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Multiple Solutions for the $p(x)-$ Laplace Operator with Critical Growth

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Abstract

The aim of this paper is to extend previous results regarding the multiplicity of solutions for quasilinear elliptic problems with critical growth to the variable exponent case. We prove, in the spirit of [4], the existence of at least three nontrivial solutions to the quasilinear elliptic equation $-\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda f(x, u)$ in a smooth bounded domain Ω of $\mathbb{R}^N$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. We assume that $\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical Sobolev exponent for variable exponents and $\Delta_{p(x)}u = \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$laplacian. The proof is based on variational arguments and the extension of concentration compactness method for variable exponent spaces.

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1 Introduction

Let us consider the following nonlinear elliptic problem:

$$\begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$laplacian, $1 < p(x) < N$. On the exponent $q(x)$ we assume that it is critical in the sense that $\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical exponent in the Sobolev embedding, λ is a positive parameter and the nonlinear term f is a subcritical perturbation with some precise assumptions that we state below.

The purpose of this paper is to extend the results obtained in [4] where the same problem but with constant p was treated. Namely, in [4], problem (1.1) was analyzed in the case $p(x) \equiv p$ constant and $q(x) \equiv p^*$.

To be more precise, the result in [4] proves the existence of at least three nontrivial solutions for (1.1), one positive, one negative and one that changes sign, under adequate assumptions on the source term f and the parameter λ.

The method in the proof used in [4] consists of restricting the functional associated to (1.1) to three different Banach manifolds, one consisting of positive functions, one consisting of negative functions and the third one consisting of sign-changing functions, all of them under a normalization condition. Then, by means of a suitable version of the Mountain Pass Theorem due to Schwartz [15] and the concentration-compactness principle of P.L. Lions [12] the authors can prove the existence of a critical point of each restricted functional and, finally, the authors were able to prove the that critical points of each restricted functional are critical points of the unrestricted one.

This method was introduced by M. Struwe [16] where the subcritical case (in the sense of the Sobolev embeddigs) for the $p-$Laplacian was treated. A related result for the $p-$Laplacian under nonlinear boundary condition can be found in [8]. Also, a similar problem in the case of the $p(x)-$Laplacian, but with subcritical nonlinearities was analyzed in [5].

In all the above mentioned works, the main feature on the nonlinear term f is that no oddness condition is imposed. Very little is known about critical growth nonlinearities for variable exponent problems, since one of the main techniques used in order to deal with such issues is the concentration-compactness principle. This result was recently obtained for the variable exponent case independently in [9] and [10]. In both of these papers the proofs are similar and both relate to that of the original proof of P.L. Lions. However, the arguments in [9] are a little more subtle and allow the authors to deal with the case where the exponent $q(x)$ is critical only in some part of the domain, while the results in [10] require $q(x)$ to be identically $p^*(x)$. So we will rely on the concentration-compactness principle proved in [9] in this work.

The concentration compactness method to deal with the $p-$Laplacian has been used by so many authors before that it is almost impossible to give a complete list of contributions. However we want to refer to the work of J. García Azorero and I. Peral in [11] from where we borrow some ideas.

Throughout this work, by (weak) solutions of (1.1) we understand critical points of the associated energy functional acting on the Sobolev space $W_0^{1,p(x)}(\Omega)$:

$$\Phi(v) = \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx - \lambda\int_\Omega F(x,v)\,dx, \qquad (1.2)$$

where $F(x,u) = \int_0^u f(x,z)\,dz$.

To end this introduction, let us comment on different applications where the $p(x)-$Laplacian has appeared.

Up to our knowledge there are two main fields where the $p(x)-$Laplacian has been proved to be extremely useful in applications:

- Image Processing
- Electrorheological Fluids

For instance, Y. Chen, S. Levin and R. Rao [1] proposed the following model in image processing

$$E(u) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} + f(|u(x) - I(x)|)\, dx \to \min$$

where $p(x)$ is a function varying between 1 and 2 and f is a convex function. In their application, they chose $p(x)$ close to 1 where there is likely to be edges and close to 2 where it is unlikely to be edges. The electrorheological fluids application is much more developed and we refer to the monograph by M. Ružička, [14], and its references.

