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Subharmonic Solutions of Planar Hamiltonian Systems: a Rotation Number Approach

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Abstract

We prove the existence of infinitely many subharmonic solutions, with prescribed nodal properties, for a planar Hamiltonian system $Jz' = \nabla_z H(t,z)$, with H periodic in the first variable. The goal is achieved by performing estimates of the rotation numbers with respect to deformed polar coordinates and applying Ding's version of the Poincaré-Birkhoff fixed point theorem.

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1 Introduction

In this paper we deal with the problem of the existence and multiplicity of subharmonic solutions for a planar Hamiltonian system of the type

$$Jz' = \nabla_z H(t,z) \qquad z = (x,y) \in \mathbb{R}^2, \tag{1.1}$$

being $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard symplectic matrix and $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ regular enough and such that:

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- H is T-periodic in the first variable, with $T > 0$ fixed,
- $\nabla_z H(t, 0) \equiv 0$.

Incidentally, we remark that the last condition is often unrestrictive. In fact, a typical preliminary step is given by the proof of the existence of at least one T-periodic solution; then, via the obvious change of variable which sends a periodic solution into the origin, the original problem is reduced to the existence of subharmonics for a planar system of the same type and for which the condition $\nabla_z H(t, 0) \equiv 0$ holds true.

The problem of the existence of (harmonic and) subharmonic solutions for a planar Hamiltonian system, and in particular for a conservative scalar second order equation like

$$u'' + g(t, u) = 0, \tag{1.2}$$

is classical and widely studied. Due to the variational structure of equation (1.1), tools from critical point theory like linking theorems can be successfully applied, even in dimension greater than two: we refer to the book [15] for a standard reference on the subject and to the more recent contributions [1], [4], [11], [12] for the specific problem of the existence of subharmonic solutions.

On the other hand, a more classical dynamical approach can be followed. In fact, if the uniqueness and the global continuability of the solutions of the initial value problems associated to the equation (1.1) are guaranteed, the Poincaré map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined (and it turns out to be a global homeomorphism of the plane) and, as well known, kT-periodic solutions ($k \in \mathbb{N}_0$) of (1.1) correspond to the fixed points of the k-th iterate Φ^k. In the particular case of an Hamiltonian flow, Liouville's theorem implies that Φ^k is area preserving and some refined versions of the classical Poincaré-Birkhoff fixed point theorem can be applied, the crucial point of the proof consisting, of course, in performing some careful estimates of the rotations of the solutions in order to show a "twist condition" for the Poincaré map.

For a detailed analysis of the relationship between these two different approaches, and in particular for a comparison between the classical twist condition in the Poincaré-Birkhoff theorem and tools of critical point theory like Morse and Maslov indexes, we refer to the paper [14]. We only remark that, with respect to variational techniques, the Poincaré-Birkhoff theorem seems to be particularly well suited in order to obtain fine *multiplicity* results for the existence of subharmonics, as information about nodal properties (and hence multiplicity) are intrinsic in the method itself.

From here on, we will focus on this dynamical approach, which will be used in the paper. In the particular case of the second order equation (1.2), Poincaré-Birkhoff theorem has been applied in many works (concerning the existence of both harmonic and subharmonic solutions), in order to cover a very wide set of nonlinearities: we refer in particular to [8] for the sublinear case and to [7], [17] for the superlinear one.

In this paper we consider the general Hamiltonian system (1.1), together with hypotheses (H_∞) on the Hamiltonian which generalize a (possibly one-side) sublinearity condition for the second order equation; on the other hand, a positive mean condition (H_0) is required near the equilibrium. In this way, small solutions in the phase-plane wind around the origin many times, while large solutions do not: as consequence, we get (via Poincaré-Birkhoff theorem) the existence of infinitely many subharmonic solutions (Theorems 3.1 and 4.1). As corollaries, some results for the second order equation (1.2) are obtained (Corollaries 3.1 and 4.1). With respect to previous works, the standard linearization hypotheses at zero and at infinity (as in [18]) are substituted by suitable inequalities

((g_0) and (g_∞)) only; in particular, no differentiability condition on the nonlinearity is required. The conclusion is analogous to that of the main result of [8], but no sign condition is required; moreover, while in this latter work a time-map approach is performed in order to study the behavior of large norm solutions, our proof relies only on some explicit computations of the rotation numbers.

The plan of the article is the following. In Section 2 we introduce a "modified rotation number", suitable for our problem, and relate it to the classical one, completing in this way the work of [19]. It is worth noticing that these systems of "deformed" polar coordinates are used not only as a useful trick in order to simplify some estimates (as in [17]), but play an essential role in the formulation of the hypotheses on the Hamiltonian at infinity.

In Section 3 and 4 we prove our main results and some related corollaries.

In Section 5, finally, an outline of a possible application to a class of Lotka-Volterra type planar systems is presented.

We end this introduction by recalling, for the reader's convenience, the main tools which will be used in the paper. For a (at least absolutely continuous) path $z = (x, y) : [s_1, s_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every t, we define the rotation number

$$\mathrm{Rot}(z; [s_1, s_2]) := \frac{1}{2\pi} \int_{s_1}^{s_2} \frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2} dt. \tag{1.3}$$

As well known, it represents an algebraic count of the *clockwise* windings around the origin of the path $z(t)$ in the time interval $[s_1, s_2]$; details will be given in Section 2. Based on the rotation number, we employ the following generalized version of the Poincaré-Birkhoff twist theorem, where $k \in \mathbb{N}_0$ is fixed.

Application of the Poincaré-Birkhoff fixed point theorem

Let us assume the uniqueness and the global continuability of the solutions of the Cauchy problems associated to equation (1.1) and denote by $z(\cdot; z_0)$ the solution with $z(0; z_0) = z_0$.
Let us suppose that there exist two circumferences $\Gamma_i = r\mathbb{S}^1$ and $\Gamma_o = R\mathbb{S}^1$, with $0 < r < R$, and $j \in \mathbb{N}_0$ such that:

- $\mathrm{Rot}(z(t; z_0); [0, kT]) > j$ *for every* $z_0 \in \Gamma_i$;
- $\mathrm{Rot}(z(t; z_0); [0, kT]) < j$ *for every* $z_0 \in \Gamma_o$.

Then, denoting with $\mathcal{A}$ the closed annulus having as inner and outer boundaries the circumferences Γ_i and Γ_o respectively, equation (1.1) has at least two kT-periodic solutions z_1, z_2 with $z_1(0), z_2(0) \in \mathcal{A}$ and such that

$$\mathrm{Rot}(z_1; [0, kT]) = \mathrm{Rot}(z_2; [0, kT]) = j.$$

This result can be proved by applying the Poincaré-Birkhoff theorem in the version given by W.Y. Ding in [5], to the k-th iterate of the Poincaré map as an area-preserving homeomorphism

$$\Phi^k : \mathcal{A} \subset \mathbb{R}^2 \setminus \{0\} \to \Phi^k(\mathcal{A}) \subset \mathbb{R}^2 \setminus \{0\}.$$

Observe that the condition $\nabla_z H(t, 0) \equiv 0$ implies that $z(t; z_0) \neq 0$ for every t whenever $z_0 \neq 0$; hence the rotation numbers are well defined. For a complete proof of the statements and more details, we refer to [17]. For the present paper it is sufficient to have stated the theorem for a standard annulus. However, the result holds even in greater generality for some topological planar annuli.

2 Modified rotation numbers

The use of a suitable system of “polar coordinates”, and the consequent definition of an associated rotation number, is an essential tool in the qualitative theory of ordinary differential equations in the plane. The most standard choice is given by the classical polar coordinates, which lead to the definition of the standard rotation number (1.3). Different choices, however, are possible and can be useful; we refer to the work [17] for a unifying approach on this matter.
The aim of this section is to introduce a system of deformed polar coordinates along a strictly star-shaped Jordan curve surrounding the origin, which allows the definition of a modified rotation number which will be extensively used in all the paper. The definition goes back to the work [19].

Definition 2.1 Let $\mathcal{P}$ be the set of all C^1 functions $V : \mathbb{R}^2 \to \mathbb{R}$ such that:

- V is positively homogeneous of degree 2, i.e. $V(\lambda z) = \lambda^2 V(z)$ for every $\lambda > 0$ and for every $z \in \mathbb{R}^2$:
- $V(z) > 0$ for every $z \neq 0$.

Let us recall that, if $V \in \mathcal{P}$, then $\lim_{|z|\to\infty} V(z) = +\infty$ and the Euler's formula

$$2V(z) = \nabla V(z) \cdot z$$

holds true. These properties easily imply that the open set $\{z \in \mathbb{R}^2 \mid V(z) < 1\}$ is a bounded neighborhood of the origin, with boundary

$$\Gamma_V = V^{-1}(1) \subset \mathbb{R}^2 \setminus \{0\}$$

which turns out to be a Jordan curve (i.e. a subset of $\mathbb{R}^2$ homeomorphic to the circle $\mathbb{S}^1$) surrounding the origin and strictly star-shaped with respect to it, in the sense that every ray emanating from $0 \in \mathbb{R}^2$ intersects the curve in exactly one point. We want to make $\mathbb{R}$ a (universal) covering space of Γ_V.

The following preliminary lemma is useful; we remark that it follows by the result of [16], but here we propose an independent proof.

Lemma 2.1 *The uniqueness and the global continuability of the solutions of the initial value problems associated to the autonomous planar Hamiltonian system*

$$Jz' = \frac{1}{2}\nabla V(z) \tag{2.1}$$

are ensured.

