, and lower semicontinuous. Consider the following multi-valued operator

$$\mathcal{A} + \partial I_K : V \to 2^{V^*}, \tag{3.21}$$

where $\mathcal{A}$ is given by (3.17), and ∂I_K is the usual subdifferential of I_K which is known to be a maximal monotone operator, cf., e.g., [9]. Taking into account Lemma 3.4, we may apply the surjectivity result of Theorem 3.1, which implies the existence of $u \in K$ such that $h \in \mathcal{A}(u) + \partial I_K(u)$. By definition of $\mathcal{A}$ and ∂I_K the latter inclusion implies the existence of $\eta^* \in \Phi_1(u)$, $\xi^* \in \Phi_2(u)$, and $\theta^* \in \partial I_K(u)$ with $\eta^* = i^*\tilde{\eta}$ and $\xi^* = \gamma^*\tilde{\xi}$ such that equation

$$Au + \tilde{B}(u) + i^*\tilde{\eta} + \gamma^*\tilde{\xi} + \theta^* = h, \quad \text{in } V^* \tag{3.22}$$

holds, where in view of (3.11)–(3.12) we have $\tilde\eta \in L^q(\Omega)$ with $\tilde\eta(x) \in \partial \tilde j_1(x, u(x))$, for a.e. $x \in \Omega$, and $\tilde\xi \in L^q(\partial\Omega)$ with $\tilde\xi(x) \in \partial \tilde j_2(x, \gamma u(x))$, for a.e. $x \in \partial\Omega$. By using the definition of $\partial I_K(u)$, the solution u of equation (3.22) is seen to be a solution of the following multi-valued variational inequality

$$u \in K : \ \langle Au - h + \tilde B(u) + i^*\tilde\eta + \gamma^*\tilde\xi, v - u\rangle \ge 0, \ \forall\, v \in K, \tag{3.23}$$

which is equivalent to

$$\begin{aligned} u \in K : \ \langle Au - h, v - u\rangle \quad &+ \quad \int_\Omega b(x,u)\,(v-u)\,dx + \int_\Omega \tilde\eta\,(v-u)\,dx \\ &+ \int_{\partial\Omega} \tilde\xi\,(\gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in K. \end{aligned} \tag{3.24}$$

We are going to show that any solution u of (3.24) is a solution of the multi-valued variational inequality (1.1) satisfying $u \in [\underline u, \overline u]$. To this end we first show that u indeed satisfies $\underline u \le u \le \overline u$. To prove the inequality $u \le \overline u$ we apply the special test function $v = \overline u \vee u = \overline u + (u - \overline u)^+$ in Definition 3.2 (iii), and $v = \overline u \wedge u = u - (u - \overline u)^+ \in K$ in (3.24), and get by adding the resulting inequalities the following one:

$$\begin{aligned} \langle A\overline u - Au, (u - \overline u)^+\rangle \quad &- \quad \int_\Omega b(x,u)\,(u-\overline u)^+\,dx + \int_\Omega (\overline\eta - \tilde\eta)\,(u - \overline u)^+\,dx \\ &+ \int_{\partial\Omega} (\overline\xi - \tilde\xi)\,(\gamma u - \gamma\overline u)^+\,d\sigma \ge 0. \end{aligned} \tag{3.25}$$

With the notation $\{u > \overline u\} = \{x \in \Omega : u(x) > \overline u(x)\}$ and $\{\gamma u > \gamma\overline u\} = \{x \in \partial\Omega : \gamma u(x) > \gamma\overline u(x)\}$, and by applying the results (P1)(iii) of Lemma 3.1 and (P2)(iii) of Lemma 3.2 we obtain

$$\int_\Omega (\overline\eta - \tilde\eta)\,(u-\overline u)^+\,dx = \int_{\{u > \overline u\}} (\overline\eta - \tilde\eta)\,(u - \overline u)\,dx = 0, \tag{3.26}$$

because $\tilde\eta(x) = \overline\eta(x)$ for $x \in \{u > \overline u\}$, and

$$\int_{\partial\Omega} (\overline\xi - \tilde\xi)\,(\gamma u - \gamma\overline u)^+\,d\sigma = \int_{\{\gamma u > \gamma\overline u\}} (\overline\xi - \tilde\xi)\,(\gamma u - \gamma\overline u)\,d\sigma = 0, \tag{3.27}$$