2 Assumptions and statement of the results.

Throughout this paper the following notation will be used: Given $q\colon \Omega \to \mathbb{R}$ bounded, we denote

$$q^+ := \sup_\Omega q(x), \qquad q^- := \inf_\Omega q(x).$$

The precise assumptions on the source term f are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x, 0) = 0$ for every $x \in \Omega$.

(F2) There exist constants $c_1 > 1/(q^- - 1)$, $c_2 \in (p^+, q^-)$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$ and $p^- \leq p^+ < r^- \leq r^+ < q^- \leq q^+$.

$$\begin{aligned} c_3\rho_r(u) &\leq c_2 \int_\Omega F(x,u)\, dx \leq \int_\Omega f(x,u)u\, dx \\ &\leq c_1 \int_\Omega f_u(x,u)u^2\, dx \leq c_4\rho_r(u) \end{aligned}$$

where $\rho_r(u) := \int_\Omega |u|^{r(x)}\, dx$

Remark 2.1 Observe that this set of hypotheses on the nonlinear term f are similar than the ones considered by [4].

Remark 2.2 We exhibit now one example of nonlinearities that fulfills all of our hypotheses. $f(x,u) = |u|^{r(x)-2}u + |u_+|^{s(x)-2}u_+$, if $s(x) < r(x)$, $q^- - 1 > s^- > p^+$. Hypotheses (F1)–(F2) are clearly satisfied.

Remark 2.3 p is a Log-Hölder continuity function if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|} \quad \forall x, y \quad \text{such that} \qquad |x - y| < \frac{1}{2}$$

So the main result of the paper reads:

Theorem 2.1 *Let $q(x)$, $p(x)$ and $r(x)$ be log-Hölder functions such that*

$$1 < \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) < N \qquad \text{and} \qquad 1 \le q(x) \le p^*(x) \quad \text{in } \Omega$$

where the set $\mathcal{A} = \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$. Under assumptions (F1)–(F2)*, there exists $\lambda^* > 0$ depending only on N, p, q and the constant c_3 in* (F2)*, such that for every $\lambda > \lambda^*$, there exist three different, nontrivial, (weak) solutions of problem (1.1). Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

3 Results on variable exponent Sobolev spaces

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \colon \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

This space is endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in W^{1,1}_{loc}(\Omega) \colon u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

The corresponding norm for this space is

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega).}$$

Define $W^{1,p(x)}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the $W^{1,p(x)}(\Omega)$ norm. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces when $1 < \inf_\Omega p \le \sup_\Omega p < \infty$.

As usual, we denote $p'(x) = p(x)/(p(x) - 1)$ the conjugate exponent of $p(x)$. Define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} \text{ if } p(x) < N \text{ or} \\ \infty \text{ if } p(x) \ge N. \end{cases}$$

The following results are proved in [7]

Proposition 3.1 (Hölder-type inequality) *Let $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$. Then the following inequality holds*

$$\int_\Omega |f(x)g(x)|\, dx \le C_p \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{p'(x)}(\Omega)}$$

Proposition 3.2 (Sobolev embedding) *Let $p, q \in C(\overline{\Omega})$ be such that $1 \le q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$. Assume moreover that the functions p and q are log-Hölder continuous. Then there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Moreover, if $\inf_\Omega(p^* - q) > 0$ then, the embedding is compact.

Proposition 3.3 (Poincaré inequality) *There is a constant $C > 0$, such that*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\| \, |\nabla u| \, \|_{L^{p(x)}(\Omega)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 3.1 By Proposition 3.3, we know that $\| \, |\nabla u| \, \|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

4 Proof of the theorem 1.

The proof uses the same approach as in [16]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of C^1-manifolds $M_i \subset W^{1,p(x)}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$J(v) = \int_\Omega |\nabla v|^{p(x)} - |v|^{q(x)} dx$$
$$M_1 = \left\{ u \in W_0^{1,p(x)}(\Omega) \colon \int_\Omega u_+ > 0 \text{ and } J(u_+) = \int_\Omega \lambda f(x,u) u_+ dx \right\},$$
$$M_2 = \left\{ u \in W_0^{1,p(x)}(\Omega) \colon \int_\Omega u_- > 0 \text{ and } J(u_-) = -\int_\Omega \lambda f(x,u) u_- dx \right\},$$
$$M_3 = M_1 \cap M_2.$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u. We define

$$K_1 = \{u \in M_1 \mid u \geq 0\},$$
$$K_2 = \{u \in M_2 \mid u \leq 0\},$$
$$K_3 = M_3.$$

First, we need a Lemma to show that these sets are nonempty and, moreover, give some properties that will be useful in the proof of the result.