Proof. Let us suppose, by contradiction, that there exist two solutions z_1, z_2 of (2.1) defined in a neighborhood of 0 and such that $z_1(0) = z_2(0) = z_0 \in \mathbb{R}^2 \setminus \{0\}$; by the conservation of energy relations $V(z_1(t)) = V(z_2(t)) = V(z_0)$ and Euler's formula, we get

$$Jz_1'(t) \cdot z_1(t) = Jz_2'(t) \cdot z_2(t) = V(z_0) \neq 0. \tag{2.2}$$

Let us define, for (r, s) in a neighborhood of $(1, 0)$, the C^1 function

$$P(r, s) = rz_2(s) \in \mathbb{R}^2;$$

we have that $P(1, 0) = z_0$, while (2.2) implies that the Jacobian matrix $DP(1, 0)$ is invertible. Then, the local inversion theorem implies that there exist C^1 maps $r(t)$, $s(t)$, defined in a neighborhood of 0 and with values in a neighborhood of 1 and 0 respectively, such that

$$z_1(t) = P(r(t), s(t)) = r(t)z_2(s(t)).$$

The conservation of the energy and the homogeneity of V imply that $r(t) = 1$; hence $z_1(t) = z_2(s(t))$. On the other hand, differentiating this last equality and using relation (2.2), we obtain $s'(t) = 1$; as $s(0) = 0$, we get $s(t) = t$. In conclusion, $z_1(t) = z_2(t)$. The observation that every positive energy level set does not contain the origin implies the uniqueness for $z_0 = 0$ too, while the global continuability follows from the compactness of the energy level set $V^{-1}(c)$ for every $c \geq 0$.

So, denoting by z_V the (unique and globally defined) solution of the Cauchy problem

$$\begin{cases} Jz' = \frac{1}{2}\nabla V(z) \\ z(0) = V^{-1}(1) \cap (]0, +\infty[\times\{0\}) \end{cases} \qquad z = (x, y),$$

and by τ_V its minimal period, we get that the map $z_V : [0, \tau_V] \to \mathbb{R}^2$ gives a simple *clockwise* parametrization of Γ_V; hence Stokes' theorem and Euler's formula imply that

$$\begin{aligned} A_V \quad := \quad & \int_{\{V<1\}} dxdy = \frac{1}{2}\int_{\partial\{V<1\}^+} (xdy - ydx) = \\ = \quad & \frac{1}{2}\int_0^{\tau_V} Jz_V'(t) \cdot z_V(t)dt = \\ = \quad & \frac{1}{2}\int_0^{\tau_V} dt = \frac{\tau_V}{2}. \end{aligned}$$

We can finally define a natural covering projection $\Pi_V : \mathbb{R} \to \Gamma_V$ letting

$$\Pi_V(\theta) := z_V\left(\frac{\tau_v}{2\pi}\theta\right).$$

By the standard theory of covering spaces, for every absolutely continuous path $z : [s_1, s_2] \to \mathbb{R}^2$ such that $z(t) \neq 0$ for every $t \in [s_1, s_2]$, the path $[s_1, s_2] \to \Gamma_V$ given by

$$t \mapsto \frac{z(t)}{\sqrt{V(z(t))}}$$

can be lifted to the covering space $(\mathbb{R}, \Pi_V)$ of Γ_V, i.e. there exists an absolutely continuous path $\theta_V : [s_1, s_2] \to \mathbb{R}$ such that

$$\frac{z(t)}{\sqrt{V(z(t))}} = \Pi_V(\theta_V(t)).$$

Moreover, standard calculations show that, for almost every t,

$$\theta_V'(t) = \frac{2\pi}{\tau_V}\frac{Jz'(t) \cdot z(t)}{V(z(t))} = \frac{2\pi}{\tau_V}\left(\frac{y(t)x'(t) - x(t)y'(t)}{V(z(t))}\right).$$

Finally, we are led to give the following definition.

Definition 2.2 The (clockwise) *V-rotation number* of an absolutely continuous path $z : [s_1, s_2] \to \mathbb{R}^2$, such that $z(t) \neq 0$ for every $t \in [s_1, s_2]$, is the number

$$\begin{aligned} \mathrm{Rot}_V(z; [s_1, s_2]) &:= \frac{\theta_V(s_2) - \theta_V(s_1)}{2\pi} = \frac{1}{\tau_V} \int_{s_1}^{s_2} \frac{Jz'(t) \cdot z(t)}{V(z(t))} dt = \\ &= \frac{1}{2A_V} \int_{s_1}^{s_2} \frac{Jz'(t) \cdot z(t)}{V(z(t))} dt. \end{aligned}$$

Note that, as usual, the definition does not depend on the choice of the lifting θ_V.

Remark 2.1 In the language of differential forms, we have that

$$\mathrm{Rot}_V(z; [s_1, s_2]) = \frac{1}{2A_V} \int_z \omega_V,$$

where ω_V is the closed (by Euler's identity) differential form

$$\omega_V(x, y) = \frac{Jdz \cdot z}{V(z)} = \frac{ydx - xdy}{V(z)}.$$

This point of view can be useful in the proof of some properties of homotopy invariance for the V-rotation number, but will not be used in the sequel.

Note at this point that the standard (clockwise) rotation number as defined in (1.3) corresponds to the choice $V(x, y) = x^2 + y^2$, i.e. to the standard covering space $(\mathbb{R}, e^{-i\theta})$ of $\mathbb{S}^1 = V^{-1}(1)$. We will continue to denote this number simply by $\mathrm{Rot}(z; [s_1, s_2])$.

The next goal of the section is to investigate the relation of a V-rotation number as defined before with the standard one. We begin with the following lemma.

Lemma 2.2 *The map* $\Psi_V : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\Psi_V(\theta) = \frac{2\pi}{\tau_V} \int_0^\theta \frac{ds}{V(\cos s, -\sin s)}$$

is an increasing C^1*-homeomorphism of* $\mathbb{R}$*, such that, for every* $\theta \in \mathbb{R}$ *and for every* $k \in \mathbb{Z}$*,*

$$\Psi_V(\theta + 2k\pi) = \Psi_V(\theta) + 2k\pi. \tag{2.3}$$

In particular, for every $k \in \mathbb{Z}$*,*

$$\Psi_V(2k\pi) = 2k\pi. \tag{2.4}$$

Proof. As

$$\Psi_V'(\theta) = \frac{2\pi}{\tau_V} \frac{1}{V(\cos\theta, -\sin\theta)} > 0,$$

we have that Ψ_V is strictly increasing. Moreover, by the 2π-periodicity of the integrand, for every $\theta \in \mathbb{R}$ and for every $k \in \mathbb{Z}$,

$$\Psi_V(\theta + 2k\pi) = \Psi_V(\theta) + k\Psi_V(2\pi);$$

so the computation in (clockwise) polar coordinates

$$\tau_V = 2\int_{\{V\leq 1\}} dxdy = \int_0^{2\pi} \frac{ds}{V(\cos s, -\sin s)} = \frac{\tau_V}{2\pi}\Psi_V(2\pi)$$

implies (2.3). On the other hand, passing to the limit in (2.3) we conclude that $\Psi_V(\theta) \to \pm\infty$ as $\theta \to \pm\infty$ and so Ψ_V is a homeomorphism of $\mathbb{R}$. Finally, (2.4) follows from (2.3) and the fact that $\Psi_V(0) = 0$.

The next proposition shows a concrete way to compute a V-rotation number starting from the knowledge of the standard one.

Proposition 2.1 *Let $z : [s_1, s_2] \to \mathbb{R}^2$ be an absolutely continuous path, such that $z(t) \neq 0$ for every $t \in [s_1, s_2]$, and $\theta : [s_1, s_2] \to \mathbb{R}$ a lifting to the covering space $(\mathbb{R}, e^{-i\theta})$ of the path*

$$t \mapsto \frac{z(t)}{|z(t)|} \in \mathbb{S}^1.$$

Then

$$\mathrm{Rot}_V(z; [s_1, s_2]) = \frac{\Psi_V(\theta(s_2)) - \Psi_V(\theta(s_1))}{2\pi}. \tag{2.5}$$

Proof. Let $\Theta_V : [s_1, s_2] \to \mathbb{R}$ be the path defined by

$$\Theta_V(t) = \Psi_V(\theta(t)).$$

Since θ is absolutely continuous and Ψ_V is of class C^1, standard properties of absolutely continuous functions imply that Θ_V is absolutely continuous too; moreover, for a.e. $t \in [s_1, s_2]$,

$$\Theta_V'(t) = \Psi_V'(\theta(t))\theta'(t).$$

Since $z(t) = |z(t)|e^{-i\theta(t)}$, we get

$$\begin{aligned}\Theta_V'(t) &= \Psi_V'(\theta(t))\theta'(t) = \frac{2\pi}{\tau_V}\frac{1}{V(\cos\theta(t), -\sin\theta(t))}\frac{Jz'(t)\cdot z(t)}{|z(t)|^2} = \\ &= \frac{2\pi}{\tau_V}\frac{Jz'(t)\cdot z(t)}{V(z(t))}.\end{aligned}$$

This finally implies that

$$\begin{aligned}\mathrm{Rot}_V(z; [s_1, s_2]) &= \frac{1}{\tau_V}\int_{s_1}^{s_2} \frac{Jz'(t)\cdot z(t)}{V(z(t))}dt = \\ &= \frac{1}{2\pi}\int_{s_1}^{s_2} \Theta_V'(t)dt = \frac{\Theta_V(s_2) - \Theta_V(s_1)}{2\pi}.\end{aligned}$$

We easily deduce the following fundamental properties.