because $\tilde\xi(x) = \overline\xi(x)$ for $x \in \{\gamma u > \gamma\overline u\}$. Taking the definition of the cut-off function b into account we get

$$\int_\Omega b(x,u)\,(u - \overline u)^+\,dx = \int_\Omega \left((u - \overline u)^+\right)^p\,dx, \tag{3.28}$$

and by means of (A2) we obtain the estimate

$$\langle A\overline u - Au, (u-\overline u)^+\rangle = -\langle Au - A\overline u, (u - \overline u)^+\rangle \le 0. \tag{3.29}$$

Applying the results (3.26)–(3.29) to (3.25), we finally get

$$\int_\Omega \left((u - \overline u)^+\right)^p\,dx = 0,$$

which implies $(u - \overline{u})^+ = 0$, and thus $u \le \overline{u}$. The proof for $\underline{u} \le u$ can be done in a similar way. So far we have shown that any solution u of the auxiliary multi-valued variational inequality (3.24) belongs to the interval $[\underline{u}, \overline{u}]$, and thus satisfies: $u \in K$, $b(x, u(x)) = 0$, $\tilde{\eta} \in L^q(\Omega)$, $\tilde{\xi} \in L^q(\partial\Omega)$ and

$$\tilde{\eta}(x) \in \partial \tilde{j}_1(x, u(x)), \text{ a.e. } x \in \Omega, \ \ \tilde{\xi}(x) \in \partial \tilde{j}_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \tag{3.30}$$

$$\langle Au - h, v - u\rangle + \int_\Omega \tilde{\eta}\,(v - u)\,dx + \int_{\partial\Omega} \tilde{\xi}\,(\gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in K. \tag{3.31}$$

From (P1)(iii) of Lemma 3.1 we see that $\partial \tilde{j}_1(x, u(x)) \subseteq \partial j_1(x, u(x))$ for any $u \in [\underline{u}, \overline{u}]$, and from (P2)(iii) of Lemma 3.2 we see that $\partial \tilde{j}_2(x, \gamma u(x)) \subseteq \partial j_2(x, \gamma u(x))$ for $\gamma u \in [\gamma \underline{u}, \gamma \overline{u}]$, and therefore we also have

$$\tilde{\eta}(x) \in \partial j_1(x, u(x)), \text{ a.e. } x \in \Omega, \ \ \tilde{\xi}(x) \in \partial j_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega,$$

which shows that the solution $u \in [\underline{u}, \overline{u}]$ of the auxiliary problem is in fact a solution of the original multi-valued variational inequality (1.1). This completes the proof. □

Remark 3.4 Without difficulties, the existence and comparison result of Theorem 3.2 can be extended to more general elliptic operators A of Leray–Lions type such as

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u(x)) + a_0(x, u, \nabla u(x)).$$

4 Variational-hemivariational inequalities

Keeping the notations and hypotheses of the preceding sections, here we provide a short account of the sub-supersolution method for the variational-hemivariational inequality (1.2) which has been obtained recently in [3, 5]. Consider (1.2), which is: Find $u \in K$ such that

$$\langle Au - h, v - u\rangle + \int_\Omega j_1^o(x, u; v - u)\,dx + \int_{\partial\Omega} j_2^o(x, \gamma u; \gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in K.$$

By specifying the closed convex set $K \subseteq V$ one can see, in a similar way as in Sect. 3, that the variational-hemivariational inequality (1.2) includes various elliptic boundary value problems as special cases such as, e.g., the following one.

Example 4.1 If $K = V$, $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and if j_k, $k = 1, 2$, are the primitives of f_k given by (3.2), then $j_k^o(x, s; r) = \frac{\partial j_k}{\partial s}(x, s)\, r = f_k(x, s) r$, and thus (1.2) reduces to the following quasilinear elliptic BVP

$$\langle Au - h, v\rangle + \int_\Omega f_1(x, u)\, v\,dx + \int_{\partial\Omega} f_2(x, \gamma u)\,\gamma v\,d\sigma = 0, \quad \forall\, v \in V, \tag{4.1}$$

which is the formulation of the weak solution of the BVP

$$Au + f_1(x, u) = h \ \text{ in } \ \Omega, \ \ \frac{\partial u}{\partial \nu} + f_2(x, u) = 0 \ \text{ on } \ \partial\Omega, \tag{4.2}$$

where $\partial/\partial\nu$ denotes the outward pointing conormal derivative associated with A.