Lemma 4.1 *For every $w_0 \in W_0^{1,p(x)}(\Omega)$, $w_0 > 0$ ($w_0 < 0$), there exists $t_\lambda > 0$ such that $t_\lambda w_0 \in M_1$($\in M_2$). Moreover, $\lim_{\lambda\to\infty} t_\lambda = 0$. As a consequence, given $w_0, w_1 \in W_0^{1,p(x)}(\Omega)$, $w_0 > 0$, $w_1 < 0$, with disjoint supports, there exists $\bar{t}_\lambda, \underline{t}_\lambda > 0$ such that $\bar{t}_\lambda w_0 + \underline{t}_\lambda w_1 \in M_3$. Moreover $\bar{t}_\lambda, \underline{t}_\lambda \to 0$ as $\lambda \to \infty$.*

Proof. We prove the lemma for M_1, the other cases being similar. For $w \in W_0^{1,p(x)}(\Omega)$, $w \geq 0$, we consider the functional

$$\varphi_1(w) = \int_\Omega |\nabla w|^{p(x)} - |w|^{q(x)} - \lambda f(x,w) w \, dx.$$

Given $w_0 > 0$, in order to prove the lemma, we must show that $\varphi_1(t_\lambda w_0) = 0$ for some $t_\lambda > 0$. Using hypothesis (F2), if $t < 1$, we have that:

$$\varphi_1(tw_0) \geq At^{p^+} - Bt^{q^-} - \lambda c_4 C t^{r^-}$$

and

$$\varphi_1(tw_0) \le At^{p^-} - Bt^{q^+} - \lambda c_3 Ct^{r^+},$$

where the coefficients A, B and C are given by:

$$A = \int_\Omega |\nabla w_0|^{p(x)}\,dx, \quad B = \int_\Omega |w_0|^{q(x)}\,dx, \quad C = \int_\Omega |w_0|^{r(x)}\,dx.$$

Since $p^- \le p^+ < r^- \le r^+ < q^- \le q^+$ it follows that $\varphi_1(tw_0)$ is positive for t small enough, and negative for t big enough. Hence, by Bolzano's theorem, there exists some $t = t_\lambda$ such that $\varphi_1(t_\lambda u) = 0$. (This t_λ need not to be unique, but this does not matter for our purposes).

In order to give an upper bound for t_λ, it is enough to find some t_1, such that $\varphi_1(t_1 w_0) < 0$. We observe that:

$$\varphi_1(tw_0) \le \max\{At^{p^-} - \lambda c_3 Ct^{r^+}; At^{p^+} - \lambda c_3 Ct^{r^-}\}$$

so it is enough to choose t_1 such that $\max\{At_1^{p^-} - \lambda c_3 Ct_1^{r^+}; At_1^{p^+} - \lambda c_3 Ct_1^{r^-}\} = 0$, i.e.,

$$t_1 = \left(\frac{A}{c_3\lambda C}\right)^{1/(r^+-p^-)} \text{ or } t_1 = \left(\frac{A}{c_3\lambda C}\right)^{1/(r^--p^+)}.$$

Hence, again by Bolzano's theorem, we can choose $t_\lambda \in [0, t_1]$, which implies that $t_\lambda \to 0$ as $\lambda \to +\infty$.

Lemma 4.2 *There exists $C_1, C_2 > 0$ depending on $p(x)$ and on c_2 such that, for every $u \in K_i$, $i = 1, 2, 3$, we have*

$$\int_\Omega |\nabla u|^{p(x)}\,dx = \left(\lambda \int_\Omega f(x,u)u\,dx + \int_\Omega |u|^{q(x)}\,dx\right) \le C_1\Phi(u) \le C_2\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right).$$

Proof. The equality is clear since $u \in K_i$. Now, by (F2), $F(x,u) \ge 0$; so

$$\begin{aligned}\Phi(u) \quad &= \textstyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} - \frac{1}{q(x)}|u|^{q(x)} - \lambda F(x,u)\,dx \\ &\le \textstyle\frac{1}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx.\end{aligned}$$