Proposition 2.2 *Let $z : [s_1, s_2] \to \mathbb{R}^2$ be an absolutely continuous path, such that $z(t) \neq 0$ for every $t \in [s_1, s_2]$, and $j \in \mathbb{Z}$. Then:*

$$\mathrm{Rot}_V(z; [s_1, s_2]) = j \Longleftrightarrow \mathrm{Rot}(z; [s_1, s_2]) = j; \tag{2.6}$$

$$\mathrm{Rot}_V(z; [s_1, s_2]) < j \Longleftrightarrow \mathrm{Rot}(z; [s_1, s_2]) < j; \tag{2.7}$$

$$\mathrm{Rot}_V(z; [s_1, s_2]) > j \Longleftrightarrow \mathrm{Rot}(z; [s_1, s_2]) > j. \tag{2.8}$$

Proof. Let $\theta : [s_1, s_2] \to \mathbb{R}$ be a lifting to the covering space $(\mathbb{R}, e^{-i\theta})$ of the path $t \mapsto \frac{z(t)}{|z(t)|}$; we begin to prove (2.6). If $\mathrm{Rot}_V(z; [s_1, s_2]) = j$ we deduce from Proposition 2.1 that

$$\Psi_V(\theta(s_2)) - \Psi_V(\theta(s_1)) = 2\pi j.$$

Defining $\theta_1^* = \theta(s_1) \bmod 2\pi$, $\theta_2^* = \theta(s_2) \bmod 2\pi$, (2.3) implies that

$$\Psi_V(\theta_2^*) - \Psi_V(\theta_1^*) \in 2\pi\mathbb{Z};$$

as $\Psi_V([0, 2\pi[) \subset [0, 2\pi[$ we conclude that $\theta_2^* = \theta_1^*$. Again by (2.3), we finally deduce that

$$\theta(s_2) - \theta(s_1) = 2\pi j,$$

that is $\mathrm{Rot}(z; [s_1, s_2]) = j$. Conversely, if $\mathrm{Rot}(z; [s_1, s_2]) = j$, then

$$\theta(s_2) - \theta(s_1) = 2\pi j$$

and Proposition 2.1 and (2.3) imply that $\mathrm{Rot}_V(z; [s_1, s_2]) = j$. We prove (2.7), the proof of (2.8) being similar. We have that $\mathrm{Rot}(z; [s_1, s_2]) < j$ if and only if

$$\theta(s_2) - (\theta(s_1) + 2\pi j) < 0;$$

being Ψ_V strictly increasing this is equivalent to the condition

$$\Psi_V(\theta(s_2)) - \Psi_V(\theta(s_1) + 2\pi j) < 0,$$

which by (2.3) is the same as

$$\Psi_V(\theta(s_2)) - \Psi_V(\theta(s_1)) < 2\pi j.$$

But this is the same as $\mathrm{Rot}(z; [s_1, s_2]) < j$, by Proposition 2.1.

Remark 2.2 A very useful choice of V is given by the diagonal quadratic form

$$V(x, y) = \frac{x^2}{c} + \frac{y^2}{d}$$

for some $c, d > 0$; in this case, clearly, $A_V = (\sqrt{cd})\pi$. The asymmetric situation

$$V(x, y) = \left(\frac{x^+}{c_1} - \frac{x^-}{c_2}\right)^2 + \left(\frac{y^+}{d_1} - \frac{y^-}{d_2}\right)^2$$

for some $c_1, c_2, d_1, d_2 > 0$ can be considered as well (see [19] for an application). In particular, in the case of a diagonal quadratic form, the symmetries of V imply that the homeomorphism Ψ_V satisfies the extra property

$$\Psi_V\left(k\frac{\pi}{2}\right) = k\frac{\pi}{2} \tag{2.9}$$

for every $k \in \mathbb{Z}$. From this fact we can easily deduce that

$$\left|\mathrm{Rot}_V(z; [s_1, s_2]) - \mathrm{Rot}(z; [s_1, s_2])\right| < \frac{1}{4}. \tag{2.10}$$

Rotation numbers of this kind have been often considered in literature (at least implicitly, as in [9]) and a systematic treatment is given in [17], where relations (2.6), (2.7), (2.8), (2.10) are proved with different arguments.

We end this section with a remark about the possibility of describing an arbitrary Jordan curve surrounding the origin as the level set $V^{-1}(1)$ for a (unique) $V \in \mathcal{P}$. We will use some concepts and results of differential geometry, for which we refer to [3].

We have already noticed that, if $\Gamma_V = V^{-1}(1)$ for $V \in \mathcal{P}$, then Γ_V is a Jordan curve around the origin and strictly star-shaped with respect to it. Moreover, since 1 is a regular value of V, the preimage theorem implies that Γ_V is a one dimensional C^1 embedded submanifold of $\mathbb{R}^2$. Finally, relation

$$J\Pi_V'(\theta) \cdot \Pi_V(\theta) = \text{const} \neq 0$$

implies that

$$\Pi_V(\theta) \notin \mathrm{T}_{\Pi_V(\theta)}\Gamma_V.$$

This last relation shows that every ray intersects Γ_V *transversally*. These conditions turn out to be sufficient, as the following proposition shows.

Proposition 2.3 *Let $\Gamma \subset \mathbb{R}^2 \setminus \{0\}$ be a Jordan curve surrounding the origin and strictly star-shaped with respect to it. Moreover, suppose that Γ is a one dimensional C^1 embedded submanifold such that for every $z \in \Gamma$ the transversality condition*

$$z \notin \mathrm{T}_z\Gamma \tag{2.11}$$

holds. Then there exists a unique $V \in \mathcal{P}$ such that $\Gamma_V = V^{-1}(1)$.

Proof. Uniqueness is quite obvious. On the other hand, we have to construct $V \in \mathcal{P}$. By definition of strictly star-shapedness of Γ with respect to the origin, for every $z \in \mathbb{R}^2 \setminus \{0\}$ there exists a unique $t(z) \in \,]0, +\infty[$ such that $t(z)z \in \Gamma$; by construction, the function $t(z)$ is also continuous on $\mathbb{R}^2 \setminus \{0\}$. Define $V : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$V(z) = \frac{1}{t(z)^2}.$$

It is clear that V is positively homogeneous of degree 2 and strictly positive; we claim that it is of class C^1 on $\mathbb{R}^2 \setminus \{0\}$. To see this, fix $z_0 \in \mathbb{R}^2 \setminus \{0\}$; it is well known that there exist an open neighborhood U of $t(z_0)z_0 \in \Gamma$ and a C^1 function $G : U \to \mathbb{R}$ with $\nabla G(t(z_0)z_0) \neq 0$ such that

$$\Gamma \cap U = G^{-1}(0). \tag{2.12}$$

Moreover, as $T_{t(z_0)z_0}\Gamma = \{w \in \mathbb{R}^2 \mid \nabla G(t(z_0)z_0) \cdot w = 0\}$, the transversality condition (2.11) implies that

$$\nabla G(t(z_0)z_0) \cdot z_0 \neq 0.$$

Define, for s in a neighborhood of $t(z_0)$ and z in a neighborhood of z_0, the map

$$F(s, z) = G(sz),$$

which is clearly of class C^1; moreover, $F(t(z_0), z_0) = 0$ and

$$\frac{\partial F}{\partial s}(t(z_0), z_0) = \nabla G(t(z_0)z_0) \cdot z_0 \neq 0.$$

Then the implicit function theorem implies that there exists a C^1 function $s(z)$, defined in a neighborhood of z_0 and with values in a neighborhood of $t(z_0)$, such that, in a neighborhood of $(t(z_0), z_0)$,

$$F(s, z) = 0 \iff s = s(z).$$

As $t(z)z \in \Gamma$ and $t(z)$ is continuous, (2.12) implies that $F(t(z), z) = G(t(z)z) = 0$ and hence $t(z) = s(z)$ in a neighborhood of z_0. We deduce that V is of class C^1 in a neighborhood of z_0 and hence, being a local property, on $\mathbb{R}^2 \setminus \{0\}$. Setting $V(0) = 0$, the positive homogeneity of degree 2 implies that V is of class C^1 on $\mathbb{R}^2$ concluding the proof.

Remark 2.3 Given a 2π periodic function $\gamma : \mathbb{R} \to \mathbb{R}^2$ of class C^1 such that, for every $s \in \mathbb{R}$,

$$J\gamma'(s) \cdot \gamma(s) > 0 \tag{2.13}$$

and that

$$\mathrm{Rot}(\gamma; [0, 2\pi]) = 1, \tag{2.14}$$

we claim that the image $\Gamma = \{\gamma(s) \mid s \in [0, 2\pi[\}$ verifies the hypotheses of Proposition 2.3.

To this aim, we denote by $\mathbb{R}/(2\pi\mathbb{Z})$ the interval $[0, 2\pi[$ with the topology which identifies the extreme points and with the natural differentiable structure and recall that γ can be viewed as a map $\gamma : \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbb{R}^2$ of class C^1 in the sense of the differentiable manifolds. Moreover, we observe that relation (2.13) implies that for every $[s, t] \subset [0, 2\pi]$

$$\mathrm{Rot}(\gamma; [s, t]) = \frac{1}{2\pi} \int_s^t \frac{J\gamma'(s) \cdot \gamma(s)}{|\gamma(s)|^2} ds > 0. \tag{2.15}$$

The fact that $\gamma'(s) \neq 0$ implies that γ is an immersion; we claim that it is injective. In fact, if $[t_1, t_2] \subset [0, 2\pi[$ are such that $\gamma(t_1) = \gamma(t_2)$, then $\mathrm{Rot}(\gamma; [t_1, t_2])$ is an integer number, strictly positive by relation (2.15); on the other hand, (2.15) implies that

$$\mathrm{Rot}(\gamma; [t_1, t_2]) < \mathrm{Rot}(\gamma; [0, 2\pi]) = 1$$

which is a contradiction.

Since $\mathbb{R}/(2\pi\mathbb{Z})$ is compact and connected, γ is actually an embedding; we deduce that $\Gamma = \gamma(\mathbb{R}/2\pi\mathbb{Z})$ is a compact, connected, one dimensional C^1 embedded submanifold of $\mathbb{R}^2$. In particular, Γ is a Jordan curve and, by (2.14), it surrounds the origin. Finally, the transversality condition (2.11) is clearly satisfied. It remains to show the condition of strictly star-shapedness.

1. We prove that every ray intersects Γ in at least one point. This is quite obvious; a formal proof can be sketched as follows. Consider the path

$$t \mapsto \tilde{\gamma}(t) = \frac{\gamma(t)}{|\gamma(t)|} \in \mathbb{S}^1$$

and let $\theta(t)$ a lifting with respect to the standard polar system; then relation (2.14) implies that $\theta(2\pi) - \theta(0) = 2\pi$. By the intermediate value theorem, every value in the interval $[\theta(0), \theta(2\pi)]$ is assumed by θ, that is $\tilde{\gamma}(t)$ is onto $\mathbb{S}^1$.

2. We prove that every ray intersects Γ in at most one point, that is $\tilde{\gamma}$ is injective on $[0, 2\pi[$. But this can be proved with the same argument used to show that γ is injective.