Next let us introduce the notion of sub- and supersolution, see [2].

Definition 4.1 *A function* $\underline{u} \in V$ *is called a* **subsolution** *of (1.2) if the following holds:*

$$(i)\ \underline{u} \vee K \subseteq K,$$

$$(ii)\ \langle A\underline{u} - h, v - \underline{u}\rangle + \int_\Omega j_1^o(x, \underline{u}; v - \underline{u})\, dx + \int_{\partial\Omega} j_2^o(x, \gamma\underline{u}; \gamma v - \gamma\underline{u})\, d\sigma \geq 0$$

for all $v \in \underline{u} \wedge K$.

Definition 4.2 *A function* $\overline{u} \in V$ *is called a* **supersolution** *of (1.2) if the following holds:*

$$(i)\ \overline{u} \wedge K \subseteq K,$$

$$(ii)\ \langle A\overline{u} - h, v - \overline{u}\rangle + \int_\Omega j_1^o(x, \overline{u}; v - \overline{u})\, dx + \int_{\partial\Omega} j_2^o(x, \gamma\overline{u}; \gamma v - \gamma\overline{u})\, d\sigma \geq 0$$

for all $v \in \overline{u} \vee K$.

Remark 4.1 Note again that the notion for sub- and supersolution defined in Definition 4.1 and Definition 4.2 have a symmetric structure, i.e., one obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}$ in Definition 4.1 by $\overline{u}$, and interchanging $\vee$ by $\wedge$. If we apply Definitions 4.1 and 4.2 to the Example 4.1 we'll see that the above definitions are in fact natural extensions of the usual notions of sub-supersolutions for elliptic boundary value problems.

As for the following existence and comparison result we refer to [3, 5].

Theorem 4.1 *Let (A1)–(A3) be satisfied. Assume the existence of sub- and supersolutions* $\underline{u}$ *and* $\overline{u}$*, respectively, of the variational-hemivariational inequality (1.2) with* $\underline{u} \leq \overline{u}$ *such that (j1)–(j2) is fulfilled. Then there exist solutions of (1.2) within the ordered interval* $[\underline{u}, \overline{u}]$.

5 Proof of the main result–Theorem 2.1

Let the hypotheses of Theorem 2.1 be satisfied. Our aim is to show that any solution of the multi-valued variational inequality (1.1) is a solution of the variational-hemivariational inequality (1.2), and vice versa. The basic tools to achieve our goal are Theorem 3.2 and Theorem 4.1 on the sub-supersolution method for problems (1.1) and (1.2), respectively.

Let $u \in K$ be a solution of (1.1), i.e., there is an $\eta \in L^q(\Omega)$, and a $\xi \in L^q(\partial\Omega)$ such that $\eta(x) \in \partial j_1(x, u(x))$, for a.e. $x \in \Omega$, $\xi(x) \in \partial j_2(x, \gamma u(x))$, for a.e. $x \in \partial\Omega$, and the following variational inequality is satisfied:

$$\langle Au - h, v - u\rangle + \int_\Omega \eta\,(v - u)\, dx + \int_{\partial\Omega} \xi\,(\gamma v - \gamma u)\, d\sigma \geq 0,\ \forall\, v \in K. \tag{5.1}$$

By the definition of Clarke's generalized gradient we readily obtain for any $v \in K$:

$$j_1^o(x, u(x); v(x) - u(x)) \geq \eta(x)\,(v(x) - u(x)),\ \text{for a.e. } x \in \Omega, \tag{5.2}$$

$$j_2^o(x, \gamma u(x); \gamma v(x) - \gamma u(x)) \geq \xi(x)\,(\gamma v(x) - \gamma u(x)),\ \text{for a.e. } x \in \partial\Omega. \tag{5.3}$$