To prove the final inequality, we proceed as follows. Using the norming condition of K_i and hypothesis (F2):

$$\begin{aligned}\Phi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} - \frac{1}{q(x)}|u|^{q(x)} - \lambda F(x,u)dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega |u|^{q(x)}\,dx + \lambda\int_\Omega\left(\frac{1}{p^+}f(x,u)u - F(x,u)\right)dx \\ &\ge \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega |u|^{q(x)}\,dx + \left(\frac{1}{p^+} - \frac{1}{c_2}\right)\lambda\int_\Omega f(x,u)udx.\end{aligned}$$

(Recall that $q^- > p^+$). This finishes the proof.

Lemma 4.3 *There exists $c > 0$ such that*

$$\|\nabla u_+\|_{L^{p(x)}(\Omega)} \geq c \quad \forall\, u \in K_1, \tag{3.1}$$

$$\|\nabla u_-\|_{L^{p(x)}(\Omega)} \geq c \quad \forall\, u \in K_2, \tag{3.2}$$

$$\|\nabla u_+\|_{L^{p(x)}(\Omega)}, \|\nabla u_-\|_{L^{p(x)}(\Omega)} \geq c \quad \forall\, u \in K_3. \tag{3.3}$$

Proof. Suppose that $\|\nabla u_\pm\|_{L^{p(x)}(\Omega)} < 1$. By the definition of K_i,by (F2) and the Poincaré inequality we have that

$$\begin{aligned}\|\nabla u_\pm\|^{p^+}_{L^{p(x)}(\Omega)} &\leq \rho_p(\nabla u_\pm) = \int_\Omega \lambda f(x,u)u_\pm + |u_\pm|^{q(x)} dx \\ &\leq C\rho_r(u_\pm) + \rho_q(u_\pm) \leq C\|u_\pm\|^{r^-}_{L^{r(x)}(\Omega)} + \|u_\pm\|^{q^-}_{L^{q(x)}(\Omega)} \\ &\leq c_1\|\nabla u_\pm\|^{r^-}_{L^{p(x)}(\Omega)} + c_2\|\nabla u_\pm\|^{q^-}_{L^{p(x)}(\Omega)}.\end{aligned}$$

As $p^+ < r^- < q^-$, this finishes the proof.

The following lemma describes the properties of the manifolds M_i.

Lemma 4.4 *M_i is a C^1 sub-manifold of $W_0^{1,p(x)}(\Omega)$ of co-dimension 1 ($i = 1, 2$), 2 ($i = 3$) respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u W_0^{1,p(x)}(\Omega) = T_u M_i \oplus span\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. Let us denote

$$\begin{aligned}\bar{M}_1 &= \left\{u \in W_0^{1,p(x)}(\Omega) \colon \int_\Omega u_+\, dx > 0\right\}, \\ \bar{M}_2 &= \left\{u \in W_0^{1,p(x)}(\Omega) \colon \int_\Omega u_-\, dx > 0\right\}, \\ \bar{M}_3 &= \bar{M}_1 \cap \bar{M}_2.\end{aligned}$$

Observe that $M_i \subset \bar{M}_i$. The set $\bar{M}_i$ is open in $W^{1,p(x)}(\Omega)$. Therefore it is enough to prove that M_i is a C^1 sub-manifold of $\bar{M}_i$. In order to do this, we will construct a C^1 function $\varphi_i : \bar{M}_i \to \mathbb{R}^d$ with $d = 1$ ($i = 1, 2$), $d = 2$ ($i = 3$) respectively and M_i will be the inverse image of a regular value of φ_i.