It is worth noticing that if we drop condition (2.14), then the result is false, as Γ is not even a Jordan curve. To see this, it is enough to consider the C^1 curve $\gamma : \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbb{R}^2$ given by

$$\gamma(t) = (\sin(2t), (t^2(t - 2\pi)^2 + 4)\cos(2t)),$$

which satisfies condition (2.13). The set $\Gamma = \gamma(\mathbb{R}/(2\pi\mathbb{Z}))$ is plotted in the figure below with MAPLE® software.

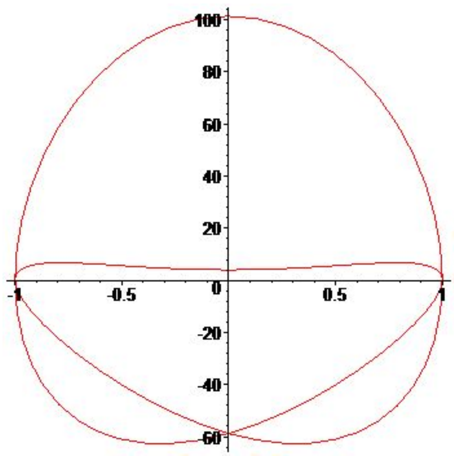

Figure 1: The set $\Gamma = \gamma(\mathbb{R}/(2\pi\mathbb{Z}))$

3 A first result of multiplicity

In this section we prove a first result of multiplicity about the existence of subharmonic solutions for equation (1.1). The goal will be achieved by performing some estimates of the modified rotation numbers introduced in the previous section and applying Ding's version of the Poincaré-Birkhoff fixed point theorem, as stated in the Introduction. Throughout the section, we will assume that $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 Carathéodory function, that is:

- $t \mapsto H(t,z)$ is measurable for every $z \in \mathbb{R}^2$;
- $z \mapsto H(t,z)$ is of class C^1 for almost every $t \in \mathbb{R}$;
- for every $r > 0$ there exists $\zeta_r \in L^1(]0,T[)$ such that $|\nabla_z H(t,z)| \leq \zeta_r(t)$ for a.e. $t \in [0,T]$ and for every $z \in \mathbb{R}^2$ with $|z| \leq r$.

Accordingly, solutions of (1.1) will be considered in the Carathéodory sense, that is (locally) absolutely continuous functions that solve the differential equation for a.e t.

Remark 3.1 According to [6], [7], [17], by subharmonic solution of order k of equation (1.1) we mean a kT-periodic solution which is not lT-periodic for $l = 1,\dots,k-1$; this does not imply in general that kT is the minimal period. We recall that other definitions of subharmonics are possible, like the one in [15]. Moreover, by periodicity class of a subharmonic solution z we mean the set $\{z, z(\cdot + T), z(\cdot + 2T), \dots, z(\cdot + (k-1)T)\}$; by the T-periodicity of H, these functions are all (subharmonic) solutions of (1.1).

Hereafter, we will always suppose that *the uniqueness and the global continuability of the solutions of the initial value problems associated to (1.1) are guaranteed.* We recall that this assumption, together with the fact that $\nabla_z H(t,0) \equiv 0$, implies the following well known "elastic property":

- for every $s, T^* > 0$, there exists $0 < r < s$ such that
$$|z_0| \leq r \Longrightarrow |z(t;z_0)| \leq s \text{ for every } t \in [0,T^*];$$
- for every $S, T^* > 0$, there exists $0 < S < R$ such that
$$|z_0| \geq R \Longrightarrow |z(t;z_0)| \geq S \text{ for every } t \in [0,T^*].$$

For a proof and more comments on this classical subject, see [13], [19].

We can state our first main result.

Theorem 3.1 *Let us suppose that:*

(H_0) *there exist $V_0 \in \mathcal{P}$, $a_0 \in L^1(]0,T[)$ with $\int_0^T a_0(t)dt > 0$ such that*
$$\liminf_{z \to 0} \frac{\nabla_z H(t,z) \cdot z}{V_0(z)} \geq a_0(t)$$
uniformly for a.e. $t \in [0,T]$;

(H_∞) *there exist two sequences $(V_\infty^n) \subset \mathcal{P}$, $(a_\infty^n) \subset L^1(]0,T[)$ such that:*

i)
$$\inf_n \frac{\int_0^T a_\infty^n(t)dt}{A_{V_\infty^n}} \leq 0;$$

ii) for every $n \in \mathbb{N}_0$

$$\limsup_{|z|\to+\infty} \frac{\nabla_z H(t,z)\cdot z}{V_\infty^n(z)} \le a_\infty^n(t)$$

uniformly for a.e. $t \in [0,T]$.

Then, for every $j \in \mathbb{N}_0$ *there exists* $m_j \in \mathbb{N}_0$ *such that, for every* $k \ge m_j$ *with k prime with j, equation (1.1) has at least two subharmonic solutions* $z_{j,k}^1, z_{j,k}^2$ *of order k, not belonging to the same periodicity class, with*

$$\mathrm{Rot}(z_{j,k}^1;[0,kT]) = \mathrm{Rot}(z_{j,k}^2;[0,kT]) = j.$$

Remark 3.2 Note that the definition of subharmonics adopted before implies that the above solutions are pairwise different. It is worth noticing, moreover, that, taking $j = 1$, Theorem 3.1 ensures the existence of subharmonics of order k for every sufficiently large integer k.

Proof. Let us fix $j \in \mathbb{N}_0$; we claim that there exists $m_j \in \mathbb{N}_0$ such that, for every $k \ge m_j$:

- there exists a circumference Γ_i^k centered at the origin such that, for every $z_0 \in \Gamma_i^k$,

$$\mathrm{Rot}(z(t;z_0);[0,kT]) > j;$$

- there exists a circumference Γ_o^k centered at the origin such that, for every $z_0 \in \Gamma_o^k$,

$$\mathrm{Rot}(z(t;z_0);[0,kT]) < 1 \le j.$$

In fact, let $\epsilon_1 > 0$ be so small that $a_0^{\epsilon_1}(t) = a_0(t) - \epsilon_1$ has positive mean and $V_0 \in \mathcal{P}$ as in hypothesis (H_0) and define m_j as the smallest integer strictly greater than $\frac{2A_{V_0}}{\int_0^T a_0^{\epsilon_1}(t)dt} j$. By hypothesis (H_0) there exists $s_{\epsilon_1} > 0$ such that

$$\frac{\nabla_z H(t,z)\cdot z}{V_0(z)} \ge a_0(t) - \epsilon_1 = a_0^{\epsilon_1}(t)$$

for a.e. $0 \le t \le T$ and $0 < |z| \le s_{\epsilon_1}$ and by the elastic property there exists $0 < r_{\epsilon_1} \le s_{\epsilon_1}$ such that

$$|z_0| \le r_{\epsilon_1} \Longrightarrow |z(t;z_0)| \le s_{\epsilon_1}$$

for every $0 \le t \le kT$. Define $\Gamma_i^k = \{z \in \mathbb{R}^2 \mid |z| = r_{\epsilon_1}\}$; we have that, if $z_0 \in \Gamma_i^k$,

$$\begin{aligned}
\mathrm{Rot}_{V_0}(z(t;z_0);[0,kT]) &= \frac{1}{2A_{V_0}}\int_0^{kT} \frac{Jz'(t;z_0)\cdot z(t;z_0)}{V_0(z(t;z_0))}dt = \\
&= \frac{1}{2A_{V_0}}\int_0^{kT} \frac{\nabla_z H(t,z(t;z_0))\cdot z(t;z_0)}{V_0(z(t;z_0))}dt \ge \\
&\ge \frac{1}{2A_{V_0}}\int_0^{kT} a_0^{\epsilon_1}(t)dt = \\
&= \frac{k}{2A_{V_0}}\int_0^{T} a_0^{\epsilon_1}(t)dt \ge \\
&\ge \frac{m_j}{2A_{V_0}}\int_0^{T} a_0^{\epsilon_1}(t)dt > j
\end{aligned}$$

and by Proposition 2.2 this implies that

$$\mathrm{Rot}(z(t;z_0);[0,kT]) > j.$$

On the other hand, by hypothesis (H_∞) there exist $V_\infty^n \in \mathcal{P}$, $a_\infty^n \in L^1(]0,T[)$ such that

$$\frac{\int_0^T a_\infty^n(t)dt}{A_{V_\infty^n}} < \frac{1}{k}$$

and

$$\limsup_{|z|\to+\infty} \frac{\nabla_z H(t,z)\cdot z}{V_\infty^n(z)} \le a_\infty^n(t), \qquad \text{uniformly for a.e. } t\in[0,T].$$

So, taken $0 < \epsilon_2 < \frac{A_{V_\infty^n}}{kT}$ there exists $S_{\epsilon_2} > 0$ such that

$$\frac{\nabla_z H(t,z)\cdot z}{V_\infty^n(z)} \le a_\infty^n(t) + \epsilon_2$$

for a.e. $0 \le t \le T$ and for every $|z| \ge S_{\epsilon_2}$ and by the elastic property again there exists $0 < S_{\epsilon_2} < R_{\epsilon_2}$ such that

$$|z_0| \ge R_{\epsilon_2} \Longrightarrow |z(t;z_0)| \ge S_{\epsilon_2}$$

for every $0 \le t \le kT$. Define $\Gamma_o^k = \{z \in \mathbb{R}^2 \mid |z| = R_{\epsilon_2}\}$; we have that, if $z_0 \in \Gamma_o^k$,

$$\begin{aligned}
\mathrm{Rot}_{V_\infty^n}(z(t;z_0),[0,kT]) &= \frac{1}{2A_{V_\infty^n}}\int_0^{kT}\frac{Jz'(t;z_0)\cdot z(t;z_0)}{V_\infty^n(z(t;z_0))}dt = \\
&= \frac{1}{2A_{V_\infty^n}}\int_0^{kT}\frac{\nabla_z H(t,z(t;z_0))\cdot z(t;z_0)}{V_\infty^n(z(t;z_0))}dt \le \\
&\le \frac{1}{2A_{V_\infty^n}}\left(\int_0^{kT} a_\infty^n(t)dt + kT\epsilon_2\right) = \\
&= \frac{k}{2}\frac{\int_0^T a_\infty^n(t)dt}{A_{V_\infty^n}} + \frac{kT\epsilon_2}{2A_{V_\infty^n}} < \\
&< \frac{1}{2}+\frac{1}{2} = 1 \le j
\end{aligned}$$

and by Proposition 2.2 again we get

$$\mathrm{Rot}(z(t;z_0);[0,kT]) < j.$$

Denoting with $\mathcal{A}^k$ the closed annulus having Γ_i^k and Γ_o^k as inner and outer boundaries, we conclude that there exist two kT-periodic solutions $z_{j,z}^1, z_{j,k}^2$ (not belonging to the same periodicity class by a remark of Neumann to the Poincaré-Birkhoff theorem, see [17] for some details) with $z_{j,z}^1(0), z_{j,z}^2(0) \in \mathcal{A}^k$ and such that

$$\mathrm{Rot}(z_{j,k}^1;[0,kT]) = \mathrm{Rot}(z_{j,k}^2;[0,kT]) = j.$$

If k is prime with j, then it can be proved in a standard manner that these solutions are actually subharmonics of order k.