Since the functions $(s,r) \mapsto j_k^o(x,s;r)$, $k = 1,2$, are superpositionally measurable, and because of (H1)(ii) and (H2)(ii), the left-hand sides of (5.2) and (5.3) can be estimated by

$$|j_1^o(x,u(x);v(x)-u(x))| \leq (k_\Omega(x)+c|u(x)|^{p-1})|v(x)-u(x)|, \tag{5.4}$$
$$|j_2^o(x,\gamma u(x);\gamma v(x)-\gamma u(x))| \leq (k_{\partial\Omega}(x)+c|\gamma u(x)|^{p-1})|\gamma v(x)-\gamma u(x)|. \tag{5.5}$$

Hence it follows that the left-hand sides of (5.2) and (5.3) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively. Thus from (5.2) and (5.3) we obtain

$$\int_\Omega j_1^o(x,u(x);v(x)-u(x))\,dx \geq \int_\Omega \eta(x)\,(v(x)-u(x))\,dx, \tag{5.6}$$
$$\int_{\partial\Omega} j_2^o(x,\gamma u(x);\gamma v(x)-\gamma u(x))\,dx \geq \int_{\partial\Omega} \xi(x)\,(\gamma v(x)-\gamma u(x))\,d\sigma. \tag{5.7}$$

The variational inequality (5.1) along with (5.6) and (5.7) implies that u is a solution of the variational-hemivariational inequality (1.2). One readily observes that this direction of the proof basically follows from the definition of Clarke's generalized gradient.

To prove the reverse, let u be any solution of (1.2), i.e., $u \in K$ and u satisfies (1.2) which is

$$\langle Au-h, v-u\rangle + \int_\Omega j_1^o(x,u;v-u)\,dx + \int_{\partial\Omega} j_2^o(x,\gamma u;\gamma v-\gamma u)\,d\sigma \geq 0,\ \forall\, v \in K. \tag{5.8}$$

The lattice condition (2.1) implies that u is both a subsolution and a supersolution for (1.2), i.e., of (5.8). Next we are going to show that u must be both a subsolution and a supersolution for (1.1). Let us show first that u is a subsolution for (1.1). Since u is a subsolution of (1.2), the inequality (5.8) is satisfied, in particular, for all $v \in u \wedge K$, i.e., $v = u-(u-\varphi)^+$ with $\varphi \in K$ which yields

$$\begin{aligned}\langle Au-h, -(u-\varphi)^+\rangle &+ \int_\Omega j_1^o(x,u;-(u-\varphi)^+)\,dx \\ &+ \int_{\partial\Omega} j_2^o(x,\gamma u;-(\gamma u-\gamma\varphi)^+)\,d\sigma \geq 0,\ \forall\, \varphi \in K.\end{aligned}$$

Because $r \mapsto j_k^o(\cdot,s;r)$ is positively homogeneous, the last inequality is equivalent to

$$\begin{aligned}\langle Au-h, -(u-\varphi)^+\rangle &+ \int_\Omega j_1^o(x,u;-1)(u-\varphi)^+\,dx \\ &+ \int_{\partial\Omega} j_2^o(x,\gamma u;-1)(\gamma u-\gamma\varphi)^+\,d\sigma \geq 0,\ \forall\, \varphi \in K.\end{aligned} \tag{5.9}$$

Using again for any $v \in u \wedge K$ its representation in the form $v = u-(u-\varphi)^+$ with $\varphi \in K$, (5.9) is equivalent to

$$\begin{aligned}\langle Au-h, v-u\rangle &- \int_\Omega j_1^o(x,u;-1)(v-u)\,dx \\ &- \int_{\partial\Omega} j_2^o(x,\gamma u;-1)(\gamma v-\gamma u)\,d\sigma \geq 0,\ \forall\, v \in u \wedge K.\end{aligned} \tag{5.10}$$