In fact, we define: For $u \in \bar{M}_1$,

$$\varphi_1(u) = \int_\Omega |\nabla u_+|^{p(x)} - |u_+|^{q(x)} - \lambda f(x,u)u_+\, dx.$$

For $u \in \bar{M}_2$,

$$\varphi_2(u) = \int_\Omega |\nabla u_-|^{p(x)} - |u_-|^{q(x)} - \lambda f(x,u)u_-\, dx.$$

For $u \in \bar{M}_3$,

$$\varphi_3(u) = (\varphi_1(u), \varphi_2(u)).$$

Obviously, we have $M_i = \varphi_i^{-1}(0)$. From standard arguments (see [3], or the appendix of [13]), φ_i is of class C^1. Therefore, we only need to show that 0 is a regular value for φ_i. To this end we compute, for $u \in M_1$,

$$\begin{aligned}\langle \nabla\varphi_1(u), u_+\rangle &\le p^+\rho_p(\nabla u_+) - q^-\rho_q(u_+) - \lambda \int_\Omega f(x,u)u_+ - f_u(x,u)u_+^2\,dx \\ &\le q^-\left(\rho_p(\nabla u_+) - \rho_q(u_+)\right) - \lambda \int_\Omega f(x,u)u_+ - f_u(x,u)u_+^2\,dx \\ &\le (q^-\lambda - \lambda)\int_\Omega f(x,u)u_+\,dx - \int_\Omega f_u(x,u)u_+^2\,dx.\end{aligned}$$

By (F2) the last term is bounded by

$$\begin{aligned}(q^-\lambda - \lambda - \frac{\lambda}{c_1})\int_\Omega f(x,u)u_+\,dx &= \left(q^- - 1 - \frac{1}{c_1}\right)\left(\rho_p(\nabla u_+) - \rho_q(u_+)\right) \\ &\le \left(q^- - 1 - \frac{1}{c_1}\right)\rho_p(\nabla u_+).\end{aligned}$$

Recall that $c_1 < 1/(q^- - 1)$. Now, the last term is strictly negative by Lemma 4.3. Therefore, M_1 is a C^1 sub-manifold of $W^{1,p(x)}(\Omega)$. The exact same argument applies to M_2. Since trivially

$$\langle \nabla\varphi_1(u), u_-\rangle = \langle \nabla\varphi_2(u), u_+\rangle = 0$$

for $u \in M_3$, the same conclusion holds for M_3.

To see that K_i is complete, let u_k be a Cauchy sequence in K_i, then $u_k \to u$ in $W^{1,p(x)}(\Omega)$. Moreover, $(u_k)_\pm \to u_\pm$ in $W^{1,p(x)}(\Omega)$. Now it is easy to see, by Lemma 4.3 and by continuity that $u \in K_i$.

Finally, by the first part of the proof we have the decomposition

$$T_u W^{1,p(x)}(\Omega) = T_u M_i \oplus \operatorname{span}\{u_+\}$$

where $M_1 = \{u : \varphi_1(u) = 0\}$ and $T_u M_1 = \{v : \langle \nabla\varphi_1(u), v\rangle = 0\}$. Now let $v \in T_u W_0^{1,p(x)}(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where $v_2 = \alpha u_+$ and $v_1 = v - v_2$. Let us take α as

$$\alpha = \frac{\langle \nabla\varphi_1(u), v\rangle}{\langle \nabla\varphi_1(u), u_+\rangle}.$$

With this choice, we have that $v_1 \in T_u M_1$. Now

$$\langle \varphi_1(u), v_1\rangle = 0.$$

The very same argument to show that $T_u W^{1,p(x)}(\Omega) = T_u M_2 \oplus \langle u_-\rangle$ and $T_u W^{1,p(x)}(\Omega) = T_u M_3 \oplus \langle u_+, u_-\rangle$. From these formulas and from the estimates given in the first part of the proof, the uniform continuity of the projections onto $T_u M_i$ follows.

Now, we need to check the Palais-Smale condition for the functional Φ restricted to the manifold M_i. We begin by proving the Palais-Smale condition for the functional Φ unrestricted, below certain level of energy.

Lemma 4.5 *Assume that $r \leq q$. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ a Palais-Smale sequence. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. By definition

$$\Phi(u_j) \to c \qquad \text{and} \qquad \Phi'(u_j) \to 0.$$

Now, we have

$$c+1 \geq \Phi(u_j) = \Phi(u_j) - \frac{1}{c_2}\langle \Phi'(u_j), u_j\rangle + \frac{1}{c_2}\langle \Phi'(u_j), u_j\rangle,$$

where

$$\langle \Phi'(u_j), u_j\rangle = \int_\Omega |\nabla u_j|^{p(x)} - |u_j|^{q(x)} - \lambda f(x,u_j)u_j\, dx.$$