Remark 3.3 Some remarks about the condition (H_∞) are in order. We first remark that an admissible choice for *i*) is given by $a^n_\infty \equiv 0$; in this case, as for every $V \in \mathcal{P}$ there exist $m, M > 0$ such that

$$m|z|^2 \leq V(z) \leq M|z|^2,$$

condition *ii*) does not depend on $V \in \mathcal{P}$. In particular, Theorem (3.1) holds in the sublinear case

$$\limsup_{|z|\to+\infty} \frac{\nabla_z H(t,z) \cdot z}{|z|^2} \leq 0, \qquad \text{uniformly for a.e. } t \in [0,T]. \tag{3.1}$$

On the other hand, the Theorem covers situations for which (3.1) is not satisfied, but such that $\limsup_{|z|\to+\infty} \frac{\nabla_z H(t,z)\cdot z}{V(z)}$ can be made "arbitrarily small", in some L^1 sense, with suitable choices of $V \in \mathcal{P}$. It is worth noticing that this kind of condition has to be invariant under the dilatation

$$\begin{cases} a_\infty \mapsto \lambda a_\infty \\ V_\infty \mapsto \frac{1}{\lambda} V_\infty, \end{cases}$$

since the second requirement of hypothesis (H_∞) is. This is indeed the case, as $A_{\frac{1}{\lambda}V} = \lambda A_V$ for every $V \in \mathcal{P}$ and $\lambda > 0$.

This last consideration can be successfully applied in order to get a more familiar result in the case of the conservative scalar second order equation (1.2), with $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ a Carathéodory function, T-periodic in the first variable and such that $g(t,0) = 0$ for a.e. $t \in \mathbb{R}$. As usual, we assume uniqueness and global continuability for the solutions of the Cauchy problems associated to (1.2).

Corollary 3.1 *Let us suppose that for every $r > 0$ there exists $C_r > 0$ such that $|g(t,x)| \leq C_r$ for a.e. $t \in [0,T]$ and for every $x \in \mathbb{R}$ with $|x| \leq r$; moreover suppose that:*

(g_0) *there exists $q_0 \in L^1(]0,T[)$ with $\int_0^T q_0(t)dt > 0$ such that*

$$\liminf_{x\to 0} \frac{g(t,x)}{x} \geq q_0(t)$$

uniformly for a.e. $t \in [0,T]$;

(g_∞)

$$\limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \leq 0$$

uniformly for a.e. $t \in [0,T]$.

Then, for every $j \in \mathbb{N}_0$ there exists $m_j \in \mathbb{N}_0$ such that, for every $k \geq m_j$ with k prime with j, equation (1.2) has at least two subharmonic solutions $u^1_{j,k}, u^2_{j,k}$ of order k, not belonging to the same periodicity class, with exactly $2j$ zeros in the interval $[0,kT[$.

Proof. Write (1.2) as the equivalent first order system

$$\begin{cases} x' = y \\ y' = -g(t,x), \end{cases}$$

which is of the form (1.1) with

$$H(t,x,y) = \frac{1}{2}y^2 + \int_0^x g(t,s)ds.$$

Clearly, H is a C^1 Carathéodory function, T-periodic in the first variable and such that $\nabla_z H(t,0) \equiv 0$. We claim that (H_0) and (H_∞) are satisfied. Indeed, take $\rho, \sigma > 0$ so small that $a_0(t) = \min(q_0(t), \frac{1}{\rho}) - \sigma$ has positive mean and define $V_0(x,y) = \rho y^2 + x^2$. By hypothesis (g_0), for a.e. $t \in [0,T]$ and x in a sufficiently small neighborhood of 0 we have

$$\frac{g(t,x)}{x} \geq q_0(t) - \sigma \geq a_0(t);$$

moreover

$$a_0(t) \leq a_0(t) + \sigma \leq \frac{1}{\rho}.$$

Hence we get

$$\begin{aligned}\liminf_{z\to 0} \frac{\nabla_z H(t,z)\cdot z}{V_0(z)} &= \liminf_{z\to 0} \frac{y^2 + g(t,x)x}{\rho y^2 + x^2} \geq \\ &\geq \liminf_{z\to 0} \frac{\rho a_0(t) y^2 + a_0(t)x^2}{\rho y^2 + x^2} \geq \\ &\geq a_0(t)\end{aligned}$$

uniformly for a.e $t \in [0,T]$.
On the other hand, define $V_\infty^n(x,y) = x^2 + 2nTy^2$ and $a_\infty^n(t) = \frac{1}{nT}$, so that

$$\inf_n \frac{\int_0^T a_\infty^n(t)dt}{A_{V_\infty^n}} = \inf_n \sqrt{\frac{2T}{n}}\frac{1}{\pi} = 0.$$

By hypothesis (g_∞) there exists $R_n^1 > 0$ such that for a.e. $t \in [0,T]$ and for $|x| \geq R_n^1$

$$\frac{g(t,x)}{x} \leq \frac{1}{2nT};$$

moreover, there exists $R_n^2 > 0$ such that for a.e. $t \in [0,T]$ and for $|x| \leq R_n^1$, $|y| \geq R_n^2$

$$\frac{g(t,x)x}{x^2 + 2nTy^2} \leq \frac{C_{R_n^1} R_n^1}{2nTy^2} \leq \frac{1}{2nT}$$

We deduce that, for a.e. $t \in [0,T]$ and for every $z \in \mathbb{R}^2 \setminus ([-R_n^1, R_n^1] \times [-R_n^2, R_n^2])$, we have

$$\begin{aligned}\frac{\nabla_z H(t,z)\cdot z}{V_\infty^n(z)} &= \frac{y^2}{x^2 + 2nTy^2} + \frac{g(t,x)x}{x^2 + 2nTy^2} \leq \\ &\leq \frac{1}{2nT} + \frac{1}{2nT} = a_\infty^n(t);\end{aligned}$$

so we get

$$\limsup_{|z|\to+\infty} \frac{\nabla_z H(t,z)\cdot z}{V^n_\infty(z)} \le a^n_\infty(t)$$

uniformly for a.e $t \in [0,T]$. We conclude by applying Theorem 3.1 and recalling that, as usual for second order scalar equations (see [17]), the fact that

$$\mathrm{Rot}((x(t;z_0), y(t;z_0)); [0,kT]) = j$$

implies that $u = x$ has exactly $2j$ zeros in the interval $[0,kT[$.

Remark 3.4 We recall that the global continuability of the solutions of the initial value problems is ensured if, instead of (g_∞), we suppose that:

(g^*_∞) there exists $m \in L^1(]0,T[)$ such that

$$-m(t) \le \liminf_{|x|\to+\infty} \frac{g(t,x)}{x} \le \limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \le 0$$

uniformly for a.e. $t \in [0,T]$.

Remark 3.5 Note that, with respect to some well known results about rotations of second order scalar equations, no sign condition is required in the hypothesis at 0. On the other hand, hypothesis at infinity cannot be improved to the mean condition

$$\limsup_{|x|\to+\infty} \frac{g(t,x)}{x} \le q_\infty(t) \quad \text{with} \int_0^T q_\infty(t)dt \le 0.$$

In fact, it is enough to consider the linear Hill's equation

$$u'' + q(t)u = 0 \tag{3.2}$$

with a two-step periodic potential

$$q(t) = \begin{cases} 1 & 0 < t < S \\ -\omega & S < t < T, \end{cases} \qquad \omega > 0.$$

We claim that if

$$S > 2\arctan\sqrt{\omega} \tag{3.3}$$

then

$$\lim_{t\to+\infty} \mathrm{Rot}(z(t;z_0); [0,t]) = +\infty \tag{3.4}$$

uniformly for $z_0 \in \mathbb{R}^2 \setminus \{0\}$. In fact, elementary considerations about the linear autonomous equation $u'' + u = 0$ show that,

$$\mathrm{Rot}(z(t;z_0); [kT, kT+S]) \ge \frac{S}{2\pi} \quad \text{for every } k \in \mathbb{N}; \tag{3.5}$$

on the other hand, since the standard angular speed θ' can be negative only in the regions

$$\mathcal{I}^+ = \{(x,y) \in \mathbb{R}^2 \mid -\sqrt{\omega}x \le y \le \sqrt{\omega}x\}$$

$$\mathcal{I}^- = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{\omega}x \leq y \leq -\sqrt{\omega}x\},$$

which are both positively invariant for the linear autonomous equation $u'' - \omega u = 0$, we see that

$$\mathrm{Rot}(z(t; z_0); [kT + S, (k+1)T]) \geq -\frac{\arctan \sqrt{\omega}}{\pi} \quad \text{for every } k \in \mathbb{N}. \tag{3.6}$$

Clearly, (3.5) and (3.6) imply (3.4). As (3.3) is compatible with a (arbitrarily large) negative mean for $q(t)$ (note in particular that it is always satisfied when $S \geq 2\pi$) the conclusion follows.