By [6, Proposition 2.1.2] we have

$$\begin{aligned} j_1^o(x,u(x);-1) &= \max\{-\theta(x) : \theta(x) \in \partial j_1(x,u(x))\} \\ &= -\min\{\theta(x) : \theta(x) \in \partial j_1(x,u(x))\} = -\underline{\eta}(x), \end{aligned} \tag{5.11}$$

where

$$\underline{\eta}(x) = \min\{\theta(x) : \theta(x) \in \partial j_1(x,u(x))\} \in \partial j_1(x,u(x)), \quad \text{for a.e. } x \in \Omega. \tag{5.12}$$

Since $x \mapsto j_1^o(x,u(x);-1)$ is a measurable function, it follows that $x \mapsto \underline{\eta}(x)$ is measurable in Ω too, and in view of the growth condition (H1)(ii), we infer $\eta \in L^q(\Omega)$.

In a similar way one can show that there is a $\underline{\xi} \in L^{\overline{q}}(\partial\Omega)$ with

$$\underline{\xi}(x) = \min\{\theta(x) : \theta(x) \in \partial j_2(x,\gamma u(x))\} \in \partial j_2(x,\gamma u(x)), \quad \text{for a.e. } x \in \partial\Omega. \tag{5.13}$$

such that

$$j_2^o(x,\gamma u(x);-1) = -\underline{\xi}(x), \quad \text{for a.e. } x \in \partial\Omega. \tag{5.14}$$

Taking (5.11)–(5.14) into account, (5.10) yields

$$\langle Au - h, v - u\rangle + \int_\Omega \underline{\eta}(v-u)\,dx + \int_{\partial\Omega} \underline{\xi}(\gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in u \wedge K, \tag{5.15}$$

which proves that u is a subsolution of (1.1). By applying similar arguments, one shows that u is also supersolution of (1.1), i.e., there is an $\overline{\eta} \in L^q(\Omega)$, and a $\overline{\xi} \in L^q(\partial\Omega)$ such that $\overline{\eta}(x) \in \partial j_1(x,u(x))$, for a.e. $x \in \Omega$, $\overline{\xi}(x) \in \partial j_2(x,\gamma u(x))$, for a.e. $x \in \partial\Omega$, and the following inequality is satisfied:

$$\langle Au - h, v - u\rangle + \int_\Omega \overline{\eta}\,(v-u)\,dx + \int_{\partial\Omega} \overline{\xi}\,(\gamma v - \gamma u)\,d\sigma \ge 0, \ \forall\, v \in u \vee K. \tag{5.16}$$

So far we have shown that any solution u of the variational-hemivariational inequality (1.2) is both a subsolution and a supersolution of the multi-valued variational inequality (1.1). Therefore, Theorem 3.2 ensures the existence of a solution $\tilde{u}$ of (1.1) within the interval $[u,u] = \{u\}$ which implies $u = \tilde{u}$. This proves that any solution u of (1.2) must be a solution of (1.1), which completes the proof. □

Remark 5.1 The equivalence of problems (1.1) and (1.2) ensured by Theorem 2.1 remains true also in case that the operator A is replaced by a more general Leray–Lions operator of the form as given in Remark 3.4.

References

[1] S. Carl, *Existence and comparison results for noncoercive and nonmonotone multivalued elliptic problems*, Nonlinear Anal. **65** (8)(2006), 1532–1546.

[2] S. Carl, V.K. Le, D. Motreanu, Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications, Springer, New York, 2007.

[3] S. Carl, *The sup-supersolution method for variational-hemivariational inequalities*, Nonlinear Anal. **69** (2008), 816–822.

[4] S. Carl, D. Motreanu, *General comparison principle for quasilinear elliptic inclusions*, Nonlinear Anal. **70** (2009), 1105–1112.

[5] S. Carl, P. Winkert, *General comparison principle for variational-hemivariational inequalities*, J. Inequal. Appl. **2009** (2009), Article ID 184348, 29 pages.

[6] F.H. Clarke, Optimization and Nonsmooth Analysis, SIAM, Philadelphia, 1990.

[7] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, New York, 1995.

[8] P.D. Panagiotopoulos, Hemivariational Inequalities and Applications in Mechanics and Engineering, Springer-Verlag, New York, 1993.

[9] E. Zeidler, Nonlinear Functional Analysis and its Applications, Vols. II B, Springer-Verlag, Berlin, 1990.