Then, if $c_2 < q^-$ we conclude

$$c+1 \geq \left(\frac{1}{p+} - \frac{1}{c_2}\right)\int_\Omega |\nabla u_j|^{p(x)}\, dx - \frac{1}{c_2}|\langle \Phi'(u_j), u_j\rangle|.$$

We can assume that $\|u_j\|_{W_0^{1,p(x)}(\Omega)} \geq 1$. As $\|\Phi'(u_j)\|$ is bounded we have that

$$c+1 \geq \left(\frac{1}{p+} - \frac{1}{c_2}\right)\|u_j\|^{p^-}_{W_0^{1,p(x)}(\Omega)} - \frac{C}{c_2}\|u_j\|_{W_0^{1,p(x)}(\Omega)}.$$

We deduce that u_j is bounded. This finishes the proof.

From the fact that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence it follows, by Lemma 4.5, that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Hence, by The Concentration-Compactness method for variable exponent (See[9]), we have

$$|u_j|^{q(x)} \rightharpoonup \nu = |u|^{q(x)} + \textstyle\sum_{i\in I} \nu_i \delta_{x_i} \quad \nu_i > 0,$$
$$|\nabla u_j|^{p(x)} \rightharpoonup \mu \geq |\nabla u|^{p(x)} + \textstyle\sum_{i\in I} \mu_i \delta_{x_i} \quad \mu_i > 0,$$
$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}$$

where S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S := \inf_{\phi\in C_0^\infty(\mathbb{R}^N)} \frac{\||\nabla\phi|\|_{L^{p(x)}(\mathbb{R}^N)}}{\|\phi\|_{L^{p^*(x)}(\mathbb{R}^N)}}.$$

Note that if $I = \emptyset$ then $u_j \to u$ strongly in $L^{q(x)}(\Omega)$. We know that $\{x_i\}_{i\in I} \subset \mathcal{A} := \{x : q(x) = p^*(x)\}$. We define $q^-_{\mathcal{A}} := \inf_{\mathcal{A}} q(x)$.

Let us show that if $c < \left(\frac{1}{p^+} - \frac{1}{q^-_{\mathcal{A}}}\right)S^N$ and $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence, with energy level c, then $I = \emptyset$. In fact, suppose that $I \neq \emptyset$. Then let $\phi \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball of $\mathbb{R}^N$. Consider the rescaled functions $\phi_{i,\varepsilon}(x) = \phi(\frac{x-x_i}{\varepsilon})$. As $\Phi'(u_j) \to 0$ in $(W_0^{1,p(x)}(\Omega))'$, we obtain that

$$\lim_{j\to\infty}\langle \Phi'(u_j), \phi_{i,\varepsilon}u_j\rangle = 0.$$

On the other hand,

$$\langle \Phi'(u_j), \phi_{i,\varepsilon}u_j\rangle = \int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon}u_j) - \lambda f(x,u_j)u_j\phi_{i,\varepsilon} - |u_j|^{q(x)}\phi_{i,\varepsilon}\, dx.$$

Then, passing to the limit as $j \to \infty$, we get

$$0 = \lim_{j\to\infty}\left(\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon})u_j\,dx\right) + \int_\Omega \phi_{i,\varepsilon}\,d\mu - \int_\Omega \phi_{i,\varepsilon}\,d\nu - \int_\Omega \lambda f(x,u)u\phi_{i,\varepsilon}\,dx.$$

By Hölder inequality, it is easy to check that

$$\lim_{j\to\infty}\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon})u_j\,dx = 0.$$

On the other hand,

$$\lim_{\varepsilon\to 0}\int_\Omega \phi_{i,\varepsilon}\,d\mu = \mu_i\phi(0), \qquad \lim_{\varepsilon\to 0}\int_\Omega \phi_{i,\varepsilon}\,d\nu = \nu_i\phi(0)$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda f(x,u)u\phi_{i,\varepsilon}\,dx = 0.$$