4 A second result of multiplicity

In this section we prove an improvement of Theorem 3.1. Roughly speaking, the ideas are the following:

- in order to have a positive angular speed for the small solutions, it would be sufficient to have a partition of the plane into several angular sectors, each with a positive weight for hypothesis (H_0), in such a way that the crossing between two adjacent regions is possible only "clockwise";
- in order to have a low angular speed for the large solutions, it would be sufficient that the sublinearity-like condition (H_∞) holds in a small angular sector.

Conditions of this kind go back to [2], Th. 2.10.1; we also refer to [10] for some recent contributions in the same spirit.

We preliminarily introduce some notations; recall that every point of $\mathbb{R}^2 \setminus \{0\}$ can be expressed in (clockwise) polar coordinates $z = \rho e^{-i\theta}$ for a unique $\rho > 0$ and $\theta \in [0, 2\pi[$.

Definition 4.1 Let $\theta \in [0, 2\pi[$. We denote by $\mathcal{L}(\theta)$ the open half-line

$$\mathcal{L}(\theta) := \{\rho e^{-i\theta} \mid \rho > 0\} \subset \mathbb{R}^2 \setminus \{0\}.$$

Definition 4.2 Let $\theta_1, \theta_2 \in [0, +\infty[$ with $0 < \theta_2 - \theta_1 \leq 2\pi$. We denote by $\mathcal{R}(\theta_1, \theta_2)$ the open angular region

$$\mathcal{R}(\theta_1, \theta_2) := \{\rho e^{-i\theta} \mid \rho > 0,\ \theta_1 < \theta < \theta_2\} \subset \mathbb{R}^2 \setminus \{0\}.$$

An ordered p-uple $(\theta_1, \ldots, \theta_p)$ $(p \geq 2)$ with

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 2\pi$$

determines a subdivision $\mathcal{S}(\theta_1, \ldots, \theta_p)$ of the plane into p angular regions

$$\mathcal{R}_1 = \mathcal{R}(\theta_1, \theta_2),$$

$$\mathcal{R}_2 = \mathcal{R}(\theta_2, \theta_3),$$

$$\vdots$$

$$\mathcal{R}_p = \mathcal{R}(\theta_p, \theta_1 + 2\pi).$$

Moreover, in order to formulate a general condition on the Hamiltonian near infinity, it is essential to introduce the following definition.

Definition 4.3 Let $\theta_1, \theta_2 \in [0, +\infty[$ with $0 < \theta_2 - \theta_1 < 2\pi$ and $\mathcal{F} \subset \mathcal{P}$. We say that $\mathcal{F}$ is *admissible* with respect to (θ_1, θ_2) if

$$\inf_{V \in \mathcal{F}} \frac{\Psi_V(\theta_2) - \Psi_V(\theta_1)}{2\pi} > 0. \tag{4.1}$$

Remark 4.1 We emphasize that, even if we proved that for every $V \in \mathcal{P}$

$$\frac{\Psi_V(\theta + 2\pi) - \Psi_V(\theta)}{2\pi} = 1,$$

there is in general no positive lower bound for the quantity $\frac{\Psi_V(\theta_2) - \Psi_V(\theta_1)}{2\pi}$ for arbitrary θ_1, θ_2. In other words, the size of an angular sector can become arbitrarily small when measured with suitable $V \in \mathcal{P}$; as we want to formulate a condition on the Hamiltonian involving only the behavior in an angular sector, a condition like (4.1) seems necessary. Note that, as

$$\frac{\Psi_V(\theta_2) - \Psi_V(\theta_1)}{2\pi} = \frac{1}{2A_V} \int_{\theta_1}^{\theta_2} \frac{d\theta}{V(\cos\theta, -\sin\theta)},$$

a class $\mathcal{F} \subset \mathcal{P}$ is certainly admissible with respect to (θ_1, θ_2) if there exists $C > 0$ such that

$$A_V V(e^{-i\theta}) \leq C \tag{4.2}$$

for every $V \in \mathcal{F}$ and for every $\theta_1 < \theta < \theta_2$.

On the other hand, a class $\mathcal{F} \subset \mathcal{P}$ can be admissible even if relation (4.2) is not satisfied; a crucial example, which will be used in the sequel, is given by the set $\mathcal{D}$ of all diagonal quadratic forms with respect to the angles $(\theta_1, \theta_2) = (j\frac{\pi}{2}, (j+1)\frac{\pi}{2})$ for some $j = 0, \ldots, 3$. In this case, indeed, relation (2.9) implies that $\mathcal{D}$ is admissible with respect to (θ_1, θ_2).

We are now in a position to state and prove our result. Throughout the section, we will assume that $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, of class C^1 in the variable $z = (x, y)$; accordingly, Carathéodory solutions will be of class C^1 and, hence, classical. We emphasize in particular, that, for every $z_0 \in \mathbb{R}^2 \setminus \{0\}$ and $V \in \mathcal{P}$, any lifting $\theta_V(t; z_0)$ of $\frac{z(t;z_0)}{\sqrt{V(z(t;z_0))}}$ to the covering space $(\mathbb{R}, \Pi_V)$ is of class C^1 and for *every t*

$$\theta_V'(t; z_0) = \frac{\pi}{A_V} \frac{\nabla_z H(t, z(t; z_0)) \cdot z(t; z_0)}{V(z(t; z_0))}.$$

Theorem 4.1 *Let us suppose that:*

(H_0') *there exist a subdivision* $\mathcal{S}(\theta_0^1, \ldots, \theta_0^p)$ *of the plane into p angular regions,* $V_0 \in \mathcal{P}$, $a_0^1, \ldots, a_0^p \in L^1(]0, T[)$ *with* $\int_0^T a_0^i(t)dt > 0$ *for* $i = 1, \ldots, p$ *such that:*

i) for every $i = 1, \ldots, p$

$$\nabla_z H(t, z) \cdot z > 0 \qquad \textit{for every } t \in [0, T], \textit{ for every } z \in \mathcal{L}(\theta_0^i);$$

ii) for every $i = 1, \dots, p$

$$\liminf_{\substack{z \to 0 \\ z \in \mathcal{R}_i}} \frac{\nabla_z H(t,z) \cdot z}{V_0(z)} \geq a_0^i(t)$$

uniformly in $t \in [0, T]$;

(H'_∞) *there exist an angular region* $\mathcal{R}(\theta_\infty^1, \theta_\infty^2)$ *and two sequences* $(V_\infty^n) \subset \mathcal{P}$, $(a_\infty^n) \subset L^1(]0, T[)$ *with* $a_\infty^n \geq 0$ *such that:*

i) the class $\mathcal{F} = \{V_\infty^n\}_{n \in \mathbb{N}_0}$ *is admissible with respect to* $(\theta_\infty^1, \theta_\infty^2)$;

ii)

$$\inf_n \frac{\int_0^T a_\infty^n(t) dt}{A_{V_\infty^n}} = 0;$$

iii) for every $n \in \mathbb{N}_0$,

$$\limsup_{\substack{|z| \to +\infty \\ z \in \mathcal{R}(\theta_\infty^1, \theta_\infty^2)}} \frac{\nabla_z H(t,z) \cdot z}{V_\infty^n(z)} \leq a_\infty^n(t)$$

uniformly in $t \in [0, T]$.

Then, for every $j \in \mathbb{N}_0$ *there exists* $m_j \in \mathbb{N}_0$ *such that, for every* $k \geq m_j$ *with k prime with j, equation (1.1) has at least two subharmonic solutions* $z_{j,k}^1, z_{j,k}^2$ *of order k, not belonging to the same periodicity class, with*

$$\mathrm{Rot}(z_{j,k}^1; [0, kT]) = \mathrm{Rot}(z_{j,k}^2; [0, kT]) = j.$$

Proof. The proof follows the same line of that of Theorem 3.1; in particular, fixed $j \in \mathbb{N}_0$, we will prove that there exists $m_j \in \mathbb{N}_0$ such that, for every $k \geq m_j$:

- there exists a circumference Γ_i^k centered at the origin such that, for every $z_0 \in \Gamma_i^k$,

$$\mathrm{Rot}(z(t; z_0); [0, kT]) > j;$$

- there exists a circumference Γ_o^k centered at the origin such that, for every $z_0 \in \Gamma_o^k$,

$$\mathrm{Rot}(z(t; z_0); [0, kT]) < 1 \leq j.$$

Let $V_0 \in \mathcal{P}$ be as in hypothesis (H_0) and define

$$\Theta_* = \min(\Psi_{V_0}(\theta_0^2) - \Psi_{V_0}(\theta_0^1), \Psi_{V_0}(\theta_0^3) - \Psi_{V_0}(\theta_0^2), \dots, \Psi_{V_0}(\theta_0^1 + 2\pi) - \Psi_{V_0}(\theta_0^p)),$$

$$\Theta^* = \max(\Psi_{V_0}(\theta_0^2) - \Psi_{V_0}(\theta_0^1), \Psi_{V_0}(\theta_0^3) - \Psi_{V_0}(\theta_0^2), \dots, \Psi_{V_0}(\theta_0^1 + 2\pi) - \Psi_{V_0}(\theta_0^p));$$

moreover, let $\epsilon_1 > 0$ be so small that, for every $i = 1, \dots, p$, $a_0^{i,\epsilon_1}(t) = a_0^i(t) - \epsilon_1$ has positive mean and define m_j as the smallest integer strictly greater than

$$\frac{2\pi j + 2\Theta_* + \Theta^*}{\Theta_*} \left(\frac{\Theta^* \frac{A_{V_0}}{\pi} + \max_{1 \leq i \leq p} \int_0^T (a_0^{i,\epsilon_1})^-(t) dt}{\min_{1 \leq i \leq p} \int_0^T a_0^{i,\epsilon_1}(t) dt} + 1 \right).$$

By *ii*) of hypothesis (H_0') there exists $s_{\epsilon_1} > 0$ such that, for every $i = 1, \dots, p$,

$$\frac{\nabla_z H(t,z)\cdot z}{V_0(z)} \geq a_0^i(t) - \epsilon_1 = a_0^{i,\epsilon_1}(t)$$

for every $0 \leq t \leq T$ and for every $z \in \mathcal{R}_i$ with $0 < |z| \leq s_{\epsilon_1}$, moreover by the elastic property there exists $0 < r_{\epsilon_1} \leq s_{\epsilon_1}$ such that

$$|z_0| \leq r_{\epsilon_1} \Longrightarrow |z(t;z_0)| \leq s_{\epsilon_1}$$

for every $0 \leq t \leq kT$. Define $\Gamma_i^k = \{z \in \mathbb{R}^2 \mid |z| = r_{\epsilon_1}\}$ and let $z_0 \in \Gamma_i^k$. By *i*) of hypothesis (H_0') we have that, if $z(t;z_0) \in \mathcal{L}(\theta_0^i)$ for some $i = 1, \dots, p$, then

$$\theta_{V_0}'(t;z_0) = \frac{\pi}{A_{V_0}} \frac{\nabla_z H(t, z(t;z_0))\cdot z(t;z_0)}{V_0(z(t;z_0))} > 0; \tag{4.3}$$

hence we deduce that the set

$$S = [0,kT] \cap z(\cdot;z_0)^{-1}\left(\bigcup_{i=1,\dots,p} \mathcal{L}(\theta_0^i)\right)$$

is finite: let l denote its cardinality.