So, we conclude that $(\mu_i - \nu_i)\phi(0) = 0$, i.e, $\mu_i = \nu_i$. Then,

$$S\nu_i^{1/p^*(x_i)} \le \nu_i^{1/p(x_i)},$$

so it is clear that $\nu_i = 0$ or $S^N \le \nu_i$. On the other hand, we consider the $\delta-$tubular neighborhood of $\mathcal{A}$, namely

$$\mathcal{A}_\delta := \bigcup_{x\in\mathcal{A}}(B_\delta(x)\cap\Omega).$$

So, as $c_2 > p^+$,

$$\begin{aligned} c = \lim_{j\to\infty}\Phi(u_j) &= \lim_{j\to\infty}\Phi(u_j) - \tfrac{1}{p+}\langle\Phi'(u_j),u_j\rangle \\ &= \lim_{j\to\infty}\int_\Omega\left(\tfrac{1}{p(x)} - \tfrac{1}{p+}\right)|\nabla u_j|^{p(x)}\,dx + \int_\Omega\left(\tfrac{1}{p+} - \tfrac{1}{q(x)}\right)|u_j|^{q(x)}\,dx \\ &\quad -\lambda\int_\Omega F(x,u_j)\,dx + \tfrac{\lambda}{p^+}\int_\Omega f(x,u_j)u_j\,dx \\ &\ge \lim_{j\to\infty}\int_\Omega\left(\tfrac{1}{p+} - \tfrac{1}{q(x)}\right)|u_j|^{q(x)}\,dx \\ &\ge \lim_{j\to\infty}\int_{\mathcal{A}_\delta}\left(\tfrac{1}{p+} - \tfrac{1}{q(x)}\right)|u_j|^{q(x)}\,dx \\ &\ge \lim_{j\to\infty}\int_{\mathcal{A}_\delta}\left(\tfrac{1}{p+} - \tfrac{1}{q^-_{\mathcal{A}_\delta}}\right)|u_j|^{q(x)}\,dx. \end{aligned}$$

But

$$\begin{aligned} \lim_{j\to\infty}\int_{\mathcal{A}_\delta}\left(\frac{1}{p+} - \frac{1}{q^-_{\mathcal{A}_\delta}}\right)|u_j|^{q(x)}\,dx &= \left(\tfrac{1}{p+} - \tfrac{1}{q^-_{\mathcal{A}_\delta}}\right)\left(\int_{\mathcal{A}_\delta}|u|^{q(x)}\,dx + \textstyle\sum_{j\in I}\nu_j\right) \\ &\ge \left(\tfrac{1}{p+} - \tfrac{1}{q^-_{\mathcal{A}_\delta}}\right)\nu_i \\ &\ge \left(\tfrac{1}{p+} - \tfrac{1}{q^-_{\mathcal{A}_\delta}}\right)S^N. \end{aligned}$$

As $\delta > 0$ is arbitrary, and q is continuous, we get

$$c \ge \left(\frac{1}{p+} - \frac{1}{q^-_{\mathcal{A}}}\right)S^N.$$

Therefore, if

$$c < \left(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^{-}}\right) S^N,$$

the index set I is empty.

Now we are ready to prove the Palais-Smale condition below level c.

Lemma 4.6 *Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, with energy level c. If $c < \left(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^{-}}\right) S^N$, then there exist $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_{j_k}\}_{k\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ a subsequence such that $u_{j_k} \to u$ strongly in $W_0^{1,p(x)}(\Omega)$.*

Proof. We have that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. Then, for a subsequence that we still denote $\{u_j\}_{j\in\mathbb{N}}$, $u_j \to u$ strongly in $L^{q(x)}(\Omega)$. We define $\Phi'(u_j) := \phi_j$. By the Palais-Smale condition, with energy level c, we have $\phi_j \to 0$ in $(W_0^{1,p(x)}(\Omega))'$.

By definition $\langle \Phi'(u_j), z\rangle = \langle \phi_j, z\rangle$ for all $z \in W_0^{1,p(x)}(\Omega)$, i.e,

$$\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla z\, dx - \int_\Omega |u_j|^{q(x)-2} u_j z\, dx - \int_\Omega \lambda f(x,u_j) z\, dx = \langle \phi_j, z\rangle.$$

Then, u_j is a weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u_j = |u_j|^{q(x)-2} u_j + \lambda f(x,u_j) + \phi_j =: f_j \text{in } \Omega, \\ u_j = 0 \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

We define $T\colon (W_0^{1,p(x)}(\Omega))' \to W_0^{1,p(x)}(\Omega)$, $T(f) := u$ where u is the weak solution of the following equation.

$$\begin{cases} -\Delta_{p(x)} u = f \text{in } \Omega, \\ u = 0 \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Then T is a continuous invertible operator.

It is sufficient to show that f_j converges in $(W_0^{1,p(x)}(\Omega))'$. We only need to prove that $|u_j|^{q(x)-2}u_j \to |u|^{q(x)-2}u$ strongly in $(W_0^{1,p(x)}(\Omega))'$. In fact,

$$\begin{aligned} \langle |u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u, \psi\rangle &= \textstyle\int_\Omega (|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\psi\, dx \\ &\le \|\psi\|_{L^{q(x)}(\Omega)} \|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{L^{q'(x)}(\Omega)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{(W_0^{1,p(x)}(\Omega))'} \\ &= \sup_{\{\psi\in W_0^{1,p(x)}(\Omega) : \|\psi\|_{W_0^{1,p(x)}(\Omega)}=1\}} \textstyle\int_\Omega (|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\psi\, dx \\ &\le \|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{L^{q'(x)}(\Omega)} \end{aligned}$$

and now, by the Dominated Convergence Theorem this last term goes to zero as $j \to \infty$.

Now, we can prove the Palais-Smale condition for the restricted functional.

Lemma 4.7 *The functional $\Phi|_{K_i}$ satisfies the Palais-Smale condition for energy level c for every $c < \left(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\right) S^N$.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \to 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i.

Let $v_j \in T_{u_j} W_0^{1,p(x)}(\Omega)$ be a unit tangential vector such that

$$\langle \nabla\Phi(u_j), v_j\rangle = \|\nabla\Phi(u_j)\|_{(W_0^{1,p(x)}(\Omega))'}.$$

Now, by Lemma 4.4, $v_j = w_j + z_j$ with $w_j \in T_{u_j} M_i$ and $z_j \in \text{span}\{(u_j)_+, (u_j)_-\}$.

Since $\Phi(u_j)$ is uniformly bounded, by Lemma 4.2, u_j is uniformly bounded in $W_0^{1,p(x)}(\Omega)$ and hence w_j is uniformly bounded in $W_0^{1,p(x)}(\Omega)$. Therefore

$$\|\nabla\Phi(u_j)\|_{(W_0^{1,p(x)}(\Omega))'} = \langle \nabla\Phi(u_j), v_j\rangle = \langle \nabla\Phi|_{K_i}(u_j), v_j\rangle \to 0.$$

As w_j is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \to 0$ strongly, the inequality converges strongly to 0. Now the result follows by Lema 4.6.

The following lemma now follows easily.

Lemma 4.8 *Let $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Then u is also a critical point of the unrestricted functional Φ and hence a weak solution to (1.1).*

Proof. To prove the Theorem, we need to check that the functional $\Phi\ |_{K_i}$ verifies the hypotheses of the Ekeland's Variational Principle [2]. The fact that Φ is bounded below over K_i is a direct consequence of the construction of the manifold K_i. Then, by Ekeland's Variational Principle, there existe $v_k \in K_i$ such that

$$\Phi(v_k) \to c_i := \inf_{K_i} \Phi \qquad \text{and} \qquad (\Phi\ |_{K_i})'(v_k) \to 0.$$

We have to check that if we choose λ large, we have that $c_i < \left(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\right) S^N$,. This follows easily from Lemma 4.1. For instance, for c_1, we have that choosing $w_0 \geq 0$, if $t_\lambda < 1$

$$c_1 \leq \Phi(t_\lambda w_0) \leq \frac{1}{p^-} t_\lambda^{p^+} \int_\Omega |\nabla w_0|^{p(x)}\, dx$$

Hence $c_1 \to 0$ as $\lambda \to \infty$. Moreover, it follows from the estimate of t_λ in Lemma 4.1, that $c_i < \left(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\right) S^n$ for $\lambda > \lambda^*(p, q, n, c_3)$. The other cases are similar.

From Lemma 4.7, it follows that v_k has a convergent subsequence, that we still call v_k. Therefore Φ has a critical point in K_i, $i = 1, 2, 3$ and, by construction, one of them is positive, other is negative and the last one changes sign.

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