We note that the case $l = 0$ is not excluded at this point; moreover, if $l > 0$ we set $S = \{t_1, \dots, t_l\}$ with $0 \leq t_1 < \dots < t_l \leq kT$. Setting $t_0 = 0$ and $t_{l+1} = kT$, we can define, for every $h = 0, \dots, l$ such that $]t_h, t_{h+1}[\neq \emptyset$, an integer $i(h) \in \{1, \dots, p\}$ such that

$$z(t;z_0) \in \mathcal{R}_{i(h)} \qquad \text{for every } t \in]t_h, t_{h+1}[.$$

Then relation (2.5) implies that, for every $h = 0, \dots, l$,

$$\mathrm{Rot}_{V_0}(z(t;z_0);[t_h,t_{h+1}]) \leq \frac{\Theta^*}{2\pi}, \tag{4.4}$$

which, denoting with $[r]$ the integer part of a real number r, implies that

$$\begin{aligned}
\Theta^*\frac{A_{V_0}}{\pi} &\geq \int_{t_h}^{t_{h+1}} \frac{\nabla_z H(t,z(t;z_0))\cdot z(t;z_0)}{V_0(z(t;z_0))}dt \geq \\
&\geq \int_{t_h}^{t_{h+1}} a_0^{i(h),\epsilon_1}(t)dt \geq \\
&\geq \left[\frac{t_{h+1}-t_h}{T}\right]\int_0^T a_0^{i(h),\epsilon_1}(t)dt - \int_0^T (a_0^{i(h),\epsilon_1})^-(t)dt \geq \\
&\geq \left[\frac{t_{h+1}-t_h}{T}\right]\min_{1\leq i\leq p}\int_0^T a_0^{i,\epsilon_1}(t)dt - \max_{1\leq i\leq p}\int_0^T (a_0^{i,\epsilon_1})^-(t)dt.
\end{aligned}$$

Hence, we get that, for every $h = 0, \dots, l$,

$$\begin{aligned}
t_{h+1} - t_h &< T\left(\left[\frac{t_{h+1}-t_h}{T}\right] + 1\right) \leq \\
&\leq T\left(\frac{\Theta^*\frac{A_{V_0}}{\pi} + \max_{1\leq i\leq p}\int_0^T (a_0^{i,\epsilon_1})^-(t)dt}{\min_{1\leq i\leq p}\int_0^T a_0^{i,\epsilon_1}(t)dt} + 1\right) \leq \\
&\leq \frac{\Theta_*}{2\pi j + 2\Theta_* + \Theta^*} m_j T,
\end{aligned}$$

which implies that

$$m_jT \leq kT = \sum_{h=0}^{l}(t_{h+1} - t_h) \leq (l+1)\frac{\Theta_*}{2\pi j + 2\Theta_* + \Theta^*}m_jT.$$

In conclusion we obtain

$$(l-1) \geq \frac{2\pi j + \Theta^*}{\Theta_*} > 0,$$

which in particular excludes the case $S = \emptyset$. On the other hand relation (4.3) implies that, for every $h = 1, \dots, l-1$

$$i(h+1) = (i(h) \bmod p) + 1,$$

which yields

$$\mathrm{Rot}_{V_0}(z(t; z_0); [t_h, t_{h+1}]) > \frac{\Theta_*}{2\pi}.$$

Moreover, by relation (4.3) again, it is easy to see that

$$\mathrm{Rot}_{V_0}(z(t; z_0); [0, t_1]) \geq 0,$$

and that

$$\mathrm{Rot}_{V_0}(z(t; z_0); [t_l, kT]) > -\frac{\Theta^*}{2\pi};$$

hence we finally obtain that

$$\begin{aligned} \mathrm{Rot}_{V_0}(z(t; z_0); [0, kT]) &= \sum_{h=0}^{l}\mathrm{Rot}_{V_0}(z(t; z_0); [t_h, t_{h+1}]) > \\ &> (l-1)\frac{\Theta_*}{2\pi} - \frac{\Theta^*}{2\pi} \geq j. \end{aligned}$$

By Proposition 2.2, we conclude that

$$\mathrm{Rot}(z(t; z_0); [0, kT]) > j.$$

On the other hand, being $\mathcal{R}(\theta^1_\infty, \theta^2_\infty)$, (V^n_∞) and (a^n_∞) as in hypothesis (H'_∞) and setting

$$m = \inf_n \frac{\Psi_{V^n_\infty}(\theta^2_\infty) - \Psi_{V^n_\infty}(\theta^1_\infty)}{2\pi} > 0, \tag{4.5}$$

we have that there exists n such that

$$\frac{\int_0^T a^n_\infty(t)dt}{A_{V^n_\infty}} < \frac{1}{mk}$$

and

$$\limsup_{\substack{|z|\to+\infty \\ z\in\mathcal{R}(\theta^1_\infty,\theta^2_\infty)}} \frac{\nabla_z H(t,z)\cdot z}{V^n_\infty(z)} \leq a^n_\infty(t), \qquad \text{uniformly in } t \in [0,T].$$

So, taken $0 < \epsilon_2 < m\frac{A_{V^n_\infty}}{kT}$ there exists $S_{\epsilon_2} > 0$ such that

$$\frac{\nabla_z H(t,z)\cdot z}{V^n_\infty(z)} \le a^n_\infty(t) + \epsilon_2$$

for every $0 \le t \le T$ and for every $z \in \mathcal{R}(\theta^1_\infty, \theta^2_\infty)$ with $|z| \ge S_{\epsilon_2}$ and by the elastic property again there exists $0 < S_{\epsilon_2} < R_{\epsilon_2}$ such that

$$|z_0| \ge R_{\epsilon_2} \Longrightarrow |z(t;z_0)| \ge S_{\epsilon_2}$$

for every $0 \le t \le kT$. Define $\Gamma^k_o = \{z \in \mathbb{R}^2 \mid |z| = R_{\epsilon_2}\}$; we claim that, if $z_0 \in \Gamma^k_o$, then

$$\mathrm{Rot}_{V^n_\infty}(z(t;z_0);[0,kT]) < 1,$$

which by Proposition 2.2 implies the conclusion. In fact, suppose by contradiction that

$$\mathrm{Rot}_{V^n_\infty}(z(t;z_0);[0,kT]) \ge 1$$

for some $z_0 \in \Gamma^k_o$. Then, by standard connectivity arguments, we get the existence of two disjoint open intervals

$$I_1 =]a_1, b_1[, \qquad (0 \le a_1 \le b_1 \le kT),$$

$$I_2 =]a_2, b_2[, \qquad (0 \le a_2 \le b_2 \le kT),$$

(the choice can be, of course, non unique, and one of them can be taken empty if $z_0 \notin \mathcal{R}(\theta^1_\infty, \theta^2_\infty)$) such that $z(t;z_0) \in \mathcal{R}(\theta^1_\infty, \theta^2_\infty)$ for every $t \in I_1 \cup I_2$ and that

$$\theta_{V^n_\infty}(b_1;z_0) - \theta_{V^n_\infty}(a_1;z_0) + \theta_{V^n_\infty}(b_2;z_0) - \theta_{V^n_\infty}(a_2;z_0) = \Psi_{V^n_\infty}(\theta^2_\infty) - \Psi_{V^n_\infty}(\theta^1_\infty).$$

Then, by relation (4.5), we get

$$\begin{aligned}
m &\le \mathrm{Rot}_{V^n_\infty}(z(t;z0);\bar{I}_1) + \mathrm{Rot}_{V^n_\infty}(z(t;z0);\bar{I}_2) = \\
&= \frac{1}{2A_{V^n_\infty}} \int_{I_1 \cup I_2} \frac{Jz'(t;z_0)\cdot z(t;z_0)}{V(z(t;z_0))} dt = \\
&= \frac{1}{2A_{V^n_\infty}} \int_{I_1 \cup I_2} \frac{\nabla_z H(t, z(t;z_0))\cdot z(t;z_0)}{V(z(t;z_0))} dt \le \\
&\le \frac{1}{2A_{V^n_\infty}} \int_{I_1 \cup I_2} (a^n_\infty(t) + \epsilon_2) dt \\
&\le \frac{1}{2A_{V^n_\infty}} \int_0^{kT} (a^n_\infty(t) + \epsilon_2) dt \le \\
&\le \frac{k}{2}\frac{\int_0^T a^n_\infty(t)dt}{A_{V^n_\infty}} + \frac{kT\epsilon_2}{2A_{V^n_\infty}} < \\
&< \frac{m}{2} + \frac{m}{2} = m
\end{aligned}$$

which is a contradiction.

Remark 4.2 Clearly, if there exists an angular region $\mathcal{R}(\theta^1_\infty, \theta^2_\infty)$ such that

$$\limsup_{\substack{|z|\to+\infty \\ z\in\mathcal{R}(\theta^1_\infty,\theta^2_\infty)}} \frac{\nabla_z H(t,z)\cdot z}{|z|^2} \le 0, \qquad \text{uniformly in } t \in [0,T],$$

then hypothesis (H'_∞) is satisfied with $a^n_\infty \equiv 0$ and $V^n_\infty(x,y) = x^2 + y^2$ for every $n \in \mathbb{N}_0$. Moreover, we remark that it is possible to prove a variant of Theorem 4.1 by keeping hypothesis (H'_∞) and assuming (H_0) of Theorem 3.1, instead of (H'_0); it is worth noticing that, in this case, the result is still true in the Carathéodory setting.

Again, we get a corollary for the second order equation (1.2), in order to cover an asymmetric behavior at 0 and a one-side sublinearity condition at infinity. Here we assume that $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, T-periodic in the first variable and such that $g(t,0) \equiv 0$; moreover, we suppose that the uniqueness and global continuability of the solutions of the initial value problems are guaranteed.

Corollary 4.1 *Let us suppose that*

$(g^\pm_0)$ *there exist* $q^+_0, q^-_0 \in L^1(]0,T[)$ *with* $\int_0^T q^\pm_0(t)dt > 0$ *such that*

$$\liminf_{x\to 0^\pm} \frac{g(t,x)}{x} \ge q^\pm_0(t)$$

uniformly in $t \in [0,T]$;

moreover suppose that one of the conditions

(g^-_∞)

$$\limsup_{x\to-\infty} \frac{g(t,x)}{x} \le 0$$

uniformly in $t \in [0,T]$,

(g^+_∞)

$$\limsup_{x\to+\infty} \frac{g(t,x)}{x} \le 0$$

uniformly in $t \in [0,T]$,

is satisfied. Then, for every $j \in \mathbb{N}_0$ *there exists* $m_j \in \mathbb{N}_0$ *such that, for every* $k \ge m_j$ *with k prime with j, equation (1.2) has at least two subharmonic solutions* $u^1_{j,k}, u^2_{j,k}$ *of order k, not belonging to the same periodicity class, with exactly* $2j$ *zeros in the interval* $[0,kT[$.

Proof. Hypothesis (H'_0) is satisfied for the subdivision $\mathcal{S}(\frac{\pi}{2}, \frac{3}{2}\pi)$: in fact, for $z \in \mathcal{L}(\frac{\pi}{2}) \cup \mathcal{L}(\frac{3}{2}\pi)$

$$\nabla_z H(t,z)\cdot z = y^2 > 0,$$

while computations analogous to those of Corollary 3.1 show that with $a^1_0(t) = \min(q^-_0(t), \frac{1}{\rho}) - \sigma$, $a^2_0(t) = \min(q^+_0(t), \frac{1}{\rho}) - \sigma$ (for $\sigma, \rho > 0$ small enough) the second condition in (H'_0) holds true.
On the other hand, hypothesis (H'_∞) is satisfied for one of the half-planes $\mathcal{R}(\frac{\pi}{2}, \frac{3}{2}\pi)$ or $\mathcal{R}(\frac{3}{2}\pi, \frac{5}{2}\pi)$ and $V^n_\infty(x,y) = x^2 + 2nTy^2$. As $\mathcal{F} = \{V^n_\infty\}$ is admissible with respect to each half-plane, Theorem 4.1 can be applied to get the conclusion.

Remark 4.3 Note that Corollary 4.1 applies to nonlinearity which near 0 are of the form

$$g(t,x) \sim q_0^+(t)x^+ - q_0^-(t)x^-,$$

being $q_0^+, q_0^- \in L^1(]0,T[)$ with $\int_0^T q_0^\pm(t)dt > 0$; this does not in general imply that $\min(q_0^+(t), q_0^-(t))$ has positive mean, so Corollary 3.1 does not apply.

5 An application to a class of Lotka-Volterra planar systems

In this section we give a very brief sketch of a possible application of Theorem 4.1 to some Hamiltonian planar systems. In particular, we start our considerations by considering a generalized Lotka-Volterra system

$$\begin{cases} p' = p(a(t) - b(t)q) \\ q' = q(-c(t) + d(t)p) \end{cases} \qquad (p,q) \in \mathbb{R}^2, \tag{5.1}$$

with $a, b, c, d : \mathbb{R} \to \mathbb{R}$ continuous and T-periodic, b and d strictly positive. Of course, system (5.1) is not Hamiltonian; but, as already observed in many papers, the change of variables

$$u = \log p \qquad v = \log q$$

permits to establish a one-to-one correspondence between positive solutions of (5.1) and solutions of the planar Hamiltonian system

$$\begin{cases} u' = a(t) - b(t)e^v \\ v' = -c(t) + d(t)e^u \end{cases} \qquad (u,v) \in \mathbb{R}^2. \tag{5.2}$$

With a topological degree argument, it is shown in [6] that, under the conditions

$$\int_0^T a(t)dt > 0, \qquad \int_0^T c(t)dt > 0,$$

system (5.2) has a T-periodic solution $(u^*(t), v^*(t))$; then, via the change of variables

$$x = v - v^*(t) \qquad y = u - u^*(t),$$

the problem of the existence of subharmonic solutions of (5.2) is reduced to the existence of subharmonic solutions for the planar Hamiltonian system

$$\begin{cases} x' = d(t)e^{u^*(t)}(e^y - 1) := X(t,y) \\ y' = b(t)e^{v^*(t)}(1 - e^x) := Y(t,x), \end{cases} \tag{5.3}$$

which is of type (1.1) with

$$H(t,x,y) = \int_0^y X(t,s)ds - \int_0^x Y(t,s)ds.$$

Actually, system (5.3) has infinitely many subharmonic solutions, as shown in [6], with a long and careful phase-plane analysis. Here we want to show that a similar result can be obtained, in a very direct way, as a consequence of our main results.

The smoothness of H implies the uniqueness for the solutions of the Cauchy problems associated to (5.3), while the global existence is proved in [6]; we claim that (H_0) and (H'_∞) hold, too. In fact, since

$$\frac{\nabla_z H(t,z)\cdot z}{|z|^2} = \frac{d(t)e^{u^*(t)}y(e^y-1)+b(t)e^{v^*(t)}x(e^x-1)}{x^2+y^2} \geq$$
$$\geq \min(d(t)e^{u^*(t)}, b(t)e^{v^*(t)})\left(\frac{y(e^y-1)+x(e^x-1)}{x^2+y^2}\right),$$

hypothesis (H_0) is satisfied with $a_0(t) = \min(d(t)e^{u^*(t)}, b(t)e^{v^*(t)})$ and $V_0(z) = |z|^2$; on the other hand, since

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathcal{R}(\frac{\pi}{2},\pi)}} \frac{\nabla_z H(t,z)\cdot z}{|z|^2} = 0$$

uniformly in $t \in [0,T]$, hypothesis (H'_∞) is satisfied for the angular region $\mathcal{R}(\frac{\pi}{2},\pi)$.
In conclusion, the variant of Theorem 4.1 described in Remark 4.2 can be applied, giving the existence of infinitely many subharmonic solutions for system (5.3).

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References

[1] A. Abbondandolo, *Sub-harmonics for two-dimensional Hamiltonian systems*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), 341–355.

[2] S.R. Bernfeld and V. Lakshmikantham, An Introduction to Nonlinear Boundary Value Problems, Academic Press, New York, 1968.

[3] W. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press, New York, 1975.

[4] N.P. Cac and A.C. Lazer, *On second order, periodic, symmetric, differential systems having subharmonics of all sufficiently large orders*, J. Differential Equations **127** (1996), 426–438.

[5] W.Y. Ding, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **20** (1983), 341–346.

[6] T. Ding and F. Zanolin, *Periodic Solutions and Subharmonic Solutions for a Class of Planar Systems of Lotka-Volterra Type*, World Congress of Nonlinear Analysts '92, Vol. **I-IV** (Tampa, FL, 1992) (Berlin), de Gruyter, Berlin, (1996), 95–106.

[7] T. Ding and F. Zanolin, *Periodic solutions of Duffing's equation with superquadratic potential*, J. Differential Equations **97** (1992), 328–378.

[8] T. Ding and F. Zanolin, *Subharmonic solutions of second order nonlinear equations: a time-map approach*, Nonlinear Anal. **20** (1993), 509–532.

[9] C. Fabry, *Landesman - Lazer conditions for periodic boundary value problems with asymmetric nonlinearities*, J. Differential Equations **116** (1995), 405–418.

[10] A. Fonda, *Positively homogeneous Hamiltonian systems in the plane*, J. Differential Equations **200** (2004), 162–184.

[11] A. Fonda, R. Manasevich and F. Zanolin, *Subharmonic solutions for some second-order differential equations with singularities*, Siam J. Math. Anal. **24** (1993), 1294–1311.

[12] A. Fonda and M. Ramos, *Large amplitude subharmonic oscillations for scalar second order differential equations with asymmetric nonlinearities*, J. Differential Equations **109** (1994), 354–372.

[13] J.K. Hale, Ordinary Differential Equations, Krieger, New York, 1980.

[14] A. Margheri, C.Rebelo and F. Zanolin, *Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, J. Differential Equations **183** (2002), 342–367.

[15] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences **74**, Springer-Verlag, New York, 1989.

[16] C. Rebelo, *A note on uniqueness of Cauchy problems associated to planar Hamiltonian systems*, Portugal. Math. **57** (2000), 415–419.

[17] C. Rebelo and F. Zanolin, *Multiplicity results for periodic solutions of second order ODEs with asymmetric nonlinearities*, Trans. Amer. Math. Soc. **348** (1996), 2349–2389.

[18] C. Zanini, *Rotation numbers, eigenvalues, and the Poincaré-Birkhoff theorem*, J. Math. Anal. Appl. **279** (2003), 290–307.

[19] F. Zanolin, *Continuation theorems for the periodic problem via the translation operator*, Rend. Sem. Mat. Univ. Politec. Torino **54** (1996), 1–